Information and Computation 207 (2009) 1315–1328

Contents lists available at ScienceDirect

Information and Computation

journal homepage: www.elsevier.com/locate/ic

On periodicity of generalized two-dimensional infinite words ☆

S.A. Puzynina *

Sobolev Institute of Mathematics, pr. Koptyuga 4, Novosibirsk 630090, Russia

A R T I C L E I N F O

Article history:
Received 5 June 2008
Revised 13 February 2009
Available online 28 March 2009

Keywords:
Bidimensional word
Periodicity
Perfect coloring
Equitable partition
Nivat's conjecture

A B S T R A C T

A generalized two-dimensional word is a function on $\mathbb{Z}^2$ with a finite number of values. The main problem we are interested in is periodicity of two-dimensional words satisfying some local conditions. Let A be a matrix of order n. The function $\varphi : \mathbb{Z}^2 \to \mathbb{R}^n$ is a *generalized centered function of radius r with the matrix A* if

$$\sum_{\mathbf{y}\in\mathbb{Z}^2:0<|\mathbf{y}-\mathbf{x}|\leqslant r} \varphi(\mathbf{y}) = \varphi(\mathbf{x})A$$

for every $\mathbf{x} \in \mathbb{Z}^2$, where for $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ we have $|\mathbf{y} - \mathbf{x}| = |y_1 - x_1| + |y_2 - x_2|$. We prove that every generalized centered function of radius $r > 1$ with a finite number of values is periodic. For $r = 1$ the existence of non-periodic generalized centered functions depends on the spectrum of the matrix A. Similar results are obtained for the infinite triangular and hexagonal grids.

1. Introduction

The study of local conditions ensuring periodicity is an important problem for the combinatorics on words, both in the one-dimensional and two-dimensional case. One of the principle results of combinatorics on words, the Critical Factorization Theorem, relates local periodicities of a word to its global periodicity. It was first proved by Cesari and Vincent, and in the present form it is due to Duval [3,5]. Connections between local and global periodicity are studied well, see also [11,9] and [7] for a multidimensional extension.

Some questions about periodicity become much more difficult in dimensions higher than one. One of such questions is the connection between local complexity and periodicity. For a k-dimensional word ω its complexity function $p_\omega(n_1, \ldots, n_k)$ counts the number of distinct $n_1 \times \cdots \times n_k$ blocks in ω. In the one-dimensional case it is known that if there exist an integer n such that $p_\omega(n) \leqslant n$, then ω is periodic (Morse and Hedlund [10]). For $k = 2$ the following hypothesis is known as Nivat's conjecture [12]: if there exists a pair (n, m) such that the complexity function $p_\omega(n, m)$ of a two-dimensional word ω satisfies the condition $p_\omega(n, m) \leqslant mn$, then ω has at least a periodicity vector. Weak forms of the conjecture for $p_\omega(n, m) \leqslant mn/144$ and for $p_\omega(n, m) \leqslant mn/16$ were proved by Epifanio, Koskas, Mignosi in [6] and by Quas and Zamboni in [17], respectively. In [2] V. Berthe and L. Vuillon explore the notion of minimal complexity for two-dimensional sequences, in particular, they give an example of two-dimensional sequence of complexity $p_\omega(n, m) = mn + m$, for every (m, n), which is uniformly recurrent and which has no rational periodic direction.

We consider some special types of two-dimensional words as functions on vertices of the graphs of the infinite rectangular, triangular and hexagonal grids. Let $G = (V, E)$ be a graph. The distance between two vertices $\mathbf{x}$ and $\mathbf{y}$, denoted by $d(\mathbf{x}, \mathbf{y})$,

☆ This work was supported in part by Russian Foundation of Basic Research (grants 07-01-00248 and 09-01-00244).
* Current address: Department of Mathematics, University of Turku, FI-20014 Turku, Finland.
Email address: puzynina@math.nsc.ru

0890-5401/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.ic.2009.03.005

is the usual graph metric, i.e., the number of edges in the shortest path connecting these vertices. A ball $B_r(\mathbf{x})$ of radius r centered at the vertex $\mathbf{x}$ is defined in the following way: $B_r(\mathbf{x}) = \{\mathbf{y} \in V \mid d(\mathbf{x},\mathbf{y}) \leqslant r\}$.

Let $A = (a_{ij})_{i,j=1}^n$ be an integer matrix. A function $\varphi : V \to \mathbb{R}^n$ is called *generalized centered function of radius r* with the matrix A if

$$\sum_{\mathbf{y}\in B_r(\mathbf{x}),\, \mathbf{y}\neq\mathbf{x}} \varphi(\mathbf{y}) = \varphi(\mathbf{x})A$$

for every $\mathbf{x} \in V$.

Notice that if $n = 1$, then A is an eigenvalue of the graph G and generalized centered function is an eigenvector corresponding to eigenvalue A [4].

In fact, the notion of a generalized centered function is a generalization of the notions of a perfect coloring and of an ordinary centered function. A coloring of the graph G into n colors is a function $f : V \to \{1,..,n\}$. Let $A = (a_{ij})_{i,j=1}^n$ be an integer nonnegative matrix, r an integer, $r \geq 1$. A coloring of vertices of a graph G with n colors is called *perfect* of radius r with the matrix A, if for every vertex of a color i the number of vertices of a color j at distance at most r from the vertex of a color i does not depend on the vertex and is equal to a_{ij}. A perfect coloring with n colors with the matrix A can be considered as a generalized centered function with the value area $\{e_1,\ldots,e_n\}$, where e_i is a unit vector with 1 in its i-th coordinate. The notion of a perfect coloring naturally arises in different fields of mathematics, such as algebraic combinatorics, graph theory and coding theory. Properties of perfect colorings had been studied under different names, e.g. equitable partitions (see, for example, [8]).

A function $f : V(G) \to \mathbb{R}$ is called *centered of radius r* if the sum of its values in every ball of radius r is equal to 0. An ordinary centered function can be considered as a generalized centered function for $n = 1, A = -1$. The notion of a centered function was introduced as a generalization of the notion of a perfect code in the hypercube $\mathbb{H}^n$ [1]. The notion of a perfect coloring also generalizes the notion of a perfect code and several other well-known codes, such as the Preparata code, a completely regular code, a uniformly packed code. Namely, these codes can be interpreted as perfect colorings into two or more colors. This means that generalized centered functions can be used as an instrument for studying perfect colorings and different codes.

In this paper, the periodicity of generalized centered functions is studied. In the one-dimensional case generalized centered functions are periodic, the question is non-trivial in dimensions higher than 1. We prove that every generalized centered function of radius $r > 1$ with a finite number of values on the infinite rectangular grid is periodic. For $r = 1$, generalized centered function with the matrix A such that $\det A \neq 0$ is also periodic. If $r = 1$ and $\det A = 0$, then there exist non-periodic and periodic generalized centered functions. Similar results are obtained for the infinite triangular and hexagonal grids. These results are obtained using the method of R-prolongable words, which had been earlier used for obtaining some results about periodicity of perfect colorings and ordinary centered functions [16,13,14]. The results of this paper generalize some previous results, present them in new general notation and explain them in a convenient linear-algebraic form. The property of periodicity in different meanings have been studied for a long time. There exist many methods of proving periodicity and many theorems about periodicity of different types of words. In this paper we develop the method of R-prolongable words for proving periodicity of words with local conditions.

2. The infinite rectangular grid

The graph of the infinite rectangular grid is 4-regular, its vertices are all possible ordered pairs of integers. Two vertices $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are adjacent if $|x_1 - y_1| + |x_2 - y_2| = 1$.

Let φ be a function from $\mathbb{Z}^2$ to $\mathbb{R}^n$, i.e., $\varphi(\mathbf{x})$ is a vector of length n. Denote the i-th coordinate of $\varphi(\mathbf{x})$ by $\varphi_i(\mathbf{x})$: $\varphi(\mathbf{x}) = (\varphi_1(\mathbf{x}),\ldots,\varphi_n(\mathbf{x}))$. We study generalized centered functions on $\mathbb{Z}^2$ with a finite number of values. These functions can be considered as two-dimensional words over the finite alphabet Σ of the values of such functions, $\Sigma \subset \mathbb{R}^n$.

Define the operation of addition in the usual way: $\mathbf{x} + \mathbf{z} = (x_1 + z_1, x_2 + z_2)$. A two-dimensional word ω is $\mathbf{v}$*-periodic* (or $\mathbf{v}$ is a vector of periodicity of the word ω) if $\omega(\mathbf{x} + \mathbf{v}) = \omega(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{Z}^2$. A two-dimensional word that is $\mathbf{v}$- and $\mathbf{u}$-periodic for some noncollinear $\mathbf{v}$ and $\mathbf{u}$ is called *periodic*. It is easy to show that we can take $\mathbf{v} = (p, p)$, $\mathbf{u} = (q, -q)$ for some integers p and q.

We say that a two-dimensional word ω is *R-prolongable* if for any $\mathbf{x},\mathbf{y} \in \mathbb{Z}^2$ an equality $\omega|_{B_R(\mathbf{x})} = \omega|_{B_R(\mathbf{y})}$ implies $\omega|_{B_{R+1}(\mathbf{x})} = \omega|_{B_{R+1}(\mathbf{y})}$. The notation $\omega|_{B_R(\mathbf{x})} = \omega|_{B_R(\mathbf{y})}$ means that $\omega(\mathbf{x} + \mathbf{z}) = \omega(\mathbf{y} + \mathbf{z})$ for $|\mathbf{z}| \leqslant R$.

The following lemma gives the idea of the method of R-prolongable words:

Lemma 1. *Let ω be a two-dimensional word on a finite alphabet. If ω is R-prolongable for some $R \geq 0$, then ω is periodic.*

The proof of this lemma is simple and can be found in [16]. This lemma means that we can prove R-prolongability of two-dimensional words instead of periodicity.

Let $\varphi : V \to \mathbb{R}^n$ be a generalized centered function with a finite number of values. Denote by Σ the alphabet of values of the function φ, $\Sigma \subset \mathbb{R}^n$, $|\Sigma| < \infty$. We will use the following notation:

$$\Sigma_1 = \{\boldsymbol{v} \mid \boldsymbol{v} = \boldsymbol{v}^1 - \boldsymbol{v}^2,\quad \boldsymbol{v}^i \in \Sigma,\quad i = 1, 2\},$$

$$\Sigma' = \{\boldsymbol{v} \mid \boldsymbol{v} = \boldsymbol{v}^1 + \boldsymbol{v}^2 - \boldsymbol{v}^3, \quad \boldsymbol{v}^i \in \Sigma, \quad i = 1, 2, 3\},$$

$$\Sigma'' = \{\boldsymbol{v} \mid \boldsymbol{v} = \boldsymbol{v}^1 + \boldsymbol{v}^2 - \boldsymbol{v}^3, \quad \boldsymbol{v}^i \in \Sigma', \quad i = 1, 2, 3\}.$$

Notice that the sum of generalized centered functions is a generalized centered function with the same matrix.

In the further text we will write "an (A, r)-function" instead of "a generalized centered function of radius r with the matrix A with a finite number of values" for short.

In this section, we study the periodicity of (A, r)-functions on the infinite rectangular grid depending on the radius r and the matrix A.

Remark. Note that in the one-dimensional case every (A, r)-function is periodic. If we consider a subword $\varphi(i - r) \ldots \varphi(i + r - 1)$ of length $2r$, then the values $\varphi(i - r - 1)$ and $\varphi(i + r)$ are uniquely determined by the definition of (A, r)-function. The alphabet is finite, so the periodicity follows. Thus the question is trivial in the one-dimensional case.

Proposition 1. *Every (A, r)-function on the infinite rectangular grid is periodic for $r > 1$.*

This fact is a generalization of results obtained in [16,15].

Proof. We need some notation to prove the proposition.

A sphere $S_\rho(\mathbf{x})$ of radius ρ with the center at the vertex $\mathbf{x}$ is defined in the following way: $S_\rho(\mathbf{x}) = \{\mathbf{y} \in V \mid d(\mathbf{x}, \mathbf{y}) = \rho\}$. Every sphere consists of five sets of vertices: $S_\rho(\mathbf{x}) = \bigcup_{i=1}^{5} S_\rho^i(\mathbf{x})$, where

$$S_\rho^1(\mathbf{x}) = \{(x_1, x_2) + (j, j - \rho) | j = 1, 2, \ldots, \rho\},$$

$$S_\rho^2(\mathbf{x}) = \{(x_1, x_2) + (-j, j - \rho) | j = 1, 2, \ldots, \rho\},$$

$$S_\rho^3(\mathbf{x}) = \{(x_1, x_2) + (j, \rho - j) | j = 1, 2, \ldots, \rho\},$$

$$S_\rho^4(\mathbf{x}) = \{(x_1, x_2) + (-j, \rho - j) | j = 1, \ldots, \rho\},$$

$$S_\rho^5(\mathbf{x}) = \{(x_1, x_2) + (0, -\rho), (x_1, x_2) + (0, \rho), (x_1, x_2) + (\rho, 0), (x_1, x_2) + (-\rho, 0)\}.$$

In Fig. 1, we can see a ball $B_5(\mathbf{x})$ of radius 5, its boundary is marked by bold. Vertices from each set $S_6^i(\mathbf{x})$ of the sphere $S_6(\mathbf{x})$ are marked by i. A function on vertices of a flat graph can be considered as a function on faces of dual graph. Notice that the graph of the infinite rectangular grid is self-dual, so the pictures illustrate functions on faces instead of vertices.

Denote $m = \min\{|v_i| \mid \boldsymbol{v} = (v_1, .., v_n) \in \Sigma_1, v_i \neq 0\}$, $M = \max\{|v_i| \mid \boldsymbol{v} = (v_1, .., v_n) \in \Sigma_1\}$.

Due to Lemma 1 it is sufficient to prove that φ is R-prolongable for some $R > r$. We will prove it for $R > (2r + 1)2rM/m + 2r$.

Consider two arbitrary balls $B_R(\mathbf{x})$ and $B_R(\mathbf{y})$ such that $\varphi|_{B_R(\mathbf{x})} = \varphi|_{B_R(\mathbf{y})}$. It suffices to prove that $\varphi|_{S_{R+1}(\mathbf{x})} = \varphi|_{S_{R+1}(\mathbf{y})}$. Consider the function

$$\psi(\mathbf{t}) = \varphi(\mathbf{x} + \mathbf{t}) - \varphi(\mathbf{y} + \mathbf{t}).$$

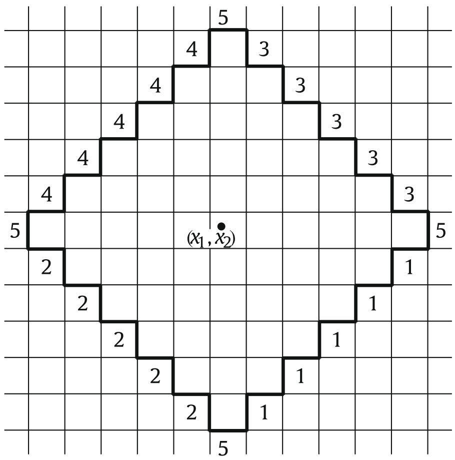

Fig. 1. A ball $B_5(\mathbf{x})$ of radius 5 and the sets $S_6^i(\mathbf{x})$, $i = 1, \ldots, 5$.

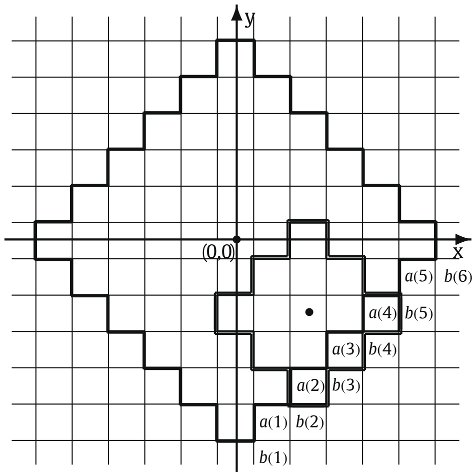

Fig. 2. The illustration for the proof of Proposition 1 for $R = 5$, $r = 2$.

We have $\psi|_{B_R(\mathbf{0})} = \boldsymbol{0}$ by definition of the function ψ. To prove that φ is R-prolongable it suffices to prove that $\psi|_{S_{R+1}(\mathbf{0})} = \boldsymbol{0}$. Notice that the function ψ has a finite number of values, the alphabet of its values is Σ_1.

First we will prove that $\psi(\mathbf{z}) = \boldsymbol{0}$ for $\mathbf{z} \in S^1_{R+1}$.

Denote $a(i) = \psi(i, i-1-R)$, $i = 1, \ldots, R$. Fig. 2 illustrates our reasoning for the case $R = 5$, $r = 2$. By definition of generalized centered function we have

$$\psi(j, j-1+r-R)A = \sum_{\substack{(x,y) \in B_r(j, j-1+r-R), \\ (x,y) \neq (j, j-1+r-R)}} \psi(x,y).$$

It holds $\psi|_{B_R(\mathbf{0})} = \boldsymbol{0}$, so $\psi(j, j-1+r-R) = \boldsymbol{0}$, thus we get $\sum_{i=j}^{j+r} a(i) = \boldsymbol{0}$, $j = 1, \ldots, R-r$. Therefore $a(j+r+1) = a(j)$ for every $1 \leqslant j \leqslant R-r-1$, i.e., the sequence $a(i)$ is periodic with period $(r+1)$.

Suppose that there exists $\mathbf{z} \in S^1_{R+1}$ such that $\psi(\mathbf{z}) \neq \boldsymbol{0}$. This means that there exists $i : 1 \leqslant i \leqslant R$ such that $a(i) \neq \boldsymbol{0}$. Denote $a(i) = \boldsymbol{d}$. Therefore $a(i + k(r+1)) = \boldsymbol{d}$,

$$\sum_{j=i+1+k(r+1)}^{i+r+k(r+1)} a(j) = -\boldsymbol{d} \tag{1}$$

for every k with $1 \leqslant i + k(r+1) \leqslant R - r$.

Now we consider elements of the sphere $S_{R+2}(\mathbf{0})$, more precisely, the set $S^1_{R+2}(\mathbf{0})$. Denote the values of the function ψ in the vertices of this set in the following way: $b(i) = \psi(i, i-2-R)$, $i = 1, \ldots, R+1$ (see Fig. 2). Consider the balls $B_r(\mathbf{v^k})$, $\mathbf{v^k} = (i+1+k(r+1),\ i-1-R+r+k(r+1))$, where $k \in \mathbb{Z}$, $1 \leqslant i+1+k(r+1) \leqslant R-r+2$. In Fig. 3, we can see a part of the ball $B_R(\mathbf{0})$ of radius $R = 17$ and five balls $B_r(\mathbf{v^k})$ of radius $r = 2$. Boundaries of balls are marked by bold line.

Now we are going to apply the definition of a generalized centered function to the vertex $\mathbf{v^k}$. The ball $B_r(\mathbf{v^k})$ consists of vertices which have values zero in the ball $B_R(\mathbf{0})$, vertices from $B_r(\mathbf{v^k}) \bigcap S_{R+1}(\mathbf{0})$, which are marked by black circles in Fig. 3 (vertices which have the values $a(i+1+k(r+1)), \ldots, a(i+r+k(r+1))$), and vertices from $A_k = B_r(\mathbf{v^k}) \bigcap S_{R+2}(\mathbf{0})$, which are marked by white circles in Fig. 3 (vertices which have the values $b(i+1+k(r+1)), \ldots, b(i+(k+1)(r+1))$). By definition of generalized centered function we have

$$\psi(\mathbf{v^k})A = \sum_{\mathbf{x} \in B_r(\mathbf{v^k}) \setminus \mathbf{v^k}} \psi(\mathbf{x}).$$

Combining it with (1), we get

$$\sum_{j=i+1+k(r+1)}^{i+r+1+k(r+1)} b(j) = \boldsymbol{d}. \tag{2}$$

The set $S^1_{R+2}(\mathbf{0})$ can be represented as a union of disjoint sets A_k and the set D of vertices in $S^1_{R+2}(\mathbf{0})$ that do not belong to one of the sets A_k (boundary effects):

$$S^1_{R+2}(\mathbf{0}) = \bigcup_k A_k \cup D, \quad 1 \leqslant i + k(r+1) \leqslant R - r + 2.$$

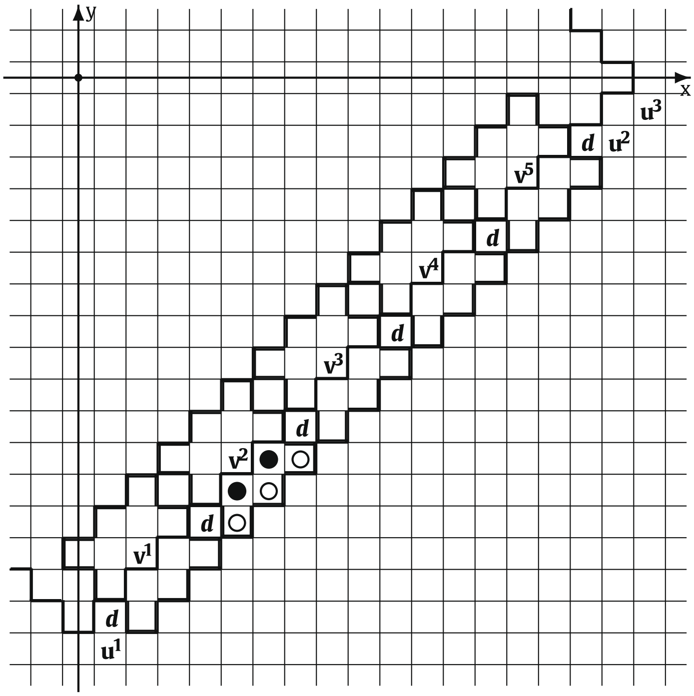

Fig. 3. The illustration for the proof of Proposition 1: part of the ball $B_R(\mathbf{0})$ and five balls $B_r(\mathbf{v^i})$, $R = 17$, $r = 2$. Black circles mark $B_r(\mathbf{v^2}) \bigcap S_{R+1}(\mathbf{0})$, white circles mark $B_r(\mathbf{v^2}) \bigcap S_{R+2}(\mathbf{0})$.

In Fig. 3, the set D consists of three vertices $\mathbf{u^1}$, $\mathbf{u^2}$, $\mathbf{u^3}$. The number of elements in the set D is not more than $2r$: $|D| \leqslant 2r$, denote $\sum_{\mathbf{x}\in D} \psi(\mathbf{x}) = \boldsymbol{c}$.

Let us calculate the sum of the values in the set $S^1_{R+2}(\mathbf{0})$:

$$\sum_{\mathbf{x}\in S^1_{R+2}(\mathbf{0})} \psi(\mathbf{x}) = \sum_{j=1}^{R} b(j) = \sum_k \sum_{\mathbf{x}\in A_k} \psi(\mathbf{x}) + \sum_{\mathbf{x}\in D} \psi(\mathbf{x}) = k\boldsymbol{d} + \boldsymbol{c}.$$

Consider a nonzero coordinate of $\boldsymbol{d} = (d_1, \ldots, d_n)$, denote its number by l: $d_l \neq 0$. Without loss of generality we can assume that $d_l \geq m$, $c_l \geq -|D| \max\{|v_i| | \boldsymbol{v} = (v_1, .., v_n) \in \Sigma_1\} \geq -2rM$. So, if we take $k > 2rM/m$ (therefore, $R > (2r+1)2rM/m + 2r$), then the vector $\sum_{j=1}^{R} b(j)$ is greater than 0 in its l-th coordinate.

We have that $\sum_{i=j}^{j+r} a(i) = \boldsymbol{0}$, so there exists k such that $a(k) = \boldsymbol{f} = (f_1, \ldots, f_n)$ and $f_l \leqslant 0$. Arguing as above we get that the l-th coordinate of the vector $\sum_{j=1}^{R} b(j)$ is less than 0. A contradiction.

Thus $a(i) = \boldsymbol{0}$ for $i = 1, \ldots, R$.

So, we proved that $\psi|_{S^1_{R+1}} = \boldsymbol{0}$. The proof is similar for the sets $S^2_{R+1}, S^3_{R+1}, S^4_{R+1}$. Now, $\psi(0, -R-1) = \boldsymbol{0}$, because ψ is equal to $\boldsymbol{0}$ in all other vertices of the ball $B_r(0, r-R-1)$. Similarly for the other elements of the set S^5_{R+1}. Thus, we have $\varphi|_{S_{R+1}(\mathbf{x})} = \varphi|_{S_{R+1}(\mathbf{y})}$, therefore, φ is R-prolongable for $R \geq (2r+1)2rM/m + 2r$. Now, by Lemma 1, φ is periodic. □

Now we proceed to the case $r = 1$.

Example 1. Here we give an example of generalized centered function of radius 1 for every degenerate matrix A. Let $\boldsymbol{v}$ be its left eigenvector corresponding to the eigenvalue $\lambda = 0$: $\boldsymbol{v}A = \boldsymbol{0}$. In the further text we omit the word "left" but always mean left eigenvectors.

The function $\theta^{\boldsymbol{v}} : \mathbb{Z}^2 \to \mathbb{R}^n$, given by the formula

$$\theta^{\boldsymbol{v}}(x_1, x_2) = \begin{cases} \boldsymbol{0}, & \text{if } x_1 \neq x_2, \\ \boldsymbol{v}, & \text{if } x_1 = x_2 \text{ is even}, \\ -\boldsymbol{v}, & \text{if } x_1 = x_2 \text{ is odd} \end{cases}$$

is a non-periodic generalized centered function with the matrix A of radius 1. Indeed, this function satisfies the definition, because for all balls of radius 1 both left and right side of the equality from the definition are equal to $\boldsymbol{0}$.

Denote by $\chi_{\mathbf{y}}^{\boldsymbol{v}}$ a function which is obtained from $\theta^{\boldsymbol{v}}$ by translation by a vector $\mathbf{y}$: $\chi_{\mathbf{y}}^{\boldsymbol{v}}(\mathbf{x}) = \theta^{\boldsymbol{v}}(\mathbf{x}+\mathbf{y})$. Denote by $\chi_{\mathbf{y}}^{*\boldsymbol{v}}$ a function, which is obtained from $\chi_{\mathbf{y}}^{\boldsymbol{v}}$ by rotation by $\pi/2$: $\chi_{\mathbf{y}}^{*\boldsymbol{v}}(x_1,x_2) = \chi_{\mathbf{y}}^{\boldsymbol{v}}(-x_2,x_1)$. Functions $\chi_{\mathbf{y}}^{\boldsymbol{v}}$ and $\chi_{\mathbf{y}}^{*\boldsymbol{v}}$ are called *rectangular alternating functions*.

Proposition 2. *Let φ be an $(A,1)$-function. Then there exists R_0, such that for each $R \geq R_0$ the condition $\varphi|_{B_R(\mathbf{x})} = \varphi|_{B_R(\mathbf{y})}$ implies either*

$$\varphi|_{S^1_{R+1}(\mathbf{x})} = \varphi|_{S^1_{R+1}(\mathbf{y})}$$

or

$$\varphi|_{S^1_{R+1}(\mathbf{x})} = (\varphi + \chi^{\boldsymbol{v}}_{(y_1,y_2-R-1)})|_{S^1_{R+1}(\mathbf{y})},$$

where $\boldsymbol{v}$ is an eigenvector of A corresponding to the eigenvalue $0 : \boldsymbol{v}A = \boldsymbol{0}$.

Proof. In the same way as for the proof of periodicity for $r \geq 2$ consider a function $\psi(\mathbf{t}) = \varphi(\mathbf{x}+\mathbf{t}) - \varphi(\mathbf{y}+\mathbf{t})$, it holds $\psi|_{B_R(\mathbf{0})} = \boldsymbol{0}$ by definition of the function ψ.

Consider the values of ψ on the set $S^1_{R+1}(\mathbf{0})$, $a(i)$ is defined as above (see Fig. 2). Denote $a(1) = \boldsymbol{u}$, consider a ball $B_1(1, 1-R)$. The value of ψ in its center is $\boldsymbol{0}$, the sum of values of ψ in this ball is equal to $\boldsymbol{0}$. So $a(2) = -\boldsymbol{u}$. Considering sums of values of ψ in the balls $B_1(j, j-R)$, we get $a(j+1) = -a(j) = (-1)^j\boldsymbol{u}$.

Now we consider the values of ψ on the set $S^1_{R+2}(\mathbf{0})$. Denote $b(i) = \psi(i, i-2-R), i = 1, \ldots, R+1$ (see Fig. 2). Denote $b(1) = \boldsymbol{w}$, consider a ball $B_1(1,-R)$. The value of ψ in its center is $\boldsymbol{u}$, the sum of values of ψ in this ball is equal to $\boldsymbol{u}A$: $b(1)+b(2) = a(1)A$. So $b(2) = \boldsymbol{u}A - \boldsymbol{w}$. Considering sums of values of ψ in the balls $B_1(j, j-1-R)$ in the same way as for $r > 1$, we get $b(j)+b(j+1) = a(j)A$ for $1 \leqslant j \leqslant R$, whence $b(j+1) = (-1)^{j+1}(j\boldsymbol{u}A - \boldsymbol{w})$. Notice that all these values $b(j)$ are different for odd j in the case $\boldsymbol{u}A = \boldsymbol{0}$. The alphabet Σ_1 of values of ψ is finite, so if $R \geq R_0 = 2|\Sigma_1|+1$, then $\boldsymbol{u}A = \boldsymbol{0}$. If $\boldsymbol{u} = \boldsymbol{0}$, then $\psi|_{S^1_{R+1}(\mathbf{0})} = \boldsymbol{0}$, i.e., $\varphi|_{S^1_{R+1}(\mathbf{x})} = \varphi|_{S^1_{R+1}(\mathbf{y})}$. If $\boldsymbol{u}A = \boldsymbol{0}$ and $\boldsymbol{u} \neq \boldsymbol{0}$, then $\boldsymbol{u}$ is an eigenvector of A, corresponding to the eigenvalue $\lambda = 0$. In this case, we have that $\psi|_{S^1_{R+1}(\mathbf{0})} = \chi^{\boldsymbol{u}}_{(0,-R-1)}|_{S^1_{R+1}(\mathbf{0})}$, i.e., $\varphi|_{S^1_{R+1}(\mathbf{x})} = (\varphi + \chi^{\boldsymbol{u}}_{(y_1,y_2-R-1)})|_{S^1_{R+1}(\mathbf{y})}$. The proposition is proved. □

Notice that the values of $\chi^{\boldsymbol{v}}_{(y_1,y_2-R-1)}$ are included in Σ_1.

It is easy to obtain similar assertions for the values of φ on the sets $S^i_{R+1}(\mathbf{x})$ and $S^i_{R+1}(\mathbf{y})$, $i = 2,3,4$.

Proposition 3. *Let φ be an $(A,1)$-function on the infinite rectangular grid, $R \geq 1$. If $\varphi|_{B_R(\mathbf{x})} = \varphi|_{B_R(\mathbf{y})}$ and $\varphi|_{S^i_{R+1}(\mathbf{x})} = \varphi|_{S^i_{R+1}(\mathbf{y})}$ for $i = 1,2,3,4$, then $\varphi|_{S^5_{R+1}(\mathbf{x})} = \varphi|_{S^5_{R+1}(\mathbf{y})}$.*

Proof. To prove that $\varphi(x_1-R-1, x_2) = \varphi(y_1-R-1, y_2)$ it is sufficient to compare the values of the function φ in the balls $B_1(x_1-R, x_2)$ and $B_1(y_1-R, y_2)$. Similarly for other vertices of the sets $S^5_{R+1}(\mathbf{x})$ and $S^5_{R+1}(\mathbf{y})$. □

Proposition 4. *Let φ be an $(A,1)$-function on the infinite rectangular grid. If $\det A \neq 0$, then φ is periodic. If $\det A = 0$, then either φ is periodic or it can be obtained from a periodic $(A,1)$-function by adding rectangular alternating functions.*

Proof. If $\det A \neq 0$, then $\boldsymbol{v}A = \boldsymbol{0}$ implies $\boldsymbol{v} = \boldsymbol{0}$. In this case, Propositions 2 and 3 imply that φ is R-prolongable. By Lemma 1 it is periodic.

Let φ be a non-periodic function, $\det A = 0$. Consider the set of balls $\{B_R(x,-x) | x \in \mathbb{Z}\}$, where $R > R_0+1, R_0 = 2|\Sigma_1|+1$, as defined in the proof of Proposition 2. Vertices of these balls lie in a region $\{(z_1,z_2) : |z_1+z_2| \leqslant R\}$ (see Fig. 4). This set of balls is infinite, the alphabet is finite, so we have a finite number of subwords on B_R, this number is less than or equal to $|\Sigma|^{|B_R|} = |\Sigma|^{(2R^2+2R+1)}$, where Σ is the alphabet of values of φ, $|B_R|$ is the number of vertices in a ball of radius R. This means that we have two balls $B_R(x,-x)$ and $B_R(y,-y)$, such that

$$\varphi|_{B_R(x,-x)} = \varphi|_{B_R(y,-y)}$$

and $|x-y| \leqslant |\Sigma|^{(2R^2+2R+1)}$. To be definite, assume that $x \leqslant y$. In Fig. 4, boundaries of these balls are marked by bold line.

Consider the sets $S^1_{R+1}(x,-x)$ and $S^1_{R+1}(y,-y)$. In Fig. 4, they are marked by white circles. By Proposition 2 we have either

$$\varphi|_{S^1_{R+1}(x,-x)} = \varphi|_{S^1_{R+1}(y,-y)}$$

or

$$\varphi|_{S^1_{R+1}(x,-x)} = (\varphi + \chi^{\boldsymbol{v}}_{(y,-y-R-1)})|_{S^1_{R+1}(y,-y)},$$

where $\boldsymbol{v}$ is such that $\boldsymbol{v}A = \boldsymbol{0}$. In the first case, we define $\varphi' = \varphi$. In the second case, we subtract the function $\chi^{\boldsymbol{v}}_{(y,-y-R-1)}$ from the function φ: $\varphi' = \varphi - \chi^{\boldsymbol{v}}_{(y,-y-R-1)}$. Notice that if the alphabet of the values of φ is Σ, then the alphabet of the values of φ' is Σ'. Therefore,

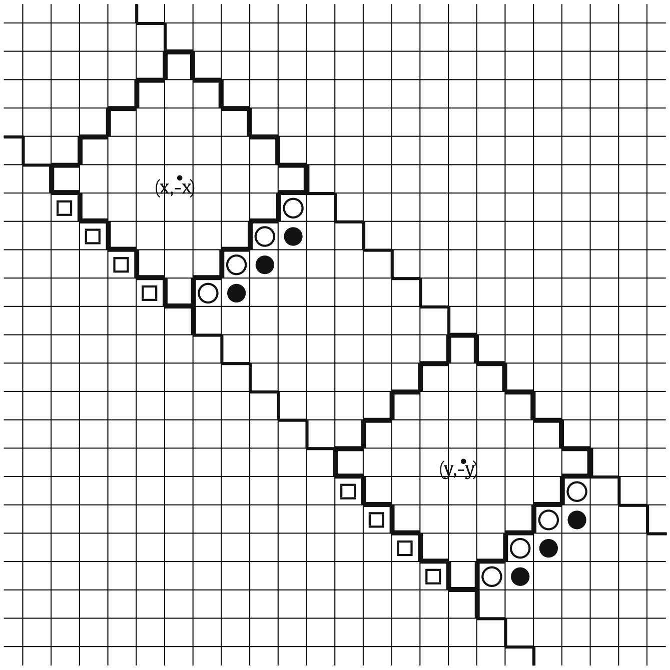

Fig. 4. The illustration for the proof of Proposition 4: the region $\{(z_1, z_2) : |z_1 + z_2| \leqslant R\}$ and two balls $B_R(x, -x)$ and $B_R(y, -y)$. White circles mark $S^1_{R+1}(x, -x)$ and $S^1_{R+1}(y, -y)$, black circles mark $S^1_R(x + 1, -x - 1)$ and $S^1_R(y + 1, -y - 1)$, squares mark the sets $S^2_{R+1}(x, -x)$ and $S^2_{R+1}(y, -y)$.

$$\varphi'|_{S^1_{R+1}(x,-x)} = \varphi'|_{S^1_{R+1}(y,-y)}.$$

We have that $\varphi'|_{B_{R-1}(x+1,-x-1)} = \varphi'|_{B_{R-1}(y+1,-y-1)}$. Now consider the sets $S^1_R(x + 1, -x - 1)$ and $S^1_R(y + 1, -y - 1)$. In Fig. 4, they are marked by black circles. By Proposition 2 we have either

$$\varphi'|_{S^1_R(x+1,-x-1)} = \varphi'|_{S^1_R(y+1,-y-1)}$$

or

$$\varphi'|_{S^1_R(x+1,-x-1)} = (\varphi' + \chi^{\boldsymbol{u}}_{(y+1,-y-1-R)})|_{S^1_R(y+1,-y-1)},$$

where $\boldsymbol{u}$ is such that $\boldsymbol{u}A = \boldsymbol{0}$. In the first case, we define $\varphi'' = \varphi'$. In the second case, we subtract the function $\chi^{\boldsymbol{u}}_{(y+1,-y-1-R)}$ from the function φ': $\varphi'' = \varphi' - \chi^{\boldsymbol{u}}_{(y+1,-y-1-R)}$. Therefore, $\varphi''|_{S^1_R(x+1,-x-1)} = \varphi''|_{S^1_R(y+1,-y-1)}$. Notice that the alphabet of the values of φ'' is also Σ'.

By Proposition 3 we obtain that

$$\varphi''|_{B_R(x+1,-x-1)} = \varphi''|_{B_R(y+1,-y-1)}.$$

Arguing as above we proceed unit by unit adding rectangular alternating functions if necessary and then obtain a function $\tilde{\varphi}$, which satisfies $\tilde{\varphi}(\mathbf{z}) = \tilde{\varphi}(\mathbf{z} + \mathbf{y} - \mathbf{x})$ for $\mathbf{z}$ such that $|z_1 + z_2| \leqslant R$. Note that we should also use an assertion that is analogous to Proposition 2 for the set S^3_{R+1}. The alphabet of the values of the function $\tilde{\varphi}$ is Σ' and this function is $(\mathbf{y} - \mathbf{x})$-periodic in the stripe $|z_1 + z_2| \leqslant R$.

If $(y - x)$ is even, then denote $t = y - x$, if $(y - x)$ is odd, then denote $t = 2(y - x)$, $\mathbf{t} = (t, -t)$ (we double period, because it will be convenient for us to deal with even period).

Now we are going to prove, that $\tilde{\varphi}$ is $\mathbf{t}$-periodic. Remind that we have $\mathbf{t}$-periodicity in the region $\{\mathbf{z} : |z_2 + z_1| \leqslant R\}$.

First we will prove that $\tilde{\varphi}$ is $\mathbf{t}$-periodic in the next diagonal $\{(s - R, -s + 1) | s \in \mathbb{Z}\}$. Suppose, by contradiction, that there exists q such that

$$\tilde{\varphi}(q - R, -q + 1) \neq \tilde{\varphi}(q - R + t, -q + 1 - t).$$

Denote $\boldsymbol{w} = \tilde{\varphi}(q - R, -q + 1) - \tilde{\varphi}(q - R + t, -q + 1 - t)$.

The function $\tilde{\varphi}$ is $\mathbf{t}$-periodic in the region $|z_1 + z_2| \leqslant R$, so we have that

$$\tilde{\varphi}|_{B_R(q,-q)} = \tilde{\varphi}|_{B_R(q+t,-q-t)}.$$

So by assertion that is analogous to Proposition 2 for the set S^2_{R+1} we have that

$$\tilde{\varphi}|_{S^2_{R+1}(q,-q)} = (\tilde{\varphi} + \chi^{\boldsymbol{w}}_{(q-R,-q+1)})|_{S^2_{R+1}(q+t,-q-t)},$$

where $\boldsymbol{w}$ is such that $\boldsymbol{w}A = \boldsymbol{0}$. The sets $S^2_{R+1}(q,-q)$ and $S^2_{R+1}(q+t,-q-t)$ (for $q = x$, $(y-x)$ even) are marked by squares in Fig. 4. In particular, this means that

$$\boldsymbol{w} = \tilde{\varphi}((q+2)-R, -(q+2)+1) - \tilde{\varphi}((q+2)-R+t, -(q+2)+1-t).$$

Now by induction we get that

$$\boldsymbol{w} = \tilde{\varphi}((q+k)-R, -(q+k)+1) - \tilde{\varphi}((q+k)-R+t, -(q+k)+1-t)$$

for every even integer k. Using this equality for $k = t, 2t, \ldots, mt$ we obtain $\tilde{\varphi}(q-R, -q+1) = \tilde{\varphi}(q+t-R, -q-t+1) + \boldsymbol{w} = \tilde{\varphi}(q+2t-R, -q-2t+1) + 2\boldsymbol{w} = \tilde{\varphi}(q+mt-R, -q-mt+1) + m\boldsymbol{w}$.

This implies that $\tilde{\varphi}$ has infinite number of values. A contradiction.

Thus we obtain **t**-periodicity in extended region (in a stripe with additional diagonal). Continuing this line of reasoning, we obtain **t**-periodicity for all $\mathbb{Z}^2$.

Therefore we obtained a $(t,-t)$-periodic function $\tilde{\varphi}$ by adding rectangular alternating functions χ and we did not use functions χ^*. (t,t)-periodicity can be organized for the function $\tilde{\varphi}$ in the same way by adding functions χ^*. Note that adding functions χ^* does not break $(t,-t)$-periodicity. The resulting function has a finite number of values, the alphabet of its values is Σ''. The proposition is proved. □

The main results of this section are summarized in the following theorem.

Theorem 1. *The necessary and sufficient condition for existence of non-periodic (A,r)-functions on the infinite rectangular grid is $r = 1, \det A = 0$. In this case, a periodic function can be obtained from a non-periodic one by adding rectangular alternating functions.*

The theorem follows from Propositions 1 and 4.

Perfect coloring and centered function are partial cases of generalized centered functions, so Theorem 1 implies the following results:

Corollary 1 [15]. *Every perfect coloring of radius $r \geq 2$ of the infinite rectangular grid is periodic.*

Corollary 2 [16]. *Every centered function with a finite number of values of radius $r \geq 1$ on the infinite rectangular grid is periodic.*

3. The infinite triangular grid

In this and the next sections, we consider the periodicity of generalized centered functions on the infinite triangular and hexagonal grids (see Figs. 5 and 6). These graphs are dual. All the pictures in what follows illustrate functions on faces of dual graphs instead of functions on vertices of graphs. A translation of the infinite grid is a translation of the plane superposing the grid and its image under the action of translation. A function on vertices of an infinite grid is called periodic, if there exist two noncollinear translations of this grid leaving the function invariant.

Theorem 2. *The necessary and sufficient condition for existence of non-periodic (A,r)-functions on the infinite triangular grid is $r = 1, \det(A+2I) = 0$.*

Proof. Proof of periodicity is very similar to the proof of periodicity in Theorem 1. Here is an example of a non-periodic $(A,1)$-function for an arbitrary matrix A such that $\det(A+2I) = 0$.

Example 2. Let A be an integer matrix, such that $\lambda = -2$ is an eigenvalue of A, let $\boldsymbol{v}$ be a corresponding eigenvector: $\boldsymbol{v}A = -2\boldsymbol{v}$. The example of non-periodic $(A,1)$-function on the infinite triangular grid (on faces of hexagonal grid) is in Fig. 7.

Indeed, for the balls centered in the vertices of the support of this function both sides of the equality in the definition of generalized centered function are equal to $2\boldsymbol{v}$ or $-2\boldsymbol{v}$. For all other balls both sides are equal to $\boldsymbol{0}$. □

Perfect coloring and centered function are partial cases of generalized centered functions, so Theorem 2 implies the following results:

Corollary 3. *Every perfect coloring of radius $r \geq 2$ of the infinite triangular grid is periodic.*

Corollary 4 [16]. *Every centered function with a finite number of values of radius $r \geq 1$ on the infinite triangular grid is periodic.*

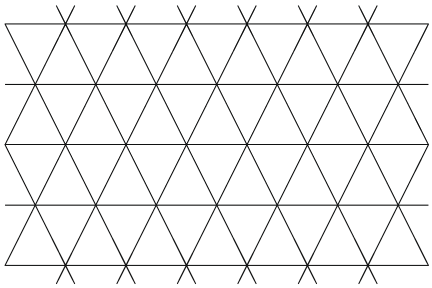

Fig. 5. The infinite triangular grid.

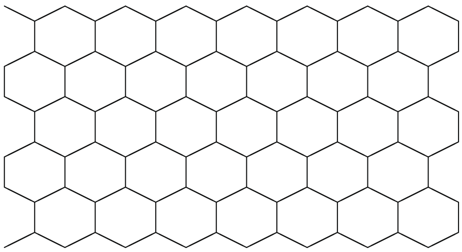

Fig. 6. The infinite hexagonal grid.

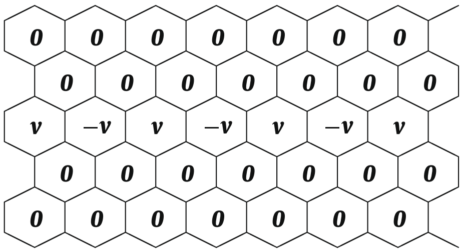

Fig. 7. An example of non-periodic generalized centered function on faces of the infinite hexagonal grid, here $\boldsymbol{v}A = -2\boldsymbol{v}$.

4. The infinite hexagonal grid

In this section, we consider the periodicity of generalized centered functions on the infinite hexagonal grid (see Fig. 6). The idea of the proof is the same as for the infinite rectangular and triangular grids, but in the case of hexagonal grid there appear some technical differences. Notice that the graph of the hexagonal grid is bipartite. Balls with centers at vertices from different partition classes cannot be obtained one from another by translations. So we consider a function on this graph to be R-prolongable if it is R-prolongable for each partition class separately. For the infinite hexagonal grid R-prolongability also implies periodicity.

Theorem 3. *Non-periodic (A, r)-functions on the infinite hexagonal grid exist only in the following two cases*:

(1) $r = 1$, $\det(A^2 - I) = 0$,
(2) $r = 2$, $\det(A^2 + 4A + 3I) = 0$.

Proof. At first we give the examples of non-periodic (A, r)-functions for these two cases.

Example 3. Let A be an integer matrix, such that $\det(A^2 - I) = 0$. Then $\lambda = 1$ is an eigenvalue of A^2, let $\boldsymbol{v}$ be corresponding eigenvector: $\boldsymbol{v}A^2 = \boldsymbol{v}$. Denote $\boldsymbol{v}A = \boldsymbol{u}$. The example of non-periodic $(A, 1)$-function on the infinite hexagonal grid (on faces on triangular grid) is in Fig. 8.

Example 4. Let A be an integer matrix, such that $\det(A^2 + 4A + 3I) = 0$. Then $\lambda = 1$ is an eigenvalue of $(A + 2I)^2$, let $\boldsymbol{v}$ be corresponding eigenvector: $\boldsymbol{v}(A + 2I)^2 = \boldsymbol{v}$. Denote $\boldsymbol{v}(A + 2I) = \boldsymbol{u}$. The example of non-periodic $(A, 2)$-function on the infinite hexagonal grid is in Fig. 8.

The fact that these functions are generalized centered functions can be checked straightforwardly in the same way as for the rectangular and triangular grids.

A ball in the infinite hexagonal grid has a little bit more complicated structure than in the rectangular and triangular grids. First, there are three types of boundary vertices in a ball in the infinite hexagonal grid. We will consider each type of vertices separately. Secondly, spheres of even and odd radii should be considered separately, because they differ by the structure of boundary. A sphere of even radius r looks like a hexagon with sides of length $\frac{r}{2}+1$. A sphere of odd radius r has sides of alternating lengths $\frac{r+1}{2}$ (side with vertices of type 2 and two additional vertices of type 3) and $\frac{r+1}{2}+1$ (side with vertices of type 1 and two additional vertices of type 3). These lengths can be written in common way: $\left\lfloor \frac{r}{2} \right\rfloor + 1$ (side with vertices of type 2) and $\left\lfloor \frac{r+1}{2} \right\rfloor + 1$ (side with vertices of type 1). Figs. 9 and 10 illustrate balls of radii 5 and 6, respectively. Thirdly, in the proof of the theorem we have to consider up to four layers instead of two layers to obtain a contradiction.

The idea of the proof is the following. Arguing as for the infinite rectangular grid, it is sufficient to prove that there exist $R \in \mathbb{Z}$ such that for every vertex $\mathbf{x}$ the condition $\psi|_{B_R(\mathbf{x})} = \boldsymbol{0}$ implies $\psi|_{S_{R+1}(\mathbf{x})} = \boldsymbol{0}$, where ψ is a function in the alphabet Σ_1. We obtain some conditions on the values of the function on the vertices of the layer i (i.e., vertices of the sphere $S_{R+i}(\mathbf{x})$) by considering balls of radius r with centers at vertices of the sphere $S_{R-r+i}(\mathbf{x})$. Figs. 11 and 12 illustrate part of the ball of radius R with zero values (its boundary is marked with bold line) and four layers. Values of the layers 1–4 are denoted by $a(i)$, $b(i)$, $c(i)$, $d(i)$, respectively. Using these conditions for the layers 1-4, we get that if $a(i) \neq 0$, then we obtain different values on one of these layers, that contradicts the condition that the alphabet of values of ψ is finite.

Now let us consider details of each case for $r \geq 4$. The cases $r = 1, 2, 3$ will be considered separately in the further text, because in these cases when we apply the definition of generalized function to obtain conditions for layers 2–4, centers of corresponding balls of radius r are outside the ball $B_R(\mathbf{x})$ and thus the argument is different from the case $r \geq 4$.

Case 1: (r is even, vertices of the sphere of radius $R+1$ are of type 1). See Fig. 11.
In this case, we should consider four layers. Denote $n = \frac{r}{2}+1$.

Values of the first layer $(a(1), a(2), \ldots, a(p),\ p = \left\lfloor \frac{R}{2} \right\rfloor)$. In the further text formulas hold for all indices for which the a-values (or b-, c-, d-values) are defined. Applying the definition of a generalized centered function for the function ψ to vertices with centers in the spheres $S_{R-r+1}(\mathbf{x})$, we get $\sum_{i=j}^{j+n-1} a(i) = \boldsymbol{0}$. Therefore $a(i+n) = a(i)$, i.e., the sequence $a(i)$ is periodic with period n.

Values of the second layer $(b(1), b(2), \ldots, b(p))$. Applying the definition of a generalized centered function for the function ψ to vertices with centers in the spheres $S_{R-r+2}(\mathbf{x})$, we get $\sum_{i=j}^{j+n-1} b(i) = \boldsymbol{0}$. Therefore $b(i+n) = b(i)$, i.e., the sequence $b(i)$ is periodic with period n.

Values of the third layer $(c(1), c(2), \ldots, c(p-1))$. Applying the definition of a generalized centered function for the function ψ to vertices with centers in the spheres $S_{R-r+3}(\mathbf{x})$, we get $\sum_{i=j}^{j+n-1} c(i) - b(j) + a(j) = \boldsymbol{0}$. So $\sum_{i=j}^{j+n-1} c(i) = -a(j) + b(j)$ and $\sum_{i=j+1}^{j+n} c(i) = -a(j+1) + b(j+1)$, therefore, $c(j+n) - c(j) = -b(j) + b(j+1) + a(j) - a(j+1)$. Therefore, we have $c(j+kn) - c(j) = k(-b(j) + b(j+1)) + k(a(j) - a(j+1))$.

Values of the fourth layer $(d(1), d(2), \ldots, d(p-1))$. Applying the definition of a generalized centered function for the function ψ to vertices with centers in the spheres $S_{R-r+3}(\mathbf{x})$, we get $\sum_{i=j}^{j+n-1} d(i) + b(j) + b(j+n) = \boldsymbol{0}$. It follows that $\sum_{i=j}^{j+n-1} d(i) = -2b(j)$ and $\sum_{i=j+1}^{j+n} d(i) = -2b(j+1)$, therefore, $d(j+n) - d(j) = -2b(j+1) + 2b(j)$. It follows that $d(j+kn) - d(j) = 2k(-b(j+1) + b(j))$. If $b(j+1) \neq b(j)$, then all the values $d(j+kn)$ are different. The fact that the alphabet Σ_1 of values of ψ is finite implies that $b(i) = b(i+1)$ for $i = 1, \ldots, p$.

Using formulas for the values $c(i)$ we conclude that $a(i) = a(i+1)$, so $a(i) = 0$ for $i = 1, \ldots, p$.

Case 2: r is even, vertices of the sphere of radius $R+1$ are of type 2. This case is similar to the case of the infinite rectangular grid.

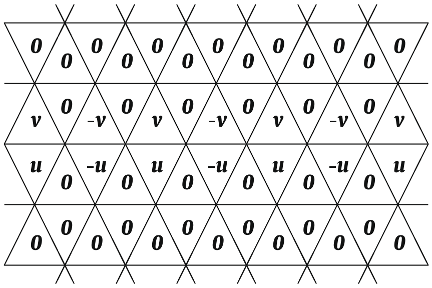

Fig. 8. An example of non-periodic generalized centered function on faces of the infinite triangular grid.

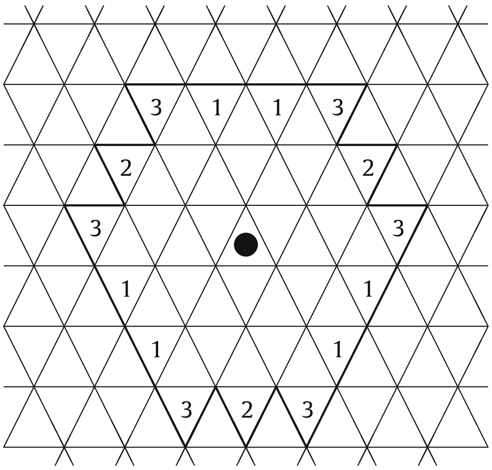

Fig. 9. A ball of radius 5 in the infinite hexagonal grid and three types of boundary vertices.

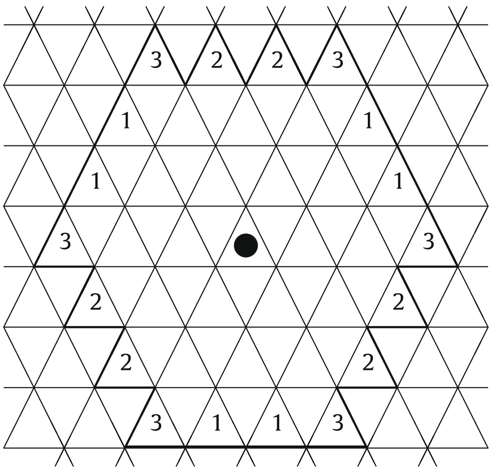

Fig. 10. The illustration for the proof of Proposition 1 for $R = 5$, $r = 2$.

Case 3: r is odd, vertices of the sphere of radius $R+1$ are of type 1 (see Fig. 11). In this case, we should consider three layers. Denote $n = \frac{r+1}{2}$. Here $p = \left\lfloor \frac{R}{2} \right\rfloor$.

Values of the first layer. Arguing as above, we get that $\sum_{i=j}^{j+n} a(i) = \mathbf{0}$. Therefore the sequence $a(i)$ is periodic with period $n+1$.

Values of the second layer. It holds $\sum_{i=j+1}^{j+n} b(i) + \sum_{i=j+1}^{j+n} a(i) = \mathbf{0}$.

Values of the third layer. Using formulas for the second level, we get $\sum_{i=j}^{j+n} c(i) + a(j) + a(j+n+1) = \mathbf{0}$, it follows that $\sum_{i=j}^{j+n} c(i) = -2a(j)$ and $\sum_{i=j+1}^{j+n+1} c(i) = -2a(j+1)$, therefore, $c(j+n+1) - c(j) = -2a(j+1) + 2a(j)$. It follows that $c(j+k(n+1)) - c(j) = 2k(-a(j+1) + a(j))$. Thus $a(i) = \mathbf{0}$ for $i = 1, \ldots, p$.

Case 4: r is odd, vertices of the sphere of radius $R+1$ are of type 2 (see Fig. 12). In this case, we should consider four layers. Here $p = \left\lfloor \frac{R+1}{2} \right\rfloor - 1$. Denote $n = \frac{r+1}{2}$.

Values of the first layer. Arguing as above, we get that $\sum_{i=j}^{j+n-1} a(i) = \mathbf{0}$. Therefore the sequence $a(i)$ is periodic with period n.

Values of the second layer. For these values we have $\sum_{i=j}^{j+n} b(i) = \mathbf{0}$. Therefore the sequence $b(i)$ is periodic with period $n+1$.

Values of the third layer. It holds $\sum_{i=j}^{j+n-1} c(i) + \sum_{i=j+1}^{j+n} b(i) + \sum_{i=j}^{j+n} a(i) = \mathbf{0}$.

Values of the fourth layer. Applying the definition of (A, r)-function for corresponding values, we get $\sum_{i=j}^{j+n} d(i) + \sum_{i=j}^{j+n-1} c(i) + \sum_{i=j}^{j+n+1} b(i) + \sum_{i=j}^{j+n} a(i) = \mathbf{0}$. Using formulas obtained for the third layer, we get that

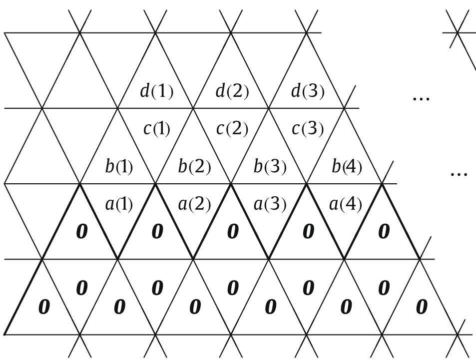

Fig. 11. Part of a ball of radius R with zero values and four layers, where the vertices of the sphere of radius $R+1$ are of type 1.

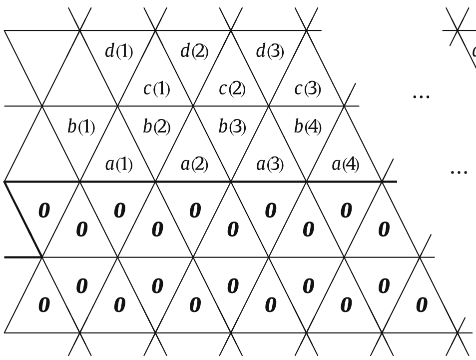

Fig. 12. Part of a ball of radius R with zero values and four layers, where the vertices of the sphere of radius $R+1$ are of type 2.

$\sum_{i=j}^{j+n} d(i) = -2b(j)$, therefore, $d(j+k(n+1)) - d(j) = 2k(-b(j+1)+b(j))$. As before, we obtain $b(i) = \boldsymbol{0}$ for $i = 1, \ldots, p+1$.

Combining formulas obtained for the c-values with the fact that $b(i) = \boldsymbol{0}$, we get $\sum_{i=j}^{j+n-1} c(i) + \sum_{i=j}^{j+n} a(i) = \boldsymbol{0}$, whence $c(j+kn) - c(j) = 2k(-a(j+1)+a(j))$. Again we obtain that $a(i) = \boldsymbol{0}$ for $i = 1, \ldots, p$.

Now we will consider the case $r = 3$ in similar way.

Case 1 ($r = 3$): vertices of the sphere of radius $R+1$ are of type 1 (see Fig. 11). In this case, the proof follows from the proof for the case $r \geq 4$ (case 3), because in this case we considered only three layers and centers of balls were inside the ball $B_R(\mathbf{x})$.

Case 2 ($r = 3$): vertices of the sphere of radius $R+1$ are of type 2 (see Fig. 12). In this case, we have to consider four layers, so when we consider the vertices of the layer 4, centres of balls of radius r are on the first layer. Here $p = \left\lfloor \frac{R+1}{2} \right\rfloor - 1$.

Values of the first layer. Arguing as above, we get that $a(j) + a(j+1) = \boldsymbol{0}$. Therefore the sequence $a(i)$ is periodic with period 2.

Values of the second layer. We have $b(j) + b(j+1) + b(j+2) = \boldsymbol{0}$, therefore the sequence $b(i)$ is periodic with period 3.

Values of the third layer. For these values we have $c(j) + c(j+1) + b(j+1) + b(j+2) + a(j) + a(j+1) + a(j+2) = \boldsymbol{0}$. Using formulas obtained for the layers 1–2, we have $c(j) + c(j+1) - b(j) + a(j) = \boldsymbol{0}$.

Values of the fourth layer. Applying the definition of a generalized centered function to vertices of the first layer, we get $d(j) + d(j+1) + d(j+2) + c(j) + c(j+1) + b(j) + b(j+1) + b(j+2) + b(j+3) + a(j) + a(j+2) = a(j+1)A$. Taking into account formulas obtained for the layers 1–3, we get $d(j) + d(j+1) + d(j+2) = -2b(j) - a(j)(A+I)$.

This equality holds for all j for which the a-, b-, c- and d-values are defined, so substitute j by $j+1$ in this formula: $d(j+1) + d(j+2) + d(j+3) = -2b(j+1) - a(j+1)(A+I) = -2b(j+1) + a(j)(A+I)$.

Subtract previous equation from this one: $d(j+3) - d(j) = 2a(j)(A+I) - 2b(j+1) + 2b(j)$.

This equality holds for all j for which the a-, b-, c- and d-values are defined, so substitute j by $j+3$ in this formula: $d(j+6) - d(j+3) = 2a(j+3)(A+I) - 2b(j+4) + 2b(j+3) = -2a(j)(A+I) - 2b(j+1) + 2b(j)$.

Addition of these two equations yields to $d(j+6) - d(j) = -4(b(j+1) - b(j))$.

Arguing as above, we get $d(j+6k) - d(j) = -4k(b(j+1) - b(j))$. As before, it follows $b(i) = \boldsymbol{0}$ for $i = 1, \ldots, p+1$.

Combining this with the formulas for the c-values, we get that $b(i) = \boldsymbol{0}$, we obtain $c(j) + c(j+1) = -a(j)$.

This equality holds for all j, so $c(j+1) + c(j+2) = -a(j+1) = a(j)$, whence $c(j+2) - c(j) = 2a(j)$ and $c(j+kn) - c(j) = -2k(-a(j+1) + a(j))$. Arguing as above we get that $a(i) = \boldsymbol{0}$ for $i = 1, \ldots, p$.

So we proved that $\psi = \mathbf{0}$ on vertices of types 1 and 2 of the sphere $S_{R+1}(\mathbf{x})$ for $r = 3$. The values of ψ on the vertices of type 3 are equal to $\mathbf{0}$, that can be proved in the same way as for the rectangular grid. Thus, we have $\psi|_{S_{R+1}(\mathbf{x})} = \mathbf{0}$. This means that the function φ is R-prolongable. So it is periodic. This completes the proof of periodicity for $r = 3$.

Now we will consider the cases $r = 1$ and $r = 2$.

Let φ be a $(A, 2)$-function, A such that $\det(A + 4A + 3I) \neq 0$. We will prove that φ is R-prolongable and thus periodic. Consider function ψ defined as above.

Case 1 ($r = 2$): vertices of the sphere of radius $R + 1$ are of type 1 (see Fig. 11). In this case, we should consider four layers.

*V*alues of the first layer. Applying the definition of $(A, 2)$-function to the vertices in the sphere $S_{R-1}(\mathbf{x})$, we get $a(j) + a(j + 1) = \mathbf{0}$. Therefore the sequence $a(i)$ is periodic with period 2.

*V*alues of the second layer. Arguing as above we obtain $b(j) + b(j + 1) = \mathbf{0}$. Therefore the sequence $b(i)$ is also periodic with period 2.

*V*alues of the third layer. For these values we have the following equality: $a(j - 1) + a(j + 1) + b(j) + c(j - 1) + c(j) = a(j)A$. Using formulas for the first layer, we obtain $c(j - 1) + c(j) = a(j)(A + 2I) - b(j)$. Thus $c(j) + c(j + 1) = a(j + 1)(A + 2I) - b(j + 1) = -a(j)(A + 2I) + b(j), c(j + 1) - c(j - 1) = 2(-a(j)(A + 2I) + b(j))$.

Whence $c(j - 1 + 2k) - c(j - 1) = 2k(-a(j)(A + 2I) + b(j))$. The function ψ has a finite number of values, so $a(j)(A + 2I) - b(j) = \mathbf{0}, b(j) = a(j)(A + 2I)$. Therefore, $c(j - 1) + c(j) = \mathbf{0}$.

*V*alues of the fourth layer. Applying the definition of a generalized centered function of radius 2 to the vertices in the second layer, we get $a(j) + b(j - 1) + b(j + 1) + c(j - 1) + c(j) + d(j - 1) + d(j) = b(j)A$.

Using formulas obtained above, we get $d(j - 1) + d(j) = b(j)(A + 2I) - a(j)$. Applying this formula for $j + 1$ instead of j and using formulas for a- and b-values, we get $d(j) + d(j + 1) = b(j + 1)(A + 2I) - a(j + 1) = -b(j)(A + 2I) + a(j)$. Arguing as above, $d(j - 1 + 2k) - d(j - 1) = -2k(b(j)(A + 2I) - a(j))$.

As before, it follows $b(j)(A + 2I) - a(j) = \mathbf{0}$, $a(j) = b(j)(A + 2I)$. Combining it with results for layer 3 we get $a(j) = a(j)(A + 2I)^2$. It holds $\det((A + 2I)^2 - I) \neq 0$ by our assumption, so $a(i) = \mathbf{0}$ for $i = 1, \ldots, p$.

Case 2 ($r = 2$): vertices of the sphere of radius $R + 1$ are of type 2 (see Fig. 12). In this case, the fact that $a(i) = \mathbf{0}$ follows from the proof for the case 2 for $r \geq 4$, because for this case only a- and b-values were used.

Therefore, the $(A, 2)$-function φ is periodic, if $\det(A^2 + 4A + 3I) \neq 0$.

Let φ be a $(A, 1)$-function, A such that $\det(A^2 - I) \neq 0$. We will prove that φ is R-prolongable and thus periodic. Consider the function ψ defined as above.

Case 1 ($r = 1$): vertices of the sphere of radius $R + 1$ are of type 1 (see Fig. 11). In this case, we should consider three layers. Here $p = \left\lfloor \frac{R}{2} \right\rfloor$.

*V*alues of the first layer. Arguing as above, we get that $a(j) + a(j + 1) = \mathbf{0}$. Therefore the sequence $a(i)$ is periodic with period 2.

*V*alues of the second layer. Considering balls with centers in the first layer, we obtain $b(j) = a(j)A$.

*V*alues of the third layer. Considering balls with centers in the first layer, we obtain $c(j - 1) + c(j) + a(j) = b(j)A$.

Using formulas for the first and the second layers, we get $c(j - 1) + c(j) + a(j) = a(j)A^2, c(j - 1) + c(j) = a(j)(A^2 - I)$, $c(j) + c(j + 1) = a(j + 1)(A^2 - I) = -a(j)(A^2 - I), c(j + 1) - c(j - 1) = -2a(j)(A^2 - I)$.

Therefore $c(j - 1 + 2k) - c(j - 1) = -2ka(j)(A^2 - I)$. So $a(j)(A^2 - I) = \mathbf{0}$. We have $\det(A^2 - I) \neq 0$ by our assumption, so $a(i) = \mathbf{0}$ for $i = 1, \ldots, p$.

Case 2 ($r = 1$): vertices of the sphere of radius $R + 1$ are of type 2 (see Fig. 12). Applying the definition of a generalized centered function of radius 1 to the vertices in the sphere $S_R(\mathbf{x})$, we immediately get $a(j) = \mathbf{0}$.

We proved, that $(A, 1)$-function is periodic, if $\det(A^2 - I) \neq 0$.

This completes the proof of the theorem. $\square$

Perfect coloring and centered function are partial cases of generalized centered functions, so Theorem 3 implies the following results:

Corollary 5. *Every perfect coloring of radius $r \geq 3$ of the infinite hexagonal grid is periodic.*

Corollary 6. [16]. *Every centered function with a finite number of values of radius $r \geq 3$ on the infinite hexagonal grid is periodic.*

Remark. It is easy to see that all theorems in this paper hold also for generalized centered functions with values in $\mathbb{C}^n$.

5. Conclusion

The results of this paper can be summarized in the following table. In each cell we write necessary and sufficient conditions for existence of non-periodic functions in the corresponding case.

Rectangular grid	Triangular grid	Hexagonal grid
$r = 1, \det A = 0$	$r = 1, \det(A + 2I) = 0$	$r = 1, \det(A^2 - I) = 0$ or $r = 2, \det(A^2 + 4A + 3I) = 0$

We suppose that the technique used for graphs of these grids can be also used for other transitive graphs, because the notion of R-prolongability and Lemma 1 can be generalized for this case.

For example, consider a group $H =< \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n >$ of translations of $\mathbb{Z}^2$, generated by the vectors $\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n$, $\boldsymbol{a}^i \in \mathbb{Z}^2$. Consider a Cayley graph of this group. Denote it by $G(\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n)$. It would be interesting to study the following problem: for what values of A, $\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n$ a generalized centered function with matrix A on the graph $G(\boldsymbol{a}^1, \ldots, \boldsymbol{a}^n)$ is periodic? Notice that the triangular grid can be interpreted as a Cayley graph $G((1, 0), (0, 1), (1, 1))$. It would be interesting to consider the periodicity of m-dimensional words of this type.

Acknowledgment

The author is grateful to S.V. Avgustinovich for useful discussions and comments.

References

[1] S.V. Avgustinovich, A.Yu. Vasiléva, Reconstruction of centered function by its values at two middle levels of the hypercube, Diskretn. Anal. Issled. Oper. 10 (2) (2003) 3–16. (Russian).
[2] V. Berthe, L. Vuillon, Tilings and rotations: a two-dimensional generalization of Sturmian sequences, Discrete Math. 223 (2000) 27–53.
[3] Y. Cesari, M. Vincent, Une Caracterisation des Mots Périodiques, C. R. Acad. Sci. Paris, Ser. A 286 (1978) 1175–1177.
[4] M. Cvetkovic, M. Doob, H. Sachs, Spectra of Graphs, VEB Deutcher Verlag der Wissenschaften, Berlin, 1980.
[5] J.P. Duval, Périodes et Répetitions des Mots du Monoide Libre, Theor. Comput. Sci. 9 (1979) 17–26.
[6] C. Epifanio, M. Koskas, F. Mignosi, On a conjecture on bidimensional words, Theor. Comput. Sci. 299 (1–3) (2003) 123–150.
[7] C. Epifanio, F. Mignosi, A multidimensional critical factorization theorem, Theor. Comput. Sci. 346 (2) (2005) 265–280.
[8] C. Godsil, Equitable partitions, in: Combinatorics, Paul Erdos is Eighty, vol. 1, 1993, János Bolyai Math. Soc., Budapest, pp. 173–192.
[9] J. Karhumäki, A. Lepisto, W. Plandowski, Locally periodic versus globally periodic infinite words, J. Comb. Theory Ser. A 100 (2002) 250–264.
[10] M. Morse, G.A. Hedlund, Symbolic dynamics II: Sturmian trajectories, Am. J. Math. 61 (1940) 1–42.
[11] F. Mignosi, A. Restivo, S. Salemi, Periodicity and the golden ratio, Theor. Comput. Sci. 204 (1998) 153–167.
[12] M. Nivat, Invited talk at ICALP'97.
[13] S.A. Puzynina, Periodicity of perfect colorings of the infinite rectangular grid, Diskretn. Anal. Issled. Oper. 11 (1) (2004) 79–92. (Russian).
[14] S.A. Puzynina, Perfect colorings of the infinite rectangular grid, Bayreuther Math. Schr. 74 (2005) 317–331.
[15] S.A. Puzynina, Perfect colorings of radius $r > 1$ of the infinite rectangular grid, Siberian Electronic Mathematical Reports (2008) 283–292.
[16] S.A. Puzynina, S.V. Avgustinovich, On periodicity of two-dimensional words, Theor. Comput. Sci. 391 (2008) 178–187.
[17] A. Quas, L. Zamboni, Periodicity and local complexity, Theor. Comput. Sci. 319 (1–3) (2004) 229–240.