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Six new Redheffer-type inequalities for circular and hyperbolic functions

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Abstract

In this paper, six new Redheffer-type inequalities involving circular functions and hyperbolic functions are established.

Keywords: Redheffer-type inequalities; Circular functions; Hyperbolic functions

1. Introduction

Redheffer [1] proposed the inequality

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2}, \quad x \in (0, \pi]. \tag{1}$$

Chen, Zhao, and Qi [2] obtained the following three Redheffer-type inequalities

$$\cos x \geq \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}, \quad x \in \left[0, \frac{\pi}{2}\right], \tag{2}$$

$$\cosh x \leq \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}, \quad x \in \left[0, \frac{\pi}{2}\right), \tag{3}$$

$$\frac{\sinh x}{x} \leq \frac{\pi^2 + x^2}{\pi^2 - x^2}, \quad x \in (0, \pi). \tag{4}$$

Recently, some extensions of inequalities (2)–(4) for Bessel functions have been shown in Baricz [3].

In this paper, we shall extend and sharpen the inequalities (1) and (2) above, and show a new Redheffer-type inequality for $\tan x$ as follows.

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Theorem 1. *Let* $0 < x < \pi$. *Then*

$$\left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{\beta} \le \frac{\sin x}{x} \le \left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{\alpha} \tag{5}$$

holds if and only if $\alpha \le \pi^2/12$ *and* $\beta \ge 1$.

Theorem 2. *Let* $0 \le x \le \pi/2$. *Then*

$$\left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^{\beta} \le \cos x \le \left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^{\alpha} \tag{6}$$

holds if and only if $\alpha \le \pi^2/16$ *and* $\beta \ge 1$.

Theorem 3. *Let* $0 < x < \pi/2$. *Then*

$$\left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\alpha} \le \frac{\tan x}{x} \le \left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\beta} \tag{7}$$

holds if and only if $\alpha \le \pi^2/24$ *and* $\beta \ge 1$.

Corresponding to Theorems 1–3, we show three new Redheffer-type inequalities for hyperbolic functions.

Theorem 4. *Let* $0 < x < r$. *Then*

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} \le \frac{\sinh x}{x} \le \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{8}$$

holds if and only if $\alpha \le 0$ *and* $\beta \ge r^2/12$.

Theorem 5. *Let* $0 \le x < r$. *Then*

$$\left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\alpha} \le \cosh x \le \left(\frac{r^2 + x^2}{r^2 - x^2}\right)^{\beta} \tag{9}$$

holds if and only if $\alpha \le 0$ *and* $\beta \ge r^2/4$.

Theorem 6. *Let* $0 < x < r$. *Then*

$$\left(\frac{r^2 - x^2}{r^2 + x^2}\right)^{\beta} \le \frac{\tanh x}{x} \le \left(\frac{r^2 - x^2}{r^2 + x^2}\right)^{\alpha} \tag{10}$$

holds if and only if $\alpha \le 0$ *and* $\beta \ge r^2/6$.

Remark 1. Let $\alpha = 0$ in (8) and (10), then

$$\tanh x \le x \le \sinh x, \quad x \ge 0, \tag{11}$$

which can be found in Bullen [4, p. 9].

2. Five lemmas

Lemma 1 (*[5, Theorem 3.4]*). *Let* B_{2n} *be the even-indexed Bernoulli numbers, and* $\zeta(\cdot)$ *the Riemann's zeta function. Then*

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|, \quad n = 1, 2, \ldots. \tag{12}$$

(For further comprehension of the even-indexed Bernoulli numbers B_{2n}*, refer to pp. 231–232 in* [6].*)*

Lemma 2. *Let* $0 \le x < \pi/2$. *Then*

$$\tan x = \sum_{n=1}^{\infty} \frac{2(2^{2n}-1)}{\pi^{2n}} \zeta(2n) x^{2n-1}. \tag{13}$$

Proof. The following power series expansion can be found in [7, 1.3.1.4 (3)]:

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} (-1)^{n-1} B_{2n} x^{2n-1} = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} |B_{2n}| x^{2n-1}, \quad |x| < \frac{\pi}{2}. \tag{14}$$

using the relational expression (12), we obtain (13). □

Lemma 3. *Let* $|x| < \pi$. *Then*

$$x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2\zeta(2n)}{\pi^{2n}} x^{2n}. \tag{15}$$

Proof. The following power series expansion can be found in [7, 1.3.1.4 (2)]:

$$x \cot x = 1 - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n}, \quad |x| < \pi. \tag{16}$$

using the relational expression (12), we obtain (15). □

Lemma 4 (*[8–11]*). *Let* $f, g : [a, b] \to \mathbb{R}$ *be two continuous functions which are differentiable on* (a, b). *Further, let* $g' \neq 0$ *on* (a, b). *If* f'/g' *is increasing (or decreasing) on* (a, b), *then the functions*

$$\frac{f(x) - f(b)}{g(x) - g(b)}$$

and

$$\frac{f(x) - f(a)}{g(x) - g(a)}$$

are also increasing (or decreasing) on (a, b).

Remark 2. This l'Hospital rule for monotonicity has become a standard tool and found wide application, reader can refer to [11] and references therein.

Lemma 5 (*[12–14]*). *Let* a_n *and* b_n $(n = 0, 1, 2, \ldots)$ *be real numbers, and let the power series* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *be convergent for* $|x| < R$. *If* $b_n > 0$ *for* $n = 0, 1, 2, \ldots$, *and if* a_n/b_n *is strictly increasing (or decreasing) for* $n = 0, 1, 2, \ldots$, *then the function* $A(x)/B(x)$ *is strictly increasing (or decreasing) on* $(0, R)$.

3. Proof of Theorem 1

Let $f(x) = \frac{\pi^2}{12} \log \frac{\pi^2 - x^2}{\pi^2 + x^2} - \log \frac{\sin x}{x}$. Then $f(0^+) = 0$, and

$$\begin{aligned} f'(x) &= \frac{\sin x - x\cos x}{x \sin x} - \frac{\pi^4 x}{3(\pi^4 - x^4)} \\ &= \frac{1}{x(\pi^2 + x^2)} \left[(\pi^2 + x^2)(1 - x\cot x) - \frac{\pi^4 x^2}{3(\pi^2 - x^2)} \right]. \end{aligned}$$

By Lemma 3, we have

$$f'(x) = \frac{1}{x(\pi^2+x^2)}\left[2(\pi^2+x^2)\sum_{n=1}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}x^{2n} - \frac{\pi^2}{3}\sum_{n=0}^{\infty}\left(\frac{x}{\pi}\right)^{2n}x^2\right]$$
$$= \frac{1}{x(\pi^2+x^2)}\left[2\zeta(2)x^2 + 2\sum_{n=2}^{\infty}\frac{\zeta(2n)+\zeta(2n-2)}{\pi^{2n-2}}x^{2n} - \frac{\pi^2}{3}x^2 - \frac{\pi^2}{3}\sum_{n=2}^{\infty}\frac{x^{2n}}{\pi^{2n-2}}\right]$$
$$= \frac{2}{x(\pi^2+x^2)}\sum_{n=2}^{\infty}\frac{\zeta(2n)+\zeta(2n-2)-\pi^2/6}{\pi^{2n-2}}x^{2n}.$$

Since $\zeta(2n)+\zeta(2n-2)-\pi^2/6 > 1+1-\pi^2/6 > 0$, we conclude that $f(x)$ is increasing on $(0,\pi)$. Then $f(x) > f(0^+) = 0$ for $x \in (0,\pi)$, and the following inequality

$$\frac{\sin x}{x} \le \left(\frac{\pi^2-x^2}{\pi^2+x^2}\right)^{\pi^2/12} \tag{17}$$

holds for $x \in (0,\pi)$.

By Redheffer's inequality (1) and the inequality (17), we have the double inequality as follows

$$\frac{\pi^2-x^2}{\pi^2+x^2} \le \frac{\sin x}{x} \le \left(\frac{\pi^2-x^2}{\pi^2+x^2}\right)^{\pi^2/12}, \quad x \in (0,\pi]. \tag{18}$$

Let $F(x) = \frac{\log\frac{\sin x}{x}}{\log\frac{\pi^2-x^2}{\pi^2+x^2}}$. Then $F(0^+) = \frac{\pi^2}{12}$, and $F(\pi^-) = 1$. So 1 and $\frac{\pi^2}{12}$ are the best constants in (18), the proof of Theorem 1 is complete.

4. Proof of Theorem 2

Let $g(x) = \frac{\pi^2}{16}\log\frac{\pi^2-4x^2}{\pi^2+4x^2} - \log\cos x$. Then $g(0) = 0$, and

$$g'(x) = \tan x - \frac{\pi^4 x}{(\pi^2-4x^2)(\pi^2+4x^2)} = \frac{1}{\pi^2+4x^2}\left[(\pi^2+4x^2)\tan x - \frac{\pi^4 x}{\pi^2-4x^2}\right].$$

By Lemma 2, we have

$$g'(x) = \frac{1}{\pi^2+4x^2}\left[(\pi^2+4x^2)\sum_{n=1}^{\infty}\frac{2(2^{2n}-1)\zeta(2n)}{\pi^{2n}}x^{2n-1} - \sum_{n=1}^{\infty}\frac{2^{2n-1}\pi^2}{\pi^{2n-2}\cdot 2}x^{2n-1}\right]$$
$$= \frac{1}{\pi^2+4x^2}\left[6\zeta(2)x + \sum_{n=2}^{\infty}\frac{2(2^{2n}-1)\zeta(2n)+8(2^{2n-2}-1)\zeta(2n-2)}{\pi^{2n-2}}x^{2n-1}\right.$$
$$\left. - \left(\pi^2 x + \sum_{n=2}^{\infty}\frac{2^{2n-1}}{\pi^{2n-2}}\frac{\pi^2}{2}x^{2n-1}\right)\right]$$
$$= \frac{1}{\pi^2+4x^2}\sum_{n=2}^{\infty}\frac{2(2^{2n}-1)\zeta(2n)+8(2^{2n-2}-1)\zeta(2n-2)-(\pi^2/2)2^{2n-1}}{\pi^{2n-2}}x^{2n-1}.$$

Since $2(2^{2n}-1)\zeta(2n)+8(2^{2n-2}-1)\zeta(2n-2)-(\pi^2/2)2^{2n-1} > 2(2^{2n}-1)+8(2^{2n-2}-1)-(\pi^2/2)2^{2n-1} = 2^{2n}(4-\frac{\pi^2}{4})-10 > 16(4-\frac{\pi^2}{4})-10 > 0$ for $n \ge 2$, we conclude that $g(x)$ is increasing on $[0,\pi/2)$. Then $g(x) \ge g(0) = 0$ for $x \in [0,\pi/2)$, and the following inequality

$$\cos x \le \left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\pi^2/16} \tag{19}$$

holds for $x \in [0,\pi/2)$.

By inequality (2) and the inequality (19), we have the double inequality

$$\frac{\pi^2-4x^2}{\pi^2+4x^2} \le \cos x \le \left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\pi^2/16}, \quad x \in [0, \pi/2]. \tag{20}$$

Let $G(x) = \frac{\log\cos x}{\log\frac{\pi^2-4x^2}{\pi^2+4x^2}}$. Then $G(0^+) = \frac{\pi^2}{16}$, and $G(\frac{\pi}{2}^-) = 1$. So 1 and $\frac{\pi^2}{16}$ are the best constants in (20), and the proof of Theorem 2 is complete.

5. Proof of Theorem 3

Let $h(x) = \log\frac{\tan x}{x} - \frac{\pi^2}{24}\log\frac{\pi^2+4x^2}{\pi^2-4x^2}$. Then $h(0^+) = 0$, and

$$\begin{aligned} h'(x) &= \frac{x\sec^2 x - \tan x}{x\tan x} - \frac{2\pi^4 x}{3(\pi^2+4x^2)(\pi^2-4x^2)} \\ &= \frac{1}{\pi^2+4x^2}\left[(\pi^2+4x^2)\left(\tan x + \frac{x\cot x - 1}{x}\right) - \frac{2}{3}\pi^4\frac{x}{\pi^2-4x^2}\right]. \end{aligned}$$

By Lemmas 2 and 3, we have

$$\begin{aligned} h'(x) &= \frac{1}{\pi^2+4x^2}\left[(\pi^2+4x^2)\left(\sum_{n=1}^{\infty}\frac{2(2^{2n}-1)\zeta(2n)}{\pi^{2n}}x^{2n-1} - 2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}x^{2n-1}\right) - \sum_{n=1}^{\infty}\frac{2^{2n-1}}{\pi^{2n-2}}\frac{\pi^2}{3}x^{2n-1}\right] \\ &= \frac{1}{\pi^2+4x^2}\left[4\zeta(2)x + 2\sum_{n=2}^{\infty}\frac{(2^{2n}-2)\zeta(2n)+4(2^{2n-2}-2)\zeta(2n-2)}{\pi^{2n-2}}x^{2n-1}\right. \\ &\quad \left. - \left(\frac{2}{3}\pi^2 x + \sum_{n=2}^{\infty}\frac{2^{2n-1}}{\pi^{2n-2}}\frac{\pi^2}{3}x^{2n-1}\right)\right] \\ &= \frac{2}{\pi^2+4x^2}\sum_{n=2}^{\infty}\frac{(2^{2n}-2)\zeta(2n)+4(2^{2n-2}-2)\zeta(2n-2)-(\pi^2/6)2^{2n-1}}{\pi^{2n-2}}x^{2n-1}. \end{aligned}$$

Since $(2^{2n}-2)\zeta(2n) + 4(2^{2n-2}-2)\zeta(2n-2) - (\pi^2/6)2^{2n-1} > 2^{2n} - 2 + 4(2^{2n-2}-2) - (\pi^2/6)2^{2n-1} = 2^{2n}(2-\frac{\pi^2}{12}) - 10 > 16(2-\frac{\pi^2}{12}) - 10 > 0$ for $n \ge 2$, we conclude that $h(x)$ is increasing on $(0, \pi/2)$. Then $h(x) > h(0^+) = 0$ for $x \in (0, \pi/2)$, and the following inequality

$$\left(\frac{\pi^2+4x^2}{\pi^2-4x^2}\right)^{\pi^2/24} \le \frac{\tan x}{x} \tag{21}$$

holds for $x \in (0, \pi/2)$.

In a similar way, the inequality

$$\frac{\tan x}{x} \le \frac{\pi^2+4x^2}{\pi^2-4x^2} \tag{22}$$

holds for $x \in (0, \pi/2)$.

Combining inequality (21) and the inequality (22), we have the double inequality as follows

$$\left(\frac{\pi^2+4x^2}{\pi^2-4x^2}\right)^{\pi^2/24} \le \frac{\tan x}{x} \le \frac{\pi^2+4x^2}{\pi^2-4x^2}, \quad x \in (0, \pi/2). \tag{23}$$

Let $H(x) = \frac{\log\frac{\tan x}{x}}{\log\frac{\pi^2+4x^2}{\pi^2-4x^2}}$. Then $H(0^+) = \frac{\pi^2}{24}$, and $H(\frac{\pi}{2}^-) = 1$. So 1 and $\frac{\pi^2}{24}$ are the best constants in (23), and the proof of Theorem 3 is complete.

6. Proof of Theorem 4

Let $I(x) = \frac{\log \frac{\sinh x}{x}}{\log \frac{r^2+x^2}{r^2-x^2}} = \frac{f_1(x)}{g_1(x)}$, where $f_1(x) = \log \frac{\sinh x}{x}$, and $g_1(x) = \log \frac{r^2+x^2}{r^2-x^2}$. Then

$$\frac{f_1'(x)}{g_1'(x)} = \frac{1}{4r^2} \frac{(r^4 - x^4)(x \cosh x - \sinh x)}{x^2 \sinh x} = \frac{1}{4r^2} \frac{A(x)}{B(x)},$$

where

$$\begin{aligned} A(x) &= (r^4 - x^4)(x \cosh x - \sinh x) \\ &= \frac{2r^4}{3!} x^3 + \frac{4r^4}{5!} x^5 + \sum_{n=3}^{\infty} \left[\frac{r^4}{(2n)!} - \frac{r^4}{(2n+1)!} - \frac{1}{(2n-4)!} + \frac{1}{(2n-3)!} \right] x^{2n+1} \\ &= \sum_{n=1}^{\infty} a_n x^{2n+1}, \\ B(x) &= x^2 \sinh x = \frac{1}{1!} x^3 + \frac{1}{3!} x^5 + \sum_{n=3}^{\infty} \frac{x^{2n+1}}{(2n-1)!} \\ &= \sum_{n=1}^{\infty} b_n x^{2n+1}, \end{aligned}$$

and

$$\begin{aligned} a_1 &= \frac{2r^4}{3!}, \qquad a_2 = \frac{4r^4}{5!}, \\ a_n &= \frac{r^4}{(2n)!} - \frac{r^4}{(2n+1)!} - \frac{1}{(2n-4)!} + \frac{1}{(2n-3)!}, \\ b_1 &= 1, \qquad b_2 = \frac{1}{3!}, \qquad b_n = \frac{1}{(2n-1)!} > 0, \quad n \geq 3, n \in \mathbb{N}^+. \end{aligned}$$

So $\frac{a_1}{b_1} > \frac{a_2}{b_2} > \frac{a_3}{b_3}$, and for $n \geq 3$ we have

$$\begin{aligned} c_n = \frac{a_n}{b_n} &= \frac{\frac{r^4}{(2n)!} - \frac{r^4}{(2n+1)!} - \frac{1}{(2n-4)!} + \frac{1}{(2n-3)!}}{\frac{1}{(2n-1)!}} \\ &= \frac{2nr^4 - (2n+1)2n(2n-1)(2n-2)(2n-4)}{2n(2n+1)}. \end{aligned}$$

We conclude that c_n is decreasing for $n = 1, 2, \ldots$, and $\frac{f_1'(x)}{g_1'(x)} = \frac{1}{4r^2} \frac{A(x)}{B(x)}$ is decreasing on $(0, r)$ by Lemma 5. Thus $I(x) = \frac{f_1(x)}{g_1(x)} = \frac{f_1(x) - f_1(0^+)}{g_1(x) - g_1(0^+)}$ is decreasing on $(0, r)$ by Lemma 4.

Furthermore, $\lim_{x \to 0^+} I(x) = \frac{r^2}{12}$, and $\lim_{x \to r} I(x) = 0$. So the proof of Theorem 4 is complete.

7. Proof of Theorem 5

Let $J(x) = \frac{\log \cosh x}{\log \frac{r^2+x^2}{r^2-x^2}} = \frac{r_1(x)}{s_1(x)}$, where $r_1(x) = \log \cosh x$, and $s_1(x) = \log \frac{r^2+x^2}{r^2-x^2}$. Then

$$\frac{r_1'(x)}{s_1'(x)} = \frac{1}{4r^2} \frac{(r^4 - x^4) \sinh x}{x \cosh x} = \frac{1}{4r^2} \frac{R(x)}{S(x)},$$

where

$$R(x) = (r^4 - x^4) \sinh x = \frac{r^4}{1!} x + \frac{r^4}{3!} x^3 + \sum_{n=3}^{\infty} \left[\frac{r^4}{(2n-1)!} - \frac{1}{(2n-5)!} \right] x^{2n-1}$$

$$= \sum_{n=1}^{\infty} a_n x^{2n-1},$$

$$S(x) = x\cosh x = x + \frac{1}{2!}x^3 + \sum_{n=3}^{\infty} \frac{x^{2n-1}}{(2n-2)!}$$

$$= \sum_{n=1}^{\infty} b_n x^{2n-1},$$

$a_1 = r^4$, $a_2 = \frac{r^4}{3!}$, $a_n = \frac{r^4}{(2n-1)!} - \frac{1}{(2n-5)!}$, $b_1 = 1$, $b_2 = \frac{1}{2!}$, $b_n = \frac{1}{(2n-2)!} > 0$, $n \geq 3$, and $n \in \mathbb{N}^+$.

So $\frac{a_1}{b_1} > \frac{a_2}{b_2} > \frac{a_3}{b_3}$, and for $n \geq 3$ we have

$$c_n = \frac{a_n}{b_n} = \frac{\frac{r^4}{(2n-1)!} - \frac{1}{(2n-5)!}}{\frac{1}{(2n-2)!}}$$

$$= \frac{r^4 - (2n-1)(2n-2)(2n-3)(2n-4)}{2n-1}.$$

We conclude that c_n is decreasing for $n = 1, 2, \ldots$, and $\frac{r_1'(x)}{s_1'(x)} = \frac{1}{4r^2}\frac{R(x)}{S(x)}$ is decreasing on $[0, r)$ by Lemma 5. Thus $J(x) = \frac{r_1(x)}{s_1(x)} = \frac{r_1(x) - r_1(0)}{s_1(x) - s_1(0)}$ is decreasing on $[0, r)$ by Lemma 4. At the same time, $\lim_{x\to 0^+} J(x) = \frac{r^2}{4}$ and $\lim_{x\to r} J(x) = 0$. So the proof of Theorem 5 is complete.

8. Proof of Theorem 6

Let $K(x) = \frac{\log\frac{\tanh x}{x}}{\log\frac{r^2-x^2}{r^2+x^2}} = \frac{l_1(x)}{m_1(x)}$, where $l_1(x) = \log\frac{\tanh x}{x}$, and $m_1(x) = \log\frac{r^2-x^2}{r^2+x^2}$. Then

$$\frac{l_1'(x)}{m_1'(x)} = \frac{1}{r^2}\frac{(r^4 - x^4)(\sinh 2x - 2x)}{(2x)^2\sinh 2x} = \frac{1}{16r^2}\frac{L(t)}{M(t)},$$

where $t = 2x$,

$$L(t) = (16r^4 - t^4)(\sinh t - t) = \frac{16r^4}{3!}t^3 + \frac{16r^4}{5!}t^5 + \sum_{n=3}^{\infty}\left[\frac{16r^4}{(2n+1)!} - \frac{1}{(2n-4)!}\right]t^{2n+1}$$

$$= \sum_{n=1}^{\infty} a_n t^{2n+1},$$

$$M(t) = t^2\sinh t = t^3 + \frac{1}{3!}t^5 + \sum_{n=3}^{\infty}\frac{t^{2n+1}}{(2n-1)!}$$

$$= \sum_{n=1}^{\infty} b_n t^{2n+1},$$

and $a_1 = \frac{16r^4}{3!}$, $a_2 = \frac{16r^4}{5!}$, $a_n = \frac{16r^4}{(2n+1)!} - \frac{1}{(2n-4)!}$, $b_1 = 1$, $b_2 = \frac{1}{3!}$, $b_n = \frac{1}{(2n-1)!} > 0$, $n \geq 3$, and $n \in \mathbb{N}^+$.

So $\frac{a_1}{b_1} > \frac{a_2}{b_2} > \frac{a_3}{b_3}$, and for $n \geq 3$ we have

$$c_n = \frac{a_n}{b_n} = \frac{\frac{16r^4}{(2n+1)!} - \frac{1}{(2n-4)!}}{\frac{1}{(2n-1)!}}$$

$$= \frac{16r^4 - (2n+1)2n(2n-1)(2n-2)(2n-3)}{2n(2n+1)}.$$

We conclude that c_n is decreasing for $n = 1, 2, \ldots$, $L(t)/M(t)$ is decreasing on $(0, 2r)$ and $\frac{l_1'(x)}{m_1'(x)}$ is decreasing on $(0, r)$ by Lemma 5. Thus $K(x) = \frac{l_1(x)}{m_1(x)} = \frac{l_1(x)-l_1(0^+)}{m_1(x)-m_1(0^+)}$ is decreasing on $(0, r)$ by Lemma 4. At the same time, $\lim_{x\to 0^+} K(x) = \frac{r^2}{6}$ and $\lim_{x\to r} K(x) = 0$. So the proof of Theorem 6 is complete.

Remark 3. New researches on Jordan's inequality which is similar in theme to Redheffer-type inequalities are in active progress, reader can refer to [15–32].

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