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Homogenization of high-contrast two-phase conductivities perturbed by a magnetic field. Comparison between dimension two and dimension three

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ABSTRACT

Homogenized laws for sequences of high-contrast two-phase non-symmetric conductivities perturbed by a parameter h are derived in two and three dimensions. The parameter h characterizes the antisymmetric part of the conductivity for an idealized model of a conductor in the presence of a magnetic field. In dimension two an extension of the Dykhne transformation to non-periodic high conductivities permits to prove that the homogenized conductivity depends on h through some homogenized matrix-valued function obtained in the absence of a magnetic field. This result is improved in the periodic framework thanks to an alternative approach, and illustrated by a cross-like thin structure. Using other tools, a fiber-reinforced medium in dimension three provides a quite different homogenized conductivity.

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1. Introduction

The mathematical theory of homogenization for second-order elliptic partial differential equations has been widely studied since the pioneer works of Spagnolo on G-convergence [1], of Murat, Tartar on H-convergence [2,3], and of Bensoussan, Lions, Papanicolaou on periodic structures [4], in the framework of uniformly bounded (both from below and above) sequences of conductivity matrix-valued functions. It is also known since the end of the seventies [5,6] (see also the extensions [7–10]) that the homogenization of the sequence of conductivity problems, in a bounded open set Ω of $\mathbb{R}^3$,

$$\begin{cases} \operatorname{div}(\sigma_n \nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

with a uniform boundedness from below but not from above for σ_n, may induce nonlocal effects. However, the situation is radically different in dimension two since the nature of problem (1.1) is shown [11,12] to be preserved in the homogenization process provided that the sequence σ_n is uniformly bounded from below.

H-convergence theory includes the case of non-symmetric conductivities in connection with the Hall effect [13] in electrodynamics (see, e.g., [14,15]). Indeed, in the presence of a constant magnetic field the conductivity matrix is modified and becomes non-symmetric. Here, we consider an idealized model of an isotropic conductivity $\sigma(h)$ depending on a parameter h which characterizes the antisymmetric part of the conductivity in the following way:

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- in dimension two,

$$\sigma(h) = \alpha I_2 + \beta h J, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1.2}$$

  where α, β are scalar and $h \in \mathbb{R}$,
- in dimension three,

$$\sigma(h) = \alpha I_3 + \beta \mathscr{E}(h), \quad \mathscr{E}(h) := \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix}, \tag{1.3}$$

  where α, β are scalar and $h \in \mathbb{R}^3$.

Since the seminal work of Bergman [16] the influence of a low magnetic field in composites has been studied for two-dimensional composites [17–19], and for columnar composites [20–24]. The case of a strong field, namely when the symmetric part and the antisymmetric part of the conductivity are of the same order, has been also investigated [25,26]. Moreover, dimension three may induce anomalous homogenized Hall effects [27–29] which do not appear in dimension two [19].

In the context of high-contrast problems the situation is more delicate when the conductivities are not symmetric. An extension in dimension two of H-convergence for non-symmetric and non-uniformly bounded conductivities was obtained in [30] thanks to an appropriate div–curl lemma. More recently, the Keller [31] and Dykhne [32] two-dimensional duality principle which claims that the mapping

$$A \mapsto \frac{A^T}{\det A} \tag{1.4}$$

is stable under homogenization, was extended to high-contrast conductivities in [33]. However, the homogenization of both high-contrast and non-symmetric conductivities has not been precisely studied in the context of the strong field magneto-transport especially in dimension three. In this paper we establish an effective perturbation law for a mixture of two high-contrast isotropic phases in the presence of a magnetic field. The two-dimensional case is performed in a general way for non-periodic and periodic microstructures. It is then compared to the case of a three-dimensional fiber-reinforced microstructure.

In dimension two, following the modelization (1.2), consider a sequence $\sigma_n(h)$ of isotropic two-phase matrix-valued conductivities perturbed by a fixed constant $h \in \mathbb{R}$, and defined by

$$\sigma_n(h) := (1-\chi_n)\big(\alpha_1 I_2 + \beta_1 h J\big) + \chi_n\big(\alpha_{2,n} I_2 + \beta_{2,n} h J\big), \tag{1.5}$$

where χ_n is the characteristic function of phase 2, with volume fraction $\theta_n \to 0$, $\alpha_1 > 0$, β_1 are the constants of the low conducting phase 1, and $\alpha_{2,n} \to \infty$, $\beta_{2,n}$ are real sequences of the highly conducting phase 2 where $\beta_{2,n}$ is possibly unbounded. The coefficients α_1 and β_1, respectively $\alpha_{2,n}$ and $\beta_{2,n}$ also have the same order of magnitude according to the strong field assumption. Assuming that the sequence $\theta_n^{-1}\chi_n$ converges weakly-$*$ in the sense of the Radon measures to a bounded function, and that $\theta_n\alpha_{2,n}, \theta_n\beta_{2,n}$ converge respectively to constants $\alpha_2 > 0, \beta_2$, we prove (see Theorem 2.2) that the perturbed conductivity $\sigma_n(h)$ converges in an appropriate sense of H-convergence (see Definition 1.1) to the homogenized matrix-valued function

$$\sigma_*(h) = \sigma_*^0\big(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h^2\big) + \beta_1 h J, \tag{1.6}$$

for some matrix-valued function σ_*^0 which depends uniquely on the microstructure χ_n in the absence of a magnetic field, and is defined for a subsequence of n. The proof of the result is based on a Dykhne transformation of the type

$$A_n \mapsto \big((p_n A_n + q_n J)^{-1} + r_n J\big)^{-1}, \tag{1.7}$$

which permits to change the non-symmetric conductivity $\sigma_n(h)$ into a symmetric one. Then, extending the duality principle (1.4) established in [33], we prove that transformation (1.7) is also stable under high-contrast conductivity homogenization.

In the periodic case, i.e. when $\sigma_n(h)(\cdot) = \Sigma_n(\cdot/\varepsilon_n)$ with $\Sigma_n Y$-periodic and $\varepsilon_n \to 0$, we use an alternative approach based on an extension of Theorem 4.1 of [12] to $\varepsilon_n Y$-periodic but non-symmetric conductivities (see Theorem 3.1). So, it turns out that the homogenized conductivity $\sigma_*(h)$ is the limit as $n \to \infty$ of the constant H-limit $(\sigma_n)_*$ associated with the periodic homogenization (see, e.g., [4]) of the oscillating sequence $\Sigma_n(\cdot/\varepsilon)$ as $\varepsilon \to 0$ and for a fixed n. Finally, the Dykhne transformation performed by Milton [17] (see also [34, Chapter 4]) applied to the local periodic conductivity Σ_n and its effective conductivity $(\sigma_n)_*$, allows us to recover the perturbed homogenized formula (1.6). An example of a periodic cross-like thin structure provides an explicit computation of $\sigma_*(h)$ (see Proposition 3.2).

To make a comparison with dimension three we restrict ourselves to the $\varepsilon_n Y$-periodic fiber-reinforced structure introduced by Fenchenko and Khruslov [5] to derive a nonlocal effect in homogenization. However, in the present context the fiber radius r_n is chosen to be super-critical, i.e. $r_n \to 0$ and $\varepsilon_n^2|\ln r_n| \to 0$, in order to avoid such an effect. Similarly to (1.5) and following the modelization (1.3), the perturbed conductivity is defined for $h \in \mathbb{R}^3$, by

$$\sigma_n(h) := (1-\chi_n)\big(\alpha_1 I_3 + \beta_1 \mathscr{E}(h)\big) + \chi_n\big(\alpha_{2,n} I_3 + \beta_{2,n} \mathscr{E}(h)\big), \tag{1.8}$$

where χ_n is the characteristic function of the fibers which are parallel to the direction e_3. The form of (1.8) ensures the rotational invariance of $\sigma_n(h)$ for those orthogonal transformations which leave h invariant. Under the same assumptions on the conductivity coefficients as in the two-dimensional case, with $\theta_n = \pi\, r_n^2$, but using a quite different approach, the homogenized conductivity is given by (see Theorem 4.1)

$$\sigma_*(h) = \alpha_1 I_3 + \left(\frac{\alpha_2^3 + \alpha_2\beta_2^2|h|^2}{\alpha_2^2 + \beta_2^2 h_3^2}\right) e_3 \otimes e_3 + \beta_1 \mathscr{E}(h). \tag{1.9}$$

The difference between formulas (1.6) and (1.9) provides a new example of gap between dimension two and dimension three in the high-contrast homogenization framework. As former examples of dimensional gap, we refer to the works [19,27] about the 2d positivity property, versus the 3d non-positivity, of the effective Hall coefficient, and to the works [12,5] concerning the 2d lack, versus the 3d appearance, of nonlocal effects in the homogenization process.

The paper is organized as follows. Sections 2 and 3 deal with dimension two. In Section 2 we study the two-dimensional general (non-periodic) case thanks to an appropriate div–curl lemma. In Section 3 an alternative approach is performed in the periodic framework. Finally, Section 4 is devoted to the three-dimensional case with the fiber-reinforced structure.

Notations

- Ω denotes a bounded open subset of $\mathbb{R}^d$;
- I_d denotes the unit matrix in $\mathbb{R}^{d\times d}$, and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;
- for any matrix A in $\mathbb{R}^{d\times d}$, A^T denotes the transpose of the matrix A, A^s denotes its symmetric part;
- for $h \in \mathbb{R}^3$, $\mathscr{E}(h)$ denotes the antisymmetric matrix in $\mathbb{R}^{3\times 3}$ defined by $\mathscr{E}(h)\,x := h \times x$, for $x \in \mathbb{R}^3$;
- for any $A, B \in \mathbb{R}^{d\times d}$, $A \le B$ means that for any $\xi \in \mathbb{R}^d$, $A\xi\cdot\xi \le B\xi\cdot\xi$; we will use the fact that for any invertible matrix $A \in \mathbb{R}^{d\times d}$, $A \ge \alpha I_d \Rightarrow A^{-1} \le \alpha^{-1} I_d$;
- $|\cdot|$ denotes both the euclidean norm in $\mathbb{R}^d$ and the subordinate norm in $\mathbb{R}^{d\times d}$;
- for any locally compact subset X of $\mathbb{R}^d$, $\mathcal{M}(X)$ denotes the space of the Radon measures defined on X;
- for any $\alpha, \beta > 0$, $\mathcal{M}(\alpha, \beta; \Omega)$ is the set of invertible matrix-valued functions $A: \Omega \to \mathbb{R}^{d\times d}$ such that

$$\forall \xi \in \mathbb{R}^d, \quad A(x)\xi\cdot\xi \ge \alpha|\xi|^2 \quad \text{and} \quad A^{-1}(x)\xi\cdot\xi \ge \beta^{-1}|\xi|^2 \quad \text{a.e. in } \Omega; \tag{1.10}$$

- C denotes a constant which may vary from a line to another one.

In the sequel, we will use the following extension of H-convergence and introduced in [33].

Definition 1.1. Let α_n and β_n be two sequences of positive numbers such that $\alpha_n \le \beta_n$, and let A_n be a sequence of matrix-valued functions in $\mathcal{M}(\alpha_n, \beta_n; \Omega)$ (see (1.10)).

The sequence A_n is said to $H(\mathcal{M}(\Omega)^2)$-converge to the matrix-valued function A_* if for any distribution f in $H^{-1}(\Omega)$, the solution u_n of the problem

$$\begin{cases} \operatorname{div}(A_n \nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies the convergences

$$\begin{cases} u_n \rightharpoonup u & \text{in } H_0^1(\Omega) \\ A_n\nabla u_n \rightharpoonup A_*\nabla u & \text{weakly-}*\ \text{in } \mathcal{M}(\Omega)^2, \end{cases}$$

where u is the solution of the problem

$$\begin{cases} \operatorname{div}(A_* \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We now give a notation for $H(\mathcal{M}(\Omega)^2)$-limits of high-contrast two-phase composites. We consider the characteristic function χ_n of the highly conducting phase, and denote $\omega_n := \{\chi_n = 1\}$.

Notation 1.1. *A sequence of isotropic two-phase conductivities in the absence of a magnetic field is denoted by*

$$\sigma_n^0(\alpha_{1,n}, \alpha_{2,n}) := (1-\chi_n)\alpha_{1,n} I_2 + \chi_n \alpha_{2,n} I_2, \tag{1.11}$$

with

$$\lim_{n\to\infty} \alpha_{1,n} = \alpha_1 > 0 \quad \textit{and} \quad \lim_{n\to\infty} |\omega_n|\,\alpha_{2,n} = \alpha_2 > 0, \tag{1.12}$$

and its $H(\mathcal{M}(\Omega)^2)$-limit is denoted by $\sigma_*^0(\alpha_1, \alpha_2)$.

2. A two-dimensional non-periodic medium

2.1. A div–curl approach

We extend the classical div–curl lemma.

Lemma 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^2$. Let $\alpha > 0$, let $\bar{a} \in L^\infty(\Omega)$ and let A_n be a sequence of matrix-valued functions in $L^\infty(\Omega)^{2\times 2}$ (not necessarily symmetric) satisfying*

$$A_n \geq \alpha I_2 \quad \text{a.e. in } \Omega \quad \text{and} \quad \frac{\det A_n}{\det A_n^s}|A_n^s| \rightharpoonup \bar{a} \in L^\infty(\Omega) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{2.1}$$

Let ξ_n be a sequence in $L^2(\Omega)^2$ and v_n a sequence in $H^1(\Omega)$ satisfying the following assumptions:

(i) *ξ_n and v_n satisfy the estimate*

$$\int_\Omega A_n^{-1}\xi_n \cdot \xi_n dx + \|v_n\|_{H^1(\Omega)} \leq C; \tag{2.2}$$

(ii) *ξ_n satisfies the classical condition*

$$\operatorname{div} \xi_n \quad \text{is compact in } H^{-1}(\Omega). \tag{2.3}$$

Then, there exist ξ in $L^2(\Omega)^2$ and v in $H^1(\Omega)$ such that the following convergences hold true up to a subsequence

$$\xi_n \rightharpoonup \xi \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2 \quad \text{and} \quad \nabla v_n \rightharpoonup \nabla v \text{ weakly in } L^2(\Omega)^2. \tag{2.4}$$

Moreover, we have the following convergence in the distribution sense

$$\xi_n \cdot \nabla v_n \rightharpoonup \xi \cdot \nabla v \quad \text{weakly in } \mathscr{D}'(\Omega).$$

Proof of Lemma 2.1. The proof consists in considering the "good-divergence" sequence ξ_n as a sum of a compact sequence of gradients ∇u_n and a sequence of divergence-free functions $J\nabla z_n$. We then use Lemma 3.1 of [33] to obtain the strong convergence of z_n in $L^2_{\text{loc}}(\Omega)$. Finally, replacing ξ_n by $\nabla u_n + J\nabla z_n$, we conclude owing to integration by parts.

First step: Proof of convergences (2.4).

An easy computation gives

$$\left(\left(A_n^{-1}\right)^s\right)^{-1} = \frac{\det A_n}{\det A_n^s}A_n^s. \tag{2.5}$$

The sequence ξ_n is bounded in $L^1(\Omega)^2$ since the Cauchy–Schwarz inequality combined with the weak-$*$ convergence of (2.1), (2.2) and (2.5) yields

$$\left(\int_\Omega |\xi_n|\, dx\right)^2 \leq \int_\Omega \left|\left(\left(A_n^{-1}\right)^s\right)^{-1}\right| dx \int_\Omega \left(A_n^{-1}\right)^s \xi_n \cdot \xi_n\, dx = \int_\Omega \frac{\det A_n}{\det A_n^s}\left|A_n^s\right| dx \int_\Omega A_n^{-1}\xi_n \cdot \xi_n\, dx \leq C.$$

Therefore, ξ_n converges up to a subsequence to some $\xi \in \mathcal{M}(\Omega)^2$ in the weak-$*$ sense of the measures. Let us prove that the vector-valued measure ξ is actually in $L^2(\Omega)^2$. Again by the Cauchy–Schwarz inequality combined with (2.1), (2.2) and (2.5) we have, for any $\Phi \in \mathscr{C}_0(\Omega)^2$,

$$\begin{aligned}\left|\int_\Omega \xi(dx)\cdot\Phi\right| &= \lim_{n\to\infty}\left|\int_\Omega \xi_n\cdot\Phi\, dx\right| \\ &\leq \limsup_{n\to\infty}\left(\int_\Omega \frac{\det A_n}{\det A_n^s}\left|A_n^s\right||\Phi|^2\, dx\right)^{\frac{1}{2}}\left(\int_\Omega A_n^{-1}\xi_n\cdot\xi_n\, dx\right)^{\frac{1}{2}} \leq C\left(\int_\Omega \bar{a}|\Phi|^2\, dx\right)^{\frac{1}{2}},\end{aligned}$$

which implies that ξ is absolutely continuous with respect to the Lebesgue measure. Since $\bar{a} \in L^\infty(\Omega)$, we also get that

$$\left|\int_\Omega \xi\cdot\Phi\, dx\right| \leq \|\Phi\|_{L^2(\Omega)^2}$$

hence $\xi \in L^2(\Omega)^2$. Therefore, the first convergence of (2.4) holds true with its limit in $L^2(\Omega)^2$. The second one is immediate.

Second step: Introduction of a stream function.

By (2.3), the sequence u_n in $H_0^1(\Omega)$ defined by $u_n := \Delta^{-1}(\operatorname{div}\xi_n)$ strongly converges in $H_0^1(\Omega)$:

$$u_n \longrightarrow u \quad \text{in } H_0^1(\Omega). \tag{2.6}$$

Let ω be a regular simply connected open set such that $\omega \subset\subset \Omega$. Since by definition $\xi_n - \nabla u_n$ is a divergence-free function in $L^2(\Omega)^2$, there exists (see, e.g., [35]) a unique stream function $z_n \in H^1(\omega)$ with zero ω-average such that

$$\xi_n = \nabla u_n + J\nabla z_n \quad \text{a.e. in } \omega. \tag{2.7}$$

Third step: Convergence of the stream function z_n.

Since ∇u_n is bounded in $L^2(\Omega)^2$ by the second step, ξ_n is bounded in $L^1(\Omega)^2$ by the first step and z_n has a zero ω-average, the Sobolev embedding of $W^{1,1}(\omega)$ into $L^2(\omega)$ combined with the Poincaré–Wirtinger inequality in ω implies that z_n is bounded in $L^2(\omega)$ and thus converges, up to a subsequence still denoted by n, to a function z in $L^2(\omega)$.

Moreover, let us define

$$S_n := \left(J^{-1}(A_n^{-1})^s J\right)^{-1}.$$

The Cauchy–Schwarz inequality gives

$$\begin{aligned}
\int_\omega S_n^{-1}\nabla z_n \cdot \nabla z_n \, dx &= \int_\omega J^{-1}(A_n^{-1})^s J\nabla z_n \cdot \nabla z_n \, dx \\
&= \int_\omega (A_n^{-1})^s J\nabla z_n \cdot J\nabla z_n \, dx \\
&= \int_\omega (A_n^{-1})^s \left[\xi_n - \nabla u_n\right] \cdot \left[\xi_n - \nabla u_n\right] dx \\
&\le 2\int_\omega (A_n^{-1})^s \xi_n \cdot \xi_n \, dx + 2\int_\omega (A_n^{-1})^s \nabla u_n \cdot \nabla u_n \, dx \\
&= 2\int_\omega A_n^{-1}\xi_n \cdot \xi_n \, dx + 2\int_\omega A_n^{-1}\nabla u_n \cdot \nabla u_n \, dx.
\end{aligned}$$

The first term is bounded by (2.2) and the last term by the inequality $A_n^{-1} \le \alpha^{-1} I_2$ and the convergence (2.6). Therefore, the sequences $v_n := z_n$ and, by (2.14), S_n satisfy all the assumptions of Lemma 3.1 of [33] since, by (2.5),

$$S_n = \frac{\det A_n}{\det A_n^s} J^{-1} A_n^s J.$$

Then, we obtain the convergence

$$z_n \longrightarrow z \quad \text{strongly in } L^2_{\mathrm{loc}}(\omega). \tag{2.8}$$

Moreover, the convergence (2.6) gives

$$\xi = \nabla u + J\nabla z \quad \text{in } \mathscr{D}'(\omega). \tag{2.9}$$

Fourth step: Integration by parts and conclusion.

We have, as $J\nabla v_n$ is a divergence-free function,

$$\xi_n \cdot \nabla v_n = (\nabla u_n + J\nabla z_n) \cdot \nabla v_n = \nabla u_n \cdot \nabla v_n - \operatorname{div}(z_n J\nabla v_n). \tag{2.10}$$

The strong convergence of ∇u_n in (2.6), the second weak convergence of (2.4) justified in the first step and (2.8) give

$$\nabla u_n \cdot \nabla v_n - \operatorname{div}(z_n J\nabla v_n) \rightharpoonup \nabla u \cdot \nabla v - \operatorname{div}(zJ\nabla v) \quad \text{in } \mathscr{D}'(\omega). \tag{2.11}$$

We conclude, by combining this convergence with (2.9) and (2.10) and integrating by parts, to the convergence

$$\xi_n \cdot \nabla v_n \rightharpoonup \nabla u \cdot \nabla v - \operatorname{div}(zJ\nabla v) = (\nabla u + J\nabla z) \cdot \nabla v = \xi \cdot \nabla v \quad \text{weakly in } \mathscr{D}'(\omega)$$

for an arbitrary open subset ω of Ω. $\square$

For the reader's convenience, we first recall in Theorem 2.1 below the main result of [33] concerning the Keller duality for high contrast conductivities. Then, Proposition 2.1 is an extension of this result to a more general transformation.

Theorem 2.1 *([33]). Let Ω be a bounded open subset of $\mathbb{R}^2$ such that $|\partial\Omega| = 0$. Let $\alpha > 0$, let β_n, $n \in \mathbb{N}$ be a sequence of real numbers such that $\beta_n \ge \alpha$, and let A_n be a sequence of matrix-valued functions (not necessarily symmetric) in $\mathcal{M}(\alpha, \beta_n; \Omega)$. Assume that there exists a function $\bar{a} \in L^\infty(\Omega)$ such that*

$$\frac{\det A_n}{\det A_n^s}|A_n^s| \rightharpoonup \bar{a} \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{2.12}$$

Then, there exist a subsequence of n, still denoted by n, and a matrix-valued function A_* in $\mathcal{M}(\alpha, \beta; \Omega)$, with $\beta = 2\|\bar{a}\|_{L^\infty(\Omega)}$, such that

$$A_n \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} A_* \quad \text{and} \quad \frac{A_n^T}{\det A_n} \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} \frac{A_*^T}{\det A_*}. \tag{2.13}$$

Proposition 2.1. *Let Ω be a bounded open subset of $\mathbb{R}^2$ such that $|\partial\Omega| = 0$. Let p_n, q_n and r_n, $n \in \mathbb{N}$ be sequences of real numbers converging respectively to $p > 0$, q and 0. Let $\alpha > 0$, let β_n, $n \in \mathbb{N}$ be a sequence of real numbers such that $\beta_n \geq \alpha$, and let A_n be a sequence of matrix-valued functions in $\mathcal{M}(\alpha, \beta_n; \Omega)$ (not necessarily symmetric) satisfying*

$$r_n A_n \quad \text{is bounded in } L^\infty(\Omega)^{2\times 2} \quad \text{and} \quad \frac{\det A_n}{\det A_n^s}|A_n^s| \rightharpoonup \bar{a} \in L^\infty(\Omega) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega), \tag{2.14}$$

and that

$$B_n = \big((p_n A_n + q_n J)^{-1} + r_n J\big)^{-1} \quad \text{is a sequence of symmetric matrices}. \tag{2.15}$$

Then, there exist a subsequence of n, still denoted by n, and a matrix-valued function A_* in $\mathcal{M}(\alpha, \beta; \Omega)$, with $\beta = 2\|\bar{a}\|_{L^\infty(\Omega)}$, such that

$$A_n \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} A_* \quad \text{and} \quad \big((p_n A_n + q_n J)^{-1} + r_n J\big)^{-1} \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} pA_* + qJ. \tag{2.16}$$

Remark 2.1. Proposition 2.1 completes Theorem 2.1 performed with the transformation

$$A \longmapsto \frac{A^T}{\det A} = J^{-1}A^{-1}J, \tag{2.17}$$

to other Dykhne transformations of type (see [34, Section 4.1]):

$$A \longmapsto \big((pA + qJ)^{-1} + rJ\big)^{-1} = \big(pA + qJ\big)\big((1 - rq)I_2 + rpJA\big)^{-1}. \tag{2.18}$$

Remark 2.2. The convergence of r_n to $r = 0$ is not necessary but sufficient for our purpose. If $r \neq 0$, different convergences are conserved but lead us to the expression

$$pA_* + qJ = B_*\big((1 - qr)I_2 + p\, rJA_*\big). \tag{2.19}$$

Proof of Proposition 2.1. The proof is divided into two steps. In the first step, we use Lemma 2.1 to show the $H(\mathcal{M}(\Omega)^2)$-convergence of $\widetilde{A}_n := p_n A_n + q_n J$ to $pA_* + qJ$. In the second step, we build a matrix Q_n which will be used as a corrector for B_n and then use again Lemma 2.1.

First step: $\widetilde{A}_* = pA_* + qJ$.

First of all, thanks to Theorem 2.2 [33], we already know that, up to a subsequence still denoted by n, A_n $H(\mathcal{M}(\Omega)^2)$-converges to A_*. We consider a corrector P_n associated with A_n in the sense of Murat–Tartar (see, e.g., [3]), such that, for $\lambda \in \mathbb{R}^2$, $P_n\lambda = \nabla w_n^\lambda$ is defined by

$$\begin{cases} \operatorname{div}(A_n \nabla w_n^\lambda) = \operatorname{div}(A_* \nabla(\lambda \cdot x)) & \text{in } \Omega \\ w_n^\lambda = \lambda \cdot x & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

Again with Theorem 2.2 of [33] and Definition 1.1, we know that $P_n\lambda$ converges weakly in $L^2(\Omega)^2$ to λ and $A_n P_n \lambda$ converges weakly-$*$ in $\mathcal{M}(\Omega)$ to $A_*\lambda$.

Since, for any $\lambda, \mu \in \mathbb{R}^2$,

$$\alpha\|\nabla w_n^\mu\|^2_{L^2(\Omega)^2} \leq \int_\Omega A_n \nabla w_n^\mu \cdot \nabla w_n^\mu \, dx = \int_\Omega A_* \mu \cdot \nabla w_n^\mu \, dx \leq 2\|\bar{a}\|_{L^\infty(\Omega)}|\mu|\, |\Omega|^{1/2}\|\nabla w_n^\mu\|_{L^2(\Omega)^2}$$

and

$$\int_\Omega A_n^{-1} A_n \nabla w_n^\lambda \cdot A_n \nabla w_n^\lambda \, dx = \int_\Omega A_n \nabla w_n^\lambda \cdot \nabla w_n^\lambda \, dx,$$

the sequences $\xi_n := A_n \nabla w_n^\lambda$ and $v_n := w_n^\mu$ satisfy (2.2) and (2.3). This combined with (2.14) implies that we can apply Lemma 2.1 to obtain

$$\forall \lambda, \mu \in \mathbb{R}, \quad A_n P_n \lambda \cdot P_n \mu \rightharpoonup A_* \lambda \cdot \mu \quad \text{in } \mathscr{D}'(\Omega). \tag{2.21}$$

We denote $\widetilde{A}_n := p_nA_n + q_nJ$ and consider δ_n such that $\delta_nJ := A_n - A_n^s$. Then, the matrix $\widetilde{A}_n$ satisfies

$$\widetilde{A}_n\xi\cdot\xi = p_nA_n\xi\cdot\xi \geq p_n\alpha|\xi|^2. \tag{2.22}$$

Moreover,

$$\det\widetilde{A}_n = p_n^2\det A_n^s + (p_n\delta_n + q_n)^2 \leq p_n^2(\det A_n^s + 2\delta_n^2) + 2q_n^2 \leq 2p_n^2\det A_n + 2q_n^2 \leq C\det A_n,$$

the last inequality being a consequence of $A_n \geq \alpha I_2$. This inequality gives, by (2.14),

$$\frac{\det\widetilde{A}_n}{\det\widetilde{A}_n^s}|\widetilde{A}_n^s| = \frac{\det\widetilde{A}_n}{p_n^2\det A_n^s}p_n|A_n^s| \leq C\frac{\det A_n}{\det A_n^s}|A_n^s| \leq C. \tag{2.23}$$

Then by (2.22) and (2.23) and [33] again, up to a subsequence still denoted by n, $\widetilde{A}_n$ $H(\mathcal{M}(\Omega)^2)$-converges to $\widetilde{A}_*$ and we have, by the classical div–curl lemma of [3] for $JP_n\lambda\cdot P_n\mu$ and (2.21),

$$\forall\lambda,\mu\in\mathbb{R},\quad (p_nA_n + q_nJ)P_n\lambda\cdot P_n\mu = p_nA_nP_n\lambda\cdot P_n\mu + q_nJP_n\lambda\cdot P_n\mu \overset{\mathscr{D}'(\Omega)}{\rightharpoonup} pA^*\lambda\cdot\mu + qJ\lambda\cdot\mu,$$

that can be rewritten

$$\widetilde{A}_* = pA_* + qJ.$$

Second step: $B_* = \widetilde{A}_*$.

Let $\theta\in\mathscr{C}_c^1(\Omega)$ and $\widetilde{P}_n$ a corrector associated with $\widetilde{A}_n$, such that, for $\lambda\in\mathbb{R}^2$, $\widetilde{P}_n\lambda = \nabla\widetilde{w}_n^\lambda$ is defined by

$$\begin{cases}\operatorname{div}(\widetilde{A}_n\nabla\widetilde{w}_n^\lambda) = \operatorname{div}\big(\widetilde{A}_*\nabla(\theta\lambda\cdot x)\big) & \text{in } \Omega\\ \widetilde{w}_n^\lambda = 0 & \text{on } \partial\Omega.\end{cases} \tag{2.24}$$

By Definition 1.1, we have

$$\begin{cases}\widetilde{w}_n^\lambda \rightharpoonup \theta\lambda\cdot x & \text{weakly in } H_0^1(\Omega),\\ \widetilde{A}_n\nabla\widetilde{w}_n^\lambda \rightharpoonup \widetilde{A}_*\nabla(\theta\lambda\cdot x) & \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2.\end{cases} \tag{2.25}$$

Let us now consider $B_n = \big(\widetilde{A}_n^{-1} + r_nJ\big)^{-1}$. B_n is symmetric and so is its inverse:

$$B_n^{-1} = \widetilde{A}_n^{-1} + r_nJ = (\widetilde{A}_n^{-1} + r_nJ)^s = (\widetilde{A}_n^{-1})^s.$$

We then have, by a little computation (like in (2.5)) and (2.23),

$$\frac{\det B_n}{\det B_n^s}|B_n^s| = |B_n| = \left|\left(\big(\widetilde{A}_n^{-1}\big)^s\right)^{-1}\right| = \frac{\det\widetilde{A}_n}{\det\widetilde{A}_n^s}|\widetilde{A}_n^s| \leq C. \tag{2.26}$$

For any $\xi\in\mathbb{R}^2$, the sequence $\nu_n := (I + r_nJ\widetilde{A}_n)^{-1}\xi$ satisfies, by (2.14),

$$|\xi| \leq \big(1 + \|r_n\widetilde{A}_n\|_{L^\infty(\Omega)^{2\times2}}\big)|\nu_n| \leq \big(1 + p_n\|r_nA_n\|_{L^\infty(\Omega)^{2\times2}} + q_nr_n\big)|\nu_n| \leq (1+C)|\nu_n|,$$

hence

$$B_n\xi\cdot\xi = \widetilde{A}_n\nu_n\cdot(I + r_nJ\widetilde{A}_n)\nu_n = \widetilde{A}_n\nu_n\cdot\nu_n = p_nA_n\nu_n\cdot\nu_n \geq p_n\alpha|\nu_n|^2 \geq \alpha\frac{p_n}{(1+C)^2}|\xi|^2 \geq C|\xi|^2 \tag{2.27}$$

with $C > 0$. Therefore, with (2.27) and (2.26), again by Theorem 2.2 of [33], up to a subsequence still denoted by n, B_n $H(\mathcal{M}(\Omega)^2)$-converges to B_*.

Let $\psi\in\mathscr{C}_c^1(\Omega)$ and R_n be a corrector associated to B_n, such that, for $\mu\in\mathbb{R}^2$, $R_n\mu = \nabla v_n^\mu$ is defined by

$$\begin{cases}\operatorname{div}\big(B_n\nabla v_n^\mu\big) = \operatorname{div}\big(B_*\nabla(\psi\mu\cdot x)\big) & \text{in } \Omega\\ v_n^\mu = 0 & \text{on } \partial\Omega.\end{cases} \tag{2.28}$$

By Definition 1.1, we have the convergences

$$\begin{cases}v_n^\mu \rightharpoonup \psi\mu\cdot x & \text{weakly in } H_0^1(\Omega),\\ B_n\nabla v_n^\mu \rightharpoonup B_*\nabla(\psi\mu\cdot x) & \text{weakly-}* \text{ in } \mathcal{M}(\Omega)^2.\end{cases} \tag{2.29}$$

Let us define the matrix $Q_n := (I + r_nJ\widetilde{A}_n)\widetilde{P}_n$. We have

$$B_nQ_n = (\widetilde{A}_n^{-1} + r_nJ)^{-1}(I + r_nJ\widetilde{A}_n)\widetilde{P}_n = (\widetilde{A}_n^{-1} + r_nJ)^{-1}(\widetilde{A}_n^{-1} + r_nJ)\widetilde{A}_n\widetilde{P}_n = \widetilde{A}_n\widetilde{P}_n. \tag{2.30}$$

We are going to pass to the limit in $\mathscr{D}'(\Omega)$ the equality given by (2.30) and the symmetry of B_n:

$$\widetilde{A}_n\widetilde{P}_n\lambda \cdot R_n\mu = B_nQ_n\lambda \cdot R_n\mu = Q_n\lambda \cdot B_nR_n\mu. \tag{2.31}$$

On the one hand, $\widetilde{A}_n$ satisfies (2.1) by (2.22) and (2.23). The sequences $\xi_n := \widetilde{A}_n\widetilde{P}_n\lambda$ and $v_n := v_n^\mu$ satisfy the hypothesis (2.3) by (2.24) and (2.2) because

$$\int_\Omega \big(\widetilde{A}_n\big)^{-1}\xi_n \cdot \xi_n\, dx + \|v_n\|_{H_0^1(\Omega)} = \int_\Omega \widetilde{A}_n\widetilde{P}_n\lambda \cdot \widetilde{P}_n\lambda\, dx + \|v_n^\mu\|_{H_0^1(\Omega)}\, dx \le C$$

by (2.24) and the convergences (2.25) and (2.29). The application of Lemma 2.1, (2.25) and (2.29) give the convergence

$$\widetilde{A}_n\widetilde{P}_n\lambda \cdot R_n\mu \rightharpoonup \widetilde{A}^*\nabla(\theta\lambda \cdot x) \cdot \nabla(\psi\mu \cdot x) \quad \text{in } \mathscr{D}'(\Omega). \tag{2.32}$$

On the other hand, we have the equality

$$Q_n\lambda \cdot B_nR_n\mu = B_nR_n\mu \cdot \widetilde{P}_n\lambda + B_nR_n\mu \cdot r_nJ\widetilde{A}_n\widetilde{P}_n. \tag{2.33}$$

The matrix B_n satisfies (2.1) by (2.27) and (2.26). The sequences $\xi_n := B_nR_n\mu$ and $v_n := \widetilde{w}_n^\lambda$ satisfy the hypothesis (2.3) by (2.28) and (2.2) of Lemma 2.1 because

$$\int_\Omega \big(B_n\big)^{-1}\xi_n \cdot \xi_n\, dx + \|v_n\|_{H_0^1(\Omega)} = \int_\Omega B_nR_n\mu \cdot R_n\mu\, dx + \|\widetilde{w}_n^\lambda\|_{H_0^1(\Omega)}\, dx \le C$$

by (2.28) and the convergences (2.25) and (2.29). The application of Lemma 2.1, (2.25) and (2.29) give the convergence

$$B_nR_n\mu \cdot \widetilde{P}_n\lambda \rightharpoonup B_*\nabla(\psi\mu \cdot x) \cdot \nabla(\theta\lambda \cdot x) \quad \text{in } \mathscr{D}'(\Omega). \tag{2.34}$$

The convergence of the right part of (2.33) is more delicate. The demonstration is the same as for Lemma 2.1. Let ω be a simply connected open subset of Ω such as $\omega \subset\subset \Omega$. The function $\widetilde{A}_n\widetilde{P}_n\lambda - \widetilde{A}_*\nabla(\theta\lambda \cdot x)$ is divergence-free and we can introduce a function z_n^λ such as

$$\widetilde{A}_n\widetilde{P}_n\lambda = \widetilde{A}_*\nabla(\theta\lambda \cdot x) + J\nabla z_n^\lambda, \tag{2.35}$$

$$z_n^\lambda \longrightarrow 0 \quad \text{strongly in } L^2_{\mathrm{loc}}(\omega). \tag{2.36}$$

The equality

$$\begin{aligned} B_nR_n\mu \cdot r_nJ\widetilde{A}_n\widetilde{P}_n\lambda &= r_nB_nR_n\mu \cdot J\widetilde{A}_*\nabla(\theta\lambda \cdot x) - r_nB_nR_n\mu \cdot \nabla z_n^\lambda \\ &= r_nB_nR_n\mu \cdot J\widetilde{A}_*\nabla(\theta\lambda \cdot x) - r_n\mathrm{div}(z_n^\lambda B_nR_n\mu) + r_nz_n^\lambda\, \mathrm{div}\big(B_*\nabla(\theta\lambda \cdot x)\big) \end{aligned}$$

leads us, by (2.29) and (2.36) and the convergence to 0 of r_n, like in the demonstration of Lemma 2.1, to

$$B_nR_n\mu \cdot r_nJ\widetilde{A}_n\widetilde{P}_n \rightharpoonup 0 \quad \text{in } \mathscr{D}'(\omega). \tag{2.37}$$

Finally, by combining (2.31), (2.32), (2.34) and (2.37), we obtain, for any simply connected open subset ω of Ω such as $\omega \subset\subset \Omega$,

$$\widetilde{A}_*\nabla(\theta\lambda \cdot x) \cdot \nabla(\psi\mu \cdot x) = B_*\nabla(\psi\mu \cdot x) \cdot \nabla(\theta\lambda \cdot x) \quad \text{in } \mathscr{D}'(\omega).$$

We conclude, by taking $\theta = 1$ and $\psi = 1$ on ω and taking into account that B_* is symmetric and ω, λ, μ are arbitrary, that:

$$B_* = \widetilde{A}_* = pA_* + qJ. \quad \square$$

2.2. An application to isotropic two-phase media

In this section, we study the homogenization of a two-phase isotropic medium with high contrast and non-necessarily symmetric conductivities. The study of the symmetric case in Proposition 2.2 permits to obtain Theorem 2.2 by applying the transformation of Proposition 2.1. We use Notation 1.1.

Proposition 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^2$ such that $|\partial\Omega| = 0$. Let ω_n, n in $\mathbb{N}$, be a sequence of open subsets of Ω with characteristic function χ_n, satisfying $\theta_n := |\omega_n| < 1$, θ_n converges to 0, and*

$$\frac{\chi_n}{\theta_n} \rightharpoonup a \in L^\infty(\Omega) \quad \text{weakly-}* \text{ in } \mathscr{M}(\Omega). \tag{2.38}$$

We assume that there exist $\alpha_1, \alpha_2 > 0$ and two positive sequences $\alpha_{1,n}, \alpha_{2,n} \ge a_0 > 0$ verifying

$$\lim_{n\to\infty} \alpha_{1,n} = \alpha_1 \quad \text{and} \quad \lim_{n\to\infty} \theta_n\alpha_{2,n} = \alpha_2, \tag{2.39}$$

and that the conductivity takes the form

$$\sigma_n^0(\alpha_{1,n},\alpha_{2,n}) = (1-\chi_n)\alpha_{1,n}I_2 + \chi_n\alpha_{2,n}I_2.$$

Then, there exist a subsequence of n, still denoted by n, and a locally Lipschitz function

$$\sigma_*^0\colon (0,\infty)^2 \longrightarrow \mathcal{M}(a_0, 2\|a\|_\infty; \Omega)$$

such that

$$\forall(\alpha_1,\alpha_2)\in(0,\infty)^2,\quad \sigma_n^0(\alpha_{1,n},\alpha_{2,n}) \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} \sigma_*^0(\alpha_1,\alpha_2). \tag{2.40}$$

Proof of Proposition 2.2. The proof is divided into two parts. We first prove the theorem for $\alpha_{1,n}=\alpha_1$, $\alpha_{2,n}=\theta_n^{-1}\alpha_2$, and then treat the general case.

First step: The case $\alpha_{1,n}=\alpha_1$, $\alpha_{2,n}=\theta_n^{-1}\alpha_2$.

In this step we denote $\sigma_n^0(\alpha) := \sigma_n^0(\alpha_1,\theta_n^{-1}\alpha_2)$, for $\alpha = (\alpha_1,\alpha_2)\in(0,\infty)^2$. Theorem 2.2 of [33] implies that for any $\alpha\in(0,\infty)^2$, there exists a subsequence of n such that $\sigma_n^0(\alpha)$ $H(\mathcal{M}(\Omega)^2)$-converges in the sense of Definition 1.1 to some matrix-valued function in $\mathcal{M}(a_0, 2\|a\|_\infty; \Omega)$.

By a diagonal extraction, there exists a subsequence of n, still denoted by n, such that

$$\forall\alpha\in\mathbb{Q}^2\cap(0,\infty)^2,\quad \sigma_n^0(\alpha) \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} \sigma_*^0(\alpha). \tag{2.41}$$

We are going to show that this convergence is true for any pair $\alpha\in(0,\infty)^2$.

We have, by (2.38), for any $\alpha\in\mathbb{Q}^2\cap(0,\infty)^2$,

$$|\sigma_n^0(\alpha)| = (1-\chi_n)\alpha_1 + \chi_n\frac{\alpha_2}{\theta_n} \rightharpoonup \alpha_1+\alpha_2\, a\in L^\infty(\Omega)\quad \text{weakly-}*\ \text{in } \mathcal{M}(\Omega) \tag{2.42}$$

and, since $\theta_n\in(0,1)$,

$$\forall\xi\in\mathbb{R}^2,\quad \sigma_n^0(\alpha)\xi\cdot\xi = \alpha_1(1-\chi_n)|\xi|^2 + \chi_n\frac{\alpha_2}{\theta_n}|\xi|^2 \geq \min(\alpha_1,\alpha_2)|\xi|^2 \quad \text{a.e. in } \Omega. \tag{2.43}$$

By applying Theorem 2.2 of [33] with (2.42), we have the inequality

$$|\sigma_*^0(\alpha)\lambda| \leq 2|\lambda|\left(\alpha_1+\alpha_2\|a\|_\infty\right). \tag{2.44}$$

For any $\alpha\in\mathbb{Q}^2\cap(0,\infty)^2$ and $\lambda\in\mathbb{R}^2$, consider the corrector $w_n^{\alpha,\lambda}$ associated with $\sigma_n^0(\alpha)$ defined by

$$\begin{cases} \operatorname{div}\left(\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\right) = \operatorname{div}\left(\sigma_*^0(\alpha)\lambda\right) & \text{in } \Omega,\\ w_n^{\alpha,\lambda} = \lambda\cdot x & \text{on } \partial\Omega, \end{cases} \tag{2.45}$$

which depends linearly on λ.

Let $\alpha\in\mathbb{Q}^2\cap(0,\infty)^2$. Let us show that the energies

$$\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,dx \tag{2.46}$$

are bounded. We have, by (2.45) and (2.44) and the Cauchy–Schwarz inequality

$$\begin{aligned} \int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,dx &= \int_\Omega \sigma_*^0(\alpha)\lambda\cdot\left(\nabla w_n^{\alpha,\lambda}-\lambda\right)dx + \int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda\,dx\\ &= \int_\Omega \sigma_*^0(\alpha)\lambda\cdot\nabla w_n^{\alpha,\lambda}\,dx - \int_\Omega \underbrace{\sigma_*^0(\alpha)\lambda\cdot\lambda}_{\geq 0}\,dx + \int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda\,dx \end{aligned}$$

which leads us to

$$\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,dx \leq \int_\Omega |\sigma_*^0(\alpha)\lambda\cdot\nabla w_n^{\alpha,\lambda}|\,dx + \int_\Omega |\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda|\,dx. \tag{2.47}$$

On the one hand, the Cauchy–Schwarz inequality gives

$$\left(\int_\Omega |\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda|\,dx\right)^2 \leq |\lambda|^2\int_\Omega |\sigma_n^0(\alpha)|\,dx \int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,dx$$

that is

$$\left(\int_\Omega |\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\lambda|\,dx\right)^2 \le |\lambda|^2|\alpha|\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,dx. \tag{2.48}$$

On the other hand, by (2.43) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_\Omega |\sigma_*^0(\alpha)\lambda\cdot\nabla w_n^{\alpha,\lambda}|\,dx &\le 2|\lambda|(\alpha_1+\alpha_2\|a\|_\infty)\sqrt{\int_\Omega |\nabla w_n^{\alpha,\lambda}|^2\,dx}\\ &\le 2|\lambda|(\alpha_1+\alpha_2\|a\|_\infty)\sqrt{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}\sqrt{\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,dx}\end{aligned}$$

that is

$$\int_\Omega |\sigma_*^0(\alpha)\lambda\cdot\nabla w_n^{\alpha,\lambda}|\,dx \le C\,|\lambda|^2\,|\alpha|\sqrt{\frac{1}{\alpha_1}+\frac{1}{\alpha_2}}\sqrt{\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,dx} \tag{2.49}$$

where C does not depend on n or α.

By combining (2.47)–(2.49), we have

$$\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,dx \le C\,|\lambda|^2\,\underbrace{\left(|\alpha|+|\alpha|^2({\alpha_1}^{-1}+{\alpha_2}^{-1})\right)}_{=:M(\alpha)} \tag{2.50}$$

where C does not depend on n nor α.

Let $\alpha'\in\mathbb{Q}^2\cap(0,\infty)^2$. The sequences $\xi_n := \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}$ and $v_n := w_n^{\alpha',\lambda}$ satisfy the assumptions (2.2) and (2.3) of Lemma 2.1. By symmetry, we have the convergences

$$\begin{cases}\sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda} \rightharpoonup \sigma_*^0(\alpha)\lambda\cdot\lambda & \text{weakly in } \mathscr{D}'(\Omega),\\ \sigma_n^0(\alpha')\nabla w_n^{\alpha',\lambda}\cdot\nabla w_n^{\alpha,\lambda} \rightharpoonup \sigma_*^0(\alpha')\lambda\cdot\lambda & \text{weakly in } \mathscr{D}'(\Omega).\end{cases} \tag{2.51}$$

As the matrices are symmetric, we have

$$\left(\sigma_n^0(\alpha)-\sigma_n^0(\alpha')\right)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda} = \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda} - \sigma_n^0(\alpha')\nabla w_n^{\alpha',\lambda}\cdot\nabla w_n^{\alpha,\lambda},$$

hence

$$\left(\sigma_n^0(\alpha)-\sigma_n^0(\alpha')\right)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda} \rightharpoonup \left(\sigma_*^0(\alpha)-\sigma_*^0(\alpha')\right)\lambda\cdot\lambda \quad \text{weakly in } \mathscr{D}'(\Omega). \tag{2.52}$$

Let $\lambda\in\mathbb{R}^2$. We have, by the Cauchy–Schwarz inequality, with the Einstein convention

$$\begin{aligned}&\int_\Omega \left|\left(\sigma_n^0(\alpha)-\sigma_n^0(\alpha')\right)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda}\right|\,dx\\ &\quad= \int_{\Omega\setminus\omega_n} |\alpha_1-\alpha_1'|\,\left|\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda}\right|\,dx + \fint_{\omega_n} |\alpha_2-\alpha_2'|\,\left|\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda}\right|\,dx\\ &\quad\le |\alpha_1-\alpha_1'|\sqrt{\int_{\Omega\setminus\omega_n}|\nabla w_n^{\alpha,\lambda}|^2\,dx}\sqrt{\int_{\Omega\setminus\omega_n}|\nabla w_n^{\alpha',\lambda}|^2\,dx} + |\alpha_2-\alpha_2'|\sqrt{\fint_{\omega_n}|\nabla w_n^{\alpha,\lambda}|^2\,dx}\sqrt{\fint_{\omega_n}|\nabla w_n^{\alpha',\lambda}|^2\,dx}\\ &\quad\le |\alpha_i-\alpha_i'|\sqrt{\frac{1}{\alpha_i}\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha,\lambda}\,dx}\sqrt{\frac{1}{\alpha_i'}\int_\Omega \sigma_n^0(\alpha)\nabla w_n^{\alpha',\lambda}\cdot\nabla w_n^{\alpha',\lambda}\,dx}.\end{aligned}$$

This combined with (2.50) yields

$$\int_\Omega \left|\left(\sigma_n^0(\alpha)-\sigma_n^0(\alpha')\right)\nabla w_n^{\alpha,\lambda}\cdot\nabla w_n^{\alpha',\lambda}\right| \le C|\lambda|^2\,\frac{|\alpha_i-\alpha_i'|}{\sqrt{|\alpha_i||\alpha_i'|}}\,M(\alpha)\,M(\alpha').$$

The sequence of (2.52) is thus bounded in $L^1(\Omega)^2$ which implies that (2.52) holds weakly-$*$ in $\mathcal{M}(\Omega)$. Hence, we get, for any $\varphi\in\mathscr{C}_c(\Omega)$, that

$$\int_\Omega \left|\left(\sigma_*^0(\alpha)-\sigma_*^0(\alpha')\right)\lambda\cdot\lambda\right|\,\varphi\,dx \le C\,|\lambda|^2\,\frac{|\alpha_i-\alpha_i'|}{\sqrt{|\alpha_i||\alpha_i'|}}\,M(\alpha)\,M(\alpha')\,\|\varphi\|_\infty. \tag{2.53}$$

Then, the Riesz representation theorem implies that

$$\left|\left|\sigma_*^0(\alpha)-\sigma_*^0(\alpha')\right|\right|_{L^1(\Omega)^{2\times 2}}\le C\,\frac{|\alpha_i-\alpha_i'|}{\sqrt{|\alpha_i||\alpha_i'|}}\,M(\alpha)\,M(\alpha').$$

Therefore, by the definition of M in (2.50), for any compact subset $K\subset(0,\infty)^2$,

$$\exists C>0,\forall\alpha,\alpha'\in\mathbb{Q}^2\cap K,\quad \left\|\sigma_*^0(\alpha)-\sigma_*^0(\alpha')\right\|_{L^1(\Omega)^{2\times 2}}\le C\,|\alpha-\alpha'|. \tag{2.54}$$

This estimate permits to extend the definition (2.41) of σ_*^0 on $(0,\infty)^2$ by

$$\forall\alpha\in(0,\infty)^2,\qquad \sigma_*^0(\alpha)=\lim_{\substack{\alpha'\to\alpha\\ \alpha'\in\mathbb{Q}^2\cap(0,\infty)^2}}\sigma_*^0(\alpha')\ \text{strongly in}\ L^1(\Omega)^{2\times 2}. \tag{2.55}$$

Let $\alpha\in(0,\infty)^2$. Theorem 2.2 of [33] implies that there exist a subsequence of n, denoted by n', and a matrix-valued function $\widetilde{\sigma}_*\in\mathcal{M}(a_0,2\|a\|_\infty;\Omega)$ such that

$$\sigma_{n'}(\alpha)\overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup}\widetilde{\sigma}_*. \tag{2.56}$$

Repeating the arguments leading to (2.54), for any positive sequence of rational pair $(\alpha^q)_{q\in\mathbb{N}}$ converging to α, we have

$$\exists C>0,\quad \left\|\widetilde{\sigma}_*-\sigma_*^0(\alpha^q)\right\|_{L^1(\Omega)^{2\times 2}}\le C\,|\alpha-\alpha^q|, \tag{2.57}$$

hence, by (2.55), $\widetilde{\sigma}_*=\sigma_*^0(\alpha)$. Therefore by the uniqueness of the limit in (2.56), we obtain for the whole sequence satisfying (2.41)

$$\forall\alpha\in(0,\infty)^2,\quad \sigma_n(\alpha)\overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup}\sigma_*^0(\alpha). \tag{2.58}$$

In particular, the function σ_*^0 satisfies (2.54) and (2.55), i.e. σ_*^0 is a locally Lipschitz function on $(0,\infty)^2$.

Second step: The general case.

We denote $\alpha^n=(\alpha_{1,n},\alpha_{2,n})$ and $\sigma_n^0(\alpha^n)=\sigma_n^0(\alpha_{1,n},\alpha_{2,n})$. Theorem 2.2 of [33] implies that there exists a subsequence of n, denoted by n', such that $\sigma_{n'}^0(\alpha^{n'})$ $H(\mathcal{M}(\Omega)^2)$-converges to some $\widetilde{\sigma}_*\in\mathcal{M}(a_0,2\|a\|_\infty;\Omega)$ in the sense of Definition 1.1.

As in the first step, for any $\alpha^{n'}\in(0,\infty)^2$ and $\lambda\in\mathbb{R}^2$, we can consider the corrector $w_{n'}^{\alpha^{n'},\lambda}$ associated with $\sigma_{n'}^0(\alpha^{n'})$ defined by

$$\begin{cases}\operatorname{div}\left(\sigma_{n'}^0(\alpha^{n'})\nabla w_{n'}^{\alpha^{n'},\lambda}\right)=\operatorname{div}(\widetilde{\sigma}_*\lambda) & \text{in }\Omega,\\ w_{n'}^{\alpha^{n'},\lambda}=\lambda\cdot x & \text{on }\partial\Omega,\end{cases} \tag{2.59}$$

which depends linearly on λ. Proceeding as in the first step, we obtain like in (2.52), with $\alpha=(\alpha_1,\alpha_2)$ the limit of α^n according to (2.39),

$$\left(\sigma_{n'}^0(\alpha)-\sigma_{n'}^0(\alpha^{n'})\right)\nabla w_{n'}^{\alpha^{n'},\lambda}\cdot\nabla w_{n'}^{\alpha,\lambda}\ \rightharpoonup\ \left(\sigma_*^0(\alpha)-\widetilde{\sigma}_*\right)\lambda\cdot\lambda\quad\text{weakly in }\mathscr{D}'(\Omega). \tag{2.60}$$

Moreover, by the energy bound (2.50), which also holds for $\alpha^{n'}$, we have, for any $\varphi\in\mathscr{D}(\Omega)$,

$$\int_\Omega\left(\sigma_{n'}^0(\alpha)-\sigma_{n'}^0(\alpha^{n'})\right)\nabla w_{n'}^{\alpha^{n'},\lambda}\cdot\nabla w_{n'}^{\alpha,\lambda}\,\varphi\,dx\underset{n'\to\infty}{\longrightarrow}0.$$

This combined with (2.60), yields

$$\int_\Omega\left(\sigma_*^0(\alpha)-\widetilde{\sigma}_*\right)\lambda\cdot\lambda\,\varphi\,dx=0,$$

which implies that $\sigma_*^0(\alpha)=\widetilde{\sigma}_*$. We conclude by a uniqueness argument. □

We can now obtain a result for (perturbed) non-symmetric conductivities. Then, we will use a Dykhne transformation to recover the symmetric case following the Milton approach [34, pp. 61–65]. This will allow us to apply Proposition 2.2.

Theorem 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^2$ such that $|\partial\Omega|=0$. Let ω_n, $n\in\mathbb{N}$, be a sequence of open subsets of Ω and denote by χ_n their characteristic function. We assume that $\theta_n=|\omega_n|<1$ converges to 0 and*

$$\frac{\chi_n}{\theta_n}\rightharpoonup a\in L^\infty(\Omega)\quad\text{weakly-}*\ \text{in}\ \mathcal{M}(\Omega). \tag{2.61}$$

Consider the conductivity defined by

$$\sigma_n(h) = (1-\chi_n)\sigma_1(h) + \frac{\chi_n}{\theta_n}\sigma_2(h) \tag{2.62}$$

where for $j = 1, 2$, $\sigma_j(h) = \alpha_j + h\beta_j J \in \mathbb{R}^{2\times 2}$ *with* $\alpha_1, \alpha_2 > 0$ *and* $(\beta_1, \beta_2) \neq (0, 0)$.

Then, there exist a subsequence of n*, still denoted by* n*, and a locally Lipschitz function*

$$\sigma_*^0 : (0,\infty)^2 \longrightarrow \mathcal{M}\Big(\min(\alpha_1,\alpha_2), 2\big(|\sigma_1| + |\sigma_2|\,\|a\|_\infty\big); \Omega\Big)$$

such that

$$\sigma_n(h) \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} \sigma_*^0\big(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h^2\big) + h\beta_1 J.$$

Proof of Theorem 2.2. We have

$$\forall \xi \in \mathbb{R}^2, \quad \sigma_n(h)\xi\cdot\xi = (1-\chi_n)\alpha_1|\xi|^2 + \frac{\chi_n}{\theta_n}\alpha_2|\xi|^2 \geq \min(\alpha_1,\alpha_2)|\xi|^2 \quad \text{a.e. in } \Omega$$

and, by (2.61),

$$|\sigma_n(h)| = (1-\chi_n)|\sigma_1(h)| + \frac{\chi_n}{\theta_n}|\sigma_2(h)| \rightharpoonup |\sigma_1(h)| + a|\sigma_2(h)| \in L^\infty(\Omega) \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).$$

In order to make a Dykhne transformation like in p. 62 of [34], we consider two real coefficients a_n and b_n in such a way that

$$B_n := \big(a_n\sigma_n(h) + b_n J\big)\big(a_n I_2 + J\sigma_n(h)\big)^{-1} = \big((p_n\sigma_n(h) + q_n J)^{-1} + r_n J\big)^{-1}$$

is symmetric. An easy computation shows that the previous equality holds when

$$p_n := \frac{a_n^2}{a_n^2 + b_n}, \qquad q_n := \frac{a_n b_n}{a_n^2 + b_n} \quad \text{and} \quad r_n := \frac{1}{a_n}.$$

On the one hand, the estimates (3.39) and (3.40) with $\alpha_{2,n} = \theta_n^{-1}\alpha_2$, $\beta_{2,n} = \theta_n^{-1}\beta_2$, yield (note that they are independent of χ_n)

$$p_n \underset{n\to\infty}{\sim} 1, \qquad q_n \underset{n\to\infty}{\longrightarrow} -h\beta_1, \qquad r_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{and} \quad \|r_n\sigma_n(h)\|_\infty \leq C\big(|\sigma_1(h)| + |\sigma_2(h)|\big). \tag{2.63}$$

On the other hand, as in Section 3.2, with Notation 1.1 and (3.34), we have

$$B_n = \sigma_n^0\big(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\big), \tag{2.64}$$

where

$$\alpha'_{1,n}(h) = \frac{a_n(\alpha_1 + ih\beta_1) + ib_n}{a_n + i(\alpha_1 + ih\beta_1)} \quad \text{and} \quad \alpha'_{2,n}(h) = \frac{a_n(\alpha_2/\theta_n + ih\beta_2/\theta_n) + ib_n}{a_n + i(\alpha_2/\theta_n + ih\beta_2/\theta_n)}. \tag{2.65}$$

Hence, like in (3.41), we have

$$\lim_{n\to\infty}\alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \lim_{n\to\infty}\theta_n\alpha'_{2,n}(h) = \alpha_2 + \alpha_2^{-1}{\beta_2}^2h^2. \tag{2.66}$$

We can first apply Proposition 2.2 with the conditions (2.64) and (2.66) to have the $H(\mathcal{M}(\Omega)^2)$-convergence of B_n. Then, by virtue of Proposition 2.1, with (2.63) we get that

$$\sigma_n(h) \overset{H(\mathcal{M}(\Omega)^2)}{\rightharpoonup} \sigma_*^0\big(\alpha_1, \alpha_2 + \alpha_2^{-1}\beta_2^2 h^2\big) + h\beta_1 J. \quad \square$$

3. A two-dimensional periodic medium

In this section we consider a sequence Σ_n of matrix valued functions (not necessarily symmetric) in $L^\infty(\mathbb{R}^2)^{2\times 2}$, which satisfies the following assumptions:

1. Σ_n is Y-periodic, where $Y := (0, 1)^2$, i.e.,

$$\forall n \in \mathbb{N}, \forall \kappa \in \mathbb{Z}^2, \quad \Sigma_n(\cdot + \kappa) = \Sigma_n(\cdot) \quad \text{a.e. in } \mathbb{R}^2, \tag{3.1}$$

2. Σ_n is equi-coercive in $\mathbb{R}^2$, i.e.,

$$\exists\alpha > 0 \text{ such that } \forall n \in \mathbb{N}, \forall \xi \in \mathbb{R}^2, \quad \Sigma_n\xi \cdot \xi \geq \alpha|\xi|^2 \quad \text{a.e. in } \mathbb{R}^2. \tag{3.2}$$

Let ε_n be a sequence of positive numbers which tends to 0. From the sequences Σ_n and ε_n we define the highly oscillating sequence of matrix-valued functions σ_n by

$$\sigma_n(x) = \Sigma_n\left(\frac{x}{\varepsilon_n}\right), \quad \text{a.e. } x \in \mathbb{R}^2. \tag{3.3}$$

By virtue of (3.1) and (3.2), σ_n is an equi-coercive sequence of ε_n-periodic matrix-valued functions in $L^\infty(\mathbb{R}^2)^{2\times 2}$. For a fixed $n \in \mathbb{N}$, let $(\sigma_n)_*$ be the constant matrix defined by

$$\forall\lambda, \mu \in \mathbb{R}^2, \quad (\sigma_n)_*\lambda \cdot \mu = \int_Y \Sigma_n \nabla W_n^\lambda \cdot \nabla W_n^\mu \, dy, \tag{3.4}$$

where, for any $\lambda \in \mathbb{R}^2$, $W_n^\lambda \in H^1_\sharp(Y)$, the set of Y-periodic functions belonging to $H^1_{\text{loc}}(\mathbb{R}^2)$, is the solution of the auxiliary problem

$$\int_Y \big(W_n^\lambda - \lambda \cdot y\big)\, dy = 0 \quad \text{and} \quad \operatorname{div}\big(\Sigma_n \nabla W_n^\lambda\big) = 0 \text{ in } \mathscr{D}'(\mathbb{R}^2) \tag{3.5}$$

or equivalently

$$\begin{cases} \displaystyle\int_Y \Sigma_n \nabla W_n^\lambda \cdot \nabla\varphi \, dy = 0, & \forall\varphi \in H^1_\sharp(Y) \\ \displaystyle\int_Y \big(W_n^\lambda(y) - \lambda \cdot y\big)\, dy = 0. \end{cases} \tag{3.6}$$

Set

$$w_n^\lambda(x) := \varepsilon_n W_n^\lambda\left(\frac{x}{\varepsilon_n}\right), \quad \text{for } x \in \Omega, \tag{3.7}$$

and

$$w_n := \big(w_n^{e_1}, w_n^{e_2}\big) = \big(w_n^1, w_n^2\big). \tag{3.8}$$

3.1. A uniform convergence result

Theorem 3.1. *Let Ω be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz boundary. Consider a highly oscillating sequence of matrix-valued functions σ_n satisfying (3.1)–(3.3) and the constant matrix $(\sigma_n)_*$ defined by (3.4). We assume that*

$$(\sigma_n)_* \longrightarrow \sigma_* \quad \text{in } \mathbb{R}^{2\times 2}. \tag{3.9}$$

Consider, for $f \in H^{-1}(\Omega) \cap W^{-1,q}(\Omega)$ with $q > 2$, the solution u_n of the problem

$$\mathscr{P}_n \begin{cases} -\operatorname{div}(\sigma_n \nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Then, u_n converges uniformly to the solution $u \in H^1_0(\Omega)$ of

$$\mathscr{P} \begin{cases} -\operatorname{div}(\sigma_* \nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Moreover we have the corrector result, with the $\varepsilon_n Y$-periodic sequence w_n defined in (3.8):

$$\nabla u_n - \sum_{i=1}^{2} \partial_i u \, \nabla w_n^i \longrightarrow 0 \quad \text{in } L^1(\Omega)^2. \tag{3.12}$$

Remark 3.1. The first point of Theorem 3.1 is an extension to the non-symmetric case of the results of [12,36]. The uniform convergence of u_n is a straightforward consequence of Theorem 2.7 of [36] taking into account that in the present case $\sigma_n \in L^\infty(\Omega)^{2\times 2}$ for a fixed n. The fact that $f \in W^{-1,q}(\Omega)$ with $q > 2$ ensures the uniform convergence.

Proof of Theorem 3.1. *Derivation of the limit problem $\mathscr{P}$.*

We only have to show that u is the solution of $\mathscr{P}$ in (3.11). We consider a corrector $D\widetilde{w}_n: \mathbb{R}^2 \longrightarrow \mathbb{R}^{2\times 2}$ associated with σ_n^T defined by

$$\widetilde{w}_n(x) := \varepsilon_n \widetilde{W}_n\left(\frac{x}{\varepsilon_n}\right) = \left(\varepsilon_n \widetilde{W}_n^1\left(\frac{x}{\varepsilon_n}\right), \varepsilon_n \widetilde{W}_n^2\left(\frac{x}{\varepsilon_n}\right)\right)$$

where for $i = 1, 2$, $\widetilde{W}_n^i \in H^1_\sharp(Y)$ is the solution of the auxiliary problem

$$\int_Y \left(\widetilde{W}_n^i - e_i \cdot x\right) dx = 0 \quad \text{and} \quad \operatorname{div}\left(\Sigma_n^T \nabla \widetilde{W}_n^i\right) = 0 \text{ in } \mathscr{D}'(\mathbb{R}^2). \tag{3.13}$$

Again, thanks to Theorem 2.7 of [36], $\widetilde{w}_n$ converges uniformly to the identity in Ω by the integral condition (3.13). Let $\varphi \in \mathscr{D}(\Omega)$. We have, using the Einstein convention, by integrating by parts and by the Schwarz theorem ($\partial^2_{i,j}\varphi = \partial^2_{j,i}\varphi$)

$$\begin{aligned}
\int_\Omega \sigma_n \nabla u_n \cdot \nabla\left(\varphi(\widetilde{w}_n)\right)\, dx &= \int_\Omega \nabla u_n \cdot \sigma_n^T \nabla \widetilde{w}_n^i (\partial_i \varphi)(\widetilde{w}_n)\, dx \\
&= \underbrace{\int_\Omega \sigma_n^T \nabla \widetilde{w}_n^i \cdot \nabla (u_n \partial_i \varphi(\widetilde{w}_n))\, dx}_{=0} - \int_\Omega \sigma_n^T \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j \;\; \partial^2_{i,j}\varphi(\widetilde{w}_n)\, u_n\, dx \\
&= -\int_\Omega \sigma_n \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^i \;\; \partial^2_{i,i}\varphi(\widetilde{w}_n)\, u_n\, dx - \int_\Omega \sigma_n^T \nabla \widetilde{w}_n^2 \cdot \nabla \widetilde{w}_n^1 \;\; \partial^2_{2,1}\varphi(\widetilde{w}_n)\, u_n\, dx \\
&\quad - \int_\Omega \sigma_n^T \nabla \widetilde{w}_n^1 \cdot \nabla \widetilde{w}_n^2 \;\; \partial^2_{1,2}\varphi(\widetilde{w}_n)\, u_n\, dx \\
&= -\int_\Omega \sigma_n \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^i \;\; \partial^2_{i,i}\varphi(\widetilde{w}_n)\, u_n\, dx - \int_\Omega \sigma_n \nabla \widetilde{w}_n^1 \cdot \nabla \widetilde{w}_n^2 \;\; \partial^2_{1,2}\varphi(\widetilde{w}_n)\, u_n\, dx \\
&\quad - \int_\Omega \sigma_n^T \nabla \widetilde{w}_n^1 \cdot \nabla \widetilde{w}_n^2 \;\; \partial^2_{1,2}\varphi(\widetilde{w}_n)\, u_n\, dx \\
&= -\int_\Omega \sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^i \;\; \partial^2_{i,i}\varphi(\widetilde{w}_n)\, u_n\, dx - 2\int_\Omega \sigma_n^s \nabla \widetilde{w}_n^1 \cdot \nabla \widetilde{w}_n^2 \;\; \partial^2_{1,2}\varphi(\widetilde{w}_n)\, u_n\, dx.
\end{aligned}$$

This leads us to the equality

$$\langle f, \varphi(\widetilde{w}_n)\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \int_\Omega \sigma_n \nabla u_n \cdot \nabla\left(\varphi(\widetilde{w}_n)\right)\, dx = -\int_\Omega \sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j \;\; \partial^2_{i,j}\varphi(\widetilde{w}_n)\, u_n\, dx. \tag{3.14}$$

To study the convergence of the last term of (3.14), we first show that $\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j$ is bounded in $L^1(\Omega)$. We have, by periodicity and the Cauchy–Schwarz inequality

$$\begin{aligned}
\int_\Omega |\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j|\, dx &= \int_\Omega |\Sigma_n^s \nabla \widetilde{W}_n^i \cdot \nabla \widetilde{W}_n^j| \left(\frac{x}{\varepsilon_n}\right)\, dx \\
&\le C \int_Y |\Sigma_n^s \nabla \widetilde{W}_n^i \cdot \nabla \widetilde{W}_n^j|\, dx \\
&\le C \sqrt{\int_Y \left|\Sigma_n^s \nabla \widetilde{W}_n^i \cdot \nabla \widetilde{W}_n^i\right|\, dx} \sqrt{\int_Y \left|\Sigma_n^s \nabla \widetilde{W}_n^j \cdot \nabla \widetilde{W}_n^j\right|\, dx} \\
&\le C \sqrt{(\sigma_n)_* e_i \cdot e_i} \sqrt{(\sigma_n)_* e_j \cdot e_j}
\end{aligned}$$

which is bounded by the hypothesis (3.9). Therefore,

$$\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j \quad \text{is bounded in } L^1(\Omega). \tag{3.15}$$

Due to the periodicity, we know that for $i, j = 1, 2$,

$$2\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j = \sigma_n^T \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j + \sigma_n^T \nabla \widetilde{w}_n^j \cdot \nabla \widetilde{w}_n^i \rightharpoonup (\sigma_*)^T e_i \cdot e_j + (\sigma_*)^T e_j \cdot e_i = 2\,(\sigma_*)^s e_i \cdot e_j$$

weakly-$*$ in $\mathscr{M}(\Omega)$. Hence, we get that

$$\sigma_n^s \nabla \widetilde{w}_n^i \cdot \nabla \widetilde{w}_n^j \rightharpoonup (\sigma_*)^s e_i \cdot e_j \quad \text{weakly-}* \text{ in } \mathscr{M}(\Omega). \tag{3.16}$$

Moreover, $\partial^2_{i,j}\varphi(\widetilde{w}_n)\, u_n$ converges uniformly to $\partial^2_{i,j}\varphi\, u$. Thus, by passing to the limit in (3.14), we have, again with the Einstein convention

$$\langle f, \varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} = -\int_\Omega (\sigma_*)^s\, e_i \cdot e_j\ \ \partial^2_{i,j}\varphi\, u\, dx = -\int_\Omega \sigma_*: \nabla^2\varphi\, u\, dx.$$

Therefore, by integrating by parts and using $\varphi = 0$ on $\partial\Omega$,

$$\int_\Omega \sigma_* \nabla u \cdot \nabla\varphi\, dx = \langle f, \varphi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}. \tag{3.17}$$

Proof of the corrector result

First of all, we show that the corrector function w_n is bounded in $H^1(\Omega)^2$. By the definition (3.8) of w_n, the Y-periodicity of $W_n^{e_i}$ and the equi-coercivity of Σ_n, we have, for $i = 1, 2$,

$$\alpha\, \|\nabla w_n^i\|^2_{L^2(\Omega)^2} \le C\alpha\, \|\nabla W_n^{e_i}\|^2_{L^2(Y)^2} \le C\int_Y \Sigma_n \nabla W_n^i \cdot \nabla W_n^i\, dx = C\, (\sigma_n)_* e_i \cdot e_i \tag{3.18}$$

which is bounded. This inequality combined with the uniform convergence of w_n yields to the boundedness of w_n in $H^1(\Omega)^2$.

Let us consider an approximation $u^\delta \in \mathscr{D}(\Omega)$ of u such that

$$\|u - u^\delta\|_{H^1_0(\Omega)} \le \delta. \tag{3.19}$$

On the one hand, we have

$$\int_\Omega \sigma_n \nabla u_n \cdot \nabla\big(u_n - u^\delta(w_n)\big)\, dx = \big\langle f, \big(u_n - u^\delta(w_n)\big)\big\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}.$$

Since w_n converges uniformly to identity on Ω and is bounded in $H^1(\Omega)$ (see (3.18)), with $u^\delta \in \mathscr{D}(\Omega)$, $u^\delta(w_n)$ converges weakly to u^δ in $H^1_0(\Omega)$. Hence, by the weak convergence of u_n to u in $H^1_0(\Omega)$ and (3.19), we can pass to the limit the previous inequality and obtain, for any $\delta > 0$,

$$\limsup_{n\to\infty} \left| \int_\Omega \sigma_n \nabla u_n \cdot \nabla\big(u_n - u^\delta(w_n)\big)\, dx \right| = \left| \langle f, u - u^\delta\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \right| \le C\delta. \tag{3.20}$$

On the other hand, similarly to the proof of the first point (3.14), we are led to the equality

$$\int_\Omega \sigma_n \nabla\big(u^\delta(w_n)\big) \cdot \nabla\big(u_n - u^\delta(w_n)\big)\, dx = -\int_\Omega \sigma_n^s \nabla w_n^i \cdot \nabla w_n^j\ \ \partial^2_{i,j} u^\delta(w_n)\, \big(u_n - u^\delta(w_n)\big)\, dx. \tag{3.21}$$

As in the first point, $\sigma_n^s \nabla w_n^i \cdot \nabla w_n^j$ is bounded in $L^1(\Omega)$ (see (3.15)), u_n converges uniformly to u and $\partial_{i,j} u^\delta(w_n)$ converges uniformly to $\partial_{i,j} u^\delta$ because u^δ is a $\mathscr{D}(\Omega)$ function. By passing to the limit in (3.21)

$$\int_\Omega \sigma_n \nabla\big(u^\delta(w_n)\big) \cdot \nabla\big(u_n - u^\delta(w_n)\big)\, dx \underset{n\to\infty}{\longrightarrow} -\int_\Omega (\sigma_*)^s e_i \cdot e_j\ \ \partial^2_{i,j} u^\delta\, \big(u - u^\delta\big)\, dx. \tag{3.22}$$

Moreover, like in (3.17) we have

$$\int_\Omega (\sigma_*)^s e_i \cdot e_j\ \ \partial^2_{i,j} u^\delta\, \big(u - u^\delta\big)\, dx = \int_\Omega \sigma_* \nabla u^\delta \cdot \nabla\big(u - u^\delta\big)\, dx. \tag{3.23}$$

By combining this equality with the convergence (3.22), we obtain the inequality

$$\lim_{n\to\infty} \left| \int_\Omega \sigma_n \nabla\big(u^\delta(w_n)\big) \cdot \nabla\big(u_n - u^\delta(w_n)\big)\, dx \right| \le \left| \int_\Omega \sigma_* \nabla u^\delta \cdot \nabla\big(u - u^\delta\big) \right| \tag{3.24}$$

$$\le C|\sigma_*|\, \|\nabla u^\delta\|_{L^2(\Omega)^2} \|\nabla(u - u^\delta)\|_{L^2(\Omega)^2} \le C\delta. \tag{3.25}$$

Thus, by adding (3.20) and (3.25), we have

$$\limsup_{n\to\infty} \int_\Omega \sigma_n \nabla\big(u_n - u^\delta(w_n)\big) \cdot \nabla\big(u_n - u^\delta(w_n)\big)\, dx \le C\delta$$

which leads us, by equi-coercivity, to

$$\limsup_{n\to\infty}\ \alpha\, \|\nabla(u_n - u^\delta(w_n))\|^2_{L^2(\Omega)^2} \le \limsup_{n\to\infty} \left| \int_\Omega \sigma_n \nabla\big(u_n - u^\delta(w_n)\big) \cdot \nabla\big(u_n - u^\delta(w_n)\big)\, dx \right| \le C\delta. \tag{3.26}$$

Thus, the Cauchy–Schwarz inequality, the boundedness of ∇w_n^i in $L^2(\Omega)^2$ (3.18) and the Einstein convention give, for any $\delta > 0$,

$$\begin{aligned}\|\nabla u_n - \nabla w_n^i \partial_i u\|_{L^1(\Omega)^2} &\le \|\nabla u_n - \nabla w_n^i \partial_i u^\delta\|_{L^1(\Omega)^2} + \|\nabla w_n^i \partial_i (u^\delta - u)\|_{L^1(\Omega)^2} \\ &\le \|\nabla u_n - \nabla w_n^i \partial_i u^\delta\|_{L^1(\Omega)^2} + \|\nabla w_n^i\|_{L^2(\Omega)^2} \|\partial_i (u^\delta - u)\|_{L^2(\Omega)} \\ &\le \|\nabla u_n - \nabla w_n^i \partial_i u^\delta\|_{L^1(\Omega)^2} + C\delta \\ &\le \|\nabla u_n - \nabla w_n^i \partial_i u^\delta(w_n)\|_{L^1(\Omega)^2} + \|\nabla w_n^i (\partial_i u^\delta - \partial_i u^\delta(w_n))\|_{L^1(\Omega)^2} + C\delta \\ &\le \|\nabla u_n - \nabla w_n^i \partial_i u^\delta(w_n)\|_{L^1(\Omega)^2} + \|\nabla w_n^i\|_{L^2(\Omega)^2} \|\partial_i u^\delta - \partial_i u^\delta(w_n)\|_{L^2(\Omega)} + C\delta \\ &\le \|\nabla u_n - \nabla w_n^i \partial_i u^\delta(w_n)\|_{L^1(\Omega)^2} + C\|\partial_i u^\delta - \partial_i u^\delta(w_n)\|_{L^2(\Omega)} + C\delta.\end{aligned}$$

Since $u^\delta \in \mathscr{D}(\Omega)$ and w_n converges uniformly to the identity on Ω, the second term of the last inequality converges to 0. Hence, we get that

$$\limsup_{n\to\infty} \|\nabla u_n - \nabla w_n^i \partial_i u\|_{L^1(\Omega)^2} \le \limsup_{n\to\infty} \|\nabla u_n - \nabla w_n^i \partial_i u^\delta(w_n)\|_{L^1(\Omega)^2} + C\delta. \tag{3.27}$$

Finally, this inequality combined with (3.26) gives, for any $\delta > 0$,

$$0 \le \limsup_{n\to\infty} \|\nabla u_n - \nabla w_n^i \partial_i u\|_{L^1(\Omega)^2} \le C\sqrt{\delta} + C\delta,$$

which implies the corrector result (3.12). □

Remark 3.2. If the solution u is a $\mathscr{C}^2$ function, then the convergence (3.12) holds true in $L^2_{\rm loc}(\Omega)$ since we may take $u = u^\delta$.

3.2. A two-phase result

Here, we recall a two-phase result due to Milton (see [34, pp. 61–65]) using the Dykhne transformation.

In order to apply the previous theorem, we reformulate Milton's calculus in such a way that every coefficient depends on n. We then consider, for a fixed n, the periodic homogenization of a conductivity $\sigma_n(h)$ to obtain $(\sigma_n)_*(h)$ through the link between the homogenization of the transformed conductivity and $(\sigma_n)_*(h)$ given by formula (4.16) in [34]. Finally, we study the limit of $(\sigma_n)_*(h)$ through the asymptotic behavior of the coefficients of the transformation, and apply Theorem 3.1 in the example Section 3.3.

In this section we consider a two-phase periodic isotropic medium. Let χ_n be a sequence of characteristic functions of subsets of Y. We define for any $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$, any sequences $\alpha_{2,n} > 0$, $\beta_{2,n} \in \mathbb{R}$ and any $h \in \mathbb{R}$, a parametrized conductivity $\Sigma_n(h)$:

$$\Sigma_n(h) = (1 - \chi_n)(\alpha_1 I_2 + h\beta_1 J) + \chi_n(\alpha_{2,n} I_2 + h\beta_{2,n} J) \quad \text{in } Y. \tag{3.28}$$

We still denote by $\Sigma_n(h)$ the periodic extension to $\mathbb{R}^2$ of $\Sigma_n(h)$ (which satisfies (3.1)). We assume that $\Sigma_n(h)$ satisfies (3.2), and define $\sigma_n(h)$ by (3.3) and $(\sigma_n)_*(h)$ by (3.4).

We have the following result based on an analysis of [34, pp. 61–65].

Proposition 3.1. *Let χ_n be a sequence of characteristic functions of subsets of Y, $\alpha_1, \alpha_2 > 0$, a positive sequence $\alpha_{2,n}$, $\beta_1, \beta_2 \in \mathbb{R}$, and a sequence $\beta_{2,n}$ such that*

$$\lim_{n\to\infty} \alpha_{2,n} = \infty, \qquad \liminf_{n\to\infty} |\beta_{2,n} - \beta_1| > 0, \quad \text{and} \quad \lim_{n\to\infty} \frac{\beta_{2,n}}{\alpha_{2,n}} = \frac{\beta_2}{\alpha_2}. \tag{3.29}$$

Assume that the effective conductivity in the absence of a magnetic field

$$\left(\sigma_n^0\right)_* (\gamma_{1,n}, \gamma_{2,n}) \quad \text{is bounded when } \lim_{n\to\infty} \gamma_{1,n} = \alpha_1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\gamma_{2,n}}{\alpha_{2,n}} = \gamma_2 > 0. \tag{3.30}$$

Then, there exist two parametrized positive sequences $\alpha'_{1,n}(h)$, $\alpha'_{2,n}(h)$ such that

$$\lim_{n\to\infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \alpha'_{2,n}(h) \underset{n\to\infty}{\sim} \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}\, \alpha_{2,n}, \tag{3.31}$$

and

$$(\sigma_n)_*(h) = \left(\sigma_n^0\right)_* \left(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\right) + h\beta_1 J + \underset{n\to\infty}{o}(1) \tag{3.32}$$

where $\left(\sigma_n^0\right)_* \left(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\right)$ is bounded.

Remark 3.3. In view of condition (3.29), the case where $\beta_{2,n}$ tends to β_1 corresponds to perturb the symmetric conductivity

$$\sigma_n^s = (1-\chi_n)\alpha_1 I_2 + \chi_n \alpha_{2,n} I_2$$

by

$$\sigma_n^s + \beta_1 J + \underset{n\to\infty}{o}(1).$$

Then it is clear that

$$(\sigma_n)_*(h) = \left(\sigma_n^s\right)_* + \beta_1 J + \underset{n\to\infty}{o}(1).$$

Proof of Proposition 3.1. The proof is divided into two parts. After applying Milton's computation (pp. 61–64 of [34]), we study the asymptotic behavior of different coefficients.

First step: Applying the Dykhne transformation through Milton's computations.

In order to make the Dykhne transformation following Milton [34, pp. 62–64], we consider two real coefficients a_n and b_n such that

$$\sigma_n' := \big(a_n\sigma_n(h) + b_n J\big)\big(a_n I_2 + J\sigma_n(h)\big)^{-1} = a_n\big(\sigma_n(h) + (a_n)^{-1} b_n J\big)\big(a_n I_2 + J\sigma_n(h)\big)^{-1} \tag{3.33}$$

is symmetric and, more precisely, according to Notation 1.1, reads as

$$\sigma_n' = (1-\chi_n)\alpha_{1,n}'(h) I_2 + \chi_n \alpha_{2,n}'(h) I_2 = \sigma_n^0\big(\alpha_{1,n}'(h), \alpha_{2,n}'(h)\big). \tag{3.34}$$

Then, using the complex representation

$$\alpha I_2 + \beta J \longleftrightarrow \alpha + \beta i \tag{3.35}$$

suggested by Tartar [37], the constants a_n, b_n must satisfy

$$\alpha_{1,n}'(h) = \frac{a_n(\alpha_1 + ih\beta_1) + ib_n}{a_n + i(\alpha_1 + ih\beta_1)} \in \mathbb{R} \quad \text{and} \quad \alpha_{2,n}'(h) = \frac{a_n(\alpha_{2,n} + ih\beta_{2,n}) + ib_n}{a_n + i(\alpha_{2,n} + ih\beta_{2,n})} \in \mathbb{R}, \tag{3.36}$$

which implies that

$$b_n = \frac{-a_n^2 h\beta_1 + a_n\Delta_1}{a_n - h\beta_1} = \frac{-a_n^2 h\beta_{2,n} + a_n\Delta_{2,n}}{a_n - h\beta_{2,n}}. \tag{3.37}$$

Denoting $\Delta_1 := \alpha_1^2 + h^2\beta_1^2$ and $\Delta_{2,n} := \alpha_{2,n}^2 + h^2\beta_{2,n}^2$ (thanks to (3.29), n is considered to be larger enough such that $\beta_{2,n} - \beta_1 \neq 0$ and a_n is real), the equality (3.37) provides two non-zero solutions for a_n:

$$a_n = \frac{\Delta_{2,n} - \Delta_1 + \sqrt{(\Delta_{2,n} - \Delta_1)^2 + 4h^2(\beta_{2,n} - \beta_1)(\beta_{2,n}\Delta_1 - \beta_1\Delta_{2,n})}}{2h(\beta_{2,n} - \beta_1)}, \tag{3.38}$$

and

$$a_n^- = \frac{\Delta_{2,n} - \Delta_1 - \sqrt{(\Delta_{2,n} - \Delta_1)^2 + 4h^2(\beta_{2,n} - \beta_1)(\beta_{2,n}\Delta_1 - \beta_1\Delta_{2,n})}}{2h(\beta_{2,n} - \beta_1)}.$$

The value (3.38) is associated with a positive matrix σ_n', while a_n^- leads us to the negative matrix $\sigma_n^- = -J(\sigma_n')^{-1}J^{-1}$ to exclude (see [17] for more details).

Second step: Asymptotic behavior of the coefficients and the homogenized matrix.

On the one hand, by the equality (3.38) combined with (3.29), we have

$$\lim_{n\to\infty} a_n \frac{h(\beta_{2,n} - \beta_1)}{\alpha_{2,n}^2} = \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}$$

which clearly implies that

$$a_n \underset{n\to\infty}{\sim} \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}\,\frac{\alpha_{2,n}^2}{h(\beta_{2,n} - \beta_1)} \quad \text{and} \quad a_n - h\beta_{2,n} \underset{n\to\infty}{\sim} \frac{\alpha_{2,n}^2}{h(\beta_{2,n} - \beta_1)}. \tag{3.39}$$

On the other hand, (3.29) and (3.39) and the first equality of (3.37) give

$$b_n = -a_n h\beta_1 + \Delta_1 + \underset{n\to\infty}{o}(1). \tag{3.40}$$

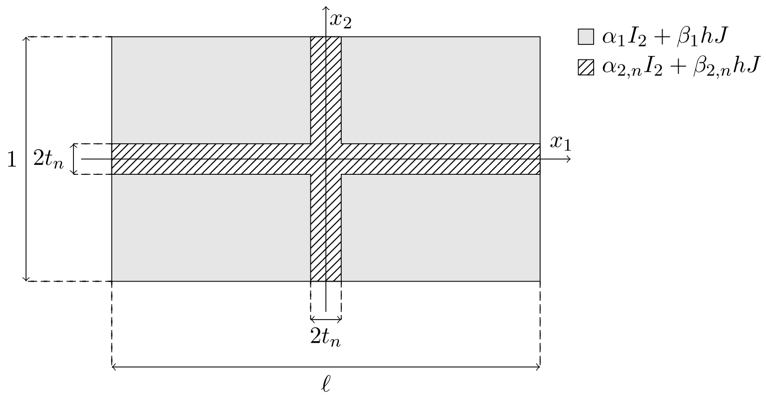


Fig. 3.1. The period of the cross-like thin structure.

From (3.29) and (3.38)–(3.40) we deduce the following asymptotic behavior for the modified phases:

$$\lim_{n\to\infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\alpha'_{2,n}(h)}{\alpha_{2,n}} = \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}. \tag{3.41}$$

To consider $\left(\sigma'_n\right)_*$, we need to verify that σ'_n is equi-coercive. We have, by denoting for any $\xi \in \mathbb{R}^2$, $\nu_n = \left(a_n I_2 + J\sigma_n(h)\right)^{-1}\xi$,

$$\forall \xi \in \mathbb{R}^2, \quad \sigma'_n\xi \cdot \xi = \left(a_n\sigma_n(h) + b_n J\right)\nu_n \cdot \left(a_n I_2 + J\sigma_n(h)\right)\nu_n = (a_n^2 + b_n)\sigma_n(h)\nu_n \cdot \nu_n$$

and, because $a_n^{-1}\sigma_n(h)$ is bounded in $L^\infty(\Omega)^{2\times 2}$ by (3.39),

$$\forall \xi \in \mathbb{R}^2, \quad |\xi| = \left|a_n\nu_n + J\sigma_n(h)\nu_n\right| \le a_n(1+C)|\nu_n|.$$

The equi-coercivity of $\sigma_n(h)$ gives

$$\exists C > 0, \forall \xi \in \mathbb{R}^2, \quad \sigma'_n\xi \cdot \xi \ge \frac{C}{(1+C)^2}\frac{a_n^2 + b_n}{a_n^2}|\xi|^2 \tag{3.42}$$

that is, for n larger enough, by (3.39) and (3.40), σ'_n is equi-coercive.

We can now apply the Keller–Dykhne duality theorem (see, e.g., [31,32]) to equality (3.33) to obtain

$$(\sigma'_n)_* = \left(a_n(\sigma_n)_* + b_n J\right)\left(a_n I_2 + J(\sigma_n)_*\right)^{-1}. \tag{3.43}$$

Moreover, by inverting this transformation, we have

$$(\sigma_n)_*(h) = \left(a_n I_2 - (\sigma'_n)_* J\right)^{-1}\left(a_n(\sigma'_n)_* - b_n J\right).$$

Considering (3.29), (3.39) and (3.40), and the boundedness of $(\sigma'_n)_*$ (as a consequence of the bound (3.30)) we get that

$$(\sigma_n)_*(h) = (\sigma'_n)_* - \frac{b_n}{a_n}J + \underset{n\to\infty}{o}(1) = (\sigma'_n)_* + h\beta_1 J + \underset{n\to\infty}{o}(1), \tag{3.44}$$

which concludes the proof taking into account (3.34). $\square$

To derive the limit of $\left(\sigma_n^0\right)_*(\alpha'_{1,n}(h), \alpha'_{2,n}(h))$, we need more information on the geometry of the high conductive phase. To this end, we study the following example.

3.3. A cross-like thin structure

We consider a bounded open subset Ω of $\mathbb{R}^2$ with a Lipschitz boundary, a real sequence ε_n converging to 0, and $f \in H^{-1}(\Omega) \cap W^{-1,q}(\Omega)$ with $q > 2$. We define, for any $h \in \mathbb{R}$, $\alpha_1, \beta_1 > 0$ and positive sequences $t_n \in (0, 1/2]$, $\alpha_{2,n}$, $\beta_{2,n}$, a parametrized matrix-valued function $\Sigma_n(h)$ from the unit rectangular cell period $Y := (-\frac{\ell}{2}, \frac{\ell}{2}) \times (-\frac{1}{2}, \frac{1}{2})$, with $\ell \ge 1$, to $\mathbb{R}^{2\times 2}$, by (cf. Fig. 3.1)

$$\Sigma_n(h) := \begin{cases} \alpha_{2,n} I_2 + \beta_{2,n} hJ & \text{in } \omega_n := \{(x_1,x_2) \in Y \mid |x_1|, |x_2| \le t_n\} \\ \alpha_1 I_2 + \beta_1 hJ & \text{in } Y \setminus \omega_n. \end{cases} \tag{3.45}$$

Denoting again by $\Sigma_n(h)$ its periodic extension to $\mathbb{R}^2$, we finally consider the conductivity

$$\sigma_n(h)(x) = \Sigma_n(h)\left(\frac{x}{\varepsilon_n}\right), \quad x \in \Omega, \tag{3.46}$$

and the associated homogenization problem:

$$\mathscr{P}_n \quad \begin{cases} -\mathrm{div}\big(\sigma_n(h)\nabla u_n\big) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.47}$$

By virtue of Theorem 3.1 and Proposition 3.1, we focus on the study of the limit of $\left(\sigma_n^0\right)_* \left(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\right)$.

Proposition 3.2. *Let $\sigma_n(h)$ be the conductivity defined by (3.45) and (3.46) and its homogenization problem (3.47). We assume that:*

$$2t_n(\ell+1)\alpha_{2,n} \underset{n\to\infty}{\longrightarrow} \alpha_2 > 0 \quad \text{and} \quad 2t_n(\ell+1)\beta_{2,n} \underset{n\to\infty}{\longrightarrow} \beta_2 > 0. \tag{3.48}$$

Then, the homogenized conductivity is given by

$$\sigma_*(h) = \begin{pmatrix} \alpha_1 + \dfrac{\alpha_2^2 + \beta_2^2 h^2}{(\ell+1)\alpha_2} & -h\beta_1 \\ h\beta_1 & \alpha_1 + \dfrac{\alpha_2^2 + \beta_2^2 h^2}{\ell(\ell+1)\alpha_2} \end{pmatrix}.$$

Remark 3.4. The previous proposition does not respect exactly the framework defined at the beginning of this section because the period cell is not the unit square $Y = (0,1)^2$: we can nevertheless extend all this section to any type of period cells.

Remark 3.5. The condition (3.48) is a condition of boundedness in $L^1(\Omega)^{2\times 2}$ of σ_n because

$$|\omega_n| = 2t_n(\ell+1) - 4t_n^2 \sim 2t_n(\ell+1),$$

which will ensure the convergence of $\left(\sigma_n^0\right)_*$.

Proof of Proposition 3.2. In order to apply Proposition 3.1, we consider two positive sequences $\alpha'_{1,n}(h), \alpha'_{2,n}(h)$ satisfying

$$\lim_{n\to\infty} \alpha'_{1,n}(h) = \alpha_1 \quad \text{and} \quad \alpha'_{2,n}(h) \underset{n\to\infty}{\sim} \frac{\alpha_2^2 + h^2\beta_2^2}{\alpha_2^2}\,\alpha_{2,n}. \tag{3.49}$$

We will study the homogenization of $\sigma'_n := \sigma_n^0\big(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\big)$.

To this end, consider a corrector $W_n^\lambda = \lambda\cdot x - X_n^\lambda$ in the Murat–Tartar sense (see, e.g., [3]) associated with

$$\Sigma'_n := \begin{cases} \alpha'_{2,n}(h)\, I_2 & \text{in } \omega_n = \{(x_1,x_2)\in Y \mid |x_1|, |x_2| \le t_n\} \\ \alpha'_{1,n}(h)\, I_2 & \text{in } Y\setminus\omega_n \end{cases} \tag{3.50}$$

and defined by

$$\begin{cases} \mathrm{div}\big(\Sigma'_n\nabla X_n^\lambda\big) = \mathrm{div}\big(\Sigma'_n\lambda\big) & \text{in } \mathscr{D}'(\mathbb{R}^2) \\ X_n^\lambda \ \text{ is } Y\text{-periodic} \\ \displaystyle\int_Y X_n^\lambda\, dy = 0. \end{cases} \tag{3.51}$$

On the one hand, the extra diagonal coefficients of $(\sigma'_n)_*$ are equal to 0 because, as Σ'_n is an even function on Y, we have, for $i = 1, 2$,

$$\begin{cases} y_i \longmapsto W_n^{e_i}(y) & \text{is an odd function,} \\ y_i \longmapsto W_n^{e_j}(y) & \text{is an even function for } i \ne j, \end{cases}$$

which implies that $y_1 \longmapsto \Sigma'_n\nabla W_n^{e_1}\cdot\nabla W_n^{e_2}$ is an odd function. Then, by symmetry of Y with respect to 0,

$$(\sigma'_n)_* e_i\cdot e_j = \int_Y \Sigma'_n\nabla W_n^{e_i}\cdot\nabla W_n^{e_j}\, dy = 0.$$

On the other hand, as Σ'_n is isotropic, for the diagonal coefficients, we use the Voigt–Reuss inequalities (see, e.g., [38, p. 44] or [39]): for any $i = 1, 2$ and $j \neq i$,

$$\left\langle \langle (\Sigma'_n e_i \cdot e_i)^{-1} \rangle_i^{-1} \right\rangle_j \leq (\sigma'_n)_* e_i \cdot e_i \leq \left\langle \langle \Sigma'_n e_i \cdot e_i \rangle_j^{-1} \right\rangle_i^{-1} \tag{3.52}$$

where $\langle \cdot \rangle_i$ denotes the average with respect to y_i at a fixed y_j for $j \neq i$.

An easy computation gives, for the direction e_1,

$$(1 - 2t_n) \left(\frac{\ell - 2t_n}{\ell \alpha'_{1,n}(h)} + \frac{2t_n}{\ell \alpha'_{2,n}(h)} \right)^{-1} + 2t_n \left(\frac{\ell}{\ell \alpha'_{2,n}(h)} \right)^{-1} \leq (\sigma'_n)_* e_1 \cdot e_1$$

and

$$(\sigma'_n)_* e_1 \cdot e_1 \leq \ell \left(\frac{\ell - 2t_n}{(1 - 2t_n)\alpha'_{1,n}(h) + 2t_n \alpha'_{2,n}(h)} + \frac{2t_n}{\alpha'_{2,n}(h)} \right)^{-1}.$$

By (3.48) and (3.49), we have the convergence

$$\lim_{n\to\infty} (\sigma'_n)_* e_1 \cdot e_1 = \alpha_1 + \frac{\alpha_2^2 + \beta_2^2 h^2}{(\ell + 1)\alpha_2}.$$

A similar computation on the direction e_2 gives the asymptotic behavior:

$$\lim_{n\to\infty} (\sigma'_n)_* = \lim_{n\to\infty} \left(\sigma_n^0\right)_* \left(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\right) = \begin{pmatrix} \alpha_1 + \dfrac{\alpha_2^2 + \beta_2^2 h^2}{(\ell + 1)\alpha_2} & 0 \\ 0 & \alpha_1 + \dfrac{\alpha_2^2 + \beta_2^2 h^2}{\ell(\ell + 1)\alpha_2} \end{pmatrix}. \tag{3.53}$$

Moreover, the matrix $\sigma_n(h)$ clearly satisfies all the hypothesis of Proposition 3.1. By Theorem 3.1 and (3.53), we have

$$\lim_{n\to\infty} (\sigma_n)_*(h) = \lim_{n\to\infty} \left(\sigma_n^0\right)_* \left(\alpha'_{1,n}(h), \alpha'_{2,n}(h)\right) + \beta_1 h J = \begin{pmatrix} \alpha_1 + \dfrac{\alpha_2^2 + \beta_2^2 h^2}{(\ell + 1)\alpha_2} & -h\beta_1 \\ h\beta_1 & \alpha_1 + \dfrac{\alpha_2^2 + \beta_2^2 h^2}{\ell(\ell + 1)\alpha_2} \end{pmatrix}.$$

We finally apply Theorem 3.1 to get that $\sigma_*(h) = \lim_{n\to\infty} (\sigma_n)_*(h)$. $\square$

4. A three-dimensional fibered microstructure

In this section we study a particular two-phase composite in dimension three. One of the phases is composed by a periodic set of high conductivity fibers embedded in an isotropic medium (Fig. 4.1(a)). The conductivity $\sigma_n(h)$ is not symmetric due to the perturbation of a magnetic field.

First, describe the geometry of the microstructure. Let $Y := \left(-\frac{1}{2}, \frac{1}{2}\right)^3$ be the unit cube centered at the origin of $\mathbb{R}^3$. For $r_n \in \left(0, \frac{1}{2}\right)$, consider the closed cylinder ω_n parallel to the x_3-axis, of radius r_n and centered in Y:

$$\omega_n := \left\{ y \in Y \mid y_1^2 + y_2^2 \leq r_n^2 \right\}. \tag{4.1}$$

Let $\Omega = \widetilde{\Omega} \times (0, 1)$ be an open cylinder of $\mathbb{R}^3$, where $\widetilde{\Omega}$ is a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary. For $\varepsilon_n \in (0, 1)$, consider the closed subset Ω_n of Ω defined by the intersection with Ω of the $\varepsilon_n Y$-periodic network in $\mathbb{R}^3$ composed by the closed cylinders parallel to the x_3-axis, centered on the points $\varepsilon_n k$, $k \in \mathbb{Z}^2$, in the x_1–x_2 plane, and of radius $\varepsilon_n r_n$, namely:

$$\Omega_n := \Omega \cap \bigcup_{\nu \in \mathbb{Z}^3} \varepsilon_n(\omega_n + \nu). \tag{4.2}$$

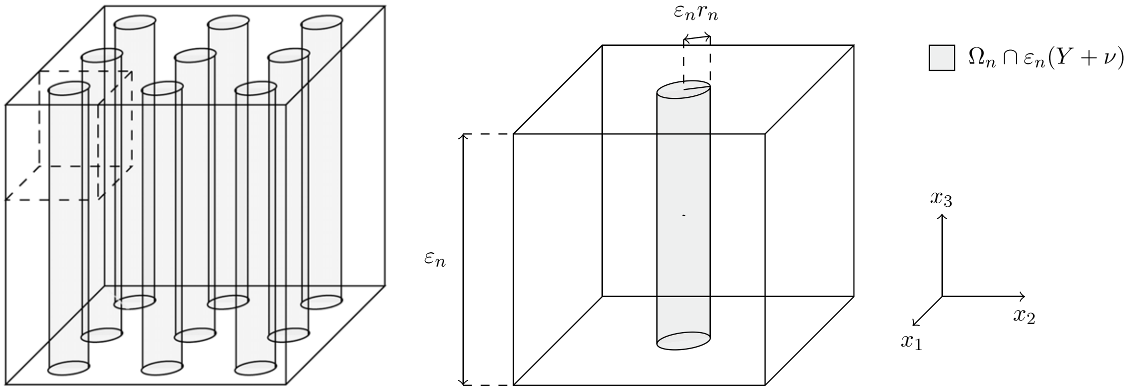
The period cell of the microstructure is represented in Fig. 4.1(b).

We then define the two-phase conductivity by

$$\sigma_n(h) = \begin{cases} \alpha_1 I_3 + \beta_1 \mathscr{E}(h) & \text{in } \Omega \setminus \Omega_n \\ \alpha_{2,n} I_3 + \beta_{2,n} \mathscr{E}(h) & \text{in } \Omega_n, \end{cases} \tag{4.3}$$

where $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$, $\alpha_{2,n} > 0$ and $\beta_{2,n}$ are real sequences, and

$$\mathscr{E}(h) := \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix}, \quad \text{for } h = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \in \mathbb{R}^3.$$



(a) The fiber lattice. (b) The period cell.

Fig. 4.1. The fibered structure in dimension three.

Our aim is to study the homogenization problem

$$\mathcal{P}_{\Omega,n}\ \begin{cases} -\mathrm{div}(\sigma_n(h)\nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Theorem 4.1. *Let $\alpha_1 > 0$, $\beta_1 \in \mathbb{R}$, and let $\varepsilon_n, r_n, \alpha_{2,n}, \beta_{2,n}$, $n \in \mathbb{N}$, be real sequences such that $\varepsilon_n, r_n > 0$ converge to 0, $\alpha_{2,n} > 0$, and*

$$\lim_{n\to\infty} \varepsilon_n^2 |\ln r_n| = 0, \qquad \lim_{n\to\infty} |\omega_n|\,\alpha_{2,n} = \alpha_2 > 0, \qquad \lim_{n\to\infty} |\omega_n|\,\beta_{2,n} = \beta_2 \in \mathbb{R}. \tag{4.5}$$

Consider, for $h \in \mathbb{R}^3$, the conductivity $\sigma_n(h)$ defined by (4.3).
Then, there exists a subsequence of n, still denoted by n, such that, for any $f \in H^{-1}(\Omega)$ and any $h \in \mathbb{R}^3$, the solution u_n of $\mathcal{P}_{\Omega,n}$ converges weakly in $H_0^1(\Omega)$ to the solution u of

$$\mathcal{P}_{\Omega,*}\ \begin{cases} -\mathrm{div}(\sigma_*(h)\nabla u_n) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

where $\sigma_*(h)$ is given by

$$\sigma_*(h) = \alpha_1 I_3 + \left(\frac{\alpha_2^3 + \alpha_2\beta_2^2|h|^2}{\alpha_2^2 + \beta_2^2 h_3^2}\right) e_3 \otimes e_3 + \beta_1 \mathcal{E}(h). \tag{4.7}$$

Remark 4.1. Theorem 4.1 can be actually extended to fibers with a more general cross-section. More precisely, we can replace the disk $r_n D$ of radius r_n by the homothetic $r_n Q$ of any connected open set Q included in the unit disk D, such that the present fiber ω_n is replaced by the new fiber $r_n Q \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ in the period cell of the microstructure.

On the one hand, this change allows us to use the same test function v_n (4.8) defined in the proof of Theorem 4.1, since v_n remains equal to 1 in the new fibers due to the inclusion $Q \subset D$. On the other hand, Lemma 4.1 allows us to replace the disk D by the open set $Q \subset D$.

Remark 4.2. We can also extend the result of Theorem 4.1 to an isotropic fibered microstructure composed by three similar periodic fiber lattices arranged in the three orthogonal directions e_1, e_2, e_3, namely

$$\omega_n := \bigcup_{j=1}^{3} \left\{ y \in Y \mid \sum_{i\neq j} y_i^2 \le r_n^2 \right\} \quad \text{and} \quad \Omega_n := \Omega \cap \bigcup_{\nu\in\mathbb{Z}^3} \varepsilon_n(\omega_n + \nu),$$

as represented in Fig. 4.2. Then, we derive the following homogenization conductivity :

$$\sigma_*(h) = \alpha_1 I_3 + \sum_{i=1}^{3} \left(\frac{\alpha_2^3 + \alpha_2\beta_2^2|h|^2}{\alpha_2^2 + \beta_2^2 h_i^2}\right) e_i \otimes e_i + \beta_1 \mathcal{E}(h).$$

Remark 4.3. We can check that when the volume fraction $\theta_n = \theta$ and the highly conducting phase of the conductivity $\alpha_{2,n} = \alpha_\theta$ and $\beta_{2,n} = \beta_\theta$ are independent of n, the explicit formula of [24] denoted by $\sigma_*(\theta, h)$, for the classical (since the

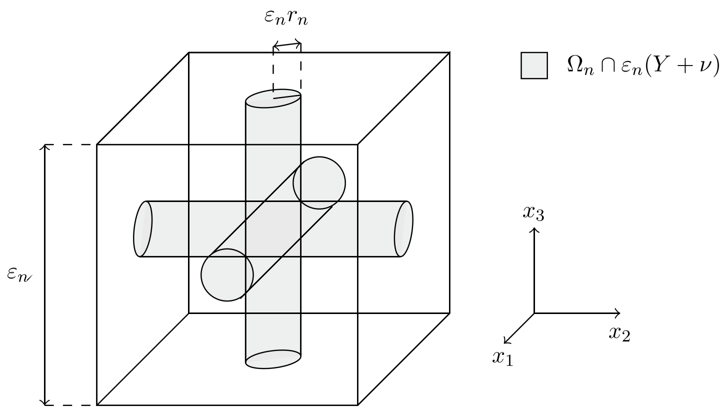


Fig. 4.2. The period cell of the isotropic fibered structure in dimension three.

period cell is now independent of n) periodically homogenized conductivity (see (3.4)) has a limit as $\theta \to 0$ when $\theta\alpha_\theta$ and $\theta\beta_\theta$ converge. Indeed, we may replace in the computations of [24] the optimal Vigdergauz shape by the circular cross-section in the previous asymptotic regime. Therefore, Theorem 4.1 validates the double process characterized by the homogenization at a fixed volume fraction θ combined with the limit as $\theta \to 0$, by one homogenization process in which both the period and the volume fraction $\theta_n = \pi r_n^2$ of the high conductivity phase tend to 0 as $n \to \infty$.

Remark 4.4. The hypothesis on the convergence of $\varepsilon_n^2|\ln r_n|$ (4.5) allows us to avoid nonlocal effects in dimension three (see [5,7]). These effects do not appear in dimension two as shown in [40]. Therefore, we can make a comparison between dimension two and dimension three based on the strong field perturbation in the absence of nonlocal effects.

Remark 4.5. If $h = h_3 e_3$, the homogenized conductivity becomes

$$\sigma_*(h) = \alpha_1 I_3 + \alpha_2\, e_3 \otimes e_3 + \beta_1 h_3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which reduces to the simplified two-dimensional case when the symmetric part of the conductivity is independent of h_3 (*i.e.* σ_*^0 in (2.40) does not depend on its second argument).

Proof of Theorem 4.1. The proof will be divided into four parts. We first prove the weak-$*$ convergence in $\mathcal{M}(\Omega)$ of $\sigma_n(h)\nabla u_n$ in Ω_n. Then we establish a linear system satisfied by the limits defined by

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_i} \rightharpoonup \xi_i \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).$$

Moreover, we deduce from Lemma 4.1 that

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3} \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).$$

We finally calculate the homogenized matrix.

We first remark that, classically, the sequence of solutions u_n of $\mathcal{P}_{\Omega,n}$ (see (4.4)) is bounded in $H_0^1(\Omega)$ because, since $\alpha_{2,n}$ diverges to ∞:

$$\|\nabla u_n\|^2_{L^2(\Omega)^3} \le C \int_\Omega (\alpha_1 \mathbb{1}_{\Omega\setminus\Omega_n} I_3 + \alpha_{2,n}\mathbb{1}_{\Omega_n} I_3)\nabla u_n \cdot \nabla u_n\, dx = \int_\Omega \sigma_n(h)\nabla u_n \cdot \nabla u_n\, dx.$$

By the Poincaré inequality, the previous inequality and (4.4) lead us to

$$\|u_n\|^2_{H_0^1(\Omega)} \le C\|\nabla u_n\|^2_{L^2(\Omega)^3} \le C\left|\langle f, u_n\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\right| \le C\|f\|_{H^{-1}(\Omega)}\|u_n\|_{H_0^1(\Omega)}$$

and then to

$$\|u_n\|_{H_0^1(\Omega)} \le C\|f\|_{H^{-1}(\Omega)}.$$

Thus, up to a subsequence still denoted by n, u_n converges weakly to some function u in $H_0^1(\Omega)$.

First step: Weak-$*$ convergence in $\mathcal{M}(\Omega)$ of the conductivity in the fibers $\mathbb{1}_{\Omega_n}\left(\alpha_{2,n}I_3+\beta_{2,n}\mathscr{E}(h)\right)\nabla u_n$. We proceed as in [8] with a suitable oscillating test function. For $R\in(0,1/2)$, define the Y-periodic (independent of y_3) function V_n by

$$V_n(y_1,y_2,y_3)=\begin{cases}1 & \text{if } \sqrt{y_1^2+y_2^2}\le r_n\\ \dfrac{\ln R-\ln\sqrt{y_1^2+y_2^2}}{\ln R-\ln r_n} & \text{if } r_n\le\sqrt{y_1^2+y_2^2}\le R\\ 0 & \text{if } \sqrt{y_1^2+y_2^2}\ge R,\end{cases}\quad \text{for } y\in Y,$$

and the rescaled function

$$v_n(x)=V_n\left(\frac{x}{\varepsilon_n}\right),\quad \text{for } x\in\mathbb{R}^3. \tag{4.8}$$

In particular, by using the cylindrical coordinates and the fact that r_n converges to 0, this function satisfies the inequalities

$$\begin{aligned}\|v_n\|^2_{L^2(\Omega)} &\le C\|V_n\|^2_{L^2(Y)}=C\left|\ln\frac{R}{r_n}\right|^{-2}\left(\pi r_n^2+\int_0^{2\pi}\int_{r_n}^R r\ln^2\frac{R}{r}\,drd\theta\right)\\ &= C\left|\ln\frac{R}{r_n}\right|^{-2}\left(\pi\frac{R^2-r_n^2}{2}-\pi r_n^2\ln^2\frac{R}{r_n}-\pi\ln\frac{R}{r_n}\right)\le C\left|\ln\frac{R}{r_n}\right|^{-2},\\ \|\nabla v_n\|^2_{L^2(\Omega)^3} &\le \frac{C}{\varepsilon_n^2}\|\nabla V_n\|^2_{L^2(Y)^3}=\frac{C}{\varepsilon_n^2}\left|\ln\frac{R}{r_n}\right|^{-2}\int_0^{2\pi}\int_{r_n}^R\frac{1}{r}\,drd\theta\le\frac{C}{\varepsilon_n^2}\left|\ln\frac{R}{r_n}\right|^{-1}\end{aligned}$$

and, consequently

$$\|v_n\|_{L^2(\Omega)}+\varepsilon_n\|\nabla v_n\|_{L^2(\Omega)^3}\le C\sqrt{\left|\ln\frac{R}{r_n}\right|^{-1}}\underset{n\to\infty}{\longrightarrow}0. \tag{4.9}$$

Let λ be a vector in $\mathbb{R}^3$ perpendicular to the x_3-axis. Define the Y-periodic function $\widetilde{X}_n$ by $\nabla\widetilde{X}_n=\lambda$ in ω_n, such that $\widetilde{X}_n\in\mathscr{D}(Y)$ and is Y-periodic, and the rescaled function X_n by

$$X_n(x)=\varepsilon_n\,\widetilde{X}_n\left(\frac{x}{\varepsilon_n}\right). \tag{4.10}$$

In particular, X_n satisfies

$$\|X_n\|_\infty=\varepsilon_n\,\|\widetilde{X}_n\|_\infty\le C\varepsilon_n,\qquad \|\nabla X_n\|_\infty=\|\nabla\widetilde{X}_n\|_\infty\le C\quad\text{and}\quad \nabla X_n=\lambda \text{ in } \Omega_n. \tag{4.11}$$

We have, by (4.9) and (4.11),

$$\begin{aligned}\|v_nX_n\|_{H^1(\Omega)} &\le \|X_n\|_\infty\|v_n\|_{L^2(\Omega)}+\|X_n\|_\infty\|\nabla v_n\|_{L^2(\Omega)^3}+\|\nabla X_n\|_\infty\|v_n\|_{L^2(\Omega)}\\ &\le C\big(\|v_n\|_{L^2(\Omega)}+\varepsilon_n\|\nabla v_n\|_{L^2(\Omega)^3}\big)\underset{n\to\infty}{\longrightarrow}0,\end{aligned}$$

which gives

$$\forall\varphi\in\mathscr{D}(\Omega),\quad \varphi\,v_nX_n\underset{n\to\infty}{\longrightarrow}0 \text{ strongly in } H_0^1(\Omega). \tag{4.12}$$

Let $\varphi\in\mathscr{D}(\Omega)$. By the strong convergence (4.12), we have

$$\int_\Omega\sigma_n(h)\nabla u_n\cdot\nabla\big(\varphi\,v_nX_n\big)\,dx=\langle f,\varphi\,v_nX_n\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\underset{n\to\infty}{\longrightarrow}0. \tag{4.13}$$

Let us decompose this integral which converges to 0, into the integral on the fiber set Ω_n and the integral on its complementary:

$$\int_\Omega\sigma_n(h)\nabla u_n\cdot\nabla\big(\varphi\,v_nX_n\big)\,dx=\int_{\Omega\setminus\Omega_n}(\alpha_1I_3+\beta_1\mathscr{E}(h))\nabla u_n\cdot\nabla\big(\varphi\,v_nX_n\big)\,dx \tag{4.14a}$$

$$+\int_{\Omega_n}(\alpha_{2,n}I_3+\beta_{2,n}\mathscr{E}(h))\nabla u_n\cdot\nabla\big(\varphi\,v_nX_n\big)\,dx. \tag{4.14b}$$

The expression (4.14a) converges to 0 since, by the Cauchy–Schwarz inequality, the boundedness of u_n in $H^1_0(\Omega)$ and (4.12), we have

$$\left|\int_{\Omega\setminus\Omega_n} (\alpha_1 I_3 + \beta_1 \mathscr{E}(h))\nabla u_n \cdot \nabla\big(\varphi\, v_n X_n\big)\, dx\right| \le |\alpha_1 I_3 + \beta_1 \mathscr{E}(h)|\, \|\nabla u_n\|_{L^2(\Omega)^3}\, \|\varphi\, v_n X_n\|_{H^1_0(\Omega)} \underset{n\to\infty}{\longrightarrow} 0. \tag{4.15}$$

Consequently, as $v_n = 1$ and $\nabla X_n = \lambda$ on Ω_n, by (4.13), (4.14a), (4.14b) and (4.15), we have

$$\int_{\Omega_n} \sigma_n(h)\nabla u_n \cdot \lambda\, \varphi\, dx + \int_{\Omega_n} \sigma_n(h)\nabla u_n \cdot \nabla\varphi\, X_n\, dx \underset{n\to\infty}{\longrightarrow} 0. \tag{4.16}$$

To prove the convergence to 0 of the right term, we now show that $\mathbb{1}_{\Omega_n}\big(\alpha_{2,n} I_3 + \beta_{2,n}\mathscr{E}(h)\big)\nabla u_n$ is bounded in $L^1(\Omega)^3$. We have, by the Cauchy–Schwarz inequality, (4.5) and the classical equivalent $|\Omega_n| \underset{n\to\infty}{\sim} |\Omega|\, |\omega_n|$,

$$\begin{aligned}\left(\int_{\Omega_n} \left|\big(\alpha_{2,n} I_3 + \beta_{2,n}\mathscr{E}(h)\big)\nabla u_n\right| dx\right)^2 &\le \left|I_3 + \alpha_{2,n}^{-1}\beta_{2,n}\mathscr{E}(h)\right|^2 |\Omega_n|\, \alpha_{2,n} \int_{\Omega_n} \alpha_{2,n}|\nabla u_n|^2\, dx \\ &\le C \int_{\Omega} \sigma_n(h)\nabla u_n \cdot \nabla u_n\, dx \\ &\le C\, \|f\|_{H^{-1}(\Omega)}\, \|u_n\|_{H^1_0(\Omega)}.\end{aligned}$$

This combined with the boundedness of u_n in $H^1_0(\Omega)$ implies that $\mathbb{1}_{\Omega_n}\big(\alpha_{2,n} I_3 + \beta_{2,n}\mathscr{E}(h)\big)\nabla u_n$ is bounded in $L^1(\Omega)^3$. This bound and the uniform convergence to 0 of X_n (see (4.11)) imply the convergence to 0 of the right term of (4.16), hence

$$\int_{\Omega_n} \big(\alpha_{2,n} I_3 + \beta_{2,n}\mathscr{E}(h)\big)\nabla u_n \cdot \lambda\, \varphi\, dx \underset{n\to\infty}{\longrightarrow} 0.$$

We rewrite this condition as

$$\forall \lambda \perp e_3, \quad \mathbb{1}_{\Omega_n}\, \big(\alpha_{2,n} I_3 + \beta_{2,n}\mathscr{E}(h)\big)\nabla u_n \cdot \lambda \rightharpoonup 0 \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega). \tag{4.17}$$

Second step: Linear relations between weak-$*$ limits of $\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_i}$.

Thanks to the Cauchy–Schwarz inequality, we have

$$\left\|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_i}\right\|_{L^1(\Omega)} \le \frac{1}{|\omega_n|}\int_{\Omega_n} |\nabla u_n|\, dx \le \frac{1}{\sqrt{\alpha_{2,n}|\omega_n|}}\sqrt{\frac{|\Omega_n|}{|\omega_n|}}\sqrt{\int_{\Omega_n}\alpha_{2,n}|\nabla u_n|^2\, dx}$$

which leads us, by (4.5) and the asymptotic behavior $|\Omega_n| \underset{n\to\infty}{\sim} |\Omega|\, |\omega_n|$, to

$$\left\|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_i}\right\|_{L^1(\Omega)} \le \frac{C}{\sqrt{\alpha_{2,n}|\omega_n|}}\int_{\Omega} \sigma_n(h)\nabla u_n \cdot \nabla u_n\, dx \le C\left|\langle f, u_n\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}\right|$$

which is bounded by the boundedness of u_n in $H^1_0(\Omega)$. This allows us to define, up to a subsequence, the following limits

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_i} \rightharpoonup \xi_i \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega), \text{ for } i = 1, 2, 3. \tag{4.18}$$

Then, by (4.17) we have

$$\big(\alpha_{2,n} I_3 + \beta_{2,n}\mathscr{E}(h)\big)\mathbb{1}_{\Omega_n}\nabla u_n \cdot \lambda = \big(\alpha_{2,n}|\omega_n| I_3 + \beta_{2,n}|\omega_n|\mathscr{E}(h)\big)\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\nabla u_n \cdot \lambda \rightharpoonup 0 \quad \text{weakly-}* \text{ in } \mathcal{M}(\Omega).$$

Therefore, putting $\lambda = e_1, e_2$ in this limit and using condition (4.5), we obtain the linear system

$$\begin{cases}\alpha_2\xi_1 + \beta_2 h_2\xi_3 - \beta_2 h_3\xi_2 = 0\\ \alpha_2\xi_2 + \beta_2 h_3\xi_1 - \beta_2 h_1\xi_3 = 0\end{cases} \quad \text{in } \mathcal{M}(\Omega),$$

which is equivalent to

$$\begin{cases}\xi_1 = \dfrac{\beta_2^2 h_1 h_3 - \alpha_2\beta_2 h_2}{\alpha_2^2 + \beta_2^2 h_3^2}\xi_3\\[2ex] \xi_2 = \dfrac{\beta_2^2 h_2 h_3 + \alpha_2\beta_2 h_1}{\alpha_2^2 + \beta_2^2 h_3^2}\xi_3\end{cases} \quad \text{in } \mathcal{M}(\Omega). \tag{4.19}$$

Third step: Proof of $\xi_3 = \frac{\partial u}{\partial x_3}$.

We need the following result which is an extension of the estimate (3.13) of [41]. The statement of this lemma is more general than necessary for our purpose but is linked to Remark 4.1.

Lemma 4.1. *Let Q be a non-empty connected open subset of the unit disk D. Then, there exists a constant $C > 0$ such that any function $U \in H^1(Y)$ satisfies the estimate*

$$\left| \frac{1}{|r_n Q|} \int_{r_n Q \times \left(-\frac{1}{2}, \frac{1}{2}\right)} U\, dy - \int_Y U\, dy \right| \le C\sqrt{|\ln r_n|}\, \|\nabla U\|_{L^2(Y)^3}. \tag{4.20}$$

Proof of Lemma 4.1. Let $U \in H^1(Y)$. To prove Lemma 4.1, we compare the average value of U on $r_n Q$ and $r_n D$. Denoting $\widetilde{y} = (y_1, y_2)$, we have, for any $y_3 \in \left(-\frac{1}{2}, \frac{1}{2}\right)$,

$$\begin{aligned} \left| \fint_{r_n Q} U(\widetilde{y}, y_3)\, d\widetilde{y} - \fint_{r_n D} U(\widetilde{y}, y_3)\, d\widetilde{y} \right| &= \left| \fint_{Q} U(r_n\widetilde{y}, y_3)\, d\widetilde{y} - \fint_{D} U(r_n\widetilde{y}, y_3)\, d\widetilde{y} \right| \\ &\le \fint_Q \left| U(r_n\widetilde{y}, y_3) - \fint_D U(r_n\widetilde{y}, y_3)\, d\widetilde{y} \right| d\widetilde{y}, \end{aligned}$$

and, since $Q \subset D$,

$$\begin{aligned} \left| \fint_{r_n Q} U(\widetilde{y}, y_3)\, d\widetilde{y} - \fint_{r_n D} U(\widetilde{y}, y_3)\, d\widetilde{y} \right| &\le \frac{|D|}{|Q|} \fint_D \left| U(r_n\widetilde{y}, y_3) - \fint_D U(r_n\widetilde{y}, y_3)\, d\widetilde{y} \right| d\widetilde{y} \\ &\le C \fint_D r_n \left(\left| \frac{\partial U}{\partial x_1} \right| + \left| \frac{\partial U}{\partial x_2} \right| \right) (r_n\widetilde{y}, y_3)\, d\widetilde{y} \\ &= \frac{C}{\pi r_n} \int_{r_n D} \left(\left| \frac{\partial U}{\partial x_1} \right| + \left| \frac{\partial U}{\partial x_2} \right| \right) (\widetilde{y}, y_3)\, d\widetilde{y}, \end{aligned}$$

the last inequality being a consequence of the Poincaré–Wirtinger inequality. Hence, integrating the previous inequality with respect to $y_3 \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and applying the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned} \left| \fint_{r_n Q \times \left(-\frac{1}{2}, \frac{1}{2}\right)} U(y)\, dy - \fint_{r_n D \times \left(-\frac{1}{2}, \frac{1}{2}\right)} U(y)\, dy \right| &\le \frac{C}{\pi r_n} \int_{r_n D \times \left(-\frac{1}{2}, \frac{1}{2}\right)} |\nabla U|\,(y)\, dy \\ &\le C \sqrt{\int_{r_n D \times \left(-\frac{1}{2}, \frac{1}{2}\right)} |\nabla U|^2(y)\, dy} \\ &\le C \|\nabla U\|_{L^2(Y)^3}. \end{aligned}$$

This combined with the estimate (3.13) of [41], i.e. (4.20) for $Q = D$, and the fact that $\sqrt{|\ln r_n|}$ diverges to ∞ give the thesis. $\square$

Let $\varphi \in \mathscr{D}(\Omega)$. A rescaling of (4.20) with $Q = D$ implies the inequality

$$\left| \frac{1}{|\omega_n|} \int_{\Omega_n} u_n \varphi\, dx - \int_\Omega u_n \varphi\, dx \right| \le C\varepsilon_n \sqrt{|\ln r_n|}\, ||\nabla(u_n\varphi)||_{L^2(\Omega)^3}\,.$$

Combining this estimate and the first condition of (4.5) with

$$||\nabla(u_n\varphi)||_{L^2(\Omega)^3} \le ||\nabla u_n||_{L^2(\Omega)^3}\, \|\varphi\|_\infty + ||u_n||_{L^2(\Omega)}\, \|\nabla\varphi\|_\infty \le C,$$

it follows that

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} u_n - u_n \rightharpoonup 0 \quad \text{in } \mathscr{D}'(\Omega).$$

This convergence does not hold true when $\varepsilon_n^2 |\ln r_n|$ converges to some positive constant. Under this critical regime, nonlocal effects appear (see Remark 4.4).

Finally, as $\mathbb{1}_{\Omega_n}$ does not depend on the x_3 variable, we have

$$\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} \frac{\partial u_n}{\partial x_3} = \frac{\partial}{\partial x_3} \frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} u_n = \frac{\partial}{\partial x_3} \left(\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|} u_n - u_n \right) + \frac{\partial u_n}{\partial x_3} \rightharpoonup \frac{\partial u}{\partial x_3} = \xi_3 \quad \text{in } \mathscr{D}'(\Omega).$$

Fourth step: Derivation of the homogenized matrix.

We now study the limit of $\sigma_n(h)\nabla u_n$ in order to obtain $\sigma_*(h)$. We have

$$\begin{aligned}\sigma_n(h)\nabla u_n\cdot e_1 &= \mathbb{1}_{\Omega\setminus\Omega_n}\left(\alpha_1\frac{\partial u_n}{\partial x_1}-\beta_1 h_3\frac{\partial u_n}{\partial x_2}+\beta_1 h_2\frac{\partial u_n}{\partial x_3}\right)\\ &\quad+\alpha_{2,n}|\omega_n|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_1}-\beta_{2,n}h_3|\omega_n|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_2}+\beta_{2,n}h_2|\omega_n|\frac{\mathbb{1}_{\Omega_n}}{|\omega_n|}\frac{\partial u_n}{\partial x_3}.\end{aligned}\tag{4.21}$$

Hence, passing to the weak-$*$ limit in $\mathcal{M}(\Omega)$ this equality and using the linear system (4.19), $\sigma_n(h)\nabla u_n\cdot e_1$ weakly-$*$ converges in $\mathcal{M}(\Omega)$ to

$$\begin{aligned}&\left(\alpha_1\frac{\partial u}{\partial x_1}-\beta_1 h_3\frac{\partial u}{\partial x_2}+\beta_1 h_2\frac{\partial u}{\partial x_3}\right)+\alpha_2\xi_1-\beta_2 h_3\xi_2+\beta_2 h_2\xi_3\\ &\quad=\big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_1+\alpha_2\frac{\beta_2^2 h_1h_3-\alpha_2\beta_2h_2}{\alpha_2^2+\beta_2^2h_3^2}\xi_3-\beta_2h_3\frac{\beta_2^2h_2h_3+\alpha_2\beta_2h_1}{\alpha_2^2+\beta_2^2h_3^2}\xi_3+\beta_2h_2\xi_3\\ &\quad=\big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_1+\underbrace{\frac{\alpha_2(\beta_2^2h_1h_3-\alpha_2\beta_2h_2)-\beta_2h_3(\beta_2^2h_2h_3+\alpha_2\beta_2h_1)+\beta_2h_2(\alpha_2^2+\beta_2^2h_3^2)}{\alpha_2^2+\beta_2^2h_3^2}}_{=0}\xi_3,\end{aligned}$$

that is

$$\sigma_n(h)\nabla u_n\cdot e_1 \rightharpoonup \big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_1\quad\text{weakly-}*\text{ in }\mathcal{M}(\Omega).\tag{4.22}$$

The same calculus leads us to

$$\sigma_n(h)\nabla u_n\cdot e_2 \rightharpoonup \big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_2\quad\text{weakly-}*\text{ in }\mathcal{M}(\Omega).\tag{4.23}$$

We have, for the last direction e_3,

$$\sigma_n(h)\nabla u_n\cdot e_3\rightharpoonup\left(\alpha_1\frac{\partial u}{\partial x_3}-\beta_1 h_2\frac{\partial u}{\partial x_1}+\beta_1 h_1\frac{\partial u}{\partial x_2}\right)+\alpha_2\xi_3+\beta_2h_2\xi_1-\beta_2h_1\xi_2\quad\text{weakly-}*\text{ in }\mathcal{M}(\Omega).$$

Hence, again with the linear system (4.19),

$$\begin{aligned}&\left(\alpha_1\frac{\partial u}{\partial x_3}-\beta_1 h_2\frac{\partial u}{\partial x_1}+\beta_1 h_1\frac{\partial u}{\partial x_2}\right)+\alpha_2\xi_3-\beta_2h_2\xi_1+\beta_2h_1\xi_2\\ &\quad=\big(\alpha_1 I_3+\beta_1\mathscr{E}(h)\big)\nabla u\cdot e_3+\alpha_2\xi_3-\beta_2h_2\frac{\beta_2^2h_1h_3-\alpha_2\beta_2h_2}{\alpha_2^2+\beta_2^2h_3^2}\xi_3+\beta_2h_1\frac{\beta_2^2h_2h_3+\alpha_2\beta_2h_1}{\alpha_2^2+\beta_2^2h_3^2}\xi_3.\end{aligned}$$

Finally, by the previous equality, (4.22) and (4.23), we get that

$$\sigma_*(h)=\alpha_1 I_3+\left(\frac{\alpha_2^3+\alpha_2\beta_2^2|h|^2}{\alpha_2^2+\beta_2^2h_3^2}\right)e_3\otimes e_3+\beta_1\mathscr{E}(h).\quad\square$$

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