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Non-Gaussian statistics, Maxwellian derivation and stellar polytropes

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A B S T R A C T

In this letter we discuss two aspects of non-Gaussian statistics. In the first, we show that Maxwell's first derivation of the stationary distribution function for a dilute gas can be extended in the context of Kaniadakis statistics. In the second, by investigating the stellar system, we study the Kaniadakis analytical relation between the entropic parameter κ and the stellar polytrope index n. We compare also the Kaniadakis relation $n = n(\kappa)$ with $n = n(q)$ proposed in the Tsallis framework.

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1. Introduction

Nonextensive statistical mechanics (NSM) [1] and extensive generalized power-law statistics [2–7] are based on the mathematical generalization of the Boltzmann exponential distribution, namely:

$$f \sim \exp(-Energy/Thermal\ energy). \tag{1}$$

Whereas in Ref. [1], the power-law distribution, $f \sim [1 - (1-q)Energy/Thermal\ energy]^{1/(1-q)}$, obeys the Tsallis distribution of NSM [1], the so-called κ-statistics [2–7] provides a power-law distribution function and the κ-entropy which emerges in the context of special relativity and in the *kinetic interaction principle*. Formally, the κ-framework is defined by considering the expressions

$$\exp_\kappa(f) = (\sqrt{1 + \kappa^2 f^2} + \kappa f)^{1/\kappa}, \tag{2}$$

$$\ln_\kappa(f) = \frac{f^\kappa - f^{-\kappa}}{2\kappa}, \tag{3}$$

where the κ-entropy associated with κ-statistics is given by [2–7]

$$S_\kappa = -\int d^3p f \ln_\kappa f = -\langle \ln_\kappa(f) \rangle, \tag{4}$$

with $\kappa^2 < 1$. The above functions reduce to the standard results in the limit $\kappa = 0$.

κ-statistics lead to a generalized framework with interesting mathematical properties [8–10], as well as a connection with the generalized Smoluchowski equation [11] and the relativistic nuclear equation of state for nuclear matter [12]. It was also shown that it is possible to obtain a consistent form for the entropy which is linked with a two-parameter deformation of the logarithm function and generalizes the Tsallis, Abe and Kaniadakis logarithmic behaviors [13]. Some

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physical systems are well approximated by a distribution that maximizes Kaniadakis' entropy, namely, cosmic ray flux, rain events in meteorology [3], quark–gluon plasma [14], kinetic models describing a gas of interacting atoms and photons [15], fracture propagation phenomena [16], and constructing financial models [17]. On the theoretical front, some studies on the canonical quantization of a classical system have also been investigated [18], as well as the *H*-theorem in relativistic and non-relativistic domain [19,20].

In the astrophysical domain, the first application has been the simulation in relativistic plasmas. In this regard, the power-law energy distribution provides a strong argument in favor of the Kaniadakis statistics [21]. Additionally, the viability of non-Gaussian statistics has been investigated from a stellar astrophysics viewpoint: the distributions of projected rotational velocity measurements of stars in the Pleiades open cluster [22], in the main sequence field stars [23], and in the estimation of the mean angle of inclination of the rotational axes of the stars in the Orion Nebula Cloud [24], as well as the strong dependence between the stellar–cluster ages and the power-law distributions [25].

The aim of this letter is twofold. First, to show that Maxwell's first derivation of the stationary distribution function for a dilute gas can be extended in the context of Kaniadakis statistics;.Second, considering the principle of maximum entropy for a stellar self-gravitating system, to investigate an analytical relation between entropic parameter κ and stellar polytrope index *n*. It is also shown that the function $n = n(\kappa)$ has a similar behavior to the Tsallis expression $n = n(q)$ [26,27].

This letter is organized as follows. In Section 2, we show the correspondence between the κ-statistics introduced by Kaniadakis and the velocity distribution for a Maxwellian gas, by assuming a non-Gaussian generalization of the separability hypothesis originally proposed by Maxwell [28]. In Section 3, we discuss a connection between Kaniadakis statistics and the polytropic index in the context of the self-gravitating system and we compare our results with ones studied in Refs. [26,27]. Finally, Section 4 is devoted to conclusions and discussion.

2. Non-Gaussian Maxwellian distribution function

In this section, in order to introduce the generalization of the Maxwell distribution in the context of the Kaniadakis framework, let us consider a spatially homogeneous gas, assumed to be in equilibrium (or in a stationary state) at temperature *T*, in such a way that $F(v)d^3v$ is the number of particles with velocity *v* in the volume element d^3v around *v*. In Maxwell's derivation, the three-dimensional distribution is factorized and depends only on the magnitude of the velocity [28,29]

$$F(v)\mathrm{d}^3v = f(v_x)f(v_y)f(v_z)\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z, \tag{5}$$

where $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ and $F(v)$ is the standard Maxwellian distribution function, given by

$$F(v) = A\exp(-mv^2/k_BT), \tag{6}$$

where $A = \left(\frac{m}{2\pi k_BT}\right)^{3/2}$ is the normalization constant.

In reality, in the κ-statistics context described by (4), the starting basic hypothesis (5), which takes into account the isotropy of all velocity directions, must be somewhat modified. From a statistical viewpoint, Maxwell's ansatz assumes that the three components of the velocity are statistically independent. However, this property does not hold in the systems endowed with long range interactions, or statistically dependent, where the Kaniadakis distribution has been observed [22,23]. Notice that the Maxwell ansatz is equivalent to expressing ln *F* as the sum of the logarithms of the one-dimensional distribution functions associated with each velocity component. A simple and natural way to generalize this procedure within the Kaniadakis framework would be to introduce statistical dependence between the velocity components, e.g., to replace the usual product between $f(v_x), f(v_y)$ and $f(v_z)$ by a κ-exponential of the sum of $\ln_\kappa$ of the $f(v_i)$, $i = x, y, z$. From a physical viewpoint, the statistical dependence allows introducing a distribution that has a better fit than the Maxwellian in the statistical description of some physical systems (see, e.g., Refs. [12,21–23,25]).[1] Therefore, in order to recover the ordinary logarithmic ansatz as a particular limiting case, it is convenient to express the power generalization in terms of the function $\ln_\kappa$ defined by Eq. (3), which is a combination of a power function plus appropriate constants. Mathematically, the consistent κ-generalization of (5) is given by

$$F\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right)\mathrm{d}^3v = \exp_\kappa[\ln_\kappa f(v_x) + \ln_\kappa f(v_y) + \ln_\kappa f(v_z)]\mathrm{d}v_x\mathrm{d}v_y\mathrm{d}v_z, \tag{7}$$

where the κ-exponential and κ-logarithm are given by identities (2) and (3). In particular, in the limit $\kappa = 0$ the standard expression (5) is recovered. Note also that $\ln_\kappa[\exp_\kappa(f)] = \exp_\kappa[\ln_\kappa(f)] = f$, and $\frac{\mathrm{d}\ln_\kappa(f)}{\mathrm{d}x} = \frac{f^\kappa + f^{-\kappa}}{2f}\frac{\mathrm{d}f}{\mathrm{d}x}$ are satisfied. The logarithmic derivative of (7) with respect to v_i is

$$\frac{\partial}{\partial v_i}\ln_\kappa F\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right) = \frac{\partial}{\partial v_i}\ln_\kappa\{\exp_\kappa[\ln_\kappa f(v_x) + \ln_\kappa f(v_y) + \ln_\kappa f(v_z)]\}, \tag{8}$$

with $i = x, y, z$.

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[1] Obviously this is not a unique generalization. For example, in the Tsallis framework it is possible to introduce statistical dependence between velocity components considering the *q*-generalization of the Maxwell ansatz (see, e.g., Refs. [30]).

Using the above mentioned properties we can write[2]

$$\frac{F^\kappa + F^{-\kappa}}{2F} \frac{\partial}{\partial v_i} F\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right) = \frac{\partial}{\partial v_i}[\ln_\kappa f(v_i)]. \tag{9}$$

Equivalently,

$$\frac{F^\kappa + F^{-\kappa}}{2F} \frac{F'(\chi)}{\chi} = \frac{1}{v_i} \frac{\partial}{\partial v_i} \ln_\kappa f(v_i), \tag{10}$$

where $\chi = \sqrt{v_x^2 + v_y^2 + v_z^2}$ and a prime represents the total derivative. Now, defining

$$\Phi(\chi) = \frac{F^\kappa + F^{-\kappa}}{2F} \frac{F'(\chi)}{\chi}, \tag{11}$$

and considering the replacement of the partial derivative by an ordinary derivative in Eq. (10) due to the partial $\kappa - ln$ differentiation of the generalized ansatz (7) with respect to any component v_i, we obtain

$$\Phi(\chi) = \frac{1}{v_i} \frac{\mathrm{d}}{\mathrm{d} v_i}[\ln_\kappa f(v_i)]. \tag{12}$$

The second member of Eq. (12) depends only on v_i, with $i = x, y$ or z. Hence, Eq. (11) can be satisfied only if all its members are equal to one and the same constant, not depending on any of the velocity components. Thus we can make $\Phi(\chi) = -m\gamma$, obtaining

$$\frac{1}{v_i} \frac{\mathrm{d}}{\mathrm{d} v_i}[\ln_\kappa f(v_i)] = -m\gamma, \tag{13}$$

where $\gamma = \frac{1}{k_B T}$.

The solution of Eq. (13) is given by

$$\ln_\kappa f(v_i) = -\frac{m\gamma v_i^2}{2}, \tag{14}$$

which provides

$$f(v_i) = \exp_\kappa\left(-\frac{m v_i^2}{2k_B T}\right). \tag{15}$$

In order to calculate the complete distribution, we insert the expressions (15) into Eq. (7) to obtain

$$F(v) = \left[\sqrt{1 + \kappa^2 \left(-\frac{m v^2}{2k_B T}\right)^2} - \frac{\kappa m v^2}{2k_B T}\right]^{\frac{1}{\kappa}}. \tag{16}$$

If we assume the function $f(v_i)$ to be normalizable, we can write

$$f(v_i) = \frac{1}{Z} \exp_\kappa\left(-\frac{m v_i^2}{2k_B T}\right), \tag{17}$$

where $1/Z$ is the κ-normalization constant.

In order to calculate the so-called κ-normalization, let us introduce the expression [3]

$$Z = \int_{\mathcal{R}} \mathrm{d}^n v \exp_\kappa\left(-\frac{m v^2}{2k_B T}\right), \tag{18}$$

where n (=1, 2, 3) is the number of degrees of freedom. Here, considering $n = 1$, $a = \frac{m}{2k_B T}$ and $x = av^2$, we obtain

$$Z = \frac{1}{\sqrt{a}} \int_0^\infty x^{-1/2} \exp_\kappa(-x)\mathrm{d}x. \tag{19}$$

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[2] It is worth mentioning that a repeated index does not mean a summation over the index.

Using the generalized gamma functions [3]

$$\int_0^\infty x^{r-1} \exp_\kappa(-x) dx = \frac{[1 + (r - 2|\kappa|)|2\kappa|^r]}{[1 - (r-1)|\kappa|]^2 - \kappa^2} \frac{\Gamma\left(\frac{1}{|2\kappa|} - \frac{r}{2}\right)}{\Gamma\left(\frac{1}{|2\kappa|} + \frac{r}{2}\right)} \Gamma\left(\frac{1}{2}\right),$$ (20)

and after some algebra, we obtain

$$Z = \sqrt{\frac{\pi k_B T}{m}} \frac{|\kappa|^{-1/2}}{[1 - \frac{1}{2}|\kappa|]} \frac{\Gamma\left(\frac{1}{|2\kappa|} - \frac{1}{4}\right)}{\Gamma\left(\frac{1}{|2\kappa|} + \frac{1}{4}\right)}.$$ (21)

It is easy to see that the standard Maxwellian result $(1/Z) = (m/2\pi k_B T)^{1/2}$ is recovered in the Gaussian limit $\kappa = 0$.

Using the expression (18) for $n = 3$, we can show that the normalization for the complete distribution is given by

$$\frac{1}{Z} = \left(\frac{m|\kappa|}{\pi k_B T}\right)^{3/2} \left[1 + \frac{3}{2}|\kappa|\right] \frac{\Gamma\left(\frac{1}{|2\kappa|} + \frac{3}{4}\right)}{\Gamma\left(\frac{1}{|2\kappa|} - \frac{3}{4}\right)}.$$ (22)

As expected, the above κ-distribution is isotropic meaning that all velocity directions are equivalent in this generalized context. Here, we emphasize that in the limit $\kappa = 0$ both the normalization and the distribution function reproduce the standard Maxwellian (6).

3. Non-Gaussian framework and stellar polytropes

As mentioned in the introduction, we shall discuss a connection between the Kaniadakis statistics and polytropic index in the context of the self-gravitating system. Following a procedure considered in Refs. [26,27], let us start with the Kaniadakis generalized entropy of index κ of the form

$$S_\kappa(f) = -\int \frac{f^{1+\kappa} - f^{1-\kappa}}{2\kappa} d\Omega,$$ (23)

where $d\Omega = d^3\mathbf{r} d^3\mathbf{v}$, and the parameter $\kappa \neq 0$ provides a possible generalization of the Boltzmann–Gibbs entropy. The extremum entropy state can be derived by varying S_κ with respect to f. Using the Lagrange multipliers α and β, the extremum solution subject to constraints[3] $M = \int f d\Omega$ and $E = K + U = \frac{1}{2}\int v^2 f d\Omega + \frac{1}{2}\int \phi f d\Omega$ is obtained from

$$\delta S_\kappa - \alpha \delta M - \beta \delta E = 0$$ (24)

which leads to

$$\int \left\{ -\frac{1}{2\kappa}[(1 + \kappa)f^\kappa - (1 - \kappa)f^{-\kappa}] - \alpha - \beta\left(\frac{v^2}{2} + \phi\right) \right\} \delta f d\Omega = 0.$$ (25)

We have used the relation $\int \delta \phi f d\Omega = \int \phi \delta f d\Omega$ for derivation of the above expression. Here, ϕ denotes gravitational potential, and Eq. (24) must be satisfied independently of the choice of δf. Thus, we obtain the following distribution function

$$f(\mathbf{r}, \mathbf{v}) = B\left[\phi_0(\epsilon) - \phi(\mathbf{r}) - \frac{v^2}{2}\right]^{1/\kappa}$$ (26)

where the constants B and ϕ_0 are defined by

$$B = \left(\frac{\kappa\beta}{(1 + \kappa)}\right)^{1/\kappa}$$

$$\phi_0(\epsilon) = \frac{-\sqrt{\kappa^2(-\alpha - \beta\epsilon)^2 + 1 - \kappa^2} - \alpha\kappa}{\beta\kappa},$$ (27)

$\epsilon = \frac{v^2}{2} + \phi$.

In order to obtain a relation between the stellar polytrope index n and entropic parameter κ, let us now introduce the polytropic sphere distribution

$$f \sim \mathcal{E}^{n-3/2},$$ (28)

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[3] The mass and energy total of the self-gravitating system governed by the Vlasov and Poisson equations. For more details, see Ref. [31].

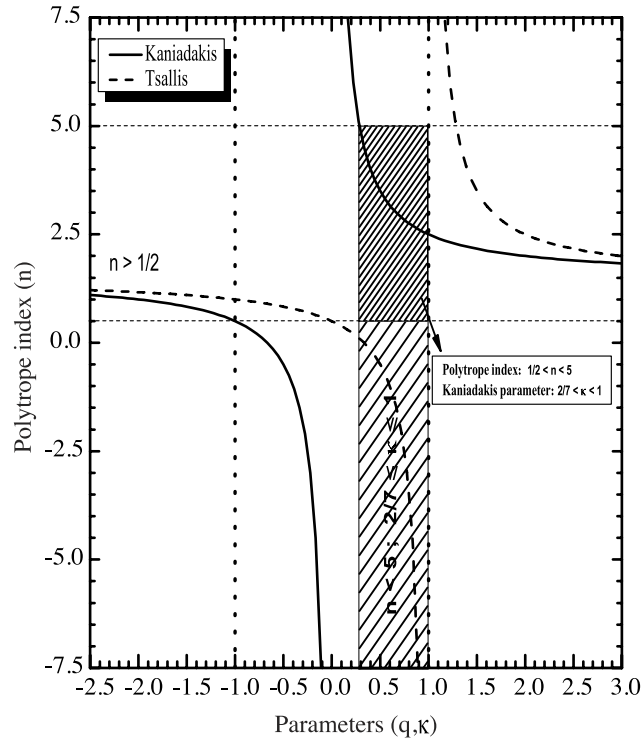**Fig. 1.** Polytrope index n as a function of the Kaniadakis (κ) and Tsallis (q) parameters. The vertical dotted lines mark the limits of validity of κ-statistical ($\kappa^2 < 1$) [2–7].

where $\mathcal{E}$ is the relative energy of a star, given by [31]

$$\mathcal{E} = \Psi - \frac{1}{2}v^2, \tag{29}$$

and Ψ is the relative potential of a star associated with ϕ. Therefore, comparing the distribution (26) with (28), we have

$$n = \frac{3}{2} + \frac{1}{\kappa}, \tag{30}$$

where the Gaussian limit $\kappa = 0$ corresponds to the so-called isothermal spheres. Let us now compare the expression $n = n(\kappa)$ with the Tsallis relation $n = n(q)$. In this regard, Plastino and Plastino [26,27] have introduced an expression given by

$$n = \frac{3}{2} + \frac{1}{q-1}, \tag{31}$$

which includes the isothermal situation for $n \to \infty$, i.e. the Gaussian limit $q = 1$. In order to guarantee the conservation of mass and energy, Tsallis' parameter q should be larger than 9/7. By comparing Eq. (30) with Eq. (31), we find a linear relation between κ and q, for a given n considered:

$$\kappa = q - 1. \tag{32}$$

In particular, the Gaussian limits $\kappa = 0$ and $q = 1$ are satisfied simultaneously in Eq. (32). Note yet that, considering the linear relation and the limit of validity of κ- statistics, i.e. $\kappa^2 < 1$, we obtain the constraint $q \in [0, 2]$ on the nonextensive parameter which is fully consistent with the results obtained from several independent investigations involving the Tsallis nonextensive approach (see, e.g. Refs. [37–39]).

In Fig. 1 we display the dependence between the entropic indexes ($\kappa; q$) and the polytropic index (n), given by Eqs. (30) and (31), respectively. We show that the polytropic index diverges for the non-Gaussian parameters $\kappa = 0$, $q = 1$, i.e. the isothermal spheres separate the polytropes into two branches. For $\kappa > 0$ and $q > 1$, n ranges from ∞ to 3/2. From $f \sim \mathcal{E}^{n-3/2}$, we see that $\mathcal{E}$ is a decreasing function of the energy and, when $n = 3/2$, the distribution function becomes a constant independent of energy. On the other hand, it is well known that, for any astrophysical system, n should be positive and higher than 1/2 [31]. Thus the intervals $0 \leq q \leq 1$ and $-1 \leq \kappa \leq 0$ represent forbidden regions, since the values of the n index tend to be smaller than 1/2 and negative.

It is also worth observing that $\kappa < -1$ and $q < 0$ also provide $n > 1/2$. However these limits on κ and q violate the validity of κ-statistics [2–7] and the nonextensive H-theorem [32], respectively. As is well known, for the polytropic index $n > 5$, the density falls off so slowly at large radii that the mass is infinite [31]. Fig. 1 shows that only the branches on the right-hand side are physically significant for $n < 5$. Therefore, the physical values of the Tsallis parameter q should be larger than 9/7,[4] and the Kaniadakis parameter κ should be constrained to the interval of validity $\kappa \in [2/7; 1]$ represented by the lower dashed rectangle. From the linear relation (32) and using the interval of validity for κ, we obtain the constraint $q \in [9/7; 2]$, which coincides with the one $q > 9/7$ calculated in Refs. [26,35]. It is also worth noting that the polytropic indices obtained from Kaniadakis' statistics are restricted to the range $5/2 \leq n \leq 5$, which excludes important stellar polytopes of index $n = 3/2$, i.e., the models of adiabatic stars supported by pressure of a non-relativistic gas [33].

Summing up, a close examination of Fig. 1 tells us that the two distributions studied here present a similar behavior for stellar polytropes, although the Kaniadakis function is more restrictive than Tsallis'. In reality, in order to know which framework is better, we should do a study based on observational data, e.g. an investigation considering the comparison between stellar polytropes and Navarro–Frenk–White halo models for the description of dark matter halos (see Ref. [34]). In this regard, an analysis considering this issue is currently under investigation.

4. Conclusions

In this work we have studied Kaniadakis statistics based on the generalized Maxwellian formulation for the κ-statistics and, as an application, we investigate the physical effect on the stellar polytropic system. In the first part, we conclude that there is a Kaniadakis velocity distribution given by Eq. (16) that is uniquely determined by the requirements of (i) isotropy and (ii) a generalization of the factorizability condition. From the physical viewpoint, this generalization introduces statistical dependence between the velocity components, when we replace the logarithm function by a power law (a similar argument was considered in Refs. [32,36–38,40] for the non-Gaussian generalization of the molecular chaos hypothesis). In particular, Maxwell expressions are recovered in the Gaussian limit, $\kappa \to 0$.

From an application perspective, we have shown that the expressions for Tsallis and Kaniadakis, given by Eqs. (30) and (31), present similar behaviors. However, the astrophysical limit on the polytropic index $1/2 < n < 5$ provides the constraint $\kappa \in [2/7; 1]$ and $q > 9/7$ for the Kaniadakis and Tsallis parameters, respectively. It is worth mentioning that the Gaussian limit $\kappa = 0$, equivalent to the Tsallis expression Eq. (31) for $q = 1$, reproduces Maxwellian isothermal spheres or, equivalently, $n = \infty$.

Acknowledgments

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