Available online at www.sciencedirect.com
SCIENCE DIRECT®

Electronic Notes in Theoretical Computer Science

ELSEVIER Electronic Notes in Theoretical Computer Science 134 (2005) 153–187
www.elsevier.com/locate/entcs

Heterogeneous Reasoning with Euler/Venn Diagrams Containing Named Constants and FOL

Nik Swoboda [1,3]

Departamento de Inteligencia Artificial
Universidad Politécnica de Madrid
Campus de Montegancedo, Boadilla del Monte,
Madrid, 28660, Spain

Gerard Allwein[2]

Naval Research Laboratory, Code 5543
4555 Overlook Avenue SW
Washington DC 20375-5337, USA

Abstract

The main goal of this paper is to present the basis for a heterogeneous Euler/Venn diagram and First Order Logic (FOL) reasoning system. We will begin by defining a homogeneous reasoning system for Euler/Venn diagrams including named constants and show this system to be sound and complete. Then we will propose a heterogeneous rule of inference allowing the extraction of formulas of FOL from an Euler/Venn diagram. In defining this rule we will attempt to capture the "explicit" information content of an Euler/Venn diagram in a way similar to the **Observe** rule in the *Hyperproof* [1] system. Two definitions for this heterogeneous rule will be presented, one syntactically based, which is intended to be intuitive and motivational, and a second based upon a framework employing information types to model heterogeneous reasoning previously presented [17]. Lastly we will explore the relationships between these two notions.

Keywords: Heterogeneous reasoning, Venn diagrams, Euler diagrams

[1] Email: nswoboda@clip.dia.fi.upm.es
[2] Email: allwein@itd.nrl.navy.mil
[3] A Ramón and Cajal research grant provided by the Spanish Ministry of Science and Technology provided partial support for this work.

doi:10.1016/j.entcs.2005.02.023

1 Introduction

A little more than a decade has passed since Shin [12] and Hammer [6] re-ignited interest in Venn diagrams and spurred their use in formal reasoning systems. Subsequently, related systems have been studied including Euler/Venn diagrams [16], constraint diagrams [10,15,4], spider diagrams [5,8] and others. A very useful survey of Euler-based diagrammatic systems can be found in [14]. Here we will focus our interest in an extended version of Euler/Venn diagrams including named constants and their use in a heterogeneous FOL and Euler/Venn reasoning system.

Euler/Venn diagrams are a modified version of the Venn diagram system presented by Shin and Hammer which include diagrams with both 'Venn-like' and 'Euler-like' features. In Euler/Venn diagrams curves can be drawn which completely contain other curves or which are completely disjoint from other curves. Furthermore, shading and anonymous constant chains in these diagrams carry information regarding the members of represented sets. Here modifications to this system will be presented which make these diagrams more suitable for reasoning along with FOL. More specifically, the system presented here will replace the anonymous constant sequences (Shin and Hammer's x-sequences) with named constant sequences and will exclude models with empty domains. [4] This change will allow information carried by FOL formulas like "Dog(fido)" to be more precisely expressed in the diagrams. An example diagram can be seen in Fig. 1. These modifications trigger further changes to the system's rules of inference to maintain soundness and completeness. More specifically, rules are added to allow the addition of constant sequences, the combination of constant sequences, and to address the two new kinds of inconsistency that can occur in these diagrams.

After presenting the homogeneous diagrammatic system we will then turn our attention to heterogeneous Euler/Venn and FOL reasoning. We will present two heterogeneous rules of inference for the extraction of FOL from Euler/Venn diagrams with constants which will be motivated by an interest in capturing the "explicit" information content of a diagram. The first of these two rules will be syntactically based in the interest of simplicity and intuitiveness. Then a second information type based approach [17] will be given and the relations between these two rules will be explored.

[4] Empty domains are not permitted in typical FOL systems due to the strange behavior that they cause when dealing with quantified expressions. For example the formula "$\forall x((A(x) \vee \neg A(x))$ is no longer universally true when empty domains are allowed. Thus to reason heterogeneously, in a natural manner, with FOL it is necessary to eliminate empty domains from the diagrammatic system.

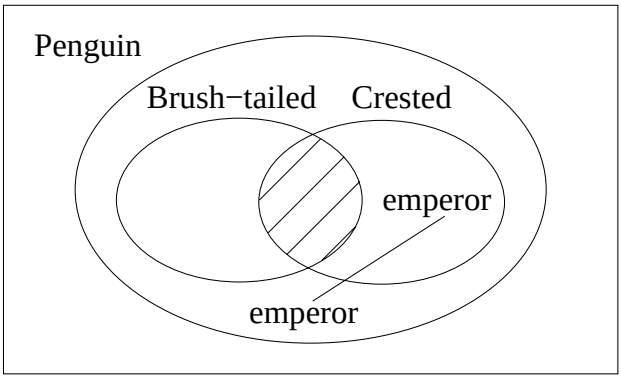


Fig. 1. An example Euler/Venn diagram with named constants

2 Euler/Venn Diagrams with Named Constants (EVc)

Here and throughout the remainder of the paper, risking confusion in the interest of readability, we will refer to diagrams of the new EVc system as simply Euler/Venn diagrams and references made to diagrams of the old EV system will be explicitly noted.

2.1 The Vocabulary of EVc

Let $\mathcal{L}$ be some set of predicates, each which can be thought of as the label of some curve of an Euler/Venn diagram, and let the set Terms be the union of a set Cons of constant symbols and a set Var of variable symbols also occurring in those diagrams. For the purposes of this project, free variables and constants in an Euler/Venn diagram will be treated almost identically. Thus, free variables will be replaced by fresh constants at the point of their evaluation.[5] Using these collections, the vocabulary of EVc consists of the following elements:

- Rectangles: Each rectangle denotes the domain of discourse to be represented by the diagram.
- Closed Curves: A countably infinite set of closed curves, each uniquely labeled with predicate symbols from $\mathcal{L}$.[6] Each of these curves is taken to

[5] A similar treatment of free variables is described in [3].

[6] This includes a very broad collection of curves, even those that self-intersect. This is done to ensure that any Euler/Venn diagram with shading can be redrawn as a logically equivalent Euler/Venn diagram with any shaded area removed. Unfortunately this also results in many diagrams which are difficult to interpret. Easy to define and stricter restrictions which would allow the removal of any shaded area while at the same time only allowing diagrams which are intuitive and easy to interpret are unknown to the authors.

represent the set which corresponds to its label.

- Shading: The shading of areas in the diagram denotes that the set represented by that area is empty.
- Constant Symbols: A countably infinite set of individual constant symbols from Terms.
- Lines: Lines are used to connect individual constants with the same name in different regions to illustrate the uncertainty regarding which set contains that constant.

2.2 The Grammar of EVc

2.2.1 Notion of region

A *region* of a diagram is any, possibly empty, area of the diagram that is completely enclosed by lines of that diagram.[7] Any region of the diagram completely enclosed by a closed curve is referred to as a *basic region*. Each basic region has a unique *label*, the label of its enclosing curve.

The collection of regions is closed under union, intersection, and complement; thus a region may contain disconnected parts. These operations are defined as follows:

- $\cup$ The union of two regions is the region containing both of those regions and no others.
- $\cap$ The intersection of two regions is the region that is common to both regions.
- $-$ The difference of two regions is the region contained in the first but not contained in the second.
- $\overline{r}$ The complement of a region is the region not contained in that region but still within the rectangle of the diagram.
- $\subset$ One region is the subset of another if that region is entirely contained within the other.

A region which can be defined using the previous operations from basic regions, but is not represented in the diagram (has no area in the diagram) will be called a *missing region*. A *minimal region* is any non-missing region which is not crossed by any of the lines of that diagram (i.e., any region that can not be thought of as the union of other non-missing regions).

[7] Here we present the notion of a region using areas, but they could be alternatively defined by thinking of the diagram as being contained in a plane with regions being open sets of points within the diagram and bound by lines of the diagram in that plane.

2.2.2 Formation rules

Formation rules for well-formed diagrams $\mathcal{V}_{EVc}$[8] of EVc:

(i) Any diagram containing only a Rectangle is a member of $\mathcal{V}_{EVc}$.

(ii) If $D \in \mathcal{V}_{EVc}$ then:
   (a) D with the addition of any closed curve C with label L (not already occurring in the diagram) completely within the rectangle of D so that all the minimal regions intersected by C are split into at most two new regions, is a member of $\mathcal{V}_{EVc}$.
   (b) D with the shading of any minimal region is a member of $\mathcal{V}_{EVc}$.
   (c) D with the addition of an individual constant symbol to any minimal region is a member of $\mathcal{V}_{EVc}$.
   (d) D with the addition of a constant symbol n to any minimal region and the inclusion of a line connecting the new n to an existing n in a different minimal region such that the new sequence doesn't have more than one link in any minimal region, is a member of $\mathcal{V}_{EVc}$.

(iii) No other diagram is in $\mathcal{V}_{EVc}$.

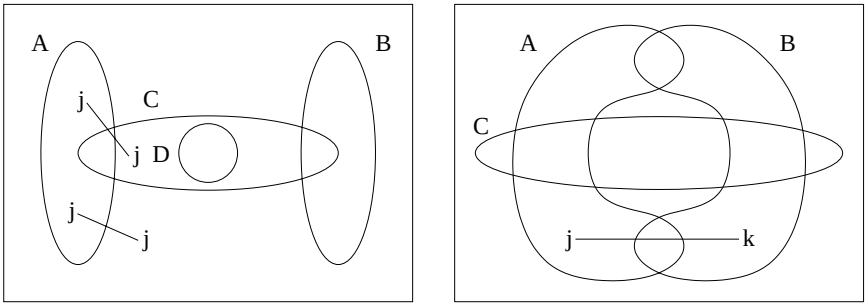
It should be noted that, as in the old Euler/Venn diagrams, no two disjoint regions in a diagram can be considered to represent the same set. One other important aspect of these new Euler/Venn diagrams is that two separate constant sequences containing the same constant are permitted. Some example diagrams in $\mathcal{V}_{EVc}$ are shown in Fig. 2. We will use the term *features of a region* to generically describe the state of a region of the diagram, i.e., that one or more links of certain constant sequences are in that region, that it is shaded, or that it is missing. Likewise, the collection of the features of all of the regions of a diagram will be referred to as the features of a diagram.

2.2.3 Notion of a tag

At times we will need to refer to the missing or non-existing regions of Eulerized Venn diagrams; tags are introduced as simply an auxiliary notion to give a mechanism for referring to these regions. Given the set $\{L_1, \ldots, L_n\}$ of labels of a diagram $D \in \mathcal{V}$, a *tag* is a subset of $\{L_1, \overline{L_1}, \ldots, L_n, \overline{L_n}\}$ not containing L_i and $\overline{L_i}$ for any i $(1 \leq i \leq n)$. A tag τ is said to be *complete* if for each label L_i of D, either $L_i \in \tau$ or $\overline{L_i} \in \tau$.

Thus for each basic region labeled L there will be a tag $\{L\}$ corresponding to it and tag $\{\overline{L}\}$ corresponding to the complement of that region. Then given two regions tagged with τ_1 and τ_2 the tag for the intersection of those

[8] Throughout the rest of this work, EVc subscripts will be omitted when this will not result in ambiguity.



(a) Well-Formed Diagram (b) Not Well-Formed Diagram

Fig. 2. Examples of diagrams of $\mathcal{V}_{EVc}$

regions will be $\tau_1 \cup \tau_2$ (provided that this tag doesn't contain L_i and $\overline{L_i}$ for some L_i). We then see that all the complete tags consisting of labels $L_1, \ldots, L_n$ correspond exactly to all of the minimal regions of a Venn diagram (a diagram with no missing regions) having curves with each of those labels. We also allow the tag $\{\}$ to refer to the region enclosed by the rectangle of the diagram. It should be noted that in diagrams with missing regions, tags can still be used to refer to the missing regions that could be re-introduced into the diagram. These intuitions are made precise by the following definition:

Definition 2.1 Tag Assignment Function
Given a diagram $D \in \mathcal{V}$ containing curves labeled $L_1, \ldots, L_n$, the function $region_D$ from the tags of D's labels to the regions of D will be defined as follows:

(i) For the region r enclosed by the rectangle of D $region_D(\{\}) = r$.

(ii) For each basic region r labeled L in D $region_D(\{L\}) = r$ and $region_D(\{\overline{L}\}) = \overline{r}$.

(iii) Assume $region_D(\tau_1) = r_1$ and $region_D(\tau_2) = r_2$ and $\tau_1 \cup \tau_2$ is a tag, if the region $r_1 \cap r_2$ is missing in D(it is not represented in the diagram) then $region_D(\tau_1 \cup \tau_2) = \emptyset$ otherwise $region_D(\tau_1 \cup \tau_2) = r_1 \cap r_2$. [9]

It is important to point out that not every region in a diagram has a tag, but rather only the region contained by the rectangle, and regions that are the intersection of basic regions with the complements of basic regions.

[9] Here the issue of missing regions is addressed for the system to be able to deal correctly with Euler type diagrams illustrating set inclusion and sets which are disjoint.

2.2.4 Notion of counterpart

Intuitively, two regions in two different diagrams are said to be counterparts if they are both interpreted as representing the same set. Given two diagrams D and D' we will say that region r of D and r' of D' are *counterparts* if there is a tag τ such that $region_D(\tau) = r$ and $region_{D'}(\tau) = r'$. *Counterparts agree with respect to shading and whether they are missing* in two diagrams when for any two regions that are counterparts one is shaded iff the other is shaded, one is missing iff the other is missing.

Counterparts agree with respect to constant sequences in two diagrams D and D' when for any constant sequence in either diagram, say in D (without loss of generality), with links in minimal regions $r_1, \ldots, r_n$ there is a constant sequence with the same name in D' with links in the minimal regions $s_1, \ldots, s_m$ and either

- for all s_i $(1 \leq i \leq m)$ there is some r_j $(1 \leq j \leq n)$ in D' with counterpart r'_j in D such that $s_i \subseteq r'_j$ and in D there are no minimal regions s_k not in $s_1, \ldots, s_m$ such that $s_k \subset r'_i$.
- for all r_i $(1 \leq i \leq m)$ there is some s_j $(1 \leq j \leq n)$ in D with counterpart s'_j in D' such that $r_i \subseteq s'_j$ and in D' there are no minimal regions r_k not in $r_1, \ldots, r_m$ such that $r_k \subset s'_i$. (This is basically same condition but with D and D' and the s's and r's swapped.)

We say that for some region r in D that its *counterparts in D' agree with respect to constant sequences* when for all constant sequences in r the previous conditions hold. For additional work on syntactically defining the notion of counterpart regions see [9].

2.3 The Deductive System of EVc

Given diagrams D and D' of $\mathcal{V}$, D' can be inferred from D if D' is the result of applying any of the following rules to D:

- **Erasure of part of a constant sequence** – D' is obtained by erasing a link n of a constant sequence of D where that link n falls within a shaded region and provided that the possibly split constant sequence is rejoined if necessary.
- **Extending a constant sequence** – D' is the result of adding a new link to a constant sequence of D in a minimal region not already containing a link of that same sequence.
- **Combining constant sequences** – D' is the result of combining two constant sequences each containing the constant n in a diagram D. Two con-

stant sequences with the same letter can be combined when they each contain links in one or more minimal regions. In the resulting diagram both sequences are replaced by one sequence of the same term with links in each and every of the minimal regions which contained links from both in the initial diagram.

- **Introduction of a Constant Sequence** – D' is the result of adding a constant sequence to D with one link in each of the minimal regions of D.
- **Erasure** – D' is obtained from D by erasing:
  (i) an entire constant sequence;
  (ii) the shading of a region;
  (iii) a closed curve (and possibly redrawing the remaining curves to keep the diagram well-formed) if the removal does not cause any counterpart regions to disagree with regard to shading, whether they are missing, and constant sequences.
- **Introduction of a new curve** – D' is the result of adding a new curve to D, whose label does not occur in D, which crosses all of the minimal regions of D once and in such a way that D' is well-formed, the other labels of D are left undisturbed, and all counterparts agree with respect to shading, whether they are missing, and constant sequences.
- **Inconsistency** – D' of any form can obtained from D if:
  (i) D contains a region that is both shaded and contains all the links of some constant sequence;
  (ii) D contains two separate constant sequences of the same term and there is no minimal region in the diagram containing links of both of the sequences.
  (iii) all of the regions in D are shaded;
- **Adding shaded regions** – D' is the result of adding a new minimal (but not basic) region which is the counterpart of a missing region in D provided that this new region is shaded and is drawn so that the region is contained within the basic regions to whose intersection it is intended to correspond.
- **Removing shaded regions** – D' is the result of removing a shaded minimal but not basic region of D. To emphasize the fact that the region has been removed the lines enclosing the now non-existing region should be smoothed where possible, and the remaining curves should be spaced out to remove points of unintended intersection.

Unification – D' can be inferred from diagrams D_1 and D_2 if it is the case that:

- The set of labels of D' is the union of the labels of D_1 and D_2.
- If a minimal region in either D_1 or D_2 is shaded or missing then there is a

counterpart region in D which is also shaded or missing. Also, if there is a region in D which is shaded or missing then there is a counterpart region in D_1 or D_2 which is shaded or missing.

- If a region in either D_1 or D_2 has a constant sequence then there are counterpart regions in D which agree with respect to that constant sequence. Also, if there is a region in D containing a constant sequence then there are counterpart regions in D_1 or D_2 which agree with respect to that constant sequence.

Examples are given in Fig. 3 to illustrate the use of the system's new and modified rules of inference.

A diagram D is provable from the set of diagrams $\mathtt{D}$ in $\mathcal{V}$, written as $\mathtt{D} \vdash_{EVc} D$, if there is a sequence of diagrams $D_1 \ldots D_n$ where D_n is equal to D and all $D_1 \ldots D_n$ are either members of $\mathtt{D}$ or the result of applying one of the above rules of inference to prior diagrams in the sequence.

2.4 The Semantics of EVc

The semantics of the system is given by the assignment of a domain to the diagram, subsets of this domain to minimal regions of the diagram, and members of the domain to each of the diagram's constants.

Definition 2.2 Hammer Models for Euler/Venn Diagrams (extension of Hammer [6])
A *model* of a Euler/Venn diagram is a triplet $M = (U_M, I_M, denotes_M)$, where U_M is a non-empty set of objects, I_M a function assigning subsets of U_M to all regions of a diagram, and $denotes_M$ a function assigning members of U_M to members of the set Terms.[10] The function I has the following properties:

- $I(r) = U$ whenever r is a region consisting of the entire interior of the diagram's rectangle.
- $I(r) = I(s)$ whenever r and s are two basic regions with the same label.
- $I(r \cap s) = I(r) \cap I(s)$, for all r and s regions of a diagram.
- $I(\overline{r}) = U - I(r)$, if r is a region of a diagram.

The collection of Euler/Venn models will be referred to as $\mathcal{M}_{EVc}$.

Next the notion of truth for Euler/Venn diagrams using these models will be defined.

Definition 2.3 Truth of Euler/Venn diagrams in Hammer Models
Given any $M \in \mathcal{M}$ with $M = (U, I, denotes)$ and $D \in \mathcal{V}$, we say that D is

[10] When clear the M subscripts will be omitted.

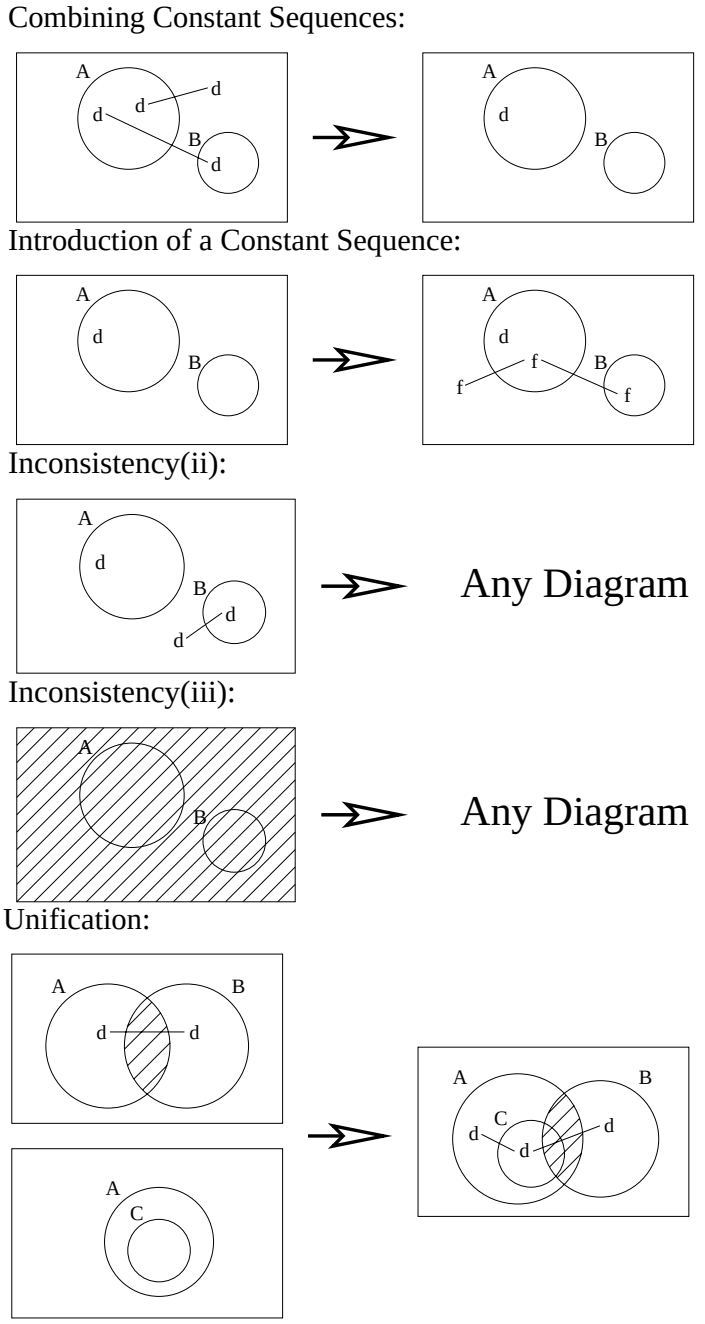


Fig. 3. Example applications of new and modified rules of inference

true in M or that $M \models D$ if the following conditions are satisfied:

(i) For each shaded region r, $I(r) = \emptyset$.

(ii) For each missing (i.e., non-existing) region r, $I(r) = \emptyset$.

(iii) For each constant symbol n which is in the region r or constant sequence n completely contained in the region r, $denotes(n) \in I(r)$.

$M \not\models D$ will be written if it is not that case that $M \models D$.

With $\mathtt{D} \cup \{D\}$ a set of diagrams, D is a *logical consequence* of $\mathtt{D}$ in EVc iff every model which makes all of $\mathtt{D}$ true in EVc also makes D true. This will be written as $\mathtt{D} \models_{EVc} D$.

2.5 Soundness and Completeness of EVc

Theorem 2.4 *Soundness of EVc (extension of Hammer [6])*
For every set of diagrams $\mathtt{D} \cup \{D\} \subseteq \mathcal{V}$, *if* $\mathtt{D} \vdash_{EVc} D$ *then* $\mathtt{D} \models_{EVc} D$.

Proof Sketch: It suffices to show that the new and modified rules of inference preserve soundness; this plus trivial changes to Hammer's soundness proof will demonstrate the soundness of EVc.

- If D' is the result of applying the rule of **Combining Constant Sequences** to D, then $D \models_{EVc} D'$. Suppose that $(U, I, denotes) \models_{EVc} D$, then for regions r and r' in D, each containing constant sequences of some term n, $denotes(n) \in I(r)$ and $denotes(n) \in I(r')$. Thus $denotes(n) \in I(r) \cap I(r')$ and then by the definition of the function I, $denotes(n) \in I(r \cap r')$. Since the resulting diagram has links of n in each and only the the minimal regions of $r \cap r'$, then $(U, I, denotes) \models_{EVc} D'$.
- If D' is the result of applying the rule of **Introduction of a Constant Sequence** to D, then $D \models_{EVc} D'$. Suppose that $(U, I, denotes) \models_{EVc} D$ and the new constant sequence contains links of the term n. Since U is non-empty, all constants have interpretations in U, the new constant sequence is contained in the region r bound by the rectangle of the diagram, and $I(r) = U$; we have that $denotes(n) \in I(r)$. Thus $(U, I, denotes) \models_{EVc} D'$.
- If D' is the result of applying the rule of **Inconsistency** to D, then $D \models_{EVc} D'$.
  - Suppose that $(U, I, denotes) \models_{EVc} D$ and all of the regions in D are shaded. Then the region r bound by the rectangle of the diagram is shaded. Therefore both $I(r) = \emptyset$ and $I(r) = U$ and U is defined to be non-empty. Since there can be no such model it is therefore the case that $D \models_{EVc} D'$ for any D'.
  - Suppose that $(U, I, denotes) \models_{EVc} D$ and that D contains two constant

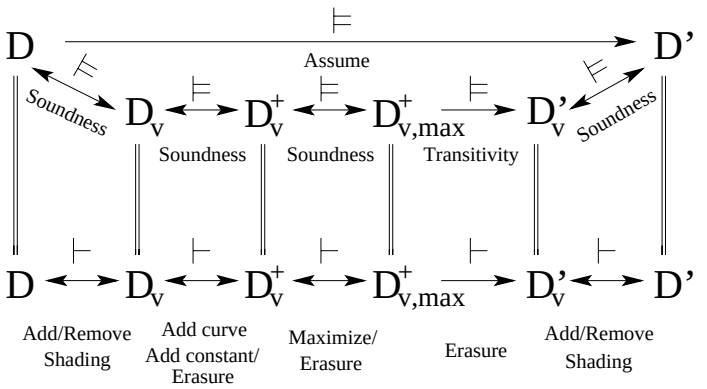


Fig. 4. Strategy for proof of completeness result

sequences of the term n without sharing links in any minimal region. Thus for some disjoint regions r, r', $denotes(n) \in I(r)$ and $denotes(n) \in I(r')$ and $r \cap r' = \emptyset$. But then $I(r) \cap I(r') = I(r \cap r') = \emptyset$ and thus $denotes(n) \in \emptyset$. Since there can be no such model it is therefore the case that $D \models_{EVc} D'$ for any D'.

- If D' is the result of applying the rule of **Adding shaded regions** to D, then $D \models_{EVc} D'$. Suppose that $(U, I, denotes) \models_{EVc} D$ then for all minimal regions r not existing in D, $I(r) = \emptyset$. Thus since the newly added region is shaded then $I(r) = \emptyset$, and $(U, I, denotes) \models_{EVc} D'$.
- If D' is the result of applying the rule of **Removing shaded regions** to D, then $D \models_{EVc} D'$. Suppose that $(U, I, denotes) \models_{EVc} D$ then for all shaded regions r in D, $I(r) = \emptyset$. Thus since the removed minimal region does not exist in the diagram then $I(r) = \emptyset$, and $(U, I, denotes) \models_{EVc} D'$.

□

Theorem 2.5 *Finite Completeness of EVc (extension of Shin [13]) For any two diagrams D, D' in $\mathcal{V}$, if $D \models_{EVc} D'$ then $D \vdash_{EVc} D'$.*

Proof Sketch: For this proof, Hammer's completeness proof of Shin's completeness result found in [6] will be greatly relied upon. We begin by assuming that $D \models_{EVc} D'$ to show that $D \vdash_{EVc} D'$. An illustration of the proof strategy that we will follow is given in Fig 4.

First, the diagram D is extended to a Venn diagram D_v, through the repeated application of the **Adding shaded regions** inference rule. The same is done to D' extending it to D'_v. From soundness we know that D and D_v as well as D' and D'_v are logically equivalent. (This also depends on the fact that when from one diagram, a second diagram can be concluded using the repeated application of **Adding shaded regions** then the second can

be concluded from the first through the repeated application of **Removing shaded regions**.)

Next we add any basic regions in D'_v that don't appear in D_v to D_v using the **Introduction of a new curve** rule of inference. At the same time we use the rule **Introduction of a Constant Sequence** to add every constant n appearing in D'_v to D_v. Constants are added even if some sequence of the same constant already exists in D_v. (The order in which these additions, curves and constants, are made is irrelevant.) We call the resulting diagram D^+_v. As in the last step, we note that from soundness D and D^+_v are logically equivalent (the added curves and constant sequences can be removed with the **Erasure** rule of inference). Then we construct Shin's *maximal diagram* of D^+_v.

To do this we begin by constructing the set Δ. We first repeatedly use the **Erasure of part of a constant sequence** rule to remove all links of constant sequences in D^+_v falling in shaded minimal regions and place the resulting diagram in Δ. Then we close Δ under the rule of **Extending a constant sequence** and use the unify rule to unify all of those diagrams into the resulting diagram $D^+_{v,max}$. Since $D^+_{v,max}$ is provable from D, D is provable from $D^+_{v,max}$ (using **Erasure** and **Extending a constant sequence** to arrive at D^+_v from which D can be proved as mentioned earlier), and D' is logically equivalent to D'_v we then know that $D^+_{v,max} \models D'_v$.

Then as in Hammer's proof, to show that we can prove D'_v from $D^+_{v,max}$ using the **Erasure** rule we need to establish two claims:

- For every minimal region r of D'_v that is shaded its counterpart in $D^+_{v,max}$ is also shaded.
- For every region r of D'_v that contains a constant sequence n its counterparts in $D^+_{v,max}$ agree with respect to that constant sequence.

To show both of these claims we will construct models that show that if the above two claims aren't the case then contradictions will occur. We begin by supposing that there is a minimal region r of $D^+_{v,max}$ that is not shaded but its counterpart r' in D'_v is shaded. We then construct a model $(U, I, denotes)$ in which $D^+_{v,max}$ is true but D'_v is false. As the domain U we take the set $\{o_1, \ldots, o_n\}$ with a unique o_k $(1 \leq k \leq n)$ for each non-shaded minimal region in $D^+_{v,max}$. We note that if there are no non-shaded minimal regions then immediately we can use the **Inconsistency** rule to conclude D' completing the construction. For the function I we simply assign a unique o_k $(1 \leq k \leq n)$ to each non-shaded minimal region of $D^+_{v,max}$ and extend this assignment appropriately to all regions of the diagram. Lastly we define the function *denotes*, but in doing so we need to consider the following possibilities:

- For every constant sequence of the constant n in $D^+_{v,max}$ there is at least

one minimal region containing links from every one of those sequences. In this case, for each constant n in $D^+_{v,max}$ we pick one unshaded minimal region r' from the minimal regions in which one link of each and every of the n sequences fall and define *denotes* so that $denotes(n) = o_k$ such that $o_k \in I(r')$ (and we know from the construction that $I(r')$ is a single object). In the case that it is not possible to pick an unshaded minimal region we can then use the **Combining Constant Sequences** and the **Inconsistency** rule to directly conclude D' completing the construction.

- There are at least two constant sequences of some constant n which do not both have links in some minimal region. In this case then we can use the **Inconsistency** rule to directly conclude D' completing the construction.

Since in all the cases where no $(U, I, denotes)$ could be constructed the **Inconsistency** rule can be employed we can then focus on just those diagrams for which a model can be constructed. We then return our attention to the region r of $D^+_{v,max}$ that is not shaded and its counterpart r' in D'_v that is shaded. In this case $I(r) = \emptyset$ in our model, and thus $I(r') = \emptyset$ since they are counterparts. Then $(U, I, denotes) \not\models D'_v$, contradiction.

To show the second claim we suppose that there is some constant sequence n in D'_v in the region r' and that counterparts in $D^+_{v,max}$ do not agree with respect to that constant sequence. We use a similar construction as in the last case to make a model such that $(U, I, denotes) \models D^+_{v,max}$. Let $n_1, \ldots, n_j$ be the constants appearing in D'_v but not in the appropriate regions in $D^+_{v,max}$. As above in the case that all regions of the diagram are shaded, there is a n_i $(1 \leq i \leq j)$ with links in only shaded regions, or some n_i $(1 \leq i \leq j)$ for which it is the case that there are two n_i sequences with no links in the same minimal region we can use the **Inconsistency** rule to directly conclude D' completing the construction. Otherwise we assign an arbitrary object o to $denotes(n_i)$ in some minimal region containing one link of each and every n_i sequence in the diagram, for each constant n_i in $D^+_{v,max}$. Then we then collect the arbitrary o into the domain U, and construct the appropriate function I assigning those o_i to the appropriate minimal regions, $\emptyset$ to all other minimal regions, and the correct combinations to all the other regions. However since $denotes(n_i) \notin I(r)$ and thus $denotes(n_i) \notin I(r')$ we then have that $(U, I, denotes) \not\models D'_v$, contradiction. □

3 Heterogeneous Rules of Inference

The general goal of this portion of the paper is to define a heterogeneous rule of inference that allows the extraction of information in an Euler/Venn diagram in the form of a formula of FOL. To accomplish this end, we will employ the

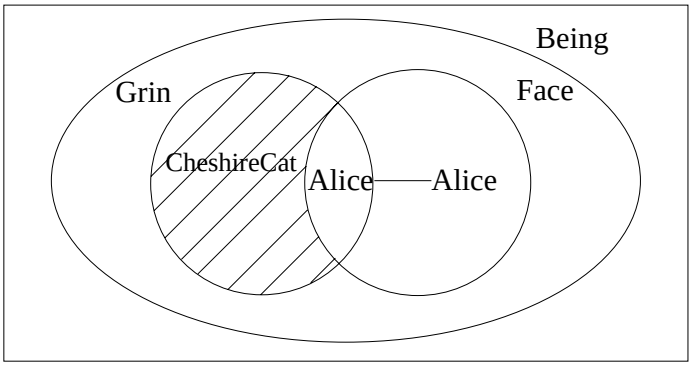


Fig. 5. Alice in Wonderland

information type based framework for defining heterogeneous rules of inference given in [17]. Furthermore we will attempt to define this rule in the spirit of the **Observe** rule presented in that same article. Stated briefly, one should only be able to observe information from a diagram when that information is explicitly expressed in that diagram. Thus we will present two such rules. The first will be syntactically based and thus defined using only the explicit information in the diagram. Then a second information type based notion will be presented and it will be shown that this notion can be characterized by the first notion.

3.1 Observe and Euler/Venn diagrams

To develop some intuitions about what should and what should not be observable from diagrams in general, let's first look at observations that we would and would not like to be able to make from the Euler/Venn diagram in Fig. 5.

- One can observe that Alice has a face, since that information is explicitly present.
- One cannot observe that Alice has a grin and one cannot observe that she does not have a grin since neither piece of information is present.
- One can observe that the Cheshire Cat has a grin and does not have a face, due to the placement of 'CheshireCat'.
- One can observe that nothing has a grin that does not have a face, due to the shading of the appropriate region.
- Nothing can be observed regarding the Mad Hatter, since he is not mentioned in the diagram.

In trying to make precise our intuitions about what can be observed from Euler/Venn diagrams we are immediately confronted with a number of inter-

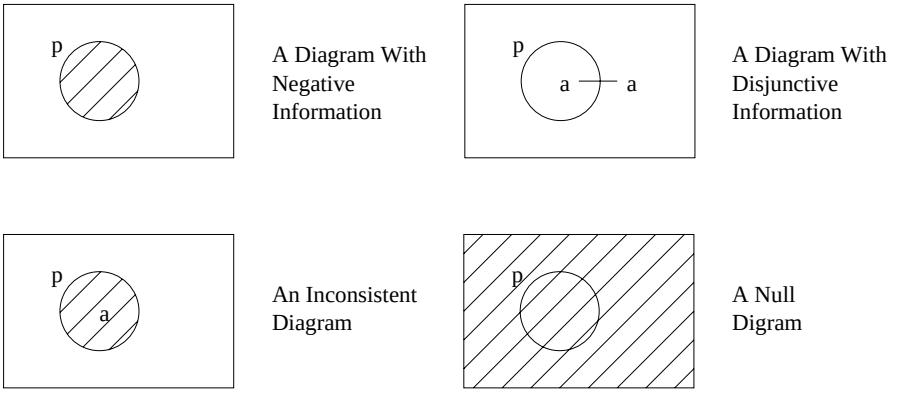


Fig. 6. Examples of problematic Euler/Venn diagrams

esting issues arising from the kinds of information that can be contained in Euler/Venn diagrams. These issues have the following as their source:

- Negative Information – Through shading one can represent negative information, that some set does not contain any members.
- Disjunctive Information – By creating chains of individual constants in a Euler/Venn diagram one can represent the uncertainty of which of the represented sets contain the denoted individual, while at the same time representing the certainty that the individual is in one of those sets.
- Open world – there is no assumption when using an Euler/Venn diagram that all objects under discussion are explicitly represented in the diagram.
- Inconsistency – One can represent inconsistent information in a single Euler/Venn diagram. An example was the placing of the constant 'Cheshire Cat' in a shaded region of Fig. 5.
- Null Diagrams – By a null diagram, a diagram in which all regions are shaded is meant. These diagrams are used to represent a domain without any members. This contravenes a basic assumption of first order logic, where one always assumes that the domain of discourse is non-empty.

Examples of diagrams demonstrating some of these problems are in Fig. 6. It is important to keep these special kinds of diagrams in mind and later refer back to them as we critically evaluate the following theories of observation.

3.2 Strong observation for Euler/Venn diagrams

In this section a very intuitive and simple definition for an observe relation between well-formed Euler/Venn diagrams and a fragment of monadic first order logic with generalized quantifiers will be stated.

The usual formulation of first order logic uses only the quantifiers $\forall$ and $\exists$. As is well known, many other forms of quantification can be be defined in terms of these, at least up to logical equivalence. However, the observation relation that will be proposed does not preserve logical equivalence. That is, there are formulas which are logically equivalent but which do not explicitly express the same information, so that one can be observed from a diagram but the other cannot. For this reason, monadic first order logic (**MFOL**) will be defined in a way that explicitly introduces some additional forms of quantification that are only implicitly introduced in the standard formulation. These new quantifiers include: $\mathsf{N}x\ \varphi(x)$ which will be read as "there is no x such that $\varphi(x)$", $\mathsf{All}\ x\ (\varphi_1(x), \varphi_2(x))$ which will be read as "all x such that $\varphi_1(x)$ then $\varphi_2(x)$", and $\mathsf{Some}\ x\ (\varphi_1(x), \neg\varphi_2(x))$ which will be read as "there is some x such that $\varphi_1(x)$ and not $\varphi_2(x)$."

For our purposes monadic first order logic will be thought of as consisting of a set of monadic predicates taken from $\mathcal{L}$ containing terms from Terms closed under negation, conjunction, disjunction, existential, and the above types of generalized quantification It will be *assumed* that each formula of **MFOL** is written so that there are no embedded quantifiers and only predicate expressions containing the quantified variable occur in the scope of a quantifier. This assumption is being made to simplify a number of the following definitions. By doing this we are not loosing any of the expressiveness of **MFOL** because any formula of **MFOL** can be translated into a logically equivalent formula adhering to this restriction.[11] Each of the basic predicate expressions $P(x)$ will be thought of as a membership relation for the set corresponding to the curve labeled P in some Euler/Venn diagram. Please note that where $\varphi(t)$ is written in the following definitions it will mean that *all* of the predicates in the formula φ contain the term t.

Definition 3.1 Euler/Venn Observational Formulas (EVOF)

(i) Basic formulas: For every predicate P in $\mathcal{L}$, and term t in Term, $P(t)$ is in **EVOF**.

(ii) Negations, Conjunctions, and Disjunctions: For every $\varphi_1(t), \ldots, \varphi_n(t)$ in **EVOF**, the following are also in **EVOF**:
$\neg\varphi_1(t)$
$(\varphi_1(t) \wedge \ldots \wedge \varphi_n(t))$
$(\varphi_1(t) \vee \ldots \vee \varphi_n(t))$

(iii) Quantifiers: For every unquantified $\varphi(x)$ in **EVOF** the following are also

[11] Quine [11] refers to such formulas as being *pure* and describes an algorithm for the purification of a formula, based upon [2] (his citation). Similar work is also presented in [3].

in **EVOF**:
$\mathsf{N}x\ \varphi(x)$
$\exists x\ \varphi(x)$

(iv) Generalized Quantifiers: For every unquantified $\varphi_1(x), \varphi_2(x)$ in **EVOF**, the following are also in **EVOF**:
$\mathsf{All}\ x\ (\varphi_1(x), \varphi_2(x))$
$\mathsf{Some}\ x\ (\varphi_1(x), \neg\varphi_2(x))$

To see some examples of formulas of **EVOF** let us translate some of the example observations from Diagram 5 into observational formulas.

- $\mathsf{Face}(\mathsf{Alice})$ can be observed.
- $\mathsf{Grin}(\mathsf{Cheshire\text{-}Cat}) \wedge \neg\mathsf{Face}(\mathsf{Cheshire\text{-}Cat})$ can be observed.
- $\mathsf{N}x[\mathsf{Grin}(x) \wedge \neg\mathsf{Face}(x)]$ can be observed.
- $\mathsf{All}\ x\ (\mathsf{Grin}(x), \mathsf{Being}(x))$ can be observed.

The basic idea behind the above definition is that each unquantified member of **EVOF** is really making a statement about features of some region of the diagram. This relation between formulas of **EVOF** and regions of a diagram is then made precise by the following function.

Definition 3.2 Region Assignment Function
Given a diagram $D \in \mathcal{V}$ containing curves labeled $P_1, \ldots, P_n$, the partial function $region_D$ from **EVOF** to the regions of D will be defined as follows:

(i) For each basic region r labeled P in D $region_D(P(t)) = r$.

(ii) If $region_D(\varphi_1(t)) = r_1, \ldots, region_D(\varphi_n(t)) = r_n$ then:
$region_D(\neg\varphi_1(t)) = \overline{r_1}$.
$region_D((\varphi_1(t) \wedge \ldots \wedge \varphi_n(t))) = r_1 \cap \ldots \cap r_n$
$region_D((\varphi_1(t) \vee \ldots \vee \varphi_n(t))) = r_1 \cup \ldots \cup r_n$

Now the main definition of this section will be stated, that of strong observation. As was alluded to earlier, our goal is to define when something can be observed, and when something can be observed to fail. These two definitions will be separate, and thus at times there will be things which can neither be observed nor observed to fail. Also in the case of inconsistent diagrams something might be able to be observed and observed to fail at the same time. The intuitions behind each of these relations of observation are given briefly below:

$\boldsymbol{D} \models\!\!\!\approx^{+} \boldsymbol{\varphi}$ or that φ *can be strongly observed* from D if it is the case that there exists explicit information in D supporting φ.

$\boldsymbol{D} \models\!\!\!\approx^{-} \boldsymbol{\varphi}$ or that φ *can be strongly observed to fail* from D if it is the case

that there exists explicit information in D denying φ.

$\boldsymbol{D \models^{?} \varphi}$ or that φ *cannot be strongly observed to hold nor to fail* from D if D contains neither explicit support for φ nor is there explicit support for it failing from D.

Definition 3.3 Euler/Venn Strong Observation
The relations $D \models^{+} \varphi(t)$ and $D \models^{-} \varphi(t)$ will be defined between diagrams of $\mathcal{V}$ and formulas of **EVOF** by induction on the complexity of $\varphi(t)$ as follows:

For unquantified formulas:

$\boldsymbol{D \models^{+} \varphi(t)}$ if the term symbol t appears in $region_D(\varphi(t))$.[12]

$\boldsymbol{D \models^{-} \varphi(t)}$ if the term symbol t appears in the complement of $region_D(\varphi(t))$.

For quantified formulas:

$\boldsymbol{D \models^{+} \mathsf{N}x\ \psi(x)}$ if the region $region_D(\psi(x))$ is shaded.

$\boldsymbol{D \models^{-} \mathsf{N}x\ \psi(x)}$ if $D \models^{+} \exists x\ \psi(x)$.

$\boldsymbol{D \models^{+} \exists x\ \psi(x)}$ if some term symbol t appears in $region_D(\psi(x))$.

$\boldsymbol{D \models^{-} \exists x\ \psi(x)}$ if $D \models^{+} \mathsf{N}x\ \psi(x)$.

$\boldsymbol{D \models^{+} \mathsf{All}\ x\ (\psi_1(x), \psi_2(x))}$ if the region $region_D(\psi_1(x))$ is a subregion of $region_D(\psi_2(x))$.

$\boldsymbol{D \models^{-} \mathsf{All}\ x\ (\psi_1(x), \psi_2(x))}$ if $D \models^{+} \mathsf{Some}\ x\ (\psi_1(x), \neg\psi_2(x))$.

$\boldsymbol{D \models^{+} \mathsf{Some}\ x\ (\psi_1(x), \neg\psi_2(x))}$ if some term symbol t appears in the region $region_D(\psi_1(x))$ and not in the region $region_D(\psi_2(x))$.

$\boldsymbol{D \models^{-} \mathsf{Some}\ x\ (\psi_1(x), \neg\psi_2(x))}$ if $D \models^{+} \mathsf{All}\ x\ (\psi_1(x), \psi_2(x))$.

$D \models^{?} \varphi$ will be written if neither $D \models^{+} \varphi$ nor $D \models^{-} \varphi$.

Now we will briefly show a few properties of strong observation: that it entails logical consequence, that it is decidable, and that any logical consequence of a diagram is a logical consequence of observations from that diagram. Before we do this a number of things need to be defined, we begin by defining relations between formulas of **MFOL** and Hammer models.

Definition 3.4 Truth of MFOL in Hammer Models
The relation $M \models \varphi(t)$ will be defined between Hammer models in $\mathcal{M}$ and formulas of **MFOL** by induction on the complexity of $\varphi(t)$ as follows:[13]

(i) $M \models P(t)$ if $denotes(t) \in I(region_D(P(t)))$.

(ii) Negation, conjunction and disjunction are defined in the natural way.

[12] The term symbol t appears in the region r if t is not part of a term sequence and t appears in r or if t is part of a term sequence and the entire sequence appears in r.

[13] Here and throughout the rest of the paper in cases where the diagram D of the function $region_D$ isn't explicitly stated, we take a canonical Venn diagram D containing the appropriate curves and no shading or constant sequences as the basis of the function $region_D$.

(iii) $M \models \exists x\ \psi(x)$ if there is some M' such that $M =_x M'$ and it is the case that $M' \models \psi(x)$.[14]

(iv) $M \models \forall x\ \psi(x)$ if for all M' such that $M =_x M'$ it is the case that $M' \models \psi(x)$.

(v) $M \models \mathsf{N}x\ \psi(x)$ if there is no M' such that $M =_x M'$ and $M' \models \psi(x)$.

(vi) $M \models \mathsf{All}\ x\ (\psi_1(x), \psi_2(x))$ if for all M' such that $M =_x M'$ and $M' \models \psi_1(x)$ we have that $M' \models \psi_2(x)$.

(vii) $M \models \mathsf{Some}\ x\ (\psi_1(x), \neg\psi_2(x))$ if there is some M' such that $M =_x M'$, and $M' \models \psi_1(x)$ and $M' \not\models \psi_2(x)$.

Given any Euler/Venn diagram D and φ a formula of **MFOL** we say that φ is a logical consequence of D (written as $D \models \varphi$) if every Hammer model which satisfies D also satisfies φ. When this is not the case we say that φ is not a logical consequence of D which we will write as $D \not\models \varphi$.

Theorem 3.5 *For all Euler/Venn diagrams D and φ a formula of **EVOF**, if φ can be strongly observed from D then φ is a logical consequence of D. Likewise if D can be strongly observed to fail from D then $\neg\varphi$ is a logical consequence of D.*

Proof Sketch: This result will be shown by induction on **EVOF**:

(i) Basic formulas -

$\boldsymbol{D \models^{+} P(t)}$ in this case t is in the region $region_D(P(t))$ and thus all M s.t. $M \models D$ we have that $denotes(t) \in I(region_D(P(t)))$ and hence $D \models P(t)$.

$\boldsymbol{D \models^{-} P(t)}$ in this case t is in the complement of the region $\overline{region_D(P(t))}$ and thus all M s.t. $M \models D$ we have that $denotes(t) \in I(\overline{region_D(P(t))})$ and hence $D \models \neg P(t)$.

(ii) Negations - Similar to last case except we take the complement of the region.

(iii) Conjunctions -

$\boldsymbol{D \models^{+} (\varphi_1(t) \wedge \ldots \wedge \varphi_n(t))}$ in this case $denotes(t) \in region_D(\varphi_1(t)) \cap \ldots \cap region_D(\varphi_n(t))$ hence $D \models (\varphi_1(t) \wedge \ldots \wedge \varphi_n(t))$

$\boldsymbol{D \models^{-} (\varphi_1(t) \wedge \ldots \wedge \varphi_n(t))}$ similar to last case.

(iv) Disjunctions - Similar to conjunctions except we take the union of the regions.

(v) $\mathsf{N}x\ \varphi(x)$

[14] In this context $M =_x M'$ if the structures M and M' are the same except for the interpretation by *denotes* for some variable x.

$\boldsymbol{D} \models^{+} \mathsf{N}\ \boldsymbol{x}\ \boldsymbol{\psi(x)}$ in this case $region_D(\varphi(x))$ is shaded. Thus any models M s.t. $M \models D$ must assign $\emptyset$ to that region. Hence $D \models \mathsf{N}\ x\ \varphi(x)$.

$\boldsymbol{D} \models^{-} \mathsf{N}\ \boldsymbol{x}\ \boldsymbol{\psi(x)}$ in this case there is some t s.t. $denotes(t) \in region_D(\varphi(x))$ hence using the same argument as before $D \models \varphi(t)$, and thus $D \models \neg\mathsf{N}\ x\ \varphi(x)$

(vi) $\exists x\ \varphi(x)$ is defined as the dual of the last case.

(vii) Generalized Quantifiers

$\boldsymbol{D} \models^{+} \mathsf{All}\ \boldsymbol{x}\ (\boldsymbol{\psi_1(x), \psi_2(x)})$ in this case $region_D(\varphi_1(x))$ is contained in $region_D(\varphi_2(x))$ and thus any model must assign a subset of $I(region_D(\varphi_2(x)))$ to $region_D(\varphi_1(x))$ thus $D \models \mathsf{All}\ x\ (\varphi_1(x), \varphi_2(x))$.

$\boldsymbol{D} \models^{-} \mathsf{All}\ \boldsymbol{x}\ (\boldsymbol{\psi_1(x), \psi_2(x)})$ in this case the intersection $region_D(\varphi_1(x))$ and the complement of $region_D(\varphi_2(x))$ contains some term t thus all models must assign a non-empty set to the interpretation of that region and thus $D \models \neg\mathsf{All}\ x\ (\varphi_1(x), \varphi_2(x))$.

(viii) $\mathsf{Some}\ x\ (\varphi_1(x), \neg\varphi_2(x))$ is defined as the dual of the last case.

□

Theorem 3.6 *There is a simple decision procedure for deciding from an arbitrary Euler/Venn diagram D and an Euler/Venn Observational Formula φ whether or not φ can be strongly observed to hold on the basis of D ($D \models^{+} \varphi$) and whether or not φ can be strongly observed to fail on the basis of D ($D \models^{-} \varphi$).*

Proof Sketch: First note that an Euler/Venn diagram can be represented by discrete objects (cf. [16] with trivial changes to add named constants) in a machine. Since on the basis of these objects the process of deciding if any region of a diagram is shaded or contains a constant or variable symbol is decidable, observation is trivially decidable. □

Lastly it will shown that a diagram is the logical consequence of all the formulas observable from it. The set of formulas observable from D will be referred to as its *observational theory*, which will be defined to be the set of all φ such that $D \models^{+} \varphi$. From this result we get as corollary that any logical consequence of a diagram is a consequence of its observational theory.

Theorem 3.7 *Every Euler/Venn diagram D is a logical consequence of its observational theory.*

Proof Sketch: Suppose this is not the case. Then there is some $M = (U, I, denotes) \in \mathcal{M}$ such that the observational theory of D is true in M but $M \not\models D$. In this case we know that there is some feature in D which disagrees with M, i.e., there is some region $r = region_D(\varphi(x))$ in D such that

one of the following is the case:

- r is shaded but $I(r) \neq \emptyset$, but then $D \models\!\!\sim^{+} \mathsf{N}x\ \varphi(x)$ and also $M \not\models \mathsf{N}x\ \varphi(x)$; contradiction.
- r is missing but $I(r) \neq \emptyset$, but then there is some missing minimal region r_1 with tag τ_1 such that $r_1 \subset r$ and another non-missing region r_2 with tag τ_2 such that that the region r_2 also has the tag $\tau_1 \cap \tau_2$. There also must be $\varphi_1(x)$ and $\varphi_2(x)$ such that $r_1 = region_D(\varphi_1(x))$ and $r_2 = region_D(\varphi_2(x))$. Then $D \models\!\!\sim^{+} \mathsf{All}\ x\ (\varphi_1(x) \wedge \varphi_2(x), \varphi_2(x))$ then $M \not\models \mathsf{All}\ x\ (\varphi_1(x) \wedge \varphi_2(x), \varphi_2(x))$; contradiction.
- r contains some constant sequence t but $denotes(t) \notin I(r)$, but then $D \models\!\!\sim^{+} \varphi(t)$ but $M \not\models \varphi(t)$; contradiction.

□

Hyperproof [1] was the first detailed case-study of a heterogeneous reasoning system; it allows students to write proofs including blocks-world diagrams and formulas of FOL. It is interesting to note that, though we have a much weaker notion of observation than that of *Hyperproof*, we get the result in Theorem 3.7 which we do not get in the case of *Hyperproof*. One of the biggest contributing factors to achieving this result is that the language of observational formulas (**EVOF**) that was chosen is rich enough to completely express all the information which can be expressed in an Euler/Venn diagram. In the case of *Hyperproof* there is certain information that can be expressed in the diagram but cannot be observed from the diagram due to the limitations of *Hyperproof*'s language.

This last proof also hints at the interesting product of our definition of strong observation that was mentioned earlier. If we are given two logically equivalent Euler/Venn diagrams one an Euler diagram and the other a Venn diagram with shading we can distinguish between them observationally though they are equivalent from the point of view of truth. To finish this section, the Corollary that any logical consequence of a diagram is the consequence of observations made from the diagram will be stated.

Corollary 3.8 *Any **MFOL** formula which is a logical consequence of an Euler/Venn diagram D is a logical consequence of the observational theory of D.*

3.3 Observe relation for monadic FOL and Euler/Venn diagrams

In the previous section a simple definition for a strong relation of observation was presented. This relation was defined recursively on the formulas of our observational language and the diagrams themselves without an appeal to

some intermediate representation. In this section a more semantic approach to defining a similar observe relation using a generalization of partial structures, which we will call information types, as our intermediate representation will be given. By using information types to define observation, will be able to more precisely capture the information content of the diagram. This definition will be based upon the framework given in [17]; we will begin with a brief overview of that work.

3.3.1 Overview

One way to model the information content of a sentence of a first order language is by means of the class of all the total structures that make that sentence true. A somewhat more fine-grained approach than this is to take the class of partial structures making the sentence true. Here a partial structure is taken to be a universe, an interpretation for each constant, and a positive and negative extension for each predicate. Modeling information content by means of partial structures gives a better model of explicit information, because sentences which are logically true are, by definition, true in all total structures but not in all the partial structures. It is also better when we are considering sentences containing inconsistent information provided we allow partial structures where the intersection of the positive and the negative information is non-null.

In fact, *Hyperproof* uses a notion of partial structures similar to that given above to evaluate the validity of observations based on the blocks-world diagram (along with a set of domain assumptions). For diagrams like these, which preserve many properties of their represented domain, this approach is particularly useful because for each diagram D there is a unique, up to isomorphism, *minimal* partial structure M_D making the diagram D true. The observation relation between formulas and diagrams of *Hyperproof* is basically taken to be the relation of truth under the Kleene evaluation scheme: we can observe φ from D if $M_D \models^+ \varphi$, where $\models^+$ is the positive part of Kleene three-valued evaluation scheme. [15]

Part of what makes this portion of the project of interest is that the traditional notion of partial structure that worked so well for capturing the information content of the diagrams of *Hyperproof* is inappropriate for capturing the information content of Euler/Venn diagrams.

The inability of the traditional notion of partial structure to act as a model of the information content of Euler/Venn diagrams stems from the inability of

[15] Note that "if and only if" is not said here because *Hyperproof* actually has some additional heuristics built into the **Observe** rule, based on certain spatial features of the diagrams. But this is the main idea of the **Observe** rule of *Hyperproof*.

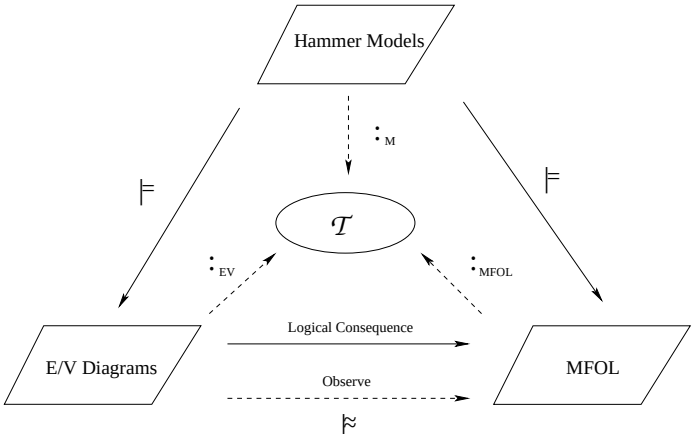


Fig. 7. The Euler/Venn distributed system

these structures to model certain kinds of information, and thereby precludes the existence of a single minimal partial structure for Euler/Venn diagrams. Some of the information that these partial structures cannot express includes: disjunctive information about individuals, negative universal information, and certain contradictory information. Another source of divergence comes from the fact that *Hyperproof* diagrams embody a *closed world assumption* (all block in the domain of discourse are depicted in the diagram) whereas no such assumption is natural with Euler/Venn diagrams. Studying these diagrams forces us to think about the modeling of partial information in new ways.

With the above intuitive notion of strong observation to guide us, a mathematical notion of an information type for Euler/Venn diagrams will be given. Using these types, relations between Euler/Venn diagrams and the types, formulas of **MFOL** and the types, and finally Hammer models and the types will be defined. These relations will then be used to define our second notion of observation. This notion of observation will be defined in a way to behave consistently with the intuitions that have been drawn out about observations from Euler/Venn diagrams (this relation is illustrated in Fig. 7 by the dashed line labeled $\models\!\approx$). It will shown that if φ is observable from D in our sense, then it is a logical consequence of D (in the sense of Shin and Hammer) but not conversely. This observe relation will also be shown to be decidable from results in the last section of the paper.

3.3.2 Euler/Venn information types

We are after a mathematical representation of the explicit information content of Euler/Venn diagrams. What will be offered, then, is a substitute for the

traditional notion of a partial structure, since these are not appropriate in this context. For lack of a better name, these objects will simply be called *information types*, or types for short.

Definition 3.9 $\mathcal{T}_{EVc}$ - Euler/Venn Information Types
Let $\mathcal{L}$ be some finite set of predicates, thought of as labels. A *type* over $L \in \mathcal{L}$ is a structure of the form $T = (U, pos, neg, denotes)$ where:

- U is a non-empty set called the domain of T. (Its members will be referred to as the objects of T.)
- pos is a function assigning to each complete tag of L some subset of U such that if $\tau_1 \neq \tau_2$ then $pos(\tau_1) \cap pos(\tau_2) = \emptyset$. We interpret $pos(\tau) = \{o_1, o_2\}$ as the information that the objects o_1, o_2 are definitely members of the set represented by the minimal region associated with the tag τ.
- neg is a function assigning to each complete tag of L either $\emptyset$ or $\bullet$. We interpret of $neg(\tau) = \bullet$ as the information that the set represented by the region associated with τ is definitely empty, whereas $neg(\tau) = \emptyset$ will be interpreted as giving us no information.
- $denotes$ is a partial function with domain the set of Terms taking values in U. We think of $denotes(t) = o$ as the information that t denotes the object o.

The collection of Euler/Venn information types will be referred to as $\mathcal{T}_{EVc}$.

A type $T = (U, pos, neg, denotes)$ is *inconsistent with respect to* a tag τ if we have conflicting positive and negative information regarding τ ($pos(\tau) \neq \emptyset$ and $neg(\tau) = \bullet$). A type T will be referred to as *maximal consistent* with respect to D if it is not inconsistent with respect to any tag τ, and we have *complete information* in T about every set and every member of the domain. By complete information it is meant that for each o in U there is some tag τ s.t. $o \in pos(\tau)$ and for no τ, $pos(\tau) = neg(\tau) = \emptyset$.

The functions pos and neg can be extended uniquely to $\overline{pos}$ and $\overline{neg}$ defined on all tags as follows: [16]

$$\overline{pos}(\tau) = \cup\{pos(\tau') : \ \tau' \text{ complete and } \tau \subseteq \tau'\}$$

[16] This extension is unique due to the fact that each tag labels a region which is comprised of a fixed number of minimal regions, each having a complete tag. Hammer proved the uniqueness of a very similar kind of extension of the assignment of sets to minimal regions in [6].

$$\overline{neg}(\tau) = \begin{cases} \bullet & \text{if } neg(\tau') = \bullet \text{ for all} \\ & \quad \text{complete } \tau' \text{ s.t. } \tau \subseteq \tau' \\ \emptyset & \text{otherwise} \end{cases}$$

3.3.3 Type relations on Euler/Venn diagrams

In this section two type membership relations ($:^{+}_{EVc}$ and $:^{-}_{EVc}$) on diagrams of $\mathcal{V}$, and types of the set $\mathcal{T}$ will be defined.[17]

Definition 3.10 Type Relations on Euler/Venn Diagrams
For any $T \in \mathcal{T}$ with $T = (U, pos, neg, denotes)$, and a diagram $D \in \mathcal{V}$, It will be said that D is of the type T, written as $D :^{+}_{EVc} T$, if the following conditions are satisfied:

(i) For each tag τ, if the region associated with τ is shaded, $\overline{neg}(\tau) = \bullet$.

(ii) For each tag τ, if the region associated with τ is missing (i.e., non-existing), $\overline{neg}(\tau) = \bullet$ and $\overline{pos}(\tau) = \emptyset$.

(iii) For each term sequence t contained in regions $region_D(\tau_1), \ldots, region_D(\tau_n)$, $denotes(t) \in \overline{pos}(\tau_1) \cup \ldots \cup \overline{pos}(\tau_n)$.

Note that though a diagram may not be inconsistent, there are many types inconsistent with respect to some τ which support it.

*3.3.4 Type relations on **MFOL** formulas*

Here type membership relations between formulas of **MFOL** and types of the set $\mathcal{T}$ will be defined ($:^{+}_{MFOL}$, $:^{-}_{MFOL}$, and $:^{?}_{MFOL}$). In the following definitions the same region assignment function $region_D$ as given in Definition 3.2 will be used. We are able to use the same function due to the assumption that in **MFOL** each formula is written so that there are no embedded quantifiers and only predicate expressions containing a quantified variable occur in the scope of a quantifier.

Definition 3.11 Type Relations on MFOL
Let φ be a formula of **MFOL**, and $T \in \mathcal{T}$ with $T = (U, pos, neg, denotes)$. $\varphi :^{+} T$ and $\varphi :^{-} T$ will be defined as follows on the structure of φ:

(i) $\boldsymbol{P(t) :^{+} T}$ if $denotes(t) \in \overline{pos}(P)$.
$\boldsymbol{P(t) :^{-} T}$ if $denotes(t) \in \overline{pos}(\overline{P})$.

(ii) $\boldsymbol{\neg\psi :^{+} T}$ if $\psi :^{-} T$.

[17] Throughout the remainder of this work the subscripts of ":" relations may be omitted when which is meant is clear from the current context.

$\boldsymbol{\neg\psi :^- T}$ if $\psi :^+ T$.

(iii) conjunctions and disjunctions are defined in the natural fashion.

(iv) $\boldsymbol{\exists x\ \psi(x) :^+ T}$ if there is some term t such that $\psi(t) :^+ T$.
$\boldsymbol{\exists x\ \psi(x) :^- T}$ if $\mathsf{N}x\ \psi(x) :^+ T$.

(v) $\boldsymbol{\mathsf{N}x\ \psi(x) :^+ T}$ if for all complete τ s.t. $region_D(\tau) \subseteq region_D(\psi(x))$ it is the case that $neg(\tau) = \bullet$.
$\boldsymbol{\mathsf{N}x\ \psi(x) :^- T}$ if $\exists x\ \psi(x) :^+ T$.

(vi) $\boldsymbol{\forall x\ \psi(x) :^+ T}$ if for all complete τ s.t. $region_D(\tau) \subseteq region_D(\neg\psi(x))$ it is the case that $neg(\tau) = \bullet$.
$\boldsymbol{\forall x\ \psi(x) :^- T}$ if $\exists x\ \neg\psi(x) :^+ T$

(vii) $\boldsymbol{\mathsf{All}\ x\ (\psi_1(x), \psi_2(x)) :^+ T}$ if for all complete τ s.t. $region_D(\tau) \subseteq region_D(\psi_1(x) \wedge \neg\psi_2(x))$ it is the case that $neg(\tau) = \bullet$.
$\boldsymbol{\mathsf{All}\ x\ (\psi_1(x), \psi_2(x)) :^- T}$ if $\mathsf{Some}\ x\ (\psi_1(x), \neg\psi_2(x)) :^+ T$.

(viii) $\boldsymbol{\mathsf{Some}\ x\ (\psi_1(x), \neg\psi_2(x)) :^+ T}$ if there is some term t such that $\psi_1(t) \wedge \neg\psi_2(t) :^+ T$.
$\boldsymbol{\mathsf{Some}\ x\ (\psi_1(x), \neg\psi_2(x)) :^- T}$ if $\mathsf{All}\ x\ (\psi_1(x), \psi_2(x)) :^+ T$

$\varphi :^? T$ will be written if neither $\varphi :^+ T$ nor $\varphi :^- T$.

Proposition 3.12 *For all $T \in \mathcal{T}$ and $\psi_1(x), \psi_2(x) \in$ **MFOL**:*

(i) $\forall x\ \psi_1(x) :^+ T$ *iff* $\mathsf{N}x\ \neg\psi_1(x) :^+ T$.

(ii) $\mathsf{All}\ x\ (\psi_1(x), \psi_2(x)) :^+ T$ *iff* $\forall x\ (\neg\psi_1(x) \vee \psi_2(x)) :^+ T$

(iii) $\mathsf{Some}\ x\ (\psi_1(x), \neg\psi_2(x)) :^+ T$ *iff* $\exists x\ (\psi_1(x) \wedge \neg\psi_2(x)) :^+ T$.

3.3.5 Second definition of observation

Next the relations $\models\!\!\approx^+$, $\models\!\!\approx^-$, and $\models\!\!\approx^?$ will be defined following the same intuitions given above for $\models\!\!\approx^+$, $\models\!\!\approx^-$, and $\models\!\!\approx^?$.

Definition 3.13 Observe Relations over MFOL ($\models\!\!\approx$)
For all $D \in \mathcal{V}$, and all $\varphi \in$ **MFOL**:

$\boldsymbol{D \models\!\!\approx^+ \varphi}$ if for all Euler/Venn types T such that $D :^+ T$ we have that $\varphi :^+ T$.

$\boldsymbol{D \models\!\!\approx^- \varphi}$ if for all Euler/Venn types T such that $D :^+ T$ we have that $\varphi :^- T$.

$\boldsymbol{D \models\!\!\approx^? \varphi}$ if it is not the case that $D \models\!\!\approx^+ \varphi$ nor is it the case that $D \models\!\!\approx^- \varphi$.

3.3.6 Properties of observation

Definition 3.14 Extension Relation on Euler/Venn Diagrams
Given diagrams D and D' in $\mathcal{V}$ it can be said that D' is an *extension* of

D, or that $D \leq D'$ if for all regions in D containing some features, their corresponding regions in D' contain the same features.

Lemma 3.15
For all $D, D' \in \mathcal{V}$ if D' is an extension of D then for all $T \in \mathcal{T}$ if $D' :^+ T$ then $D :^+ T$.

Proof Sketch: Assume that $D' :^+ T$. Since D' is an extension of D we know that all of the features of D are in D'. Suppose that it is not the case that $D :^+ T$, then T and D would have to disagree on some feature, from the definition of :, but that same feature would be in D' and then it would not be the case that $D' :^+ T$ contradiction. □

Theorem 3.16 *Any **MFOL** formula φ that can be observed to hold of an Euler/Venn diagram D can be observed to hold in any extension of D. Likewise any φ which can be observed to fail from D can be observed to fail from any extension of D.*

Proof Sketch

(i) Assume that $D \models\!\!\approx^+ \varphi$, then for all $T \in \mathcal{T}$ such that $D :^+ T$, we have that $\varphi :^+ T$. Since D' is an extension of D we know from Lemma 3.15 that for all $T' \in \mathcal{T}$ such that $D' :^+ T'$, it must be the case that $D :^+ T'$, and thus from the above assumption we have that $\varphi :^+ T'$. Hence we have that $D' \models\!\!\approx^+ \varphi$.

(ii) Assume that $D \models\!\!\approx^- \varphi$, then $D \models\!\!\approx^+ \neg\varphi$ and from the last part we have that for all extensions D' of D it is the case that $D' \models\!\!\approx^+ \neg\varphi$, and hence that $D' \models\!\!\approx^- \varphi$.

□

3.3.7 Type relations on hammer models

Let us start by recalling the definition of Hammer Models given above (Definition 2.2). As before, the collection of Hammer models will be referred to as $\mathcal{M}$. Note that since each of the basic regions of a diagram are required to be labeled by some member of our language $\mathcal{L}$ there naturally arises a tag assignment function $region_D$ from the diagram. The underlying assumption in the definition of a Hammer Euler/Venn model is that all the regions are part of some potential diagram and thus any of the above regions are taken to be in the range of some tag assignment function.

The type membership relations, $:^+_M$ and $:^-_M$, on models of $\mathcal{M}$ and types of the set $\mathcal{T}$ will be defined as follows:

Definition 3.17 Hammer Type Membership Relations ($:_M$)

Let $M = (U_M, I_M, denotes_M)$ be a Hammer model and $T = (U, pos, neg, denotes)$ be an Euler/Venn Information Type. $M :^+ T$ will be written when T is maximal consistent and for each tag τ such that $region_D(\tau) = r$ it is the case that $I_M(r) = pos(\tau)$, and for each term t, $denotes_M(t) = denotes(t)$. $M :^- T$ will be written if it is not the case that $M :^+ T$.

Proposition 3.18

(i) *There is a one-to-one and onto mapping between maximal consistent types in $\mathcal{T}$ and models in $\mathcal{M}$. These pairs of maximal consistent types and models will be referred to as matching type-model pairs.*

(ii) *For any maximal consistent type T in $\mathcal{T}$ and its matching model M we have that for all $\varphi \in$ **MFOL**, $\varphi :^+ T$ iff $M \models \varphi$.*

(iii) *For any maximal consistent type T in $\mathcal{T}$ and its matching model M we have that for all $D \in \mathcal{V}$, $D :^+ T$ iff $M \models D$.*

Proof Sketch:

(i) We will define a one-to-one and onto function f from maximal consistent types in $\mathcal{T}$ to models in $\mathcal{M}$. First recall that both our types and our models are defined using the same domain, the same collection of Terms and that for each region in the domain of a model's I_M function there is a corresponding tag τ for which the type's $\overline{pos}$ and $\overline{neg}$ functions are defined and vice-versa. Thus for a type $T = (U, pos, neg, denotes)$ we define $f(T)$ to be the model $M = (U, I_M, denotes_M)$ such that for each τ with some $region_D(\tau) = r$ it is the case that $I_M(r) = \overline{pos}(\tau)$, and for each term t, $denotes_M(t) = denotes(t)$. This function is one-to-one by definition and is onto in virtue of the relations between the domains of I_M and $\overline{pos}$, and $denotes_M$ and $denotes$.

(ii) Arbitrarily pick $T \in \mathcal{T}$ and $M \in \mathcal{M}$ that are a matching pair, and a formula $\varphi \in$ **MFOL**. ($\rightarrow$) Assume that $\varphi :^+ T$ but that $M \not\models \varphi$. Then M and φ must disagree with respect to some predicate $P(n)$ such that $region_D(P(n)) = r$ but $denotes(n) \notin I(r)$, some region $r = region_D(\varphi(x))$ such that $\mathsf{N}x\ \varphi(x)$ but $I(r) \neq \emptyset$, or some region $r = region_D(\psi_1(x) \wedge \neg\psi_2(x))$ such that $\mathsf{All}\ x\ (\psi_1(x), \psi_2(x))$ but $I(r) \neq \emptyset$. But then then there must be a τ such that $region_D(\tau) = r$ in T and from the fact that they are matching pairs the same disagreement must exist between φ and T thus $\varphi :^- T$ contradiction. ($\leftarrow$) Because of the very close relation between the M and T, and $\models$ and $:_{MFOL}$ the argument is analogous.

(iii) Arbitrarily pick $T \in \mathcal{T}$ and $M \in \mathcal{M}$ that are a matching pair, and an

Euler/Venn diagram $D \in \mathcal{V}$. ($\rightarrow$) Assume that $D :^+ T$ but that $M \not\models D$. Then M and D must disagree with respect to some region which is r which is shaded but $I(r) \neq \emptyset$, which is missing but $I(r) \neq \emptyset$, or which contains a constant n but $denotes(n) \notin I(r)$. But then then there must be a τ such that $region_D(\tau) = r$ in T and from the fact that they are matching pairs the same disagreement must exist between D and T thus $D :^- T$ contradiction. ($\leftarrow$) Because of the very close relation between the M and T, and $\models$ and $:_M$ the argument is analogous.

□

3.3.8 Relations of truth

We will use the satisfaction relation $M \models \varphi$ between Hammer Models and formulas of **MFOL** as that given above in Definition 3.4. We will also use the same definitions of truth of Euler/Venn diagrams in Hammer models and logical consequence for **MFOL** and Euler/Venn diagrams as those given in Definitions 2.3 and 3.4 respectively. Using these relations it will be shown that observation implies logical consequence, but first let us prove a couple of Lemmas used in proving this result.

Lemma 3.19 *For each $M = (U_M, I_M, denotes_M) \in \mathcal{M}$ and each diagram $D \in \mathcal{V}$, there is a unique **characterizing type** (defined in the proof) $T \in \mathcal{T}$ such that $M :^+ T$, and $M \models D$ iff $D :^+ T$.*

Proof Sketch:
To construct this characterizing type $T = (U, pos, neg, denotes)$ we take $U = U_M$, $denotes = denotes_M$, and for all complete tags τ, $pos(\tau) = I_M(region_D(\tau))$, and:

$$neg(\tau) = \begin{cases} \bullet & \text{if } I_M(region_D(\tau)) = \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

We get that $M :^+ T$ trivially. We just need to show that $M \models D$ iff $D :^+ T$. ($\rightarrow$) Assume that $M \models D$ and we don't have that $D :^+ T$. Then D and T disagree on some tag or variable. But since T was derived from M, M and D must also disagree, contradiction. ($\leftarrow$) Assume that $D :^+ T$ and $M \not\models D$. Then M and D must disagree on some region or variable. Then since T was derived from M, the same disagreement must exist between T and M, contradiction. □

Lemma 3.20 *For each $M \in \mathcal{M}$, $T \in \mathcal{T}$ and all $\varphi \in$ **MFOL**, if $M :^+ T$ and $\varphi :^+ T$ then $M \models \varphi$.*

Proof Sketch:
Arbitrarily choose $M = (U_M, I_m, denotes_M) \in \mathcal{M}$ and $T = (U, pos, neg, denotes) \in \mathcal{T}$ such that $M :^+ T$, we will proceed by induction on formulas:

Base: φ an atomic formula. Assume that $P(t) :^+ T$. Then we know that $denotes(t) \in \overline{pos}(P)$, $denotes(t) = denotes_M(t)$, and $I_M(region_D(P)) = \overline{pos}(P)$ hence $M \models P(t)$.

Induction: Assume for formulas φ and ψ.

The arguments for $\neg$, $\vee$, and $\wedge$ are omitted as they are trivial.

$\exists$ Assume that $\exists x\ \psi(x) :^+ T$. Then there is some term t such that $\psi(t) :^+ T$. Then using the hypothesis we know that $M \models \psi(t)$ and thus that $M \models \exists x\ \psi(x)$.

N Assume that $\mathsf{N}x\ \psi(x) :^+ T$. Then for all complete τ s.t. $region_D(\tau) \subseteq region_D(\psi(x))$ it is the case that $neg(\tau) = \bullet$, and then $I_M(region_D(\psi(x))) = \emptyset$. Hence $M \models \mathsf{N}x\ \psi(x)$.

The other quantified cases are all reducible to the last two cases.

□

Theorem 3.21 *For all Euler/Venn diagrams D and φ a formula of **MFOL**, if φ can be observed from D then φ is a logical consequence of D. Likewise if φ can be observed to fail from D then $\neg\varphi$ is a logical consequence of D.*

Proof Sketch

(i) Assume that $D \mathrel{|\!\approx^+} \varphi$ then for all $T \in \mathcal{T}$ such that $D :^+ T$, then $\varphi :^+ T$. We need to show that all Hammer models M such that $M \models D$ then $M \models \varphi$. From Lemma 3.19 we know that if $M \models D$ then M's characterizing type T is such that $D :^+ T$. Then using $\varphi :^+ T$ and Lemma 3.20 we have that that $M \models \varphi$.

(ii) Proof similar to the last case.

□

Theorem 3.22 *There is a simple decision procedure for deciding from an arbitrary Euler/Venn diagram D and a formula of **MFOL** φ whether or not φ can be observed to hold on the basis of D ($D \mathrel{|\!\approx^+} \varphi$) and whether or not φ can be observed to fail on the basis of D ($D \mathrel{|\!\approx^-} \varphi$.)*

Proof Sketch: We get the above result indirectly from the results in the next section. □

3.4 Bridging the Gap

Now that two relations of observation have been defined, one which is very simple and only defined in terms of diagrams and a fragment of monadic first

order logic, and the other which is more complicated and defined on the richer language of **MFOL**. Next we will relate these two definitions and show that while the second definition is more rich and complicated, we do not get a much more powerful notion of observation. In fact if we take the boolean closure of the first, and blur the distinction between the generalized quantifiers and the standard quantifiers, we get an observe relation which is equivalent to the second observe relation ($\models\!\!\approx$). Thus in the following theorem, we will show that the second relation of observation can be characterized in terms of the first.

Theorem 3.23 *For every φ, which is a formula of **MFOL** and D an Euler/Venn diagram, the positive and negative observe relations ($D\models\!\!\approx^{+}\varphi$ and $D\models\!\!\approx^{-}\varphi$) can be characterized as follows:*

(i) if φ is unquantified and of the form $\varphi(t)$ (only containing the term t) then
$\boldsymbol{D\models\!\!\approx^{+}\varphi(t)}$ if $D\models\!\!\approx^{+}\varphi(t)$.
$\boldsymbol{D\models\!\!\approx^{-}\varphi(t)}$ if $D\models\!\!\approx^{-}\varphi(t)$.

(ii) $\boldsymbol{D\models\!\!\approx^{+}\neg\varphi}$ if $D\models\!\!\approx^{-}\varphi$.
$\boldsymbol{D\models\!\!\approx^{-}\neg\varphi}$ if $D\models\!\!\approx^{+}\varphi$.

(iii) Conjunctions, and disjunctions are done in the natural way.

(iv) $\boldsymbol{D\models\!\!\approx^{+}\ \mathsf{N}x\ \psi(x)}$ if $D\models\!\!\approx^{+}\mathsf{N}x\ \psi(x)$ or if $region_D(\psi(x))$ is missing. [18]
$\boldsymbol{D\models\!\!\approx^{-}\ \mathsf{N}x\ \psi(x)}$ if $D\models\!\!\approx^{+}\exists x\ \psi(x)$.

(v) $\boldsymbol{D\models\!\!\approx^{+}\exists x\ \varphi(x)}$ if $D\models\!\!\approx^{+}\exists x\ \varphi(x)$.
$\boldsymbol{D\models\!\!\approx^{-}\exists x\ \varphi(x)}$ if $D\models\!\!\approx^{+}\mathsf{N}x\ \varphi(x)$.

(vi) $\boldsymbol{D\models\!\!\approx^{+}\forall x\ \varphi(x)}$ if $D\models\!\!\approx^{+}\mathsf{N}x\ \neg\varphi(x)$ or if $region_D(\neg\varphi(x))$ is missing.
$\boldsymbol{D\models\!\!\approx^{-}\forall x\ \varphi(x)}$ if $D\models\!\!\approx^{+}\exists x\ \neg\varphi(x)$.

(vii) $\boldsymbol{D\models\!\!\approx^{+}\ \mathsf{All}\ x\ (\psi_1(x),\psi_2(x))}$ if $D\models\!\!\approx^{+}\mathsf{N}x\ (\psi_1(x)\wedge\neg\psi_2(x))$ or if $region_D(\psi_1(x)\wedge\neg\psi_2(x))$ is missing.
$\boldsymbol{D\models\!\!\approx^{-}\ \mathsf{All}\ x\ (\psi_1(x),\psi_2(x))}$ if $D\models\!\!\approx^{+}\exists x\ (\psi_1(x)\wedge\neg\psi_2(x))$.

(viii) $\boldsymbol{D\models\!\!\approx^{+}\ \mathsf{Some}\ x\ (\psi_1(x),\neg\psi_2(x))}$ if for some term symbol t we have that $D\models\!\!\approx^{+}\psi_1(t)$ and $D\models\!\!\approx^{-}\psi_2(t)$.
$\boldsymbol{D\models\!\!\approx^{-}\ \mathsf{Some}\ x\ (\psi_1(x),\neg\psi_2(x))}$ if $D\models\!\!\approx^{+}\mathsf{All}\ x\ (\psi_1(x),\psi_2(x))$.

Before we start the main proof we will prove two lemmas:

Lemma 3.24 *For all $D\in\mathcal{V}$ the constant sequence t is contained in the the minimal regions associated with the complete tags $\tau_1\ldots\tau_k$, $region_D(\tau_1)\cup\ldots\cup region_D(\tau_k)$, of D iff for all $T=(U,pos,neg,denotes)\in\mathcal{T}$ it is the case that $denotes(t)\in pos(\tau_1)\cup\ldots\cup pos(\tau_k)$.*

[18] It is here that the distinction is being blurred between the two types of quantifiers.

Proof Sketch: ($\rightarrow$) Assume that the constant symbol t appears in the region $region_D(\tau_1) \cup \ldots \cup region_D(\tau_k)$. Thus we know that any $T = (U, pos, neg, denotes)$ in $\mathcal{T}$ s.t. $D :^+ T$ it is the case that $denotes(t) \in pos(\tau_1) \cup \ldots \cup pos(\tau_k)$ from the definition of a diagram being of a type. ($\leftarrow$) Assume that all $T = (U, pos, neg, denotes) \in \mathcal{T}$ such that $D :^+ T$, it is the case that $denotes(t) \in pos(\tau_1) \cup \ldots \cup pos(\tau_k)$ but the constant symbol t is not contained in the region $region_D(\tau_1) \cup \ldots \cup region_D(\tau_k)$ of D. Furthermore we know that the symbol t doesn't occur in D, because if it did then $denotes(t) \notin pos(\tau_1) \cup \ldots \cup pos(\tau_k)$. We can construct a type T' differing from T only in that $denotes'(t)$ is undefined. Since t is not in D and $D :^+ T$ we know that $D :^+ T'$, contradiction. □

Lemma 3.25 *For all $D \in \mathcal{V}$ the region of D associated with any complete tag τ is shaded iff for all $T = (U, pos, neg, denotes) \in \mathcal{T}$ s.t. $D :^+ T$ it is the case that $neg(\tau) = \bullet$.*

Proof Sketch: ($\rightarrow$) Assume that the region associated with the tag τ is shaded. Thus we know that any $T = (U, pos, neg, denotes)$ in $\mathcal{T}$ s.t. $D :^+ T$ it is the case that $neg(\tau) = \bullet$ from the definition of a diagram being of a type. ($\leftarrow$) Assume that all $T = (U, pos, neg, denotes) \in \mathcal{T}$ such that $D :^+ T$, it is the case that $neg(\tau) = \bullet$ and the region associated with τ of D is not shaded. We can construct a type T' differing from T only in that $neg(\tau) = \emptyset$. Since the region associated with τ is not shaded and $D :^+ T$ we know that $D :^+ T'$, contradiction. □

Now we can return to the proof of the main result of this section, Theorem 3.23.

Proof Sketch:

For the following proof, we assume that the formula φ is in conjunctive normal form (CNF). Please note that any formula can be converted into this form [11]. The proof will proceed by induction on the complexity of formulas of **MFOL** in CNF.

(i) φ is of the form $\varphi(t)$:

- $\boldsymbol{D \mid\!\approx^+ \varphi(t)}$ Thus we know that the term symbol t occurs in $region_D(\varphi(t))$, and thus from Lemma 3.24 we know that for all $T = (U, pos, neg, denotes) \in \mathcal{T}$ s.t. $D :^+ T$ it is the case that $denotes(t) \in pos(region_D(\varphi(t)))$ and thus that $\varphi(t) :^+ T$. Hence $D \mid\!\approx^+ \varphi(t)$.
- $\boldsymbol{D \mid\!\approx^- \varphi(t)}$ Argument similar to the last case.
- $\boldsymbol{D \mid\!\approx^+ \varphi(t)}$ Here we know that all $T = (U, pos, neg, denotes) \in \mathcal{T}$ s.t. $D :^+ T$ it is the case that $\varphi(t) :^+ T$ and thus that in that T, $denotes(t) \in pos(region_D(\varphi(t)))$. Using Lemma 3.24 we know that the term symbol t is in the region $region_D(\varphi(t))$ and hence that $D \mid\!\approx^+ \varphi(t)$.

$\boldsymbol{D \models\!\approx^{-} \varphi(t)}$ Argument similar to the last case.

(ii) The argument for negations is trivial.

(iii) $\boldsymbol{D \models\!\approx^{+} \psi_1}$ and $\boldsymbol{D \models\!\approx^{+} \psi_2}$ then we know that the term symbol t appears in the intersection of $region_D(\psi_1)$ and $region_D(\psi_2)$. Again using the Lemma and the induction hypothesis we conclude that $D \models\!\approx^{+} \psi_1 \wedge \psi_2$.

$\boldsymbol{D \models\!\approx^{-} \psi_1}$ and $\boldsymbol{D \models\!\approx^{-} \psi_2}$ Argument similar to the last case.

$\boldsymbol{D \models\!\approx^{+} \psi_1 \wedge \psi_2}$ Thus we know that all $T = (U, pos, neg, denotes) \in \mathcal{T}$ s.t. $D :^{+} T$ it is the case that $\psi_1 :^{+} T$ and $\psi_2 :^{+} T$. Then from the induction hypothesis we have that $D \models\!\approx^{+} \psi_1$ and $D \models\!\approx^{+} \psi_2$.

$\boldsymbol{D \models\!\approx^{-} \psi_1 \wedge \psi_2}$ Argument similar to the last case.

(iv) The argument for disjunctions is similar.

(v) $\boldsymbol{D \models\!\approx^{+} \exists x\ \psi(x)}$ Thus we know that there is some t such that the term symbol t is in the region $region_D(\psi(t))$ and hence $D \models\!\approx^{+} \psi(t)$ and thus that $D \models\!\approx^{+} \psi(t)$, from which we can conclude $D \models\!\approx^{+} \exists x\ \psi(x)$.

$\boldsymbol{D \models\!\approx^{+} \exists x\ \psi(x)}$ Thus we know that all $T = (U, pos, neg, denotes) \in \mathcal{T}$ s.t. $D :^{+} T$ there is some t s.t. $\psi(t) :^{+} T$. Thus we know that the region $region_D(\psi(t))$ has to contain the term symbol t, and hence that $D \models\!\approx^{+} \exists x\ \psi(x)$.

$\boldsymbol{D \models\!\approx^{+} \mathsf{N}\ x\ \psi(x)}$ Thus we know that the region $region_D(\psi(x))$ is shaded and then using Lemma 3.25 we know that for all T in $\mathcal{T}$ s.t. $D :^{+} T$ and all complete tags τ s.t. $region_D(\tau) \subseteq region_D(\psi(x))$ that $neg(\tau) = \bullet$. Hence $D \models\!\approx^{+} \mathsf{N}\ x\ \psi(x)$.

$\boldsymbol{D \models\!\approx^{+} \mathsf{N}\ x\ \psi(x)}$ Thus we know that all $T = (U, pos, neg, denotes) \in \mathcal{T}$ s.t. $D :^{+} T$ and all the complete tags τ such that $region_D(\tau) \subseteq region_D(\varphi(x))$ is is the case that $neg(\tau) = \bullet$. Using Lemma 3.25 we know that the region $region_D(\varphi(x))$ is shaded and hence $D \models\!\approx^{+} \mathsf{N}\ x\ \psi(x)$.

(vi) The arguments for the other quantifiers reduce to the above two cases.

□

4 Conclusions

In this paper we have presented Euler/Venn diagrams with named constants, an extension of Shin and Hammer's Venn reasoning system, and defined half of a heterogeneous FOL and Euler/Venn reasoning system. This system will serve as the basis of the implementation of an educational tool called *Openproof*, the next generation of the *Hyperproof* system.

References

[1] Barwise, J. and J. Etchemendy, "Hyperproof," CSLI Publications, Stanford, 1994.

[2] Behmann, H., *Beiträge zur algebra der logik, insbesondere zum entscheidungsproblem*, Mathematische Annalen **86** (1922), pp. 163–229.

[3] Boolos, G. S. and R. C. Jeffrey, "Computability and Logic," Cambridge University Press, 1989, third edition.

[4] Fish, A., J. Flower and J. Howse, *The semantics of augmented constraint diagrams*, Journal of Visual Languages and Computing (2005), in press.

[5] Gil, J. Y., J. Howse and S. Kent, *Formalising spider diagrams*, in: *Proceedings IEEE Symposium on Visual Languages (VL99)* (1999), pp. 130–137.

[6] Hammer, E. M., "Logic and Visual Information," CSLI and FOLLI, Stanford, 1995.

[7] Hegarty, M., B. Meyer and N. H. Narayanan, editors, "Diagrammatic Representation and Inference," Number 2317 in Lecture Notes in Artificial Intelligence, Springer-Verlag, Berlin, 2002.

[8] Howse, J., F. Molina, J. Taylor, S. Kent and J. Y. Gil, *Spider diagrams: A diagrammatic reasoning system*, Journal of Visual Languages and Computing **12** (2001), pp. 299–324.

[9] Howse, J., G. Stapleton, J. Flower and J. Taylor, *Corresponding regions in euler diagrams*, in: Hegarty et al. [7] pp. 76–90.

[10] Kent, S., *Constraint diagrams: visualizing invariants in object-oriented models*, in: *Proceedings of the 1997 ACM SIGPLAN conference on Object-oriented programming systems, languages and applications* (1997), pp. 327–341.

[11] Quine, W. v., "Methods of Logic," Routledge and Kegan Paul, 1974, third edition.

[12] Shin, S.-J., "The Logical Status of Diagrams," Cambridge University Press, Cambridge, 1995.

[13] Shin, S.-J., *Situation-theoretic account of valid reasoning with Venn diagrams*, in: G. Allwein and J. Barwise, editors, *Logical Reasoning with Diagrams*, Oxford University Press, New York, 1996 pp. 81–108.

[14] Stapleton, G., *A survey of reasoning systems based on euler diagrams*, Electronic Notes in Theoretical Computer Science **??** (2005).

[15] Stapleton, G., J. Howse and J. Taylor, *A constraint diagram reasoning system*, in: *Proceedings of the 9th International Conference on Distributed Multimedia Systems*, 2003, pp. 263–270.

[16] Swoboda, N., *Implementing Euler/Venn reasoning systems*, in: M. Anderson, B. Meyer and P. Olivier, editors, *Diagrammatic Representation and Reasoning*, Springer-Verlag, London, 2002 pp. 371–386.

[17] Swoboda, N. and G. Allwein, *Modeling heterogeneous systems*, in: Hegarty et al. [7] pp. 131–145.