Celestial mechanics of elastic bodies

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Abstract   We construct time independent configurations of two gravitating elastic bodies. These configurations either correspond to the two bodies moving in a circular orbit around their center of mass or strictly static configurations.

1 Introduction

In the year 1933 Leon Lichtenstein published a book entitled "Gleichgewichtsfiguren rotierender Flüssigkeiten" ("equilibrium figures of rotating fluids") in which he collected his work on this topic, published in Mathematische Zeitschrift in the years 1905–1928. Lichtenstein showed existence of solutions of the nonlinear PDEs describing rotating fluids in various configurations under the influence of Newtonian gravity. The simplest configurations are: one axisymmetric rigidly rotating fluid ball and two self-gravitating fluid balls in circular motion around their center of gravity. More complicated configurations treated are toroidal rings or even families of nested rings. All these problems have in common that the equations become time independent in a corotating system and that they express the balance of gravitational, centrifugal and pressure forces. Of particular importance is that the total force and torque on the bodies must vanish. Lichtensteins method is—in modern language—the implicit function theorem in the context of bifurcation theory. In one class of problems he begins with an exact non rotating solution and finds a nearby solution with small angular velocity. More interesting are the cases with several bodies. Starting from two point particles on a circular orbit around their center of gravity, one "puts two small homogeneous fluid

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bodies on these orbits": this is no solution to the equation, however for a small body the error is small enough to begin an iteration which converges to an exact solution. The purpose of this paper is to show that basically all configurations Lichtenstein treated can also be successfully studied if the bodies are made of elastic material. The case of one slowly rotating elastic body was studied in [1]. In this paper we treat: Firstly, one elastic body in circular motion in the gravitational field of some central mass which is kept fixed. Secondly, two bodies in circular motion around their common center of gravity under their mutual Newtonian gravitational field.

Thirdly we deal with a static problem, where one small body is near a non-degenerate equilibrium point of a given gravitational field—such as that in the center of a hollow triaxial cylinder, (iv) like (iii)—with the mutual interaction and self-interaction of both bodies taken into account.

The paper is organized as follows. Section 2 reviews elastostatics; quotes properties of the elasticity operator and formulates the equations for general "force functionals". In Sect. 3 we discuss various types of forces, which in particular include centifugal and gravitational forces. Section 4 contains the main application, the motion of a body in an external grav- itational field and the motion of two bodies under their gravitational attraction. Section 5 is in a way separate from the main theme of rotating bodies. It shows, that the techniques developed can be used to demonstrate the existence of Newtonian static two-body solutions for particular geometries of the bodies. This possibility had always been conjectured, but the existence of a full PDE solutions had not been demonstrated.

2 Elastostatics

In this section we recall the setup of nonrelativistic elastostatics. We are using the material representation where elastic states are decribed by deformations, i.e. maps from material space, the "body", to physical space. A fuller and more general description can be found in [1].

Let Ω be a bounded open connected subset in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The basic objects are maps $\Phi : \Omega \to \mathbb{R}^3$. We will use X^A, resp. x^i for Cartesian coordinates in Ω resp. $\mathbb{R}^3$ and δ_{AB}, resp. δ_{ij} for the metrics on the domain, resp. target space. There is a trivial or reference state, denoted by $\bar{\Phi}$, which we will always take to be the identity. We assume the Φ's are sufficiently close to $\bar{\Phi}$ (see below), so that they have a regular inverse $f : \Phi^{-1}(\Omega) \to \Omega$. We define $\Psi^A{}_i$ by the equation

$$\Psi^A{}_i(X)(\partial_B \Phi^i)(X) = \delta^A{}_B \tag{1}$$

and the tensor H^{AB} by

$$H^{AB}(X) = \Psi^A{}_i(X)\Psi^B{}_j(X)\delta^{ij}. \tag{2}$$

In other words, H^{AB} is the inverse of the pulled-back-under-Φ Euclidean metric on $\mathbb{R}^3$.

The material is described by the stored-energy function $\hat{w}(\Phi, \partial\Phi) = \hat{w}(\Phi, H^{BC})$, smooth in its arguments and subject to the conditions

$$\left(\frac{\partial \hat{w}}{\partial H^{AB}}\right)\Bigg|_{H^{CD}= \delta^{CD}} = 0 \tag{3}$$

and that L_{ABCD}, defined by

$$L_{ABCD} = \left(\frac{\partial^2 \hat{w}}{\partial H^{AB} \partial H^{CD}} \right) \Bigg|_{H^{EF} = \delta^{EF}}, \tag{4}$$

should satisfy

$$L_{ABCD} \, M^{AB} M^{CD} \geq C |M|^2 \tag{5}$$

with C a positive constant and $M^{AB} = M^{(AB)}$. Condition Eq. (3) will mean that the reference state $\bar{\Phi}$ has vanishing stress. Condition Eq. (5), called uniform pointwise stability, is an ellipticity condition satisfied by standard materials. We will sometimes need the condition that the material be homogenous and isotropic. Homogeneity means thast $\hat{w}$ is independent of Φ. Isotropy of the material means that $\hat{w}$ depends on H^{AB} only through the principal invariants of $H^A{}_B = H^{AC} \delta_{BC}$. In the latter case L_{ABCD} is of the form

$$L_{ABCD} = \lambda \, \delta_{AB} \delta_{CD} + 2\mu \, \delta_{C(A} \delta_{B)D}, \tag{6}$$

and uniform pointwise stability is then equivalent to $\mu > 0$, $3\lambda + 2\mu > 0$.

Remark Perfect fluids can be described by assuming the stored-energy to be a function solely of J, where J is the determinant of $H^A{}_B$. However in this case condition (5) is never satisfied.

The "first Piola stress tensor" is defined by

$$\hat{\sigma}_i{}^A(\Phi, \partial \Phi) = \frac{\partial \hat{w}}{\partial(\partial_A \Phi^i)} = -2 H^{AC} \Psi^B{}_i \, \frac{\partial \hat{w}}{\partial H^{BC}} \, . \tag{7}$$

Let now $\mathcal{C}$ be a neighbourhood of $\Phi = \bar{\Phi}$ in $W^{2,p}(\Omega, \mathbb{R}^3)$, $p > 3$, small enough so that each $\Phi \in \mathcal{C}$ has a C^1-inverse (see p. 224 of [4]).

Next let $\mathcal{E}$ be the quasilinear second-order operator sending each map $\Phi \in \mathcal{C}$ to

$$\mathcal{E}_i[\Phi](X) = (\partial_A \sigma_i{}^A)(X) \tag{8}$$

where $\sigma_i{}^A(X) = \hat{\sigma}_i{}^A(\Phi(X), \partial \Phi(X))$. Denote σ_i by $(\sigma_i{}^A n_A)|_{\partial\Omega}$, where n_A is the outward normal to $\partial\Omega$. Let the "load space" $\mathcal{L}$ be defined by $\mathcal{L} = W^{0,p}(\Omega, \mathbb{R}^3) \times W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$. It is then well known (see [9]), that the operator E sending $\Phi \in \mathcal{C}$ to the pair $(b = \mathcal{E}_i, \tau = \sigma_i) \in \mathcal{L}$ is well defined and C^1. Here are some properties of the map E.

Property 1 $E(\bar{\Phi}) = 0$. This is obvious, since $\hat{\sigma}_i{}^A(\bar{\Phi}, \partial \bar{\Phi}) = 0$ by Eqs. (3,7).

Property 2 If $(b, \tau) \in E(\mathcal{C})$ then there holds

$$\int_{\Omega} (\xi^i \circ \Phi)(X) \, b_i(X) \, dV(X) + \int_{\partial\Omega} (\xi^i \circ \Phi)(X) \, \tau_i(X) \, dS(X) = 0, \tag{9}$$

where $\xi^i(x)$ is any Killing vector of Euclidean $\mathbb{R}^3$. Note that (9), when required only for translations, is independent of Φ. The conditions (9)—often called "equilibration conditions"—have the meaning of vanishing total and force and total torque. If these conditions were violated, the body will be set into motion and no time independent solution exists.

Finally, the linear map $\delta E : W^{2,p}(\Omega, \mathbb{R}^3) \to \mathcal{L}$, given by the linearization of E at $\Phi = \bar{\Phi}$, has

Property 3 δE is Fredholm with (6-dimensional) kernel given by the Euclidean Killing vectors at the reference state, i.e. $\delta \Phi^i$ of the form $\delta \Phi^i(X) = (\xi^i \circ \bar{\Phi})(X)$. Furthermore δE has a range (of codimension 6) given by the subspace $\mathcal{L}_{\bar{\Phi}}$ of all (b, τ) satisfying the equilibration conditions (9) at $\Phi = \bar{\Phi}$.

A proof of Property 3 can be found in [9].

We now come to the concept of force density in the present context. Let F be a "load map", i.e. a C^1—map from $\mathcal{C}$ to $\mathcal{L}$. We are interested in solving equations of the form

$$E[\Phi] + \lambda F[\Phi] = 0 \tag{10}$$

for small λ. Our load maps will be of the form

$$F : \Phi \in \mathcal{C} \mapsto (b_i = \mathcal{F}_i[\Phi], \tau_i = 0). \tag{11}$$

We are thus seeking solutions to the elastic equations with σ_i, the normal stress at the boundary, vanishing, i.e. "freely floating bodies".

We will call a force field $\mathcal{F}$ equilibrated at Φ, if $F[\Phi]$ satisfies Eq. (9) with $\tau_i = 0$ i.e. if

$$\int_{\Omega} (\xi^i \circ \Phi)(X) \, \mathcal{F}_i[\Phi](X) \, dV(X) = 0 \tag{12}$$

for all Euclidean Killing vectors ξ. Given a functional $\mathcal{F}_i$, we will try to solve (10) by the implicit function theorem. Suppose Φ_λ is a 1-parameter family of solutions of (10) with $\Phi_0 = \bar{\Phi}$. Then (12) must hold for any λ, whence for $\lambda = 0$. Therefore the reference configuration must be equilibrated for the force functional in order for being the starting solution for an application of the implicit function theorem. Suppose we have such a $\bar{\Phi}$, then the linearisation of the elasticity operator at $\bar{\Phi}$ still has a non trivial finite dimensional kernel and range. In the spirit of bifurcation theory one could try to project the equation onto some complement of the range. Sometimes this approach works, provided one takes "sufficiently small" but finite bodies of any shape. We used this approach in [2]. In the present section we will just consider spherical bodies. Then one can use some discrete symmetries to make the space of configurations and loads smaller so that the implicit function theorem can be applied.

3 Forces

In our applications we will deal with the centrifugal force and the gravitational force under various circumstances.

3.1 Spatial force fields

Both the centrifugal and an external gravitational force give rise to maps $\mathcal{F}$ of the form

$$\mathcal{F}_i[\Phi](X) = (K_i \circ \Phi)(X), \tag{13}$$

where $K_i(x)$ is a smooth vector field on $\mathbb{R}^3$ with a potential

$$K_i = \partial_i K \tag{14}$$

satisfying

$$\Delta K = \delta^{ij} \partial_i \partial_j K = \text{const.} \tag{15}$$

We call such fields K_i harmonic. It follows that $\xi^i K_i$, where ξ^i is any Euclidean Killing vector, is a harmonic function. Let us define

$$\mu_{R,x}[\xi; \Phi] = \int_{B_R(x)} (\xi^i \circ \Phi)(X) \, (K_i \circ \Phi)(X) \, \mathrm{d}V(x), \tag{16}$$

where $B_R(x)$ is the open ball of radius R centered at the point x. Suppose x_0 is a critical point of K. Then, by the (solid form of the) mean-value theorem for harmonic functions (see [5]), there holds

$$\mu_{R,x_0}[\xi; \bar{\Phi}] = \int_{B_R(x_0)} \xi^i K_i \, \mathrm{d}V(x) = |B_R| \, (\xi^i K_i)(x_0) = 0 \tag{17}$$

This means that $\mathcal{F}_i = K_i \circ \Phi$ is equilibrated at $\bar{\Phi}$ provided that $\Omega = B_R(x_0)$.
We will also use the variational formula

$$\delta_\Phi \mu_{R,x_0}[\xi; \bar{\Phi}] \cdot \delta\Phi = \int_{B_R(x_0)} (\mathcal{L}_\xi K)_i \, \delta\Phi^i \, \mathrm{d}V(x). \tag{18}$$

Taking in (18) for $\delta\Phi = \eta$, with η another Euclidean Killing vector, we find the integrand in (18) is again harmonic. Consequently

$$\delta_\Phi \mu_{R,x_0}[\xi; \bar{\Phi}] \cdot \eta = |B_R| \, (\xi^i \eta^j \partial_i \partial_j K)(x_0) \tag{19}$$

3.2 Force fields equilibrated at all deformations

It seems remarkable at first sight, that there should exist such force fields. Suppose there is a symmetric tensor field $\Theta_{ij}[\Phi](x)$ on $\mathbb{R}^3$, having the property that

$$\partial_j \Theta_i{}^j = 0 \quad \forall x \in \mathbb{R}^3 \setminus \overline{\Phi(\Omega)}. \tag{20}$$

and that Θ_{ij} is of $O(|x|^{-4})$ at infinity. (We are here and henceforth raising and lowering indices with the Euclidean metric δ_{ij}.) Then $\mathcal{F}_i = J(\partial_j \Theta_i{}^j) \circ \Phi$ is equilibrated for all Φ. (Recall that J denotes the Jacobian determinant of Φ.) To see this, compute

$$\int_\Omega (\xi^i \circ \Phi)(X) J(X)((\partial_j \Theta_i{}^j) \circ \Phi)(X) \, \mathrm{d}V(X) = \int_{\Phi(\Omega)} \xi^i(x) \, (\partial_j \Theta_i{}^j)(x) \, \mathrm{d}V(x). \tag{21}$$

Now the integral on the right-hand side of (21) can be extended to all of $\mathbb{R}^3$. After partial integration, using that ξ diverges at most linearly at infinity, there follows

$$\int_\Omega \mathcal{F}_i[\Phi](X) \mathrm{d}V(X) = -\int_{\mathbb{R}^3} (\partial_j \xi_i)(x) \Theta^{ij}(x) \mathrm{d}V(x) = 0 \tag{22}$$

An example is afforded by the following situation: Let ρ be the function on $\mathbb{R}^3$ given by

$$\rho[\Phi](x) = [(J \circ \Phi^{-1})(x)]^{-1} \rho_0 \, \chi_{\Phi(\Omega)} \tag{23}$$

with χ be the characteristic function and ρ_0 a constant. This is the spatial density corresponding to the (constant) reference density ρ_0. Consider, then, the function U on $\mathbb{R}^3$ given by

$$\Delta U = -4\pi G \rho \tag{24}$$

with $U(x) = O(|x|^{-1})$ at infinity, i.e. the gravitational potential generated by ρ. Now take

$$\Theta_{ij} = \frac{1}{4\pi G} \left[(\partial_i U)(\partial_j U) - \frac{1}{2}\delta_{ij}\, \delta^{kl}(\partial_k U)(\partial_l U) \right]. \tag{25}$$

Then

$$\partial_j \Theta_i{}^j = -\rho \partial_i U \tag{26}$$

so that (20) is satisfied. In this case, equilibration can also be checked by first computing

$$\mathcal{F}_i[\Phi](X) = G\rho_0 \int_\Omega \frac{\Phi_i(X) - \Phi_i(Y)}{|\Phi(X) - \Phi(Y)|^3} \, dV(Y), \quad \Phi_i = \delta_{ij}\Phi^j \tag{27}$$

and using the explicit form of the Killing vectors. We have of course rediscovered the well-known fact that the total force and the total torque on a body due to its own gravitational field is zero.

4 Circular orbits

4.1 Test body in a central gravitational field

Here we consider the gravitational field of a point source of mass M, in a reference frame rigidly rotating with angular frequence ω. This leads to a spatial, harmonic force field given by

$$\bar{K}_i = \rho_0 \, \partial_i \left[\omega^2 \frac{(x^1)^2 + (x^2)^2}{2} + \frac{GM}{r} \right], \tag{28}$$

where $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. The field $\bar{K}_i$ vanishes on the circle given by $r = L, x^3 = 0$ where

$$\omega^2 L = \frac{GM}{L^2}, \tag{29}$$

which of course corresponds to circular orbits of point particles. Thus we take $\lambda \mathcal{F}_i = \lambda K_i \circ \Phi$ with

$$K_i = \partial_i \left[\frac{(x^1)^2 + (x^2)^2}{2} + \frac{L^3}{r} \right] \tag{30}$$

on $\mathbb{R}^3 \backslash \{0\}$ and solve

$$E + \lambda F = 0, \tag{31}$$

where E and F are respectively defined after Eq. (8) and in Eq. (11). For small λ near the reference state defined by

$$\bar{\Phi} = \mathrm{id}|_\Omega, \tag{32}$$

where $\Omega = B_R(l)$. Here $R < L$ and l is for concreteness taken to be $l = (L, 0, 0)$. (The body is here and henceforth assumed to have spherical shape in its relaxed configuration.) The interpretation of a solution Φ_λ for small λ will be that of an elastic body moving with constant angular frequency ω along a circle (of radius $\sim \Phi^1(L, 0, 0)$) in the central gravitational field of a mass M. Furthermore there holds

$$\lambda = \rho_o \omega^2, \quad \omega^2 = \frac{GM}{L^3} \tag{33}$$

Since the function in square brackets in Eq. (30) has Laplacian equal to a 2, it follows from our previous discussion that $\mathcal{F}$ is equilibrated at $\Phi = \bar{\Phi}$.

Our application of the implicit function theorem will be much simplified if we a priori impose the condition that the configurations be symmetric with respect to reflection at the planes $X^2 = 0$ and $X^3 = 0$. Thus we now restrict $W^{2,p}(\Omega, \mathbb{R}^3)$ to the Banach space $W^{2,p}_{\text{sym}}(\Omega, \mathbb{R}^3)$ of all maps $\Phi \in W^{2,p}(\Omega, \mathbb{R}^3)$, so that

$$\begin{aligned}
\Phi^1(X^1, X^2, X^3) &= \Phi^1(X^1, -X^2, X^3) = \Phi^1(X^1, X^2, -X^3) \\
\Phi^2(X^1, X^2, X^3) &= -\Phi^2(X^1, -X^2, X^3) = \Phi^2(X^1, X^2, -X^3) \\
\Phi^3(X^1, X^2, X^3) &= \Phi^3(X^1, -X^2, X^3) = -\Phi^3(X^1, X^2, -X^3)
\end{aligned} \tag{34}$$

These relations are the same as those satisfied by a vector field, which has mirror symmetry w.r. to both the $X^2 = 0$ and the $X^3 = 0$ plane. Since the set (34) contains the identity, there will again be an open neighbourhood of the identity, which we call $\mathcal{C}_{\text{sym}}$, so that each element of $\mathcal{C}_{\text{sym}}$ has a C^1-inverse. We will define as restricted load space the $\mathcal{L}_{\text{sym}}$ given by all $(b, \tau) \in W^{0,p}(\Omega, \mathbb{R}^3) \times W^{1-1/p,p}(\partial\Omega, \mathbb{R}^3)$, with components having the same symmetries (34). Supposing that the stored energy function is homogenous and isotropic, it follows that $\partial_A \sigma_i{}^A$ also satifies (34). Similarly, using that Ω—whence $\partial\Omega$—is invariant under the mirror symmetry, it follows that $\sigma_i{}^A n_A$ is also invariant. Thus the elasticity operator E maps $\mathcal{C}_{\text{sym}}$ into $\mathcal{L}_{\text{sym}}$. Next observe that the equilibration conditions (9) in $\mathcal{C}_{\text{sym}}$ are by symmetry all identically satisfied except for that where $\xi = \partial_1$. Thus $\mathcal{L}^e_{\text{sym}}$, the set of equilibrated loads in $\mathcal{L}_{\text{sym}}$, is simply given by all $(b, \tau) \in \mathcal{L}_{\text{sym}}$, for which

$$\int\limits_{B_R(l)} b_1(X) \, dV(X) + \int\limits_{\partial B_R(l)} \tau_1(X) \, dS(X) = 0. \tag{35}$$

In particular the set $\mathcal{L}^e_{\text{sym}}$ does not depend on Φ. It is clearly a Banach subspace of $\mathcal{L}_{\text{sym}}$ of codimension 1. Note next that the only Killing vectors in $W^{2,p}_{\text{sym}}(\Omega, \mathbb{R}^3)$ are translations in the X^1-direction. Thus the linearized operator δE at $\bar{\Phi}$ has kernel exactly given by the linear span of $\delta\Phi = \partial_1$.

Now turn to operator F given by $\Phi \in \mathcal{C}_{\text{sym}} \mapsto (b_i = K_i \circ \Phi, \tau_i = 0) \in \mathcal{L}$. By well known results on "Nemitskii operators" (see [9]), this map is C^1. Furthermore it restricts to a map from $\mathcal{C}_{\text{sym}}$ to $\mathcal{L}_{\text{sym}}$. Now consider the subset $\mathcal{C}_{\mathcal{F}} \subset \mathcal{C}_{\text{sym}}$ subject to

$$\mu_{R,l}[\partial_1; \Phi] = \int\limits_{B_R(l)} K_1 \circ \Phi \, d^3 X = 0. \tag{36}$$

As shown in Sect. 3.1, the identity map $\bar{\Phi}$ lies in $\mathcal{C}_{\mathcal{F}}$. Using (18) and Eq. (30), an easy computation shows that

$$\delta_\Phi \mu_{R,l}[\partial_1; \bar{\Phi}] \cdot \partial_1 = |B_R| \cdot 3 \tag{37}$$

Thus $\mathcal{C}_{\mathcal{F}}$ is a C^1-submanifold $\subset \mathcal{C}_{\text{sym}}$ near $\bar{\Phi}$ of codimension 1 (see e.g. [6]). Furthermore, by construction, $E + \lambda F$ maps $\mathcal{C}_{\mathcal{F}}$ into $\mathcal{L}^e_{\text{sym}}$ and its linearization at $\lambda = 0$, $\Phi = \bar{\Phi}$ is an isomorphism. The implicit function theorem immediately gives the following

Theorem 4.1 *Let the stored energy function entering the elasticity map E be homogenous and isotropic. Fix a sphere of radius R and a number $L > R$. Let ω, λ be defined by (33). Then there exists a positive number ϵ—which depends on L, R and the stored energy function $\hat{w}$—so that a solution Φ_λ of Eq. (31) exists for $\lambda \in [0, \epsilon)$.*

4.2 Test body with self-gravity

Now the load $\mathcal{F}_i$ is given by

$$\lambda \mathcal{F}_i = \lambda \left[K_i \circ \Phi + \frac{\rho_0 L^3}{M} \int_\Omega \frac{\Phi_i(X) - \Phi_i(Y)}{|\Phi(X) - \Phi(Y)|^3} \, dV(Y) \right], \tag{38}$$

with K_i given by Eq. (30) and λ given by (33). Using Sect. 3.2 and the fact that (see [2]) Eq. (27) defines a C^1-map from $\mathcal{C}_{\text{sym}}$ to $\mathcal{L}^e_{\text{sym}} \subset \mathcal{L}_{\text{sym}}$, everything literally goes through as before and we have the

Theorem 4.2 *Under the same assumptions as in Theorem 4.1 the equation $E + \lambda F = 0$ has a unique solution Φ_λ with $\Phi_0 = \bar{\Phi}$, for sufficiently small λ.*

4.3 Two identical bodies with gravitational interaction and self-gravity

Here we take for the relaxed state two spheres of radius R at distance L along the X^1 axis. Thus, for the reference configuration we take balls $\Omega = B_R(\frac{1}{2}l)$ and $\Omega' = B_R(-\frac{1}{2}l)$, where the vector l is again given by $l = (L, 0, 0)$ with $2R < L$, and deformations $\Phi : \Omega \to \mathbb{R}^3$, $\Phi' : \Omega' \to \mathbb{R}^3$ with

$$\begin{aligned}
\Phi'^1(X^1, X^2, X^3) &= -\Phi^1(-X^1, X^2, X^3) \\
\Phi'^2(X^1, X^2, X^3) &= \Phi^2(-X^1, X^2, X^3) \\
\Phi'^3(X^1, X^2, X^3) &= \Phi^3(-X^1, X^2, X^3)
\end{aligned} \tag{39}$$

Thus the relaxed bodies and their deformations have complete mirror symmetry with respect to the plane $X^1 = 0$ (see Fig. 1).

The force field is given by

$$\begin{aligned}
\lambda \mathcal{F}_i[\Phi](X) = \rho_0 \omega^2 \Bigg[&\delta_i{}^\alpha \Phi_\alpha(X) - \frac{L^3}{2|B_R|} \int_{B_R(-\frac{1}{2}l)} \frac{\Phi_i(X) - \Phi'_i(Y')}{|\Phi(X) - \Phi'(Y')|^3} \, dV(Y') \\
&- \frac{L^3}{2|B_R|} \int_{B_R(\frac{1}{2}l)} \frac{\Phi_i(X) - \Phi_i(Y)}{|\Phi(X) - \Phi(Y)|^3} \, dV(Y) \Bigg]
\end{aligned} \tag{40}$$

where $\alpha = 1, 2$. The interpretation of a solution Φ_λ is that of a pair of elastic bodies of mass $m = |B_R| \rho_0$ (reduced mass M_r), rotating about their centre of mass with angular frequency ω where

$$\lambda = \rho_0 \omega^2, \quad \omega^2 L = \frac{G M_r}{L^2}, \quad M_r = \frac{m}{2}. \tag{41}$$
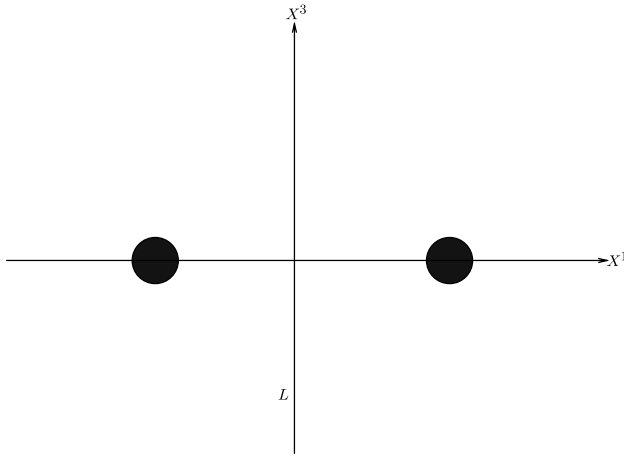
Fig. 1 Two spheres in steady rotation

The self-interaction term in (40) is again equilibrated for all Φ's. The first two terms in (40) are again of the form $K_i \circ \Phi$ with K_i a harmonic force field, where however, for the second term, the potential for this force field is in turn a functional K of Φ', namely there holds

$$\Delta K[\Phi'](x) = -4\pi\rho, \tag{42}$$

where (see (23))

$$\rho[\Phi'](x) = [(J' \circ \Phi'^{-1})(x)]^{-1} \rho_0 \, \chi_{\Phi'(\Omega)} \tag{43}$$

with J' the Jacobian of the map Φ'. We again find that $\mathcal{F}$ is equilibrated at the identity. Furthermore

$$\delta_\Phi \mu_{R, \frac{1}{2}l}[\partial_1; \bar{\Phi}] \cdot \partial_1 = |B_R| \cdot 3 \tag{44}$$

for the sum of forces in (40), of course the last term does not contribute to μ. Thus exactly as in the previous section $\mathcal{C}_\mathcal{F}$ is a C^1-submanifold $\subset \mathcal{C}_{\text{sym}}$ near $\bar{\Phi}$ of codimension 1 and $E + \lambda F$ maps $\mathcal{C}_\mathcal{F}$ into $\mathcal{L}^e_{\text{sym}}$ with its linearization at $\lambda = 0$, $\Phi = \bar{\Phi}$ an isomorphism. We have obtained the

Theorem 4.3 *Consider Eq. (10) with homogenous and isotropic stored energy function and with F of the form $F : \Phi \in \mathcal{C}' \mapsto (b_i = \mathcal{F}_i[\Phi], \tau_i = 0)$ where $\mathcal{F}_i$ is given by Eq. (40). For sufficiently small λ there exists a solution Φ_λ with $\Phi_0 = \bar{\Phi}$.*

5 Static problems

In Newtonian theory it seems obvious that there exist no static solutions wth two gravitating bodies. They should fall onto each other. This is however in general only true if the geometry is such that the bodies are separated by a plane. We will show in this section that for more involved geometries there do exist static 2-body solutions.

First we try to put a small body near a critical point of the field of some external body. The gravitational force vanishes at critical points of the gravitational potential due to this body. In Sect. 5.1 we will assume that a critical point, subject also to further restrictions, be
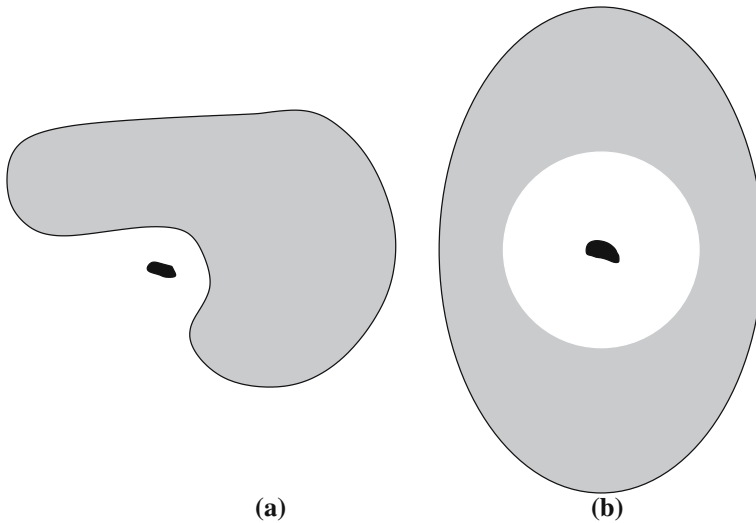
(a)   **(b)**

Fig. 2 Static configurations

given, and then show that there exists an equilibrium configuration of a small elastic body near that critical point. However such critical points are absent in those parts of the vacuum region which are separated by a plane from the support of the large body. One thus has to put the small body in the "hollow space" of a sufficiently non-convex body (Fig. 2a). We give a particular example for such a body in Sect. 5.2 corresponding to Fig. 2b, where we also consider the full 2-body problem.

5.1 Test body

We first treat the case of a body in a given potential field $K_i = \partial_i K$. We make the following assumptions:

1. K has a nondegenerate critical point at the origin.
2. The domain Ω has the origin as centre of mass.
3. The Hessian of K and the tensor of inertia of Ω are simultaneously diagonalizable (i.e. commute).
4. The eigenvalues of the inertia tensor are all different, and the eigenvalues of the Hessian are all different.

We now show that, by simultaneously translating, rotating and scaling Ω, one can satisfy the equilibration conditions (at the identity). Let $f_{ij} = f_{(ij)}$ be the Hessian of K at the origin, We first observe that the force $\mathring{K}_i = f_{ij}x^j$ is equilibrated w.r. to Ω. Namely

$$\mathring{\mu}_\Omega[d^i \partial_i; \bar{\Phi}] = \int_\Omega d^i f_{ij}x^j \, \mathrm{d}V(x) = 0, \tag{45}$$

by Assumption 2 and

$$\mathring{\mu}_\Omega[\omega^i{}_j x^j \partial_i; \bar{\Phi}] = \omega^{ij} f_{ik}\Theta_j{}^k, \tag{46}$$

where

$$\Theta^{jk} = \int_\Omega x^j x^k \, dV(x). \tag{47}$$

Expression (46) is zero on account of $\omega_{ij} = \omega_{[ij]}$ together with Assumption 3. (The quantity Θ^{ij} differs from the standard definition of the inertia tensor by a multiple of δ^{ij}, so properties 3 and 4 carry over to Θ^{ij}.) The full force K_i can be written as

$$K_i = f_{ij}x^j + f_{ijk}(x)x^j x^k. \tag{48}$$

Denote by $\bar{\Phi}_{S,D}$ the map $x^i \mapsto \bar{x}^i = S^i{}_j x^j + D^i$ with S a rotation matrix and consider the expression

$$k(d, \epsilon; S, D) = \frac{1}{\epsilon^4} \, \mu_{\Omega_\epsilon}[d^i \partial_i; \bar{\Phi}_{S, \epsilon D}], \tag{49}$$

where $\Omega_\epsilon = \{x \in \mathbb{R}^3 | \frac{1}{\epsilon}x \in \Omega\}$. After the obvious change of integration variable it follows that

$$k(d, \epsilon; S, D) = \int_\Omega d^i \big[f_{ij}(S^j{}_{j'}y^{j'} + D^j) $$
$$+ \epsilon \, f_{ijk}(\epsilon Sy + \epsilon D)(S^j{}_{j'}y^{j'} + D^j)(S^k{}_{k'}y^{k'} + D^k)\big]dV(y). \tag{50}$$

Similarly, the quantity

$$m(\omega, \epsilon; S, D) = \frac{1}{\epsilon^5} \, \mu_{\Omega_\epsilon}[\omega^i{}_j x^j \partial_i; \bar{\Phi}_{S, \epsilon D}] \tag{51}$$

is equal to

$$m(\omega, \epsilon; S, D) = \int_\Omega \omega^i{}_j (S^j{}_{j'}y^{j'} + D^j)\big[f_{ik}(S^k{}_{k'}y^{k'} + D^k) $$
$$+ \epsilon \, f_{ikl}(\epsilon Sy + \epsilon D)(S^k{}_{k'}y^{k'} + D^k)(S^l{}_{l'}y^{l'} + D^l)\big]dV(y). \tag{52}$$

Clearly we have that $k(d, 0; \mathrm{id}, 0) = 0$, $m(\omega, 0; \mathrm{id}, 0) = 0$. We can view the pair $(k(\cdot, \epsilon, S, D), m(\cdot, \epsilon, S, D))$ as defining an ϵ-dependent map from the Euclidean group $\mathbb{E}(3)$ into the dual of the Lie algebra of $\mathbb{E}(3)$, and the derivative of this map as a bilinear form V on the Lie algebra of $\mathbb{E}(3)$. We find that

$$\delta_D k(d, 0; \mathrm{id}, 0) \cdot e = |\Omega| \, d^i f_{ij}e^j \tag{53}$$
$$\delta_S k(d, 0; \mathrm{id}, 0) \cdot \mu = 0 \tag{54}$$
$$\delta_D m(\omega, 0; \mathrm{id}, 0) \cdot e = 0 \tag{55}$$
$$\delta_S m(\omega, 0; \mathrm{id}, 0) \cdot \mu = |\Omega| \, (f_{ij}\Theta^{kl} + f_{mi}\Theta^{ml}\delta^k{}_j)\omega^i{}_k \mu^j{}_l , \tag{56}$$

where $\mu_{ij} = \mu_{[ij]}$ and we have used Assumption 2 in (54) and (55). The expression in brackets on the r.h. side of (56), viewed as a bilinear form on antisymmetric tensors, is symmetric by Assumption 3. Writing $\omega^i{}_j = \epsilon^i{}_{jk}p^k$, $\mu^i{}_j = \epsilon^i{}_{jk}q^k$, this form is equivalent to $|\Omega| \, \tau_{ij} \, p^i q^j$ where $\tau^i{}_j$, by the standard ϵ - identities and using Assumption 3, is in matrix notation given by

$$\tau = 3 \, \Theta \, f - (\mathrm{tr} \, f)\Theta - (\mathrm{tr} \, \Theta) f + [(\mathrm{tr} \, \Theta)(\mathrm{tr} \, f) - 2 \, \mathrm{tr} \, (\Theta \, f)] \, \mathrm{id} \tag{57}$$

Going to a frame where both Θ and f are diagonal, we see that τ is also diagonal, with, e.g. $\tau_{11} = (f_{33} - f_{22})(\Theta_{22} - \Theta_{33})$. Thus τ is non-degenerate, using Assumption 4. Using Assumption 1, the full bilinear form V is also non-degenerate. Therefore by the implicit function theorem we conclude that there exists, for ϵ sufficiently close to 0, a unique family $(S(\epsilon), D(\epsilon))$ with $(S(0), D(0)) = (\mathrm{id}, 0)$ so that $\mu_{\Omega_\epsilon}[d^i \partial_i; \bar{\Phi}_{S(\epsilon), \epsilon D(\epsilon)}] = 0$ for all d^i and $\mu_{\Omega_\epsilon}[\omega^i{}_j x^j \partial_i; \bar{\Phi}_{S(\epsilon), \epsilon D(\epsilon)}] = 0$ for all $\omega^i{}_j$ with $\omega_{ij} = \omega_{[ij]}$.

We next define $\bar{\Omega} = \Omega_\epsilon$ for fixed small positive ϵ and, by slight abuse of notation, $\bar{\Phi} = \mathrm{id}_{\bar{\Omega}}$. Again our task is to solve (10), namely

$$E[\Phi] + \lambda F[\Phi] = 0 , \tag{58}$$

where the load map F is of the form

$$F : \Phi \in \mathcal{C} \mapsto (b_i = \mathcal{F}_i[\Phi], \tau_i = 0) \tag{59}$$

and $\mathcal{F}_i[\Phi] = K_i \circ \Phi$. Both E and F are viewed as maps from $\mathcal{C}$ to the load space $\mathcal{L}$ (see Sect. 2). We have the following

Theorem 5.1 *For sufficiently small λ equation (58) has a unique solution $\Phi_\lambda \in \mathcal{C}$ near $\bar{\Phi}$.*

Proof In a similar way as in Sect. 4.1 we define $\mathcal{C}_{\mathcal{F}}$ as a neighborhood of $\bar{\Phi}$ in $\mathcal{C}$ satisfying the six conditions

$$\mu_{\bar{\Omega}}[\xi; \Phi] = \int_{\bar{\Omega}} (\xi^i \circ \Phi)(X)\, \mathcal{F}_i[\Phi](X)\, \mathrm{d}V(X) = 0 \tag{60}$$

for all Euclidean Killing vectors ξ. Consider the expression

$$\delta_\Phi \mu_{\bar{\Omega}}[\xi; \bar{\Phi}] \cdot \delta\Phi = \int_{\bar{\Omega}} (\mathcal{L}_\xi K)_i\, \delta\Phi^i\, \mathrm{d}V(X) \tag{61}$$

If $\delta\Phi^i$ is any non-zero Killing vector η and ϵ entering the definition of $\bar{\Omega}$ is sufficiently small, it follows from the above discussion and continuity that the expression (61) can not be zero for all ξ. Thus $\mathcal{C}_{\mathcal{F}}$ is a C^1-submanifold of $\mathcal{C}$ of codimension 6.

There is an operator $P : \mathcal{L} \to \mathcal{L}_{\bar{\Phi}}$ projecting onto loads equilibrated at the identity. Such an operator is constructed by choosing a (6-dimensional) complement of $\mathcal{L}_{\bar{\Phi}}$ in $\mathcal{L}$ and requiring this complement to be the kernel of P. We now consider the modified equations

$$P \circ (E + \lambda F) = 0. \tag{62}$$

for $\Phi \in \mathcal{C}_{\mathcal{F}}$. The linearization $\delta E : W^{2,p}(\bar{\Omega}) \to \mathcal{L}_{\bar{\Phi}}$ at $\lambda = 0$ of the map in Eq. (62) is clearly an isomorphism, so there exists, for small λ, a solution $\tilde{\Phi}$ of Eq. (62) near $\bar{\Phi} \in \mathcal{C}_{\mathcal{F}}$. By property 2 of the elasticity operator in Sect. 2 we have that, for all $\Phi \in \mathcal{C}$, $E[\Phi] \in \mathcal{L}_\Phi$, the set of all loads satisfying the equilibration conditions at $\Phi = \tilde{\Phi}$, so this holds trivially for $\tilde{\Phi}$. We also have that $F[\tilde{\Phi}] \in \mathcal{L}_{\tilde{\Phi}}$, since $\tilde{\Phi} \in \mathcal{C}_{\mathcal{F}}$. Thus $(E + \lambda F)[\tilde{\Phi}] \in \mathcal{L}_{\tilde{\Phi}}$. By Eq. (62) the load $(E + \lambda F)[\tilde{\Phi}]$ lies in some complement of $\mathcal{L}_{\bar{\Phi}}$. This complement, by continuity, is also a complement of $\mathcal{L}_{\tilde{\Phi}}$. Thus $(E + \lambda F)[\tilde{\Phi}] = 0$, and the proof is complete.

5.2 A static 2-body problem

We first need a domain whose Newtonian potential (assuming constant density) has a critical point in the vacuum region with all three eigenvalues of the Hessian non-zero and different.

Consider first the Newtonian potential U_0 of a solid ellipsoid centered at the origin with axes $a_1 > a_2 > a_3$ and constant density ρ'. From Chapter 3 of [3] (Eqs. (40) and (18)) we read off that that

$$(\partial_i \partial_j U_0)(0) = -2\pi G\rho' \operatorname{diag}(A_1, A_2, A_3), \tag{63}$$

where $0 < A_1 < A_2 < A_3$. In the limit when a_3 approaches a_2 we find (using either Eq. (18) of [3] or the explicit expressions (38), (39) of [3]) that $A_1 < A < A_2 = A_3$, where A is the value of A_i when $a_1 = a_2 = a_3$ (which is equal to $\frac{2}{3}$). By continuity we have $A_1 < A < A_2 < A_3$ for a_3 less than but close to a_2. We now subtract from U_0 the potential of a solid sphere of radius $a < a_3$, centered at the origin, thus obtaining a new potential U corresponding to a hollow triaxial ellipsoid Ω'. The potential U has a critical point at the origin with Hessian having one positive and two negative eigenvalues, all different.

We now choose a domain Ω with mass center at this critical point and such that the inertia tensor satisfies Assumption's 3 and 4 in Sect. 5.1: as explained in Sect. 5.1 we can, by translating, rotating and scaling Ω, find a domain $\bar{\Omega}$ so that $\mu_{\bar{\Omega}}[\xi; \operatorname{id}_{\bar{\Omega}}] = 0$ for all Euclidean Killing vectors ξ. By slight abuse of notation we henceforth call $\bar{\Omega}$ again Ω. Using an argument identical to that in Sect. 3.2, one sees that when $\mu_\Omega[\xi; \operatorname{id}_\Omega] = 0$ there also follows that $\mu_{\Omega'}[\xi; \operatorname{id}_{\Omega'}] = 0$ for all Euclidean Killing vectors ("actio = reactio") - and this can also be checked explicitly. We now assign to the two domains Ω' and Ω the constant densities ρ' and ρ and elasticity operators E' mapping $\Phi' \in \mathcal{C}'$ into the load space $\mathcal{L}'$ and E, mapping $\Phi \in \mathcal{C}$ into the load space $\mathcal{L}$, respectively, with both stored energy functions satisfying Eqs. (3), (5). Neither isotropy nor homogeneity of the stored energy functions is needed in this section. Associated with Ω' there is the load map F' with $\mathcal{F}'$ given by

$$\lambda\mathcal{F}'_i[\Phi, \Phi'](X') = -G\Bigg[\rho \int_\Omega \frac{\Phi'_i(X') - \Phi_i(Y)}{|\Phi'(X') - \Phi(Y)|^3} \, dV(Y)$$

$$+ \rho' \int_{\Omega'} \frac{\Phi'_i(X') - \Phi'_i(Y')}{|\Phi'(X') - \Phi'(Y')|^3} \, dV(Y')\Bigg] \tag{64}$$

and with Ω there is associated $\mathcal{F}$ given by

$$\lambda\mathcal{F}_i[\Phi, \Phi'](X) = -G\Bigg[\rho' \int_{\Omega'} \frac{\Phi_i(X) - \Phi'_i(Y')}{|\Phi(X) - \Phi'(Y')|^3} \, dV(Y')$$

$$+ \rho \int_\Omega \frac{\Phi_i(X) - \Phi_i(Y)}{|\Phi(X) - \Phi(Y)|^3} \, dV(Y)\Bigg]. \tag{65}$$

We now define $\mathcal{D}_\mathcal{F} \subset \mathcal{C} \times \mathcal{C}' \subset W^{2,p}(\Omega, \mathbb{R}^3) \times W^{2,p}(\Omega', \mathbb{R}^3)$, $p > 3$ as the set of $(\Phi, \Phi') \in \mathcal{C} \times \mathcal{C}'$ for which

$$\int_{\Omega \times \Omega'} \xi^i(X) \frac{\Phi_i(X) - \Phi'_i(Y')}{|\Phi(X) - \Phi'(Y')|^3} \, dV(X) \, dV(Y') = 0 \tag{66}$$

and Φ' satifies

$$\Phi'_i(X_0) = 0, \quad \partial'_{[i}\Phi'_{j]}(X_0) = 0, \tag{67}$$

where X_0 is some given point in Ω'. Again, by Sect. 3.2 or by calculation using the explicit form of $\xi^i(X)$, one shows that Eq. (66) is equivalent to

$$\int\limits_{\Omega \times \Omega'} \xi^i(X') \, \frac{\Phi_i'(X') - \Phi_i(Y)}{|\Phi'(X') - \Phi(Y)|^3} \, dV(X) \, dV(Y') = 0 \tag{68}$$

Combining the fact that there are no non-zero Killing vectors vanishing at a point together with their curl and the discussion of the previous subsection—making ϵ entering the definition of Ω smaller if necessary, it follows that $\mathcal{D}_{\mathcal{F}}$ is a C^1-submanifold of $\mathcal{C} \times \mathcal{C}'$ near $(\bar{\Phi}, \bar{\Phi}')$ of codimension 12. We now set $\lambda = G$ and solve, using the implicit function theorem, the equations

$$P(E[\Phi] + \lambda F[\Phi, \Phi']) = 0, \quad P'(E'[\Phi'] + \lambda F'[\Phi, \Phi']) = 0 \tag{69}$$

on $\mathcal{D}_{\mathcal{F}}$, for small G and (Φ, Φ') near $(\bar{\Phi}, \bar{\Phi}')$. Here P, P' are projection operators on $\mathcal{L}$ resp. $\mathcal{L}'$, defined analogously as in Sect. 5.1. The argument showing that we have in fact also solved the "unprojected" equations proceeds as in Sect. 5.1, if it is recalled that the self force terms in (65), (64) are identically equilibrated. We thus have the

Theorem 5.2 *Consider the coupled set of equations*

$$E[\Phi] + \lambda F[\Phi, \Phi'] = 0, \quad E'[\Phi'] + \lambda F'[\Phi, \Phi'] = 0, \tag{70}$$

where $(\Phi, \Phi') \in \mathcal{C} \times \mathcal{C}'$. Then, for λ sufficiently small, there is a unique solution $(\Phi_\lambda, \Phi_\lambda')$ with $(\Phi_0, \Phi_0') = (\mathrm{id}_\Omega, \mathrm{id}_{\Omega'})$.

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