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NOTE

A NOTE ON SIMILARITY RELATIONS

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A simple proof is given for the fact that the number of nonsingular similarity relations on $\{1, 2, \ldots, n\}$, for which the transitive closure consists of k blocks, equals $\binom{2n-2k-1}{n-1} - \binom{2n-2k-1}{n}$, $1 \leq k \leq n/2$. In particular, this implies a recent result of Shapiro about Catalan numbers and Fine's sequence.

A *similarity relation* on the set $[n] = \{1, 2, \ldots, n\}$ is a reflexive and symmetric, but not necessarily transitive, binary relation R with the property

$$x < y < z \text{ and } xRz \text{ imply } xRy \text{ and } yRz, \quad \text{for } x, y, z \in [n]. \tag{1}$$

A similarity relation R will be called *nonsingular*, if for every $x \in [n]$ there is at least one $y \neq x$ such that xRy. Let $A_{n,k}$ denote the set of similarity relations on $[n]$ for which the transitive closure consists of k nonempty classes, $1 \leq k \leq n$; let $B_{n,k}$ denote the subset of those members of $A_{n,k}$ which are nonsingular. It is obvious that $B_{n,k} \neq \emptyset$ if and only if $1 \leq k \leq n/2$.

Similarity relations may be coded as integer sequences in the following manner: if R is any similarity relation on $[n]$, then an integer sequence $\alpha_R = \alpha_1\alpha_2 \cdots \alpha_n$ is defined by:

$$x - \alpha_x \text{ is the smallest } y \in [n] \text{ such that } yRx, \quad \text{for } x \in [n]. \tag{2}$$

α_R will have the following property:

$$\alpha_1 = 0, \quad 0 \leq \alpha_{x+1} \leq \alpha_x + 1, \quad \text{for } 1 \leq x \leq n-1. \tag{3}$$

Conversely, for every integer sequence $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ with property (3) there is a unique similarity relation R on $[n]$ with $\alpha_R = \alpha$: R is completely specified by rule (2), reflexivity, symmetry and property (1).

Note that the number of classes in the transitive closure of a similarity relation R equals the number of zero-components in the corresponding sequence α_R; furthermore, R is nonsingular if and only if every 0-component of α_R is followed by a 1-component, in particular: $\alpha_n \neq 0$.

Theorem. *For $n \geq 1$, $1 \leq k \leq n/2$, there is a bijection between $B_{n,k}$ and $A_{n,2k}$.*

Proof. Take any sequence $\beta = \beta_1\beta_2 \cdots \beta_n$ which represents a member of $B_{n,k}$; this sequence contains k components equal to zero. Let

$$1 = x_1 < x_2 < \cdots < x_k \leq n-1$$

denote the indices of these components, and put $x_{k+1} = n+1$ for convenience. Define y_i to be the largest index y such that $x_i < y < x_{i+1}$ and $\beta_y = 1$, $1 \leq i \leq k$. By the above remark about nonsingularity the y_i are well defined. Now a sequence $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ is defined as follows:

$$\begin{aligned} \alpha_{x_i+j} &= \beta_{x_i+j} && \text{if } 0 \leq j < y_i - x_i, && 1 \leq i \leq k; \\ \alpha_{y_i+j} &= \beta_{y_i+j} - 1 && \text{if } 0 \leq j < x_{i+1} - y_i, && 1 \leq i \leq k. \end{aligned} \tag{4}$$

The sequence α certainly has property (3) and has $2k$ components equal to zero, thus it represents a member of $A_{n,2k}$.

Conversely, given any $A_{n,2k}$-sequence $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$, let

$$1 = x_1 < y_1 < x_2 < y_2 < \cdots < x_k < y_k \leq n$$

denote the indices of the zero-components, and again put $x_{k+1} := n+1$. Use equations (4) to define a $B_{n,k}$-sequence $\beta = \beta_1\beta_2 \cdots \beta_n$ – this is just the inverse mapping. This proves the theorem.

Let now

$$a_{n,k} := \text{card } A_{n,k} \qquad (1 \leq k \leq n), \quad b_{n,k} := \text{card } B_{n,k} \qquad (1 \leq k \leq n/2).$$

Lemma.

(i) $$a_{n,k} = \binom{2n-k-1}{n-1} - \binom{2n-k-1}{n}, \qquad 1 \leq k \leq n;$$

(ii) $$b_{n,k} = \binom{2n-2k-1}{n-1} - \binom{2n-2k-1}{n}, \qquad 1 \leq k \leq n/2.$$

Proof. First note that

$$a_{n,k} = a_{n,k+1} + a_{n-1,k-1}, \qquad n \geq 2, \tag{5}$$

where it is understood that $a_{n,k} = 0$ if $k \notin [n]$.

This can be seen as follows: let $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ be any $A_{n,k}$-sequence; either $\alpha_n = 0$, then $\alpha_1\alpha_2 \cdots \alpha_{n-1}$ is a $A_{n-1,k-1}$-sequence, or $\alpha_n \neq 0$, then let m denote the largest index such that $\alpha_m = 1$ and replace α_x by $\alpha_x - 1$ for $m \leq x \leq n$, which yields a $A_{n,k+1}$-sequence. Both procedures are invertible, hence (5) holds. Now (i) follows by induction, starting with $a_{1,1} = 1$. (ii) is a consequence of (i) and the theorem. In particular, note that

$$a_{n,1} = a_{n,2} = \frac{1}{n}\binom{2n-2}{n-1} = c_{n-1},$$

the $(n-1)$-th Catalan number; from this observation and (5) it is immediate that

$$\sum_{1\leqslant k\leqslant n} a_{nk} = c_n. \tag{6}$$

Let d_n denote the number of nonsingular similarity relations on $[n]$, $n \geqslant 2$. The first few values of the d_n-sequence are: $1, 2, 6, 18, 57, 186, 622, 2120, \ldots$. This sequence is called *Fine's sequence*, [1, p. 352], [2, p. 89], [3, entry 635].

Corollary.

(i) $$d_n = \sum_{1\leqslant k\leqslant n/2} \binom{2n-2k-1}{n-1} - \binom{2n-2k-1}{n}, \qquad n \geqslant 2.$$

(ii) $$2d_n + d_{n-1} = c_n, \qquad n \geqslant 3.$$

Proof. (i) follows from the lemma; this is precisely Shapiro's result about the diagonal-sums in his "Catalan-triangle", [2].

(ii) has already been noted by Shapiro; here it is a simple consequence of (5) and (6), in loose notation:

$$c_n = \sum a_{n,2j} + \sum a_{n,2j-1} = 2\sum a_{n,2j} + \sum a_{n-1,2j-2} = 2d_n + d_{n-1}.$$

References

[1] T. Fine, Extrapolation when very little is known about the source, Information and Control 16 (1970) 331–359.

[2] L. Shapiro, A Catalan triangle, Discrete Math. 14 (1976) 83–90.

[3] N.J.A. Sloane, A Handbook of Integer Sequences (Academic Press, NY, 1973).