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An interior estimate for a nonlinear parabolic equation

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Abstract

In this paper the derivatives of the solution of an initial boundary value problem for a nonlinear uniformly parabolic equation in the interior with the total variation of the boundary data and the L^∞-norm of the initial condition are estimated.

1. Introduction

In this paper we are interested to estimate the L^1-norms of the derivatives of the solution of a nonlinear parabolic equation. More precisely, we estimate the derivatives of the solution of an initial boundary value problem in the interior with respect to the total variation of the boundary data and the L^∞-norm of the initial datum. We split the solution in three parts, one depending only on the initial datum and the other two depending only on the boundary data. Moreover these maps are solutions of a linear parabolic equation. The main tools of the proofs are the maximum principle and energy estimates.

Let $W = W(t,x)$ be the solution of the quasilinear initial boundary value problem

$$\begin{cases} W_t = a(x,W)W_{xx} & \text{for } 0 \leqslant x \leqslant 1,\ t \geqslant 0,\\ W(0,x) = \varphi(x) & \text{for } 0 \leqslant x \leqslant 1,\\ W(t,0) = g_0(t) & \text{for } t \geqslant 0,\\ W(t,1) = g_1(t) & \text{for } t \geqslant 0, \end{cases} \tag{1}$$

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where

$$a \in C^3(\mathbb{R}^2), \quad 0 < a_* \leqslant a(\cdot,\cdot) \leqslant a^* < +\infty, \quad \|a\|_{C^3} \leqslant k, \tag{2}$$

and

$$\begin{aligned} &\varphi \in C^2([0,1]), \\ &g_0, g_1 \in C^1(\mathbb{R}_+) \cap BV(\mathbb{R}_+), \quad g_0(0) = \varphi(0), \quad g_1(0) = \varphi(1). \end{aligned} \tag{3}$$

The main results of this paper are the following ones.

Theorem 1. *Let $W = W(x,t)$ be the classical solution of* (1) *with $a = a(x,y)$ satisfying* (2)*, $\varphi = \varphi(x)$, $g_0 = g_0(t)$, $g_1 = g_1(t)$ satisfying* (3)*, $c_1 > 0$. There exists $C = C(\varepsilon, c_1, k, a_*, a^*, \|\varphi\|_{L^\infty}, \|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}) > 0$ such that*

$$\int_{c_1}^{T} \bigl|W_t(t,x)\bigr|\,dt \leqslant C\|\varphi\|_{L^\infty} + \int_0^T \bigl(\bigl|g_0'(t)\bigr| + \bigl|g_1'(t)\bigr|\bigr)\,dt \tag{4}$$

for all $0 \leqslant x \leqslant 1$ and $T \geqslant c_1$.

Theorem 2. *Let $W = W(x,t)$ be the classical solution of* (1) *with $a = a(x,y)$ satisfying* (2)*, $\varphi = \varphi(x)$, $g_0 = g_0(t)$, $g_1 = g_1(t)$ satisfying* (3)*, $c_1 > 0$ and $0 < \varepsilon < 1/2$. There exists $C = C(\varepsilon, c_1, k, a_*, a^*, \|\varphi\|_{L^\infty}, \|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}) > 0$ such that*

$$\int_{c_1}^{T} dt \int_{\varepsilon}^{1-\varepsilon} \bigl|W_{tx}(t,x)\bigr|\,dx \leqslant C\left(\|\varphi\|_{L^\infty} + \int_0^T \bigl(\bigl|g_0'(t)\bigr| + \bigl|g_1'(t)\bigr|\bigr)\,dt\right) \tag{5}$$

for all $T \geqslant c_1$.

In the literature there are well-known interior estimates on the L^∞-norm of the derivatives of the solution of (1), called Schauder estimates (e.g., see [2]) and the ones on the L^2-norm (e.g., see [4]).

As a motivation and application of this results we can see [1]. There the authors prove the convergence of the vanishing viscosity solutions for a particular 2×2 system of conservation laws. They show the compactness of that family of solutions via uniform estimates on the total variation and Helly's theorem. A basic ingredient of these estimates (see [1, Lemma 3]) is proved here as Theorem 2.

Let $W = W(t,x)$ be the solution of (1) (see [3, Theorem VI 5.2]), denote

$$\bar{a}(t,x) \doteq a\bigl(x, W(t,x)\bigr), \quad 0 \leqslant x \leqslant 1,\ t \geqslant 0,$$

and consider the solution $u = u(t,x)$ of the problem

$$\begin{cases} u_t = \bar{a}(t,x)u_{xx} & \text{for } 0 \leqslant x \leqslant 1,\ t \geqslant 0, \\ u(0,x) = \varphi(x) & \text{for } 0 \leqslant x \leqslant 1, \\ u(t,0) = g_0(t) & \text{for } t \geqslant 0, \\ u(t,1) = g_1(t) & \text{for } t \geqslant 0. \end{cases} \tag{6}$$

By uniqueness, $W \equiv u$. Now fix $c_1 > 0$ and $0 < \varepsilon < 1/2$, by Schauder estimates (see [2, Chapter 3, Section 8]), there exists $K = K(c_1, \varepsilon, k, a_*, a^*) > 0$ such that

$$\sup_{(t,x)\in R} \{|W_x|, |W_t|, |W_{tx}|\} \leqslant K\big(\|\varphi\|_{L^\infty} + \|g_0\|_{L^\infty} + \|g_1\|_{L^\infty}\big),$$

and so, by the definition of $\bar{a}$ and (2),

$$\sup_{(t,x)\in R} \{|\bar{a}_t|, |\bar{a}_{tx}|\} \leqslant K_1\big(\|\varphi\|_{L^\infty} + \|g_0\|_{L^\infty} + \|g_1\|_{L^\infty}\big)$$

for some constant $K_1 = K_1(c_1, \varepsilon, k, a_*, a^*) > 0$, where

$$R \doteq \{(t,x) \in \mathbb{R}^2;\ c_1 \leqslant t,\ \varepsilon \leqslant x \leqslant 1-\varepsilon\}.$$

To simplify the notations, we shall assume also that

$$\varphi(0) = \varphi(1) = g_0(0) = g_1(0) = 0.$$

Let $u_1 = u_1(t,x)$, $u_2 = u_2(t,x)$, $u_3 = u_3(t,x)$ be the solutions of the linear equation

$$u_t = \bar{a}(t,x)u_{xx}, \quad t > 0,\ 0 < x < 1, \tag{7}$$

satisfying the initial and boundary conditions

$$\begin{aligned} &u_1(0,\cdot) \equiv \varphi, && u_1(\cdot,0) \equiv 0, && u_1(\cdot,1) \equiv 0, \\ &u_2(0,\cdot) \equiv 0, && u_2(\cdot,0) \equiv g_0, && u_2(\cdot,1) \equiv 0, \\ &u_3(0,\cdot) \equiv 0, && u_3(\cdot,0) \equiv 0, && u_3(\cdot,1) \equiv g_1, \end{aligned}$$

respectively. By (6) and the linearity of (7),

$$W(t,x) = u(t,x) = u_1(t,x) + u_2(t,x) + u_3(t,x) \tag{8}$$

is the solution of (1).

In Section 2 we prove estimates (4) and (5) for u_2 and u_3, namely we consider the case of (1) with the null initial condition. On the other side, in Section 3 we prove the same ones for u_1, namely we consider (1) with the null boundary data. In Section 4 we give the proofs of Theorems 1 and 2. Finally in Appendix A we prove two lemmas, the first one is a simply measure theory result, the second one consists of two Poincaré type inequalities. The proofs of these two lemmas are needed for the sake of the best constants.

2. The case with null initial condition and general boundary data

In this section we want to prove some estimates on the derivatives of the maps u_2 and u_3, defined in Section 1.

Lemma 3. *There results*

$$\int_0^T |u_{2,t}(t,x)|\,dt \leqslant \int_0^T |g_0'(t)|\,dt$$

for all $0 \leqslant x \leqslant 1$.

Proof. Define

$$h_0(t) \doteq \frac{1}{2}\left(\int_0^t \left|g_0'(\tau)\right| d\tau + g_0(t)\right),$$

$$k_0(t) \doteq \frac{1}{2}\left(\int_0^t \left|g_0'(\tau)\right| d\tau - g_0(t)\right), \quad t \geqslant 0.$$

Clearly $h_0, k_0 \in C^1(\mathbb{R}_+) \cap BV(\mathbb{R}_+)$, increasing in $\mathbb{R}_+$, positive, and

$$g_0(t) = h_0(t) - k_0(t),$$
$$\int_0^t \left|g_0'(\tau)\right| d\tau = \int_0^t h_0'(\tau)\, d\tau + \int_0^t k_0'(\tau)\, d\tau, \quad t \geqslant 0. \tag{9}$$

Let v_2 and ω_2 be the solutions of (7) such that

$$v_2(0,\cdot) \equiv 0, \qquad v_2(\cdot,0) \equiv h_0, \qquad v_2(\cdot,1) \equiv 0,$$
$$\omega_2(0,\cdot) \equiv 0, \qquad \omega_2(\cdot,0) \equiv k_0, \qquad \omega_2(\cdot,1) \equiv 0.$$

Since (7) is linear, by (9), we have

$$u_2 \equiv v_2 - \omega_2. \tag{10}$$

Moreover, $v_{2,xx}$ and $\omega_{2,xx}$ are solutions of the equation

$$U_t = \bar{a}(t,x)U_{xx} + 2\bar{a}_x(t,x)U_x + \bar{a}_{xx}(t,x)U \tag{11}$$

and, by the definition of h_0 and k_0,

$$v_{2,xx}(0,\cdot) \equiv v_{2,xx}(\cdot,1) \equiv 0, \qquad v_{2,xx}(\cdot,0) = \frac{h_0'}{\bar{a}(\cdot,0)} \geqslant 0,$$
$$\omega_{2,xx}(0,\cdot) \equiv \omega_{2,xx}(\cdot,1) \equiv 0, \qquad \omega_{2,xx}(\cdot,0) = \frac{k_0'}{\bar{a}(\cdot,0)} \geqslant 0,$$

by the maximum principle (see [3, Theorem I 2.1]), $v_{2,xx}$ and $\omega_{2,xx}$ are positive. So $v_2(t,\cdot)$ and $\omega_2(t,\cdot)$ are convex in [0, 1] for each $t \geqslant 0$. By (2) and (7), we have

$$v_{2,t}(t,x) \geqslant 0, \quad \omega_{2,t}(t,x) \geqslant 0, \quad t > 0,\ 0 \leqslant x \leqslant 1. \tag{12}$$

Fix $T \geqslant 0$ and $0 \leqslant x \leqslant 1$; by the maximum principle and the monotonicity of h_0, we get

$$v_2(T,x) \leqslant \max_{[0,T]} v_2(\cdot,0) = h_0(T).$$

So, by (12) and since $h_0(0) = 0$, we have

$$\int_0^T \left|v_{2,t}(t,x)\right| dt = \int_0^T v_{2,t}(t,x)\, dt = v_2(T,x) \leqslant h_0(T) = \int_0^T h_0'(t)\, dt \tag{13}$$

and analogously

$$\int_0^T \left|\omega_{2,t}(t,x)\right| dt \leqslant \int_0^T k_0'(t)\,dt. \tag{14}$$

Then, by (9), (10), (13), and (14), we get

$$\int_0^T \left|u_{2,t}(t,x)\right| dt \leqslant \int_0^T \left(\left|v_{2,t}(t,x)\right| + \left|\omega_{2,t}(t,x)\right|\right) dt$$
$$\leqslant \int_0^T \left(h_0'(t) + k_0'(t)\right) dt = \int_0^T \left|g_0'(t)\right| dt.$$

So the proof is concluded. □

In the same way we can prove the following

Lemma 4. *There results*

$$\int_0^T \left|u_{3,t}(t,x)\right| dt \leqslant \int_0^T \left|g_1'(t)\right| dt$$

for all $c_1 \leqslant T$ *and* $0 \leqslant x \leqslant 1$.

Lemma 5. *There exist two constants* $C_1, \delta_1 > 0$ *depending only on* $a_*, a^*, k, \varepsilon, c_1, \|\varphi\|_{L^\infty}$, $\|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}$ *such that if* $\varepsilon \leqslant x_1 < x_2 \leqslant 1-\varepsilon$ *and* $x_2 - x_1 < \delta_1$, *then*

$$\int_{c_1}^T dt \int_{x_1}^{x_2} \left|u_{i,xt}(x,t)\right| dx \leqslant C_1 \int_0^T \left(\left|u_{i,t}(t,x_1)\right| + \left|u_{i,t}(t,x_2)\right|\right) dt$$

for $i = 2, 3$ *and all* $T \geqslant c_1$.

Proof. Fix $i \in \{2,3\}$ and $0 \leqslant t \leqslant T$; by (7), we get

$$\frac{d}{dt}\int_{x_1}^{x_2} \frac{u_{i,tx}^2(t,x)}{2}\,dx = \int_{x_1}^{x_2} u_{i,tx}(t,x)u_{i,ttx}(t,x)\,dx$$
$$= u_{i,tx}(t,x_2)u_{i,tt}(t,x_2) - u_{i,tx}(t,x_1)u_{i,tt}(t,x_1) - \int_{x_1}^{x_2} u_{i,txx}(t,x)u_{i,tt}(t,x)\,dx$$
$$= u_{i,tx}(t,x_2)u_{i,tt}(t,x_2) - u_{i,tx}(t,x_1)u_{i,tt}(t,x_1)$$
$$\quad - \int_{x_1}^{x_2} u_{i,txx}(t,x)\left(\bar{a}_t(t,x)u_{i,xx}(t,x) + \bar{a}(t,x)u_{i,txx}(t,x)\right) dx$$

$$
\begin{aligned}
&= u_{i,tx}(t,x_2)u_{i,tt}(t,x_2) - u_{i,tx}(t,x_1)u_{i,tt}(t,x_1)\\
&\quad - \int_{x_1}^{x_2} \bar{a}_t(t,x)u_{i,txx}(t,x)u_{i,xx}(t,x)\,dx - \int_{x_1}^{x_2} \bar{a}(t,x)u_{i,txx}^2(t,x)\,dx\\
&\leqslant u_{i,tx}(t,x_2)u_{i,tt}(t,x_2) - u_{i,tx}(t,x_1)u_{i,tt}(t,x_1)\\
&\quad - \int_{x_1}^{x_2} \bar{a}_t(t,x)u_{i,txx}(t,x)u_{i,xx}(t,x)\,dx - a_* \int_{x_1}^{x_2} u_{i,txx}^2(t,x)\,dx\\
&\leqslant u_{i,tx}(t,x_2)u_{i,tt}(t,x_2) - u_{i,tx}(t,x_1)u_{i,tt}(t,x_1)\\
&\quad + \frac{1}{2a_*}\int_{x_1}^{x_2} \bar{a}_t^2(t,x)u_{i,xx}^2(t,x)\,dx - \frac{a_*}{2}\int_{x_1}^{x_2} u_{i,txx}^2(t,x)\,dx\\
&\leqslant u_{i,tx}(t,x_2)u_{i,tt}(t,x_2) - u_{i,tx}(t,x_1)u_{i,tt}(t,x_1)\\
&\quad + \frac{1}{2a_*}\int_{x_1}^{x_2} \frac{\bar{a}_t^2(t,x)}{\bar{a}^2(t,x)}u_{i,t}^2(t,x)\,dx - \frac{a_*}{2}\int_{x_1}^{x_2} u_{i,txx}^2(t,x)\,dx\\
&\leqslant u_{i,tx}(t,x_2)u_{i,tt}(t,x_2) - u_{i,tx}(t,x_1)u_{i,tt}(t,x_1)\\
&\quad + \frac{\|\bar{a}_t\|_{L^\infty(R)}^2}{2a_*^3}\int_{x_1}^{x_2} u_{i,t}^2(t,x)\,dx - \frac{a_*}{2}\int_{x_1}^{x_2} u_{i,txx}^2(t,x)\,dx.
\end{aligned}
$$

By formulas (A.2) and (A.3) of Appendix A, we have

$$
-\int_{x_1}^{x_2} u_{i,txx}^2(t,x)\,dx \leqslant -\int_{x_1}^{x_2} \frac{u_{i,tx}^2(t,x)}{2(x_2-x_1)^2}\,dx + \frac{|u_{i,t}(t,x_2)-u_{i,t}(t,x_1)|^2}{|x_2-x_1|^3},
$$

$$
\int_{x_1}^{x_2} u_{i,t}^2(t,x)\,dx \leqslant 2(x_2-x_1)^2\int_{x_1}^{x_2} u_{i,tx}^2(t,x)\,dx + 2(x_2-x_1)\big|u_{i,t}(t,x_1)\big|^2,
$$

respectively, and then

$$
\begin{aligned}
&\frac{d}{dt}\int_{x_1}^{x_2} \frac{u_{i,tx}^2(t,x)}{2}\,dx\\
&\leqslant u_{i,tx}(t,x_2)u_{i,tt}(t,x_2) - u_{i,tx}(t,x_1)u_{i,tt}(t,x_1)\\
&\quad + \left[-\frac{a_*}{4(x_2-x_1)^2} + \frac{\|\bar{a}_t\|_{L^\infty(R)}^2}{a_*^3}(x_2-x_1)^2\right]\int_{x_1}^{x_2} u_{i,tx}^2(t,x)\,dx\\
&\quad + \frac{\|\bar{a}_t\|_{L^\infty(R)}^2}{a_*^3}(x_2-x_1)\big|u_{i,t}(t,x_1)\big|^2 + \frac{a_*}{2}\frac{|u_{i,t}(t,x_2)-u_{i,t}(t,x_1)|^2}{|x_2-x_1|^3}.
\end{aligned}
\tag{15}
$$

Moreover, there exists $\delta_1 > 0$ such that, if $x_2 - x_1 < \delta$, then

$$u_{i,tx}(t,x_2)u_{i,tt}(t,x_2) - u_{i,tx}(t,x_1)u_{i,tt}(t,x_1)$$
$$+ \frac{\|\bar{a}_t\|^2_{L^\infty(R)}}{a_*^3}(x_2-x_1)^2 \int_{x_1}^{x_2} u^2_{i,tx}(t,x)\,dx + \frac{\|\bar{a}_t\|^2_{L^\infty(R)}}{a_*^3}(x_2-x_1)\bigl|u_{i,t}(t,x_1)\bigr|^2$$
$$\leqslant \frac{a_*}{8(x_2-x_1)^2}\int_{x_1}^{x_2} u^2_{i,tx}(t,x)\,dx + \frac{a_*}{2}\frac{|u_{i,t}(t,x_2)-u_{i,t}(t,x_1)|^2}{|x_2-x_1|^3},$$

and then, by (15),

$$\frac{d}{dt}\int_{x_1}^{x_2} \frac{u^2_{i,tx}(t,x)}{2}\,dx \leqslant -\frac{a_*}{8}\int_{x_1}^{x_2} \frac{u^2_{i,tx}(t,x)}{(x_2-x_1)^2}\,dx + a_*\frac{|u_{i,t}(t,x_2)-u_{i,t}(t,x_1)|^2}{|x_2-x_1|^3},$$

hence

$$\int_{x_1}^{x_2} u^2_{i,tx}(t,x)\,dx \leqslant 2a_*\frac{e^{-\frac{a_* t}{4(x_2-x_1)^2}}}{|x_2-x_1|^3}\int_0^t e^{\frac{a_*\tau}{4(x_2-x_1)^2}}\bigl|u_{i,t}(\tau,x_2)-u_{i,t}(\tau,x_1)\bigr|^2\,d\tau. \tag{16}$$

Since

$$\left(\int_{x_1}^{x_2}\bigl|u_{i,tx}(t,x)\bigr|\,dx\right)^2 \leqslant (x_2-x_1)\int_{x_1}^{x_2} u^2_{i,tx}(t,x)\,dx \tag{17}$$

and

$$\bigl|u_{i,t}(\tau,x_2)-u_{i,t}(\tau,x_1)\bigr| \leqslant \int_{x_1}^{x_2}\bigl|u_{i,tx}(\tau,x)\bigr|\,dx, \tag{18}$$

we have, by (16), (17), and (18),

$$f^2(t) \leqslant e^{-\lambda t}\int_0^t e^{\lambda\tau}h(\tau)f(\tau)\,d\tau,$$

where

$$f(t) \doteq \int_{x_1}^{x_2}\bigl|u_{i,tx}(t,x)\bigr|\,dx,$$
$$h(t) \doteq 2a_*\frac{|u_{i,t}(t,x_2)-u_{i,t}(t,x_1)|}{|x_2-x_1|^2}, \qquad \lambda \doteq \frac{a_*}{4(x_2-x_1)^2}.$$

By Lemma A.1 of Appendix A, we get

$$\int_0^T dt\int_{x_1}^{x_2}\bigl|u_{i,tx}(t,x)\bigr|\,dx \leqslant C_1\int_0^T\bigl|u_{i,t}(\tau,x_2)-u_{i,t}(\tau,x_1)\bigr|\,dt.$$

So the proof is concluded. □

3. The case with Dirichlet boundary conditions

In this section we want to prove some estimates on the derivatives of u_1, defined in Section 1.

Lemma 6. *There exists a constant $C_2 > 0$ depending only on a_*, a^*, k, such that*

$$\int_{c_1}^{T} \left|u_{1,t}(t,x)\right| dt \leqslant C_2\|\varphi\|_{L^\infty}$$

for all $0 < c_1 \leqslant T$ and $0 \leqslant x \leqslant 1$.

Proof. Let $v_1 = v_1(t,x)$ and $\omega_1 = \omega_1(t,x)$ be the solutions of (7) satisfying the following conditions:

$$v_1(c_1,x) = \frac{1}{2}\left(\int_0^x \int_0^y \left|u_{1,xx}(c_1,\xi)\right| d\xi\, dy + u_1(c_1,x)\right),$$

$$\omega_1(c_1,x) = \frac{1}{2}\left(\int_0^x \int_0^y \left|u_{1,xx}(c_1,\xi)\right| d\xi\, dy - u_1(c_1,x)\right) \tag{19}$$

for $0 \leqslant x \leqslant 1$, and

$$v_1(t,0) = v_1(t,1) = \omega_1(t,0) = \omega_1(t,1) = 0 \tag{20}$$

for $t \leqslant c_1$. By the linearity of (7), (19), and (20) there results

$$u_1 = v_1 - \omega_1, \quad 0 \leqslant x \leqslant 1,\ t \geqslant c_1. \tag{21}$$

Moreover $v_1(c_1,\cdot)$ and $\omega_1(c_1,\cdot)$ are convex. Since $v_{1,xx}$ and $\omega_{1,xx}$ are solutions of (11) and

$$v_{1,xx}(c_1,\cdot) \geqslant 0, \qquad v_{1,xx}(\cdot,0) = \frac{v_{1,t}(\cdot,0)}{\bar{a}(\cdot,0)} = 0, \qquad v_{1,xx}(\cdot,1) = \frac{v_{1,t}(\cdot,1)}{\bar{a}(\cdot,1)} = 0,$$

$$\omega_{1,xx}(c_1,\cdot) \geqslant 0, \qquad \omega_{1,xx}(\cdot,0) = \frac{\omega_{1,t}(\cdot,0)}{\bar{a}(\cdot,0)} = 0, \qquad \omega_{1,xx}(\cdot,1) = \frac{\omega_{1,t}(\cdot,1)}{\bar{a}(\cdot,1)} = 0,$$

by the maximum principle, $v_{1,xx}$ and $\omega_{1,xx}$ are positive. So $v_1(t,\cdot)$ and $\omega_1(t,\cdot)$ are convex in [0, 1] for each $t \geqslant c_1$. By (2), (7), (19), and (20), we have

$$v_{1,t}(t,x) \geqslant 0, \quad \omega_{1,t}(t,x) \geqslant 0, \quad 0 \leqslant x \leqslant 1,\ t \geqslant c_1.$$

Fix $T \geqslant c_1$ and $0 \leqslant x \leqslant 1$; there results

$$\int_{x_1}^{c_1} \left|v_{1,t}(t,x)\right| dt = \int_{c_1}^{T} v_{1,t}(t,x)\, dt = v_1(T,x) - v_1(c_1,x) \leqslant 2\|v_1\|_{L^\infty(R)} \tag{22}$$

and analogously

$$\int_{c_1}^{T} \left|\omega_{1,t}(t,x)\right| dt \leqslant 2\|\omega_1\|_{L^\infty(R)}. \tag{23}$$

By the maximum principle and the definition of u_1, v_1, ω_1 there results

$$\|v_1\|_{L^\infty(R)} + \|\omega_1\|_{L^\infty(R)} \leqslant \left\|u_{1,xx}(c_1,\cdot)\right\|_{L^\infty([0,1])} + \left\|u_1(c_1,\cdot)\right\|_{L^\infty([0,1])}.$$

By the maximum principle and the definition of u_1, there results

$$\left\|u_1(c_1,\cdot)\right\|_{L^\infty([0,1])} \leqslant \|\varphi\|_{L^\infty}.$$

Since

$$u_{1,xx}(c_1,0) = \frac{u_{1,t}(c_1,0)}{\bar{a}(c_1,0)} = 0, \qquad u_{1,xx}(c_1,1) = \frac{u_{1,t}(c_1,1)}{\bar{a}(c_1,1)} = 0,$$

there exists $0 < \bar{x} < 1$, depending on c_1, such that

$$\left\|u_{1,xx}(c_1,\cdot)\right\|_{L^\infty([0,1])} = \left|u_{1,xx}(c_1,\bar{x})\right|.$$

Let $0 < \bar{\varepsilon} < 1/2$ be such that $\bar{\varepsilon} \leqslant \bar{x} \leqslant 1-\bar{\varepsilon}$; there results

$$\left\|u_{1,xx}(c_1,\cdot)\right\|_{L^\infty([0,1])} = \left|u_{1,xx}(c_1,\bar{x})\right| = \left\|u_{1,xx}(c_1,\cdot)\right\|_{L^\infty([\bar{\varepsilon},1-\bar{\varepsilon}])}.$$

Moreover, by the Schauder estimates, there exists a constant $K_2 > 0$ such that

$$\left\|u_{1,xx}(c_1,\cdot)\right\|_{L^\infty([\bar{\varepsilon},1-\bar{\varepsilon}])} \leqslant K_2\|\varphi\|_{L^\infty}. \tag{24}$$

Finally, by (21)–(24) we can conclude

$$\begin{aligned}\int_{c_1}^{T} \left|u_{1,t}(t,x)\right| dt &\leqslant \int_{c_1}^{T} \left|v_{1,t}(t,x)\right| dt + \int_{c_1}^{T} \left|\omega_{1,t}(t,x)\right| dt \\ &\leqslant 2\left(\|v_1\|_{L^\infty(R)} + \|\omega_1\|_{L^\infty(R)}\right) \leqslant 2(K_2+1)\|\varphi\|_{L^\infty}.\end{aligned}$$

Since K_2 depends on $\bar{\varepsilon}$ that depends on c_1, the proof is done. □

Lemma 7. *There exist two constants $C_3, \delta_2 > 0$ depending only on $a_*, a^*, k, \varepsilon, c_1, \|\varphi\|_{L^\infty}$, $\|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}$ such that if $\varepsilon \leqslant x_1 < x_2 \leqslant 1-\varepsilon$ and $x_2 - x_1 < \delta_2$, then*

$$\int_{c_1}^{T} dt \int_{x_1}^{x_2} \left|u_{1,tx}(x,t)\right| dx \leqslant C_3\|\varphi\|_{L^\infty}$$

for all $T \geqslant c_1$.

Proof. Call

$$\delta_2 \doteq \frac{a_*}{8^{1/4}\|\bar{a}_t\|_{L^\infty(R)}^{1/2}}, \tag{25}$$

fix $\varepsilon \leqslant x_1 < x_2 \leqslant 1-\varepsilon$, and consider the restriction of u_1 to the strip

$$\tilde{R} \doteq \left\{(t,x) \in \mathbb{R}^2;\ t \geqslant 0,\ x_1 \leqslant x \leqslant x_2\right\}\ (\subset R).$$

There results

$$u_1 \equiv \bar{u} + \tilde{u}, \quad \text{in } \tilde{R}, \tag{26}$$

where $\bar{u}, \tilde{u}$ are the solutions of (7) such that

$$\bar{u}(0,\cdot) \equiv \varphi|_{[x_1,x_2]}, \qquad \bar{u}(\cdot,x_1) \equiv \varphi(x_1), \qquad \bar{u}(\cdot,x_2) \equiv \varphi(x_2), \tag{27}$$

$$\tilde{u}(0,\cdot) \equiv 0, \qquad \tilde{u}(\cdot,x_1) \equiv u_1(\cdot,x_1) - \varphi(x_1),$$
$$\tilde{u}(\cdot,x_2) \equiv u_1(\cdot,x_2) - \varphi(x_2). \tag{28}$$

By Lemmas 5 and 6, we get

$$\begin{aligned}\int_{c_1}^{T} dt \int_{x_1}^{x_2} \left|\tilde{u}_{tx}(t,x)\right| dx &\leqslant C_1 \int_0^T \left(\left|\tilde{u}_t(t,x_1)\right| + \left|\tilde{u}_t(t,x_2)\right|\right) dt \\ &= 2C_1 \int_0^T \left(\left|u_{1,t}(t,x_2)\right| + \left|u_{1,t}(t,x_1)\right|\right) \leqslant 4C_1 \|\varphi\|_{L^\infty}.\end{aligned} \tag{29}$$

Moreover, by the definition of $\bar{u}$, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{x_1}^{x_2} \frac{\bar{u}_{tx}^2(t,x)}{2}\,dx &= \int_{x_1}^{x_2} \bar{u}_{tx}(t,x)\bar{u}_{ttx}(t,x)\,dx = -\int_{x_1}^{x_2} \bar{u}_{txx}(t,x)\bar{u}_{tt}(t,x)\,dx \\ &= -\int_{x_1}^{x_2} \bar{u}_{txx}(t,x)\left(\bar{a}_t(t,x)\bar{u}_{xx}(t,x) + \bar{a}(t,x)\bar{u}_{txx}(t,x)\right) dx \\ &= -\int_{x_1}^{x_2} \bar{a}_t(t,x)\bar{u}_{txx}(t,x)\bar{u}_{xx}(t,x)\,dx - \int_{x_1}^{x_2} \bar{a}(t,x)\bar{u}_{txx}^2(t,x)\,dx \\ &\leqslant -\int_{x_1}^{x_2} \bar{a}_t(t,x)\bar{u}_{txx}(t,x)\bar{u}_{xx}(t,x)\,dx - a_*\int_{x_1}^{x_2} \bar{u}_{txx}^2(t,x)\,dx \\ &\leqslant \frac{1}{2a_*}\int_{x_1}^{x_2} \bar{a}_t^2(t,x)\bar{u}_{xx}^2(t,x)\,dx - \frac{a_*}{2}\int_{x_1}^{x_2} \bar{u}_{txx}^2(t,x)\,dx \\ &\leqslant \frac{1}{2a_*}\int_{x_1}^{x_2} \frac{\bar{a}_t^2(t,x)}{\bar{a}^2(t,x)}\bar{u}_t^2(t,x)\,dx - \frac{a_*}{2}\int_{x_1}^{x_2} \bar{u}_{txx}^2(t,x)\,dx \\ &\leqslant \frac{\|\bar{a}_t\|_{L^\infty(R)}^2}{2a_*^3}\int_{x_1}^{x_2} \bar{u}_t^2(t,x)\,dx - \frac{a_*}{2}\int_{x_1}^{x_2} \bar{u}_{txx}^2(t,x)\,dx.\end{aligned}$$

By (A.2) and (A.3), we have

$$-\int_{x_1}^{x_2} \bar{u}_{txx}^2(t,x)\,dx \leqslant -\int_{x_1}^{x_2} \frac{\bar{u}_{tx}^2(t,x)}{2(x_2-x_1)^2}\,dx,$$

$$\int_{x_1}^{x_2} \bar{u}_t^2(t,x)\,dx \leqslant 2(x_2-x_1)^2 \int_{x_1}^{x_2} \bar{u}_{tx}^2(t,x)\,dx,$$

respectively, and then, by (25), if $x_2 - x_1 \leqslant \delta_2$,

$$\frac{d}{dt}\int_{x_1}^{x_2} \frac{\bar{u}_{tx}^2(t,x)}{2}\,dx \leqslant \left[-\frac{a_*}{4(x_2-x_1)^2} + \frac{\|\bar{a}_t\|_{L^\infty(R)}^2}{a_*^3}(x_2-x_1)^2\right]\int_{x_1}^{x_2} \bar{u}_{tx}^2(t,x)\,dx$$

$$\leqslant \left[-\frac{a_*}{4\delta_2^2} + \frac{\|\bar{a}_t\|_{L^\infty(R)}^2}{a_*^3}\delta_2^2\right]\int_{x_1}^{x_2} \bar{u}_{tx}^2(t,x)\,dx \leqslant -\frac{a_*}{8\delta^2}\int_{x_1}^{x_2} \bar{u}_{tx}^2(t,x)\,dx.$$

Hence, by Schauder estimates, there exists a constant $K_3 > 0$ such that

$$\int_{x_1}^{x_2} \bar{u}_{tx}^2(t,x)\,dx \leqslant 2a_* \frac{e^{-\frac{a_*(t-c_1)}{4\delta_2^2}}}{|x_2-x_1|^3}\int_{x_1}^{x_2} \bar{u}_{tx}^2(c_1,x)\,dx \leqslant K_3 e^{-\frac{a_* t}{4\delta_2^2}}\|\varphi\|_{L^\infty}^2,$$

so

$$\int_{x_1}^{x_2} \left|\bar{u}_{tx}(t,x)\right|dx \leqslant (x_2-x_1)^{1/2}\left(\int_{x_1}^{x_2} \bar{u}_{tx}^2(t,x)\,dx\right)^{1/2}$$

$$\leqslant K_3^{1/2}\|\varphi\|_{L^\infty}(x_2-x_1)^{1/2} e^{-\frac{a_* t}{8\delta_2^2}} \leqslant K_3\|\varphi\|_{L^\infty}\delta_2^{1/2} e^{-\frac{a_* t}{8\delta_2^2}},$$

and integrating on $[c_1, T]$,

$$\int_{c_1}^{T} dt \int_{x_1}^{x_2} \left|\bar{u}_{tx}(t,x)\right|dx \leqslant \frac{8K_3^{1/2}\|\varphi\|_{L^\infty}\delta_2^{3/2}}{a_*}. \tag{30}$$

By (26), we have

$$\int_{c_1}^{T} dt \int_{x_1}^{x_2} \left|u_{1,tx}(t,x)\right|dx \leqslant \int_{c_1}^{T} dt \int_{x_1}^{x_2} \left(\left|\bar{u}_{tx}(t,x)\right| + \left|\tilde{u}_{tx}(t,x)\right|\right)dx,$$

then, by (29) and (30), the thesis is done □

4. Proofs of Theorems 1 and 2

In this section we give the proofs of the main results of the paper.

Proof of Theorem 1. The thesis is direct consequence of (8) and Lemmas 3, 4, and 6. □

Proof of Theorem 2. Fix $0 < c_1 \leqslant T$, $0 < \varepsilon < 1/2$ and observe that

$$\int_{c_1}^{T} dt \int_{\varepsilon}^{1-\varepsilon} \left|W_{tx}(t,x)\right| dx = \int_{c_1}^{T} dt \int_{\varepsilon}^{1-\varepsilon} \left|u_{tx}(t,x)\right| dx \leqslant \sum_{i=1,2,3} \int_{c_1}^{T} dt \int_{\varepsilon}^{1-\varepsilon} \left|u_{i,tx}(t,x)\right| dx,$$

where $W = W(t,x)$ and $u = u(t,x)$ are the solutions of (1) and (6), respectively.

Let $x_0, \ldots, x_h$ such that

$$\varepsilon = x_0 < x_1 < \cdots < x_{h-1} < x_h = 1 - \varepsilon$$

and

$$x_j - x_{j-1} < \delta, \quad j = 1, \ldots, h,$$

where $\delta \leqslant \min\{\delta_1, \delta_2\}$ and δ_1, δ_2 are the ones of Lemmas 5 and 7, respectively. By Lemma 7, we have

$$\int_{c_1}^{T} dt \int_{\varepsilon}^{1-\varepsilon} \left|u_{1,tx}(t,x)\right| dx \leqslant \sum_{j=1}^{h} \int_{c_1}^{T} dt \int_{x_{j-1}}^{x_j} \left|u_{1,tx}(t,x)\right| dx \leqslant \sum_{j=1}^{h} C_3 \|\varphi\|_{L^\infty} = hC_3 \|\varphi\|_{L^\infty}. \tag{31}$$

By Lemmas 3–5, we have

$$\begin{aligned} \sum_{i=2,3} \int_{0}^{T} dt \int_{\varepsilon}^{1-\varepsilon} \left|u_{i,tx}(t,x)\right| dx &= \sum_{i=2,3} \sum_{j=1}^{h} \int_{0}^{T} dt \int_{x_{j-1}}^{x_j} \left|u_{i,tx}(t,x)\right| dx \\ &\leqslant \sum_{i=2,3} \sum_{j=1}^{h} C_1 \int_{0}^{T} \left(\left|u_{i,t}(t,x_{j-1})\right| + \left|u_{i,t}(t,x_j)\right|\right) dt \\ &\leqslant 2hC_1 \int_{0}^{T} \left(\left|u_t(t,0)\right| + \left|u_t(t,1)\right|\right) dt = 2hC_1 \int_{0}^{T} \left(\left|g_0'(t)\right| + \left|g_1'(t)\right|\right) dt. \end{aligned} \tag{32}$$

Since h, C_1, and C_3 depends only on $a_*, a^*, k, \varepsilon, c_1, \|\varphi\|_{L^\infty}, \|g_0\|_{L^\infty}, \|g_1\|_{L^\infty}$, the thesis is direct consequence of (31) and (32). □

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Appendix A. Two technical lemmas

In this section we prove two lemmas. The proofs of these two lemmas are more or less well known. We insert here one of these ones for the sake of completeness and the best constants.

Lemma A.1. *Let $f, h \in C(\mathbb{R})$ be positive functions and fix a constant $\lambda > 0$. If*

$$f^2(t) \leqslant e^{-\lambda t} \int_0^t e^{\lambda t} h(\tau) f(\tau)\, d\tau, \quad t \geqslant 0, \tag{A.1}$$

then

$$\int_0^T f(t)\, dt \leqslant \frac{2}{\lambda} \int_0^T h(t)\, dt$$

for each $T \geqslant 0$.

Lemma A.2 (Poincaré type inequalities). *For any $f \in C^2(\mathbb{R})$ one has*

$$\int_a^b f'^2(x)\, dx \leqslant 2(b-a)^2 \int_a^b f''^2(x)\, dx + 2\frac{(f(b)-f(a))^2}{b-a}, \tag{A.2}$$

$$\int_a^b f^2(x)\, dx \leqslant 2(b-a)^2 \int_a^b f'^2(x)\, dx + 2(b-a)\big|f(a)\big|^2 \tag{A.3}$$

for each $-\infty < a < b < +\infty$.

Proof of Lemma A.1. Fix $T \geqslant 0$. Define

$$g(t) \doteq e^{(\lambda/2)t} f(t), \quad t \geqslant 0,$$

by (A.1), we have

$$g^2(t) \leqslant \int_0^t e^{(\lambda/2)\tau} h(\tau) g(\tau)\, d\tau, \quad t \geqslant 0. \tag{A.4}$$

Denote

$$M_t \doteq \sup_{0 \leqslant \tau \leqslant t} g(\tau), \quad t \geqslant 0,$$

we claim that

$$M_t \leqslant \int_0^t e^{(\lambda/2)\tau} h(\tau)\,d\tau, \quad t \geqslant 0. \tag{A.5}$$

Fix $t \geqslant 0$. If $M_t = 0$ we are done, assume that $M_t > 0$. Since g is continuous, there exists $0 \leqslant t_0 \leqslant t$ such that

$$M_t = g(t_0),$$

by (A.4) and the positivity of f and h, we get

$$\begin{aligned} M_t^2 = g^2(t_0) &\leqslant \int_0^{t_0} e^{(\lambda/2)\tau} h(\tau) g(\tau)\,d\tau \leqslant \int_0^t e^{(\lambda/2)\tau} h(\tau) g(\tau)\,d\tau \\ &\leqslant M_t \int_0^t e^{(\lambda/2)\tau} h(\tau)\,d\tau, \end{aligned}$$

so (A.5) is proved. Since

$$e^{(\lambda/2)t} f(t) = g(t) \leqslant M_t, \quad t \geqslant 0,$$

by (A.5) and the definition of g, we have

$$f(t) \leqslant e^{-(\lambda/2)t} \int_0^t e^{(\lambda/2)\tau} h(\tau)\,d\tau, \quad t \geqslant 0,$$

and then

$$\begin{aligned} \int_0^T f(t)\,dt &\leqslant \int_0^T dt \int_0^t e^{-(\lambda/2)t} e^{(\lambda/2)\tau} h(\tau)\,d\tau = \int_0^T d\tau \int_\tau^T e^{-(\lambda/2)t} e^{(\lambda/2)\tau} h(\tau)\,d\tau \\ &= \int_0^T e^{(\lambda/2)\tau} h(\tau) \left(\int_\tau^T e^{-(\lambda/2)t}\,dt \right) d\tau \\ &= \int_0^T e^{(\lambda/2)\tau} h(\tau) \frac{2}{\lambda} \left(e^{-(\lambda/2)\tau} - e^{-(\lambda/2)T}\right) d\tau \\ &= \int_0^T h(\tau) \frac{2}{\lambda} \left(1 - e^{-(\lambda/2)(T-\tau)}\right) d\tau \leqslant \frac{2}{\lambda} \int_0^T h(\tau)\,d\tau. \end{aligned}$$

So the proof is concluded. □

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