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Coloring d-Embeddable k-Uniform Hypergraphs

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Abstract This paper extends the scenario of the Four Color Theorem in the following way. Let $\mathscr{H}_{d,k}$ be the set of all k-uniform hypergraphs that can be (linearly) embedded into $\mathbb{R}^d$. We investigate lower and upper bounds on the maximum (weak) chromatic number of hypergraphs in $\mathscr{H}_{d,k}$. For example, we can prove that for $d \geq 3$ there are hypergraphs in $\mathscr{H}_{2d-3,d}$ on n vertices whose chromatic number is $\Omega(\log n/\log\log n)$, whereas the chromatic number for n-vertex hypergraphs in $\mathscr{H}_{d,d}$ is bounded by $\mathcal{O}(n^{(d-2)/(d-1)})$ for $d \geq 3$.

Keywords Hypergraphs · Coloring · Chromatic number · Embeddings · Four Color Theorem

1 Introduction

The Four Color Theorem [1,2] asserts that every graph that is embeddable in the plane has chromatic number at most four. This question has been one of the driving forces in Discrete Mathematics and its theme has inspired many variations. For example, the

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chromatic number of graphs that are embedabble into a surface of fixed genus has been intensively studied by Heawood [17], Ringel and Youngs [24], and many others.

In this paper, we consider k-uniform hypergraphs that are embeddable into $\mathbb{R}^d$ in such a way that their edges do not intersect (see Definition 1 below). For $k = d = 2$ the problem specializes to graph planarity. For $k = 2$ and $d \geq 3$ it is not a very interesting question because for any $n \in \mathbb{N}$ the vertices of the complete graph K_n can be embedded into $\mathbb{R}^3$ using the embedding

$$\varphi(v_i) = \left(i, i^2, i^3\right) \quad \forall i \in \{1, \ldots, n\}. \tag{1}$$

It is a well known property of the moment curve $t \mapsto (t, t^2, t^3)$ that any two edges between four distinct vertices do not intersect (see Proposition 14).

As a consequence, we now focus our attention on hypergraphs, which are in general not embeddable into any specific dimension. Some properties of these hypergraphs (or more generally simplicial complexes) have been investigated (see e. g. [10,11,19,20,27,31]), but to our surprise, we have not been able to find any previously established results which bound their chromatic number. However, Grünbaum and Sarkaria (see [15,26]) have considered a different generalization of graph colorings to simplicial complexes by coloring faces. They also bound this face-chromatic number subject to embeddability constraints.

Before we can state our main results, we quickly recall and introduce some useful notation. We say that $H = (V, E)$ is a *k-uniform hypergraph* if the vertex set V is a finite set and the edge set E consists of k-element subsets of V, i. e. $E \subseteq \binom{V}{k}$. For any hypergraph H, we denote by $V(H)$ the vertex set of H and by $E(H)$ its edge set. We define

$$K_n^{(k)} := \left(\{1, 2, \ldots, n\}, \binom{\{1, 2, \ldots, n\}}{k}\right)$$

and call any hypergraph isomorphic to $K_n^{(k)}$ a *complete k-uniform hypergraph of order n*.

Let H be a k-uniform hypergraph. A function $\kappa : V(H) \to \{1, \ldots, c\}$ is said to be a *weak c-coloring* if for all $e \in E(H)$ the property $|\kappa(e)| > 1$ holds. The function κ is said to be a *strong c-coloring* if $|\kappa(e)| = k$ for all $e \in E(H)$. The *weak/strong chromatic number* of H is defined as the minimum $c \in \mathbb{N}$ such that there exists a weak/strong coloring of H with c colors. The chromatic number of H is denoted by $\chi^{\mathsf{w}}(H)$ and $\chi^{\mathsf{s}}(H)$, respectively. Obviously, for graphs, weak and strong colorings are equivalent.

We next define what we mean when we say that a hypergraph is embeddable into $\mathbb{R}^d$. Here, aff denotes the affine hull of a set of points and conv the convex hull.

Definition 1 (*d-Embeddings*) Let H be a k-uniform hypergraph and $d \in \mathbb{N}$. A (linear) *embedding* of H into $\mathbb{R}^d$ is a function $\varphi : V(H) \to \mathbb{R}^d$, where $\varphi(A)$ for $A \subseteq V(H)$ is to be interpreted pointwise, such that

- $\dim \operatorname{aff} \varphi(e) = k - 1$ for all $e \in E(H)$ and
- $\operatorname{conv} \varphi(e_1) \cap \operatorname{conv} \varphi(e_2) = \operatorname{conv} \varphi(e_1 \cap e_2)$ for all $e_1, e_2 \in E(H)$.

Table 1 Currently known lower bounds for the maximum weak chromatic number of a d-embeddable k-uniform hypergraph on n vertices as $n \to \infty$

$d \diagdown k$	2	3	4	5	6	7
1	2	1	1	1	1	1
2	4	2	1	1	1	1
3	n	$\Omega\left(\frac{\log n}{\log\log n}\right)\langle 21\rangle$	1	1	1	1
4	n	$\Omega\left(\frac{\log n}{\log\log n}\right)\langle 21\rangle$	1	1	1	1
5	n	$\lceil n/2\rceil$	$\Omega\left(\frac{\log n}{\log\log n}\right)\langle 22\rangle$	1	1	1
6	n	$\lceil n/2\rceil$	$\Omega\left(\frac{\log n}{\log\log n}\right)\langle 22\rangle$	1	1	1
7	n	$\lceil n/2\rceil$	$\lceil n/3\rceil$	$\Omega\left(\frac{\log n}{\log\log n}\right)\langle 22\rangle$	1	1
8	n	$\lceil n/2\rceil$	$\lceil n/3\rceil$	$\Omega\left(\frac{\log n}{\log\log n}\right)\langle 22\rangle$	1	1

The number in chevrons indicates the theorem number where we prove this bound

The first property is needed to exclude functions mapping the vertices of one edge to affinely non-independent points. The second guarantees that the embedded edges only intersect in the convex hull of their common vertices. Note that the inclusion from left to right always holds. A k-uniform hypergraph H is said to be *d-embeddable* if there exists an embedding of H into $\mathbb{R}^d$. Also, we denote by $\mathscr{H}_{d,k}$ the set of all d-embeddable k-uniform hypergraphs.

One can easily see that our definition of 2-embeddability coincides with the classical concept of planarity [12]. Note that in general there are several other notions of embeddability. The most popular thereof are piecewise linear embeddings and general topological embeddings. A short and comprehensive introduction is given in Sect. 1 in [19]. Furthermore, there exist some quite different concepts of generalizing embeddability for hypergraphs in the literature, for example *hypergraph imbeddings* [32, Chap. 13].

We have decided to focus on linear embeddings, as they lead to a very accessible type of geometry and, at least in theory, the decision problem of whether a given k-uniform hypergraph is d-embeddable is decidable and in PSPACE [23]. One can show that the aforementioned three types of embeddings are equivalent only in the less than 3-dimensional case (see e. g. [3,4]), although piecewise linear and topological embeddability coincides if $d - k \geq 2$ or $(d, k) = (3, 3)$, see [5]. Since piecewise linear and topological embeddings are more general than linear embeddings, all lower bounds for chromatic numbers can easily be transferred. Furthermore, we prove all our results on upper bounds for piecewise linear embeddings (and thus also for topological embeddings if $d - k \geq 2$ or $(d, k) = (3, 3)$) except for one case (namely Theorem 20).

We can now give a summary of our main results in Tables 1 and 2, which contain upper or lower bounds for the maximum weak chromatic number of a d-embeddable k-uniform hypergraph on n vertices. All results which only follow non-trivially from prior knowledge are indexed with a theorem number from which they can be derived.

Table 2 Currently known upper bounds for the maximum weak chromatic number of a d-embeddable k-uniform hypergraph on n vertices as $n \to \infty$

$d \diagdown k$	2	3	4	5	6	7
1	2	1	1	1	1	1
2	4	2	1	1	1	1
3	n	$\mathcal{O}(n^{1/2})_{\langle 18\rangle}$	$\mathcal{O}(n^{1/2})_{\langle 18\rangle}$	1	1	1
4	n	$\lceil n/2\rceil$	$\mathcal{O}(n^{2/3})_{\langle 18\rangle}$	$\mathcal{O}(n^{1/2})_{\langle 20\rangle}$	1	1
5	n	$\lceil n/2\rceil$	$\mathcal{O}(n^{26/27})_{\langle 19\rangle}$	$\mathcal{O}(n^{3/4})_{\langle 18\rangle}$	$\mathcal{O}(n^{3/5})_{\langle 20\rangle}$	1
6	n	$\lceil n/2\rceil$	$\lceil n/3\rceil$	$\mathcal{O}(n^{35/36})_{\langle 19\rangle}$	$\mathcal{O}(n^{4/5})_{\langle 18\rangle}$	$\mathcal{O}(n^{1/2})_{\langle 20\rangle}$
7	n	$\lceil n/2\rceil$	$\lceil n/3\rceil$	$\mathcal{O}(n^{107/108})_{\langle 19\rangle}$	$\mathcal{O}(n^{44/45})_{\langle 19\rangle}$	$\mathcal{O}(n^{5/6})_{\langle 18\rangle}$
8	n	$\lceil n/2\rceil$	$\lceil n/3\rceil$	$\lceil n/4\rceil$	$\mathcal{O}(n^{134/135})_{\langle 19\rangle}$	$\mathcal{O}(n^{53/54})_{\langle 19\rangle}$

The number in chevrons indicates the theorem number where we prove this bound

Considering the strong chromatic number, the question whether embeddability restricts the number of colors needed can be answered negatively by the following observation.

Let $n, d \in \mathbb{N}$ such that $d \geq 3$ and $n \geq d+1$ and let $V = \{1, \dots, n\}$. Let $\varphi : \mathbb{R} \to \mathbb{R}^d$, $\varphi(x) = (x, \dots, x^{d+1})$ be the $(d+1)$-dimensional moment curve. Then $\varphi(V)$ are the vertices of a cyclic polytope $P = \operatorname{conv} \varphi(V)$ (see [6,7,21]). As $d \geq 3$, we have that P is 2-neighborly [13]. Define $H(P) = (V, E(P))$ to be the $(d+1)$-uniform hypergraph with $E(P) = \{e \subseteq V : e \text{ is the set of vertices of a facet of } P\}$. Then $H(P)$ can be linearly embedded into $\mathbb{R}^d$: for example, one can take the Schegel-Diagram [28] of P with respect to some facet.

Now, choose $k \in \mathbb{N}$ such that $2 \leq k \leq d+1$. Following [14, §7.1], for any hypergraph $H = (W, E)$, we call

$$\mathscr{S}_k(H) = (W, \{\{w_1, \dots, w_k\} : \{w_1, \dots, w_k\} \subseteq e \text{ for some } e \in E\})$$

the *k-shadow* of H. As P is 2-neighborly we have that $\mathscr{S}_2(H(P)) = K_n$ and thus $\chi^{\mathsf{S}}(H(P)) = n$. Obviously, $\mathscr{S}_2(\mathscr{S}_k(H(P))) = K_n$ and $\chi^{\mathsf{S}}(\mathscr{S}_k(H(P))) = n$, too. Thus, we have demonstrated that for any $2 \leq k \leq d+1 \leq n$ there exists a k-uniform hypergraph on n vertices that is linearly d-embeddable and has strong chromatic number n.

Thus, from now on, we restrict ourselves to the weak case and will always mean this when talking about a chromatic number. To conclude the introduction, here is a rough outline for the rest of the paper. In Sect. 2 the general concept of embedding hypergraphs into d-dimensional space is discussed. We also show the embeddability of certain structures needed later on, hereby extensively using known properties of the moment curve $t \mapsto (t, t^2, t^3, \dots, t^d)$. Then, Sect. 3 presents our current level of knowledge for the more difficult problem of weakly coloring hypergraphs.

2 Embeddability

The first part of this section gives insight into the structure of neighborhoods of single vertices in a hypergraph $H \in \mathscr{H}_{d,k}$. We will later use this information to prove upper bounds on the number of edges in our hypergraphs. This will then yield upper bounds on the weak chromatic number. However, we must first take a small technical detour into piecewise linear embeddings. As our hypergraphs are finite and of fixed uniformity we give a slightly simplified definition (for a more comprehensive introduction, see e. g. [25]).

Definition 2 (*Piecewise linear d-embeddings*) Let H be a k-uniform hypergraph and $D, d \in \mathbb{N}$. Let $\varphi : V(H) \to \mathbb{R}^D$ be a linear embedding of H and define $\varphi(H) = \bigcup_{e \in E(H)} \operatorname{conv} \varphi(e)$.

We say H is *piecewise linearly embeddable* if there exists $\psi : \varphi(H) \to \mathbb{R}^d$ such that ψ is a homeomorphism from $\varphi(H)$ onto its image and there exists a (locally finite) subdivision K of $\varphi(H)$ (seen as a geometric simplicial complex) such that ψ is affine on all elements of K. We call ψ a *piecewise linear embedding* of H into $\mathbb{R}^d$ and we denote by $\mathscr{H}^{\mathsf{PL}}_{d,k}$ the set of all piecewise linearly d-embeddable k-uniform hypergraphs.

Note that such a φ always exists, as $H \in \mathscr{H}_{2k-1,k}$ by the Menger–Nöbeling Theorem (see [20, p. 295] and [22]). Also, Definition 2 is independent of the choice of φ.

Definition 3 (*Neighborhoods*) For a k-uniform hypergraph H and a vertex $v \in V(H)$ we say the *neighborhood* of v is $N_H(v)=\{w \in V(H) : w \neq v$ and there is an edge in $E(H)$ incident with w and $v\}$. We define the *neighborhood hypergraph* (or link) of $v \in V(H)$ to be the induced $(k-1)$-uniform hypergraph

$$\mathrm{NH}_H(v) = (N_H(v), \{e \backslash \{v\} : e \in E(H), v \in e\}) .$$

The *degree* $\deg_H(v) = \deg(v)$ is the number of edges in $E(H)$ incident with v.

Lemma 4 *For a hypergraph $H \in \mathscr{H}^{\mathsf{PL}}_{d,k}$ on n vertices, $d \geq k \geq 2$, and for any vertex v we have that $\mathrm{NH}_H(v) \in \mathscr{H}^{\mathsf{PL}}_{d-1,k-1}$.*

Proof Let $d \geq k \geq 2$, $H \in \mathscr{H}^{\mathsf{PL}}_{d,k}$, $v \in V(H)$, and $V_v = N_H(v)$ nonempty. Then there exist $\varphi : V(H) \to \mathbb{R}^{2k-1}$ a linear embedding and $\psi : \varphi(H) \to \mathbb{R}^d$ a piecewise linear embedding of H for some subdivision K of $\varphi(H)$ on whose elements ψ is affine. Without restriction assume that $\varphi(v) = \mathbf{0}_{2k-1}$ and $\psi(\mathbf{0}_{2k-1}) = \mathbf{0}_d$.

Let $H_v = (V_v \cup \{v\}, \{e \in E(H) : v \in e\})$ be the sub-hypergraph of H of all edges containing v. Obviously, $\psi|\varphi(H_v)$ (the restriction of ψ onto $\varphi(H_v)$) is a piecewise linear embedding of H_v for some subdivision $K_v \subseteq K$. Let $K_v^1 = \{e \in K_v : \mathbf{0}_{2k-1} \in e\}$. Then there exists an $\varepsilon > 0$ such that

$$\varepsilon \cdot \varphi(H_v) \subseteq \bigcup_{e \in K_v^1} e,$$

i. e. all points in $\varepsilon \cdot \varphi(H_v)$ are so close to $\mathbf{0}_{2k-1}$ that they lie completely in elements of K_v that contain the origin.

Then $\varphi' : V_v \cup \{v\} \rightarrow \mathbb{R}^{2k-1}, w \mapsto \varepsilon \cdot \varphi(w)$ is a linear and thus $\psi|\varphi'(H_v)$ a piecewise linear embedding of H_v for the subdivision $K_v^2 = \{e \cap \varphi'(H_v) : e \in K_v^1\}$. Let $V_{K_v^2} \supseteq \varphi'(V_v)$ be the set of all subdivision points of K_v^2 *without* $\mathbf{0}_{2k-1}$ and let

$$\delta = \min\left\{ \|\psi(x)\| : x \in \text{conv}(e \cap V_{K_v^2}) \text{ for some } e \in K_v^2 \right\}.$$

Obviously, we have that $\delta > 0$. We take a regular d-simplex $T \subseteq \mathbb{R}^d$ centered at the origin with sides of length δ and set $C = \partial T$. Due to our choice of δ, all $\psi(w)$ for $w \in V_{K_v^2}$ lie outside of T. Further, for all $e \in K_v^2$ the intersection $\psi(e) \cap C$ is the union of finitely many at most $(k-2)$-dimensional simplices and homeomorphic to a $(k-2)$-dimensional simplex. Also, as $d \geq k$, there exists a point $x \in C$ such that $x \notin \psi(e)$ for all $e \in K_v^2$.

Thus, there exists a subdivision K_v^3 of K_v^2 such that for all $e \in K_v^3$ with dimension $k-1$ we have that $\psi(e) \cap C$ is a $(k-2)$-dimensional simplex and still $\mathbf{0}_{2k-1} \in e$. We denote the set of subdivision points *without* $\mathbf{0}_{2k-1}$ by $V_{K_v^3} \supseteq V_{K_v^2}$. Now, one can find a retraction $\rho : \psi(\varphi'(H_v)) \rightarrow \psi(\varphi'(H_v))$ that maps each $\psi(w)$, $w \in V_{K_v^3}$, to the intersection point of the line segment $[\mathbf{0}_d, \psi(w)]$ with C, such that ρ is linear on all $\psi(e)$ for $e \in K_v^3$.

Set $\hat{K} = \{\text{conv}(e \cap V_{K_v^3}) : e \in K_v^3\}$ which is now a subdivision of $\varphi'(\text{NH}_H(v)) \subseteq \varphi'(H_v)$. Then the image of $\rho \circ (\psi|\hat{K})$ lies completely in $C \backslash \{x\}$.

Finally, note that $C \backslash \{x\}$ is piecewise linearly homeomorphic to $\mathbb{R}^{d-1}$ [25, 3.20]. Let γ be such a (piecewise linear) homeomorphism . Then

$$\hat{\psi} = \gamma \circ \rho \circ (\psi|\varphi'(\text{NH}_H(v)))$$

is a piecewise linear embedding of $\text{NH}_H(v)$ into $\mathbb{R}^{d-1}$ for some subdivision of $\hat{K}$ and $\text{NH}_H(v) \in \mathscr{H}_{d-1,k-1}^{\mathsf{PL}}$. □

Note that it is quite plausible that a version of Lemma 4 for linear or general embeddings does not hold. Part (a) of the following result has previously been established by Dey and Pach for linear embeddings [8, Theorem 3.1].

Lemma 5 (a) *For a hypergraph $H \in \mathscr{H}_{k,k}^{\mathsf{PL}}$ on n vertices, $k \geq 2$, we have that*

$$|E(H)| \leq \frac{6n^{k-1} - 12n^{k-2}}{k!}.$$

(b) *For a hypergraph $H \in \mathscr{H}_{k+1,k+1}^{\mathsf{PL}}$ on n vertices, $k \geq 2$, and for any vertex v we have that*

$$\deg_H(v) \leq \frac{6n^{k-1} - 12n^{k-2}}{k!}.$$

Proof If $k = 2$, then (a) is equivalent to the fact that for G planar $|E(G)| \leq 3n - 6$. Given that (a) is true for some $k \geq 2$, we show that (b) holds for k as well. Let $H \in \mathscr{H}_{k+1,k+1}^{\mathsf{PL}}$, v one of the n vertices. By Lemma 4, $\text{NH}_H(v) \in \mathscr{H}_{k,k}^{\mathsf{PL}}$. By (a),

$$|E(\mathrm{NH}_H(v))| \leq \frac{6n^{k-1} - 12n^{k-2}}{k!}$$

which implies

$$\deg_H(v) \leq \frac{6n^{k-1} - 12n^{k-2}}{k!}.$$

Given that (b) is true for some $k \geq 2$, we show that (a) holds for $k+1$. Let $H \in \mathscr{H}^{\mathsf{PL}}_{k+1,k+1}$. Since (b) is true for every vertex v_i, we have

$$|E(H)| = \frac{\sum_{i=1}^{n} \deg_H(v_i)}{k+1} \leq \frac{n(6n^{k-1} - 12n^{k-2})}{(k+1)k!} = \frac{6n^k - 12n^{k-1}}{(k+1)!}.$$

□

Corollary 6 *For a hypergraph $H \in \mathscr{H}^{\mathsf{PL}}_{k,k}$ on n vertices, $k \geq 3$, and for any edge $e \in E(H)$ there exist at most $k\frac{6n^{k-2}-12n^{k-3}}{(k-1)!} - k$ other edges adjacent to it.*

Proof This follows from Lemma 5, since every edge has exactly k vertices and each of them has degree at most $\frac{6n^{k-2}-12n^{k-3}}{(k-1)!}$. As e itself counts for the degree as well, one can subtract k. □

We need to bound the number of edges in a d-embeddable hypergraph to prove upper bounds for the chromatic number. The following results will also help to do this. Note that there exist much stronger conjectured bounds (see [16, Conjecture 1.4.4] and [18, Conjecture 27]).

Proposition 7 (Gundert [16, Proposition 3.3.5]) *Let $k \geq 2$. For a k-uniform hypergraph on n vertices that is* topologically *embedabble into $\mathbb{R}^{2k-2}$, we have that $|E(H)| < n^{k-3^{1-k}}$.*

Corollary 8 *For a hypergraph $H \in \mathscr{H}^{\mathsf{PL}}_{2k-\ell,k}$ on n vertices, $k \geq \ell \geq 2$, we have that*

$$|E(H)| < \frac{(k-\ell+2)!}{k!} \cdot n^{k-3^{\ell-1-k}}.$$

Proof This follows from inductively applying Lemma 4 and Proposition 7. □

Corollary 9 *For a hypergraph $H \in \mathscr{H}^{\mathsf{PL}}_{2k-\ell,k}$ on n vertices, $k \geq \ell \geq 3$, and for any edge $e \in E(H)$ there exist at most $\frac{(k-\ell+2)!}{(k-1)!} \cdot n^{k-1-3^{\ell-1-k}} - k$ other edges adjacent to it.*

Proof This fact follows analogously to Corollary 6 from Corollary 8. □

Theorem 10 (Dey and Pach [8, Theorem 2.1]) *Let $k \geq 2$. For a k-uniform hypergraph on n vertices that is* linearly *embedabble into $\mathbb{R}^{k-1}$, we have that $|E(H)| < kn^{\lceil (k-1)/2 \rceil}$.*

Corollary 11 *For a hypergraph $H \in \mathscr{H}_{k-1,k}$ on n vertices, $k \geq 2$, and for any edge $e \in E(H)$ there exist at most $kn^{\lceil (k-1)/2 \rceil} - 1$ other edges adjacent to it.*

Proof This fact follows obviously from Theorem 10. □

In order to find lower bounds for the chromatic number of hypergraphs later on, we need to be able to prove embeddability. The following theorem from Shephard will turn out to be very useful when embedding vertices of a hypergraph on the moment curve.

Theorem 12 (Shephard [29]) *Let $W = \{w_1, \ldots, w_m\} \subseteq \mathbb{R}^d$ be distinct points on the moment curve in that order and $P = \text{conv}\, W$. We call a q-element subset $\{w_{i_1}, w_{i_2}, \ldots, w_{i_q}\} \subseteq W$ with $i_1 < i_2 < \cdots < i_q$* contiguous *if $i_q - i_1 = q - 1$. Then $U \subseteq W$ is the set of vertices of a $(k-1)$-face of P if and only if $|U| = k$ and for some $t \geq 0$*

$$U = Y_S \cup X_1 \cup \cdots \cup X_t \cup Y_E,$$

where all X_i, Y_S, and Y_E are contiguous *sets, $Y_S = \emptyset$ or $w_1 \in Y_S$, $Y_E = \emptyset$ or $w_m \in Y_E$, and at most $d - k$ sets X_i have odd cardinality.*

Shephard's Theorem thus says that the absolute position of points on the moment curve is irrelevant and only their relative order is important. Furthermore, note that all points in W are vertices of P. The following corollary helps in proving that two given edges of a hypergraph intersect properly.

Corollary 13 *In the setting of Theorem 12 assume that $W = U_1 \cup U_2$ where U_1 and U_2 are embedded edges of a k-uniform hypergraph. Then these edges do not intersect in a way forbidden by Definition 1, if there exists $j \in \{1, 2\}$ such that*

$$U_j = Y_S \cup X_1 \cup \cdots \cup X_t \cup Y_E$$

holds where at most $d - k$ of the contiguous sets X_i have odd cardinality.

Proof The two edges U_1 and U_2 do not intersect in a way forbidden by Definition 1 if at least one of them is a face of $P = \text{conv}\, W$, which is the case for U_j. □

Proposition 14 *Let A, B, C, and D be four distinct points on the moment curve in $\mathbb{R}^3$ in arbitrary order. Then the line segments AB and CD do not intersect.*

Proof This follows immediately from Corollary 13 for the case $k = 2$ and $d = 3$. □

In the $k = d = 3$ case Corollary 13 allows zero odd sets X_i. Thus, we can easily classify all possible configurations for two edges.

Lemma 15 *Let H be a 3-uniform hypergraph and $\varphi : V(H) \to \mathbb{R}^3$ such that φ maps all vertices one-to-one on the moment curve and for each pair of edges e and f sharing at most one vertex, the order of the points $\varphi(e \cup f)$ on the moment curve has one of the Configurations 1–12 shown in Table 3. Then φ is an embedding of H.*

Table 3 Possible configurations for two edges e and f on the moment curve in $\mathbb{R}^3$ sharing at most one vertex

The vertices of $e \backslash f$ are marked with E, those of $f \backslash e$ marked with F, and a joint vertex is marked with I. Equivalent cases, one being the reverse of the other, are only displayed once

No.	Configuration	No.	Configuration
1	E E E F F F	9	E I F F E
2	E E F F E F	10	E F I E F
3	E E F F F E	11	E F E I F
4	E F F E E F	12	E E F E F F
5	E F F E I	13	E F E F E F
6	E E I F F	14	E F E F F E
7	E I E F F	15	E F E F I
8	E E F F I	16	E F I F E

Proof Note that the relative order of edges with two common vertices is irrelevant as they always intersect according to Definition 1. Configurations 1–11 follow directly from Corollary 13 for $k = d = 3$. Thus, we are left with Configuration 12 and it is sufficient to prove the following: For $x_{0,0} < x_{1,0} < x_{0,1} < x_{2,0} < x_{1,1} < x_{2,1} \in \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}^3$, $\psi(x) = (x, x^2, x^3)$ the moment curve, and $D_i = \{x_{0,i}, x_{1,i}, x_{2,i}\}$ we have that $\operatorname{conv} \psi(D_0) \cap \operatorname{conv} \psi(D_1) = \varnothing$. Assume otherwise. Note that if two triangles intersect in $\mathbb{R}^3$ the intersection points must contain at least one point of the border of at least one of the triangles. Thus, without loss of generality, $\operatorname{conv}\{\psi(x_{j_1,0}), \psi(x_{j_2,0})\} \cap \operatorname{conv} \psi(D_1) \neq \varnothing$. However, by Theorem 12 we know that $\operatorname{conv}\{\psi(x_{j_1,0}), \psi(x_{j_2,0})\}$ is a face of the polytope $P = \operatorname{conv}(\{\psi(x_{j_1,0}), \psi(x_{j_2,0})\} \cup \psi(D_1))$ which is a contradiction. □

Note that if we have two edges with vertices on the moment curve as in Configurations 13–16 they generally *do* intersect in a way forbidden by Definition 1. Also, we have presented above all possible cases for the relative order of vertices of two edges on the moment curve. Not all of them will actually be needed in the proofs of the next section.

3 Bounding the Weak Chromatic Number

For $d, k, n \in \mathbb{N}$ we define

$$\chi^{\mathsf{w}}_{d,k}(n) = \max\{\chi^{\mathsf{w}}(H) : H \in \mathscr{H}_{d,k}, |V(H)| = n\}$$

to be the maximum weak chromatic number of a d-embeddable k-uniform hypergraph on n vertices.

In this section, we give lower and upper bounds on $\chi^{\mathsf{w}}_{d,k}(n)$. Obviously, $\chi^{\mathsf{w}}_{d,k}(n)$ is monotonically increasing in n and in d and monotonically decreasing in k if the other parameters remain fixed.

Remark 16 (a) For $k = 2$, the results in Tables 1 and 2 follow from the Four Color Theorem and the fact that all graphs are d-embeddable for $d \geq 3$.

(b) For $d \geq 2k - 1$, we have $\chi^{\mathsf{w}}_{d,k}(n) = \lceil n/(k-1) \rceil$ as $K_n^{(k)}$ is $(2k-1)$-embeddable for all $k \in \mathbb{N}$ by the Menger–Nöbeling Theorem (see [20, p. 295] and [22]) and $\chi^{\mathsf{w}}\big(K_n^{(k)}\big) = \lceil n/(k-1) \rceil$.

(c) For $d \le k-2$, we again know $\chi^{\mathsf{w}}_{d,k}(n) = 1$ as $H \in \mathscr{H}_{d,k}$ cannot have any edge.

Proposition 17 *For all $n \ge 3$ we have $\chi^{\mathsf{w}}_{2,3}(n) \le 2$. (This bound is obviously sharp.)*

Proof Let $H \in \mathscr{H}_{2,3}$ and $V = V(H)$. Then $G = \mathscr{S}(H)$ is a planar graph, thus $\chi(G) \le 4$. Let $\kappa : V \to \{1,2,3,4\}$ be a 4-coloring of G. Define

$$\kappa' : V \to \{1,2\}, v \mapsto (\kappa(v) \bmod 2) + 1.$$

In any triangle $\{u, v, w\}$ of H under the coloring κ these vertices have exactly three different colors. Therefore, under the coloring κ' at least one vertex with color 1 and one vertex with color 2 exists. Thus κ' is a valid 2-coloring of H. □

Theorem 18 *Let $d \ge 3$. Then one has*

$$\chi^{\mathsf{w}}_{d,d}(n) \le \left\lceil \left(\frac{6ed}{(d-1)!}\right)^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}} \right\rceil = \mathcal{O}\left(n^{\frac{d-2}{d-1}}\right) \quad \text{as } n \to \infty.$$

This result also holds for piecewise linear embeddings.

Proof Let $H \in \mathscr{H}^{\mathsf{PL}}_{d,d} \supseteq \mathscr{H}_{d,d}$. By Corollary 6 we know that every edge is adjacent to at most $\Delta = d(6n^{d-2} - 12n^{d-3})/(d-1)! - d$ other edges.

We want to apply the Lovász Local Lemma [9,30] to bound the weak chromatic number of H. Let $c \in \mathbb{N}$. In any c-coloring of the vertices of H an edge is called bad if it is monochromatic and good if not. In a uniformly random c-coloring the probability for any one edge to be bad is $p = \frac{1}{c^{d-1}}$. Moreover, let e be any edge in H and F be the set of edges in H not adjacent to e. Then the events of e being bad and of any edges from F being bad are independent. Thus the event whether any edge is bad is independent from all but at most Δ other such events.

The Lovász Local Lemma guarantees us that with positive probability all edges are good if $e \cdot p \cdot (\Delta + 1) \le 1$. This implies that H is weakly c-colorable. Note that

$$e \cdot p \cdot (\Delta + 1) \le 1 \Leftrightarrow \frac{ed(6n^{d-2} - 12n^{d-3})}{(d-1)!} - ed + e \le c^{d-1}.$$

Choosing an integer

$$c \ge \left(\frac{6ed}{(d-1)!}\right)^{\frac{1}{d-1}} n^{\frac{d-2}{d-1}} \ge \left(\frac{ed(6n^{d-2} - 12n^{d-3})}{(d-1)!} - ed + e\right)^{\frac{1}{d-1}},$$

the hypergraph H is c-colorable and $\chi^{\mathsf{w}}(H) \le c$. □

Theorem 19 *Let $d \ge \ell \ge 3$. Then one has*

$$\chi^{\mathsf{w}}_{2d-\ell,d}(n) \le \left\lceil \left(\frac{e(d-\ell+2)!}{(d-1)!}\right)^{\frac{1}{d-1}} n^{1-\frac{3^{\ell-1}-d}{d-1}} \right\rceil = \mathcal{O}\left(n^{1-\frac{3^{\ell-1}-d}{d-1}}\right) \quad \text{as } n \to \infty.$$

This result also holds for piecewise linear embeddings.

Proof By Corollary 9 we know that every edge is adjacent to at most $\Delta = \frac{(d-\ell+2)!}{(d-1)!} \cdot n^{d-1-3^{\ell-1-d}} - d$ other edges. The rest of the proof is now analogous to the proof of Theorem 18. □

Theorem 20 *Let $d \geq 2$. Then one has*

$$\chi^{\mathsf{w}}_{d-1,d}(n) \leq \left\lceil (ed)^{\frac{1}{d-1}} n^{\frac{\lceil (d-1)/2 \rceil}{d-1}} \right\rceil = \begin{cases} \mathcal{O}\left(n^{1/2}\right) & \text{if } d \text{ is odd} \\ \mathcal{O}\left(n^{1/2+1/(2d-2)}\right) & \text{if } d \text{ is even} \end{cases} \quad \text{as } n \to \infty.$$

Proof By Corollary 11 we know that every edge is adjacent to at most $\Delta = dn^{\lceil (d-1)/2 \rceil} - 1$ other edges. The rest of the proof is now analogous to the proof of Theorem 18. □

By monotonicity, the upper bounds presented here also hold if the uniformity of the hypergraph is larger than stated in Theorems 18 and 19. In the remaining part of this section, we now consider lower bounds for the weak chromatic number of hypergraphs.

Theorem 21 *For $n \geq 2$ we have*

$$\chi^{\mathsf{w}}_{3,3}(n) \geq \frac{\log n}{2 \log \log n} - 1 = \Omega\Big(\frac{\log n}{\log \log n}\Big) \quad \text{as } n \to \infty.$$

Proof We first define a sequence of hypergraphs H_m for $m \geq 2$ such that $\chi^{\mathsf{w}}(H_m) \geq m$. Set $H_2 = K_3^{(3)}$ which has 3 vertices. Define H_m for $m > 2$ iteratively, assuming $\chi^{\mathsf{w}}(H_{m-1}) \geq m-1$. Take m new vertices $\{v_0, \dots, v_{m-1}\}$ and $m(m-1)/2$ disjoint copies of H_{m-1}, labeled $H_{m-1}^{[0,1]}, \dots, H_{m-1}^{[m-2,m-1]}$.

The edges of H_m shall be all former edges of all $H_{m-1}^{[i,j]}$ together with all edges of the form $\{v_i, v_j, w\}$ where $i < j$ and $w \in H_{m-1}^{[i,j]}$. Assume H_m is weakly $(m-1)$-colorable. Given such a coloring, one color must occur twice in $\{v_0, \dots, v_{m-1}\}$. Say, these are the vertices v_{i_1} and v_{i_2} where $i_1 < i_2$. This color cannot occur anymore in the coloring of $H_{m-1}^{[i_1,i_2]}$. Thus, $H_{m-1}^{[i_1,i_2]}$ must be weakly $(m-2)$-colorable. This is a contradiction and H_m is at least (and obviously exactly) weakly m-chromatic.

We now claim that $H_m \in \mathscr{H}_{3,3}$ for all $m \geq 2$. For that, we give a function $f_m : V(H_m) \to \{1, \dots, n_m\}$ where n_m is the number of vertices of H_m. This function defines the order in which the vertices of H_m will be arranged on the moment curve $t \mapsto (t, t^2, t^3)$. Lemma 15 on possible configurations then guarantees that H_m is embeddable via arbitrary points on the moment curve. Note that the absolute position of vertices on the moment curve is not important, only their relative order.

The hypergraph $H_2 = K_3^{(3)}$ can be embedded into $\mathbb{R}^3$ via any three points on the moment curve, so $f_2 : V(H_2) \to \{1, 2, 3\}$ can be chosen arbitrarily. Assume that f_{m-1} has already been defined and that the vertices of H_{m-1} arranged in that order on the moment curve form an embedding. Look at the vertices of H_m as given before. We define $f_m(v_j) = n_{m-1} \cdot j(j-1)/2 + j$ for $0 \leq j \leq m-1$ and for any $w \in H_{m-1}^{[i,j]}$ with $i < j$ we set $f_m(w) = n_{m-1} \cdot (j(j-1)/2 + i) + j + f_{m-1}(w)$. This gives exactly the order shown in Fig. 1.

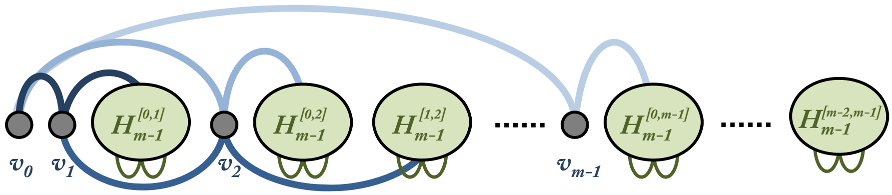


Fig. 1 Construction of H_m

Table 4 Sub-cases of Case 3 in the proof of Theorem 21 referring to the corresponding cases of Lemma 15

Relative order of v	Additional condition	Case number
$f_m(v) < f_m(w_1)$	–	1
$f_m(v) = f_m(w_1)$	–	6
$f_m(w_1) < f_m(v) < f_m(w_2)$	–	12
$f_m(v) = f_m(w_2)$	–	7
$f_m(w_2) < f_m(v) < f_m(w_3)$	–	2
$f_m(v) = f_m(w_3)$	–	8
$f_m(v) > f_m(w_3)$	$f_m(v_{i_1}) < f_m(w_3)$	3
$f_m(v) > f_m(w_3)$	$f_m(v_{i_1}) > f_m(w_3)$	1

Table 5 Sub-cases of Case 4 in the proof of Theorem 21 referring to the corresponding cases of Lemma 15

Relative order of i_1, i_2, j_1, j_2	Case number
$j_1 = j_2$ and $i_1 \neq i_2$	10
$j_1 = j_2$ and $i_1 = i_2$	Two shared vertices
$i_1 < j_1 < i_2 < j_2$	1
$i_1 < j_1 = i_2 < j_2$	7
$i_1 < i_2 < j_1 < j_2$	2
$i_1 = i_2 < j_1 < j_2$	8
$i_2 < i_1 < j_1 < j_2$	3

Now, arrange the vertices of H_m on the moment curve in that order and pick any two edges e_1 and e_2. By Lemma 15 we can assume that they do not share two vertices.

Case 1: e_1 and e_2 are from the same subhypergraph $H_{m-1}^{[i,j]}$. Then, by induction, they can only intersect according to Definition 1 as their relative order reflects that of f_{m-1}.

Case 2: e_1 and e_2 are from distinct subhypergraphs $H_{m-1}^{[i_1,j_1]}$ and $H_{m-1}^{[i_2,j_2]}$. Then we are in Case 1 in Table 3 and thus they intersect according to Definition 1.

Case 3: $e_1 = \{v_{i_1}, v_{j_1}, v\}$ where $v \in H_{m-1}^{[i_1,j_1]}$ and e_2 is from some subhypergraph $H_{m-1}^{[i_2,j_2]}$. Without loss of generality, let $e_2 = \{w_1, w_2, w_3\}$ and assume that $f_m(w_1) < f_m(w_2) < f_m(w_3)$. Then, by definition, $i_1 < j_1$ and $i_2 < j_2$ and all the possible cases of Lemma 15 are listed in Table 4.

Case 4: $e_1 = \{v_{i_1}, v_{j_1}, v\}$ and $e_2 = \{v_{i_2}, v_{j_2}, w\}$. Again, $i_1 < j_1$ and $i_2 < j_2$ holds. Without loss of generality assume $j_1 \leq j_2$. We then have one of the cases listed in Table 5.

Thus, the order given by f_m provides an embedding of H_m. To estimate n_m, we use the following recursion

$$n_2 = 3,$$
$$n_m = m + n_{m-1} \cdot m(m-1)/2 \quad \text{for } m > 2.$$

This can be bounded by $n_m \leq m^{2m} =: \hat{n}_m$. Then

$$\frac{\log \hat{n}_m}{\log\log \hat{n}_m} = 2m \cdot \frac{\log m}{\log(2m \log m)} \leq 2m$$

and we finally get that

$$m \geq \frac{\log \hat{n}_m}{2\log\log \hat{n}_m} \geq \frac{\log n_m}{2\log\log n_m}.$$

□

Note that by monotonicity also

$$\chi^{\mathsf{w}}_{4,3}(n) = \Omega\big(\frac{\log n}{\log\log n}\big)$$

holds.

Theorem 22 *Let $d \geq 3$. For $n \geq d$ we have*

$$\chi^{\mathsf{w}}_{2d-3,d}(n) \geq \frac{\log n}{2\log\log n} - \frac{d-1}{2} = \Omega\big(\frac{\log n}{\log\log n}\big) \quad \textit{as } n \to \infty.$$

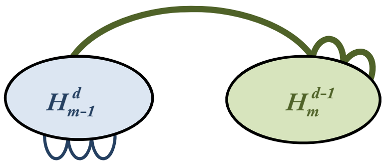
Proof Induction over d. The case $d = 3$ was shown in Theorem 21. Let $d > 3$. Suppose we have constructed a family $(H_m^{d-1})_{m\in\mathbb{N}}$ of hypergraphs in $\mathscr{H}_{2d-5,d-1}$ such that $\chi^{\mathsf{w}}(H_m^{d-1}) \geq m$ and such that all hypergraphs H_m^{d-1} are embeddable into $\mathbb{R}^{d-1}$ by vertices on the moment curve with edges intersecting according to Corollary 13 (or Lemma 15 if $d = 4$).

Let $H_2^d = K_d^{(d)}$. The hypergraph H_2^d has d vertices, one edge, and is weakly 2-colorable. Define H_m^d for $m > 2$ iteratively, given that $\chi^{\mathsf{w}}(H_{m-1}^d) \geq m-1$. For that, take one copy of H_{m-1}^d and one copy of $(d-1)$-uniform H_m^{d-1}.

The edges of H_m^d shall be all edges of H_{m-1}^d and all edges of the form $(\{v\} \cup e)$ for $v \in V(H_{m-1}^d)$ and $e \in E(H_m^{d-1})$. Assume that there exists a weak $(m-1)$-coloring of H_m^d. Then there has to be at least one monochromatic edge $e \in E(H_m^{d-1})$. No vertex of H_{m-1}^d can be colored with this color, so its edges must be weakly $(m-2)$-colored. This is a contradiction and thus $\chi^{\mathsf{w}}(H_m^d) \geq m$.

We now claim that $H_m^d \in \mathscr{H}_{2d-3,d}$ for all $m \geq 2$. As in the proof of Theorem 21, we give a function $f_m^{(d)} : V(H_m^d) \to \{1, \ldots, n_m^{(d)}\}$ where $n_m^{(d)}$ is the number of vertices of H_m^d. This defines the order in which the vertices of H_m^d will be arranged on the moment curve $t \mapsto (t, \ldots, t^{2d-3})$. We then use Corollary 13 to prove that H_m^d is embeddable

Fig. 2 Construction of H_m^d



via arbitrary points on the moment curve. As before, the absolute position of vertices on the moment curve is not important. For a fixed uniformity d and dimension $2d-3$, Corollary 13 guarantees that if for two given edges the vertices of at least one edge have at most $d-3$ odd contiguous subsets, they intersect properly according to Definition 1.

For $d=3$ we can set $f_m^{(3)} = f_m$ for all $m \geq 2$, where f_m is as in the proof of Theorem 21. For $d > 3$ we have by assumption that there exists a corresponding family of functions

$$\left(f_m^{(d-1)} : V(H_m^{d-1}) \to \{1, \ldots, n_m^{(d-1)}\}\right)_m$$

such that the vertices of H_m^{d-1} arranged in that order on the moment curve form an embedding. We then have to give an appropriate family of functions $f_m^{(d)}$ for d.

H_2^d can be embedded into $\mathbb{R}^{2d-3}$ via any d points on the moment curve, so $f_2^{(d)}$: $V(H_2^d) \to \{1, \ldots, d\}$ can be chosen arbitrarily. Assume that $f_{m-1}^{(d)}$ has already been defined and gives an embedding of H_{m-1}^d. We define $f_m^{(d)}(v) = f_{m-1}^{(d)}(v)$ for $v \in V(H_{m-1}^d)$ and for any $w \in V(H_m^{d-1})$ we set $f_m^{(d)}(w) = n_{m-1}^{(d)} + f_m^{(d-1)}(w)$. This is also shown in Fig. 2.

Arrange the vertices of H_m^d on the moment curve in that order and pick any two edges g_1 and g_2.

Case 1: *Both edges are from the subhypergraph* H_{m-1}^d. Then they intersect in accordance to Definition 1 and Corollary 13 as their relative order reflects that of $f_{m-1}^{(d)}$.

Case 2: *One edge is from* H_{m-1}^d *and the other of the form* $(\{v\} \cup e)$ *where* $v \in V(H_{m-1}^d)$ *and* $e \in E(H_m^{d-1})$. Then both edges have at most one odd contiguous subset (besides the first and last one), which is no problem for $d > 3$.

Case 3: $g_1 = (\{v_1\} \cup e_1)$ *and* $g_2 = (\{v_2\} \cup e_2)$. Then the edges e_1 and e_2 intersect according to Corollary 13 (or Lemma 15 if $d = 4$) and g_1 and g_2 have at most one more odd contiguous subset than the edges e_1 and e_2 had in the ordering of $f_m^{(d-1)}$. The last number, by assumption, was bounded from above by $(d-1)-3$ for at least one e_i, $i \in \{1, 2\}$ (unless $d = 4$ and they intersect according to Case 12 in Table 3, see below). So at least one g_i has at most $d-3$ odd contiguous subsets. Thus, the order given by $f_m^{(d)}$ provides an embedding of H_m^d.

Note that there is one small exception to Case 3 when $d = 4$. Here, e_1 and e_2 could be in the relative position of Case 12 in Table 3 and consequently have more than $(d-1)-3 = 0$ odd contiguous subsets. However, this is no problem as in all possible extensions to g_1 and g_2 at least one of the edges continues to have only one odd contiguous subset (see Table 6).

Table 6 All possible 4-uniform extensions of Case 12 in Table 3 as occurring in the construction of H_m^4

No.	Configuration
1	E F E E F E F F
2	F E E E F E F F
3	I E E F E F F

To bound the number of vertices of H_m^d we use

$$\begin{aligned} n_2^{(d)} &= d, \\ n_m^{(d)} &= n_{m-1}^{(d)} + n_m^{(d-1)} \quad \text{for } m > 2. \end{aligned}$$

Iteratively, we get that $n_m^{(d)} = d + \sum_{r=3}^{m} n_r^{(d-1)} \le m \cdot n_m^{(d-1)} \le \cdots \le m^{d-3} \cdot \hat{n}_m = m^{2m+d-3}$ and thus

$$\frac{\log n_m^{(d)}}{\log \log n_m^{(d)}} \le (2m+d-3) \cdot \frac{\log(m)}{\log\left((2m+d-3)\log(m)\right)} \le 2m+d-3.$$

Hence,

$$m \ge \frac{\log n_m^{(d)}}{2 \log \log n_m^{(d)}} - \frac{d-3}{2}.$$

□

Note that by monotonicity also

$$\chi^{\mathsf{w}}_{2d-2,d}(n) = \Omega\Big(\frac{\log n}{\log \log n}\Big)$$

holds.

4 Conclusions and Open Questions

Starting from the Four Color Theorem we have shown that it has no direct analogon for higher dimensions in general. Rather, in almost all cases, the number of colors needed to color a hypergraph embedabble in a certain dimension is unbounded. However, some questions still need to be answered.

Firstly, it would be very interesting to see whether the logarithmic-polynomial difference between lower and upper bounds for the weak coloring case can be improved substantially. If the conjectures by Gundert and Kalai mentioned in Sect. 2 were true, the upper bound for weak colorings could be lowered as follows.

Conjecture 23 *Let $k-1 \le d \le 2k-2$. Then one has*

$$\chi^{\mathsf{w}}_{d,k}(n) = \mathcal{O}\left(n^{\frac{\lceil (d-1)/2 \rceil}{k-1}}\right) \quad \text{as } n \to \infty.$$

Further, in the weak coloring case, for $k = d + 1$ no examples with an unbounded number of colors needed have yet been found and a finite bound is still possible. Also, the question whether the maximum chromatic number for some fixed k, d, and n actually differs for linear and piecewise linear embeddings, remains an open problem.

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