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The scalar curvature equation on S^3

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Abstract

We give existence results for solutions of the prescribed scalar curvature equation on S^3, when the curvature function is a positive Morse function and satisfies an index-count condition.

Keywords: Prescribed scalar curvature; Leray–Schauder degree

1. Introduction

Let S^3 be the standard sphere with round metric g_0 induced by $S^3 = \partial B_1(0) \subset \mathbb{R}^4$. We study the problem: Which functions K on S^3 occur as scalar curvature of metrics g conformally equivalent to g_0? Writing $g = \varphi^4 g_0$ and $k(\theta) := \frac{1}{6}(K(\theta) - 6)$ this is equivalent to solving for $t = 1$ (see [3])

$$-8\Delta_{S^3}\varphi + 6\varphi = 6\big(1 + tk(\theta)\big)\varphi^5, \quad \varphi > 0 \text{ in } S^3. \tag{1.1}$$

An obvious necessary condition for the existence of solutions to (1.1) is that the function K has to be positive somewhere. Moreover, there are the *Kazdan–Warner* obstructions [7,16], which imply in particular, that a monotone function of the coordinate function X_1 cannot be realized as the scalar curvature of a metric conformal to g_0.

Numerous studies have been made on Eq. (1.1) and its higher dimensional analogue and various sufficient conditions for its solvability have been found (see [2,4,6,11,12,17,18] and the

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Table 1
Degree for K_1, K_2, K_3.

	K_1	K_2	K_3
$\pm E_1$	$\Delta_{S^3} K_1(\pm E_1) > 0$	$\Delta_{S^3} K_2(\pm E_1) > 0$	$\Delta_{S^3} K_3(\pm E_1) > 0$
	$\mathrm{ind}(K_1, \pm E_1) = 0$	$\mathrm{ind}(K_2, \pm E_1) = 0$	$\mathrm{ind}(K_3, \pm E_1) = 0$
$\pm E_2$	$\Delta_{S^3} K_1(\pm E_2) < 0$	$\Delta_{S^3} K_2(\pm E_2) = 0$	$\Delta_{S^3} K_3(\pm E_2) > 0$
	$\mathrm{ind}(K_1, \pm E_2) = 1$	$\mathrm{ind}(K_2, \pm E_2) = 1$	$\mathrm{ind}(K_3, \pm E_2) = 1$
$\pm E_3$	$\Delta_{S^3} K_1(\pm E_3) < 0$	$\Delta_{S^3} K_2(\pm E_3) < 0$	$\Delta_{S^3} K_3(\pm E_3) < 0$
	$\mathrm{ind}(K_1, \pm E_3) = 2$	$\mathrm{ind}(K_2, \pm E_3) = 2$	$\mathrm{ind}(K_3, \pm E_3) = 2$
$\pm E_4$	$\Delta_{S^3} K_1(\pm E_4) < 0$	$\Delta_{S^3} K_2(\pm E_4) < 0$	$\Delta_{S^3} K_3(\pm E_4) < 0$
	$\mathrm{ind}(K_1, \pm E_4) = 3$	$\mathrm{ind}(K_2, \pm E_4) = 3$	$\mathrm{ind}(K_3, \pm E_4) = 3$
d	1	?	−1

reference therein), usually under a nondegeneracy assumption on K. On S^3 a positive function K is nondegenerate, if

$$\Delta K(\theta) \neq 0 \quad \text{if } \nabla K(\theta) = 0. \tag{nd}$$

For positive Morse functions K on S^3 it is shown in [5,10,23] that (1.1) is solvable if K satisfies (nd) and

$$d := -\left(1 + \sum_{\substack{\nabla K(\theta)=0,\\ \Delta K(\theta)<0}} (-1)^{\mathrm{ind}(K,\theta)}\right) \neq 0, \tag{1.2}$$

where $\mathrm{ind}(K,\theta)$ is the Morse index of K at θ, i.e. the number of negative eigenvalues of the Hessian. For example the simplest possible positive Morse function $K = 2 + X_1$, where we already know from the Kazdan–Warner obstructions, that there are no solutions, yields $d = 0$, as the only critical point of K with negative Laplacian is the global maximum with Morse index 3. Moreover, consider the functions $K_i \in C^\infty(S^3, \mathbb{R})$ defined by

$$\begin{aligned} K_1(X) &:= 2X_1^2 + 6X_2^2 + 7X_3^2 + 8X_4^2, \\ K_2(X) &:= 3X_1^2 + 6X_2^2 + 7X_3^2 + 8X_4^2, \\ K_3(X) &:= 4X_1^2 + 6X_2^2 + 7X_3^2 + 8X_4^2, \end{aligned}$$

where X_i for $1 \leqslant i \leqslant 4$ is the ith coordinate function of $S^3 \subset \mathbb{R}^4$. Each K_i is a positive Morse function with critical points given by

$$\left\{\pm E_i \in S^3 \subset \mathbb{R}^4 \colon\ 1 \leqslant i \leqslant 4\right\},$$

where $\{E_i,\ 1 \leqslant i \leqslant 4\}$ denotes the standard basis of $\mathbb{R}^4$. The global maximum is attained at $\pm E_4$, the global minimum at $\pm E_1$, $\pm E_2$ and $\pm E_3$ are saddle points. The sign of the Laplacian, the Morse-index, and d are collected in Table 1 below. Thus, (1.1) is solvable for $t = 1$ and $K \in \{K_1, K_3\}$. The function K_2 does not satisfy the nondegeneracy assumption (nd) at E_2 and the above result is not applicable. For the special function K_2 a different approach leads to a solution: K_2 is symmetric with respect to reflections on the sphere S^3 and the problem may be shifted to the projective space $\mathbb{RP}^3$. Since $\mathbb{RP}^3$ is not conformal to S^3 the result of [14] yields a solution on $\mathbb{RP}^3$ that may be shifted back to obtain a solution for K_2 on S^3. But, the argument breaks down for any nonsymmetric perturbation of K_2. We are interested exactly in this case, when the nondegeneracy assumption (nd) is not satisfied, and we shall give the required general existence result.

In the following, unless otherwise stated, we will assume that $K = 6(1+k) \in C^5(S^3)$ is positive. To give our main results we need the following notation. We denote by $\mathcal{S}_\theta(\cdot)$ stereographic coordinates centered at some point $\theta \in S^3$, i.e. $\mathcal{S}_\theta(0) = \theta$. We write $k_\theta = k \circ \mathcal{S}_\theta$ and for a critical point θ of k with $D^2 k_\theta(0)$ invertible we let

$$a_0(\theta) := \oint_{\mathbb{R}^3} \big(k_\theta(x) - T^2_{k_\theta,0}(x)\big)|x|^{-6},$$

$$a_1(\theta) := \Delta^2 k_\theta(0) + \nabla\big(\Delta k_\theta(0)\big) \cdot \big(D^2 k_\theta(0)\big)^{-1} \nabla\big(\Delta k_\theta(0)\big),$$

$$a_2(\theta) := k_\theta(0) a_1(\theta) - \frac{15}{8\pi} \int_{\partial B_1(0)} \big|D^2 k_\theta(0)(x)^2\big|^2,$$

where all differentiations are done in $\mathbb{R}^3$, the mth Taylor polynomial of k_θ in y is abbreviated by

$$T^m_{k_\theta,y}(x) := \sum_{\ell=0}^{m} \frac{1}{\ell!} D^\ell k_\theta(y)(x-y)^\ell,$$

and $\oint$ is the Cauchy principal value of the integral,

$$\oint_{\mathbb{R}^3} f(x) := \lim_{r\to 0} \int_{\mathbb{R}^3 \setminus B_r(0)} f(x).$$

The value $a_0(\theta)$ is well defined because of the cancellation due to symmetry. For instance expanding $T^m_{k_\theta,0}$ in spherical harmonics we get

$$\int_{\partial B_1(0)} T^m_{k_\theta,0}(x)\,dS = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \frac{2\pi}{3}\Delta k_\theta(0) & \text{if } m = 2. \end{cases}$$

The value $a_0(\theta)$ will be of interest only in points where (nd) is not satisfied, that is when $\nabla k_\theta(0)$ and $\Delta k_\theta(0)$ vanish simultaneously. In this case $a_0(\theta)$ is given by

$$a_0(\theta) = \oint_{\mathbb{R}^3} \big(k_\theta(x) - k_\theta(0)\big)|x|^{-6},$$

and measures, weighted by $|x|^{-6}$, the difference between k_θ and $k_\theta(0)$.

Denote by $\mathrm{Crit}(k)$, M, and T the sets,

$$\mathrm{Crit}(k) := \{\theta \in S^3\colon\ \nabla k(\theta) = 0\},$$
$$M := \{\theta \in \mathrm{Crit}(k)\colon\ \Delta k_\theta(0) = a_0(\theta) = 0,\ \text{and}\ a_2(\theta) \neq 0\},$$
$$T := \{-a_1(\theta)/a_2(\theta)\colon\ \theta \in M\},$$

Theorem 1.1. *Suppose* $1 + k \in C^5(S^3)$ *is a positive Morse function. Then* (1.1) *is solvable for* $t \in (0, 1] \setminus T$, *if*

$$0 \neq d(t) = -\Bigl(1 + \sum_{\theta \in \mathrm{Crit}_-(k,t)} (-1)^{\mathrm{ind}(k,\theta)}\Bigr), \tag{1.3}$$

where

$$\mathrm{Crit}_-(k,t) := \Bigl\{\theta \in S^3\colon\ \nabla k(\theta) = 0 \quad \text{and}$$
$$\lim_{\mu \to 0^+} \mathrm{sgn}\bigl(\Delta k(\theta) + a_0(\theta)\mu - \bigl(a_1(\theta) + t a_2(\theta)\bigr)\mu^2\bigr) = -1\Bigr\}.$$

The number $d(t)$ *is the Leray–Schauder degree of problem* (1.1).

We note that set of critical points of K and k are equal and for any $\theta \in \mathrm{Crit}(k)$ we have

$$\mathrm{sgn}\bigl(\Delta_{S^3} K(\theta)\bigr) = \mathrm{sgn}\bigl(\Delta_{S^3} k(\theta)\bigr) = \mathrm{sgn}\bigl(\Delta_{\mathbb{R}^3} k_\theta(0)\bigr),$$
$$\mathrm{ind}(K,\theta) = \mathrm{ind}(k,\theta) = \mathrm{ind}(k_\theta, 0).$$

Hence, the nondegeneracy condition (nd) implies that the set M is empty and the formula in (1.3) gives exactly the index-count condition in (1.2). In contrast to (1.2) the Leray–Schauder degree now depends on t and may change as t crosses some value in T. Indeed for any

$$t_* = -\frac{a_1(\theta)}{a_2(\theta)} \in T \cap (0, 1]$$

there is a "blow-up curve" $(t(s), \varphi(s))$ such that

$$\lim_{s \to 0} t(s) = t_*, \qquad \lim_{s \to 0} \bigl\|\varphi(s)\bigr\|_{L^\infty(B_\varepsilon(\theta))} = +\infty \quad \text{for all } \varepsilon > 0,$$

and $\varphi(s)$ solves (1.1) with $t = t(s)$ (see [21] and Fig. 1 below).

An inspection of the proof of Theorem 1.1 shows that the result remains valid, when k is only in $C^4(S^3)$. We state Theorem 1.1 for functions $k \in C^5(S^3)$, because we use the analysis in [20,21], which is done in this setting.

To illustrate our results we will apply Theorem 1.1 when K equals K_i for some $i \in \{1, 2, 3\}$. For $i \in \{1, 3\}$ the set M is empty, as the Laplacian does not vanish at any critical point, $d(\cdot)$ is independent of $t \neq 0$ and given by (1.2). Concerning K_2, the critical points with vanishing Laplacian are $\{\pm E_2\}$ and we need to compute $a_j(\pm E_2)$ for $j = 0, 1, 2$ and the function

$$k = k_2 := \frac{1}{6}(K_2 - 6) = \frac{1}{2}X_1^2 + X_2^2 + \frac{7}{6}X_3^2 + \frac{4}{3}X_4^2 - 1.$$

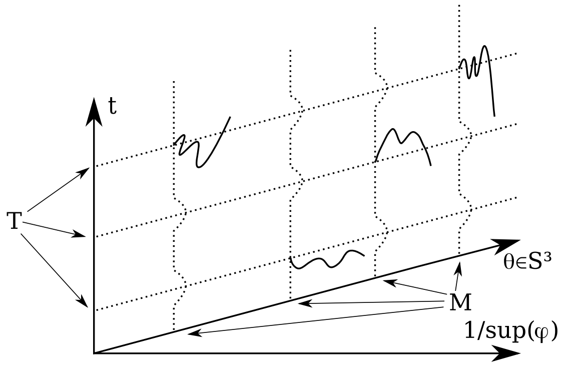

Fig. 1. Blow-up curves.

A straightforward computation (see [22]) shows

$$a_0(\pm E_2)=0, \qquad a_1(\pm E_2)=0, \qquad a_2(\pm E_2)=-\frac{224}{9}.$$

Hence, $M=\{\pm E_2\}\subset S^3$, $T=\{0\}$, and

$$d(t)=\begin{cases}-1 & \text{if } t>0,\\ 1 & \text{if } t<0.\end{cases}$$

Thus, we may replace the question mark in Table 1 by -1. Moreover, for $0\neq h\in C_c^\infty(S^3\setminus\{\pm E_2\},\mathbb{R}_{\geqslant 0})$ we consider $k_2\pm sh$, where s is a small positive parameter. Since

$$\int_{\mathbb{R}^3} h_{\pm E_2}(x)|x|^{-6}\,dx>0,$$

the sets M and T are empty for $k=k_2\pm sh$ and $0<s\ll 1$, the degree for $t\neq 0$ is given by

$$d(t)=-1 \quad \text{for } k=k_2+sh, \qquad d(t)=1 \quad \text{for } k=k_2-sh.$$

Furthermore, we consider for $0<s\ll 1$

$$k=k_2+s\big(7\big(1-X_2^2\big)^2-20\big(1-X_2^2\big)^3\big)$$

For small positive s the set of critical points of K is given by $\{\pm E_i\}$ with vanishing Laplacian only at $\pm E_2$, $a_0(\pm E_2)=0$, and

$$a_1(\pm E_2)=13440s, \qquad a_2(\pm E_2)=-\frac{224}{9}.$$

Thus, $M=\{\pm E_2\}$, $T=\{540s\}$, and for $t\neq 0$

$$d(t)=\begin{cases}-1 & \text{if } t>540s,\\ 1 & \text{if } t<540s.\end{cases}$$

The change of the degree is due to the two blow-up curves $r \mapsto (t^{\pm}(r), \varphi^{\pm}(r))$, where $t^{\pm}(r) \to 540s$ and $\varphi^{\pm}(r)$ concentrates at $\pm E_2$ as $r \to 0$. It is interesting to note that, although K is even in this case, the solutions on the blow-up curve are not even as they concentrate in a single point.

To prove our main result we embed our problem into a two-dimensional family of problems. We choose $h \in C^{\infty}(S^3, [0, \infty))$ such that

$$\operatorname{supp}(h) \cap \operatorname{Crit}(k) = \emptyset. \tag{1.4}$$

We fix $0 < t_0 \in (0, 1] \setminus T$ and consider for $s \geqslant 0$

$$-8\Delta_{S^3}\varphi + 6\varphi = 6\big(1 + t_0\big(k(\theta) + sh(\theta)\big)\big)\varphi^5, \quad \varphi > 0 \text{ in } S^3. \tag{1.5}$$

Analogously as above, we define $a_j(\theta, s)$ for $j = 0, 1, 2$ and M_s by replacing k by $k + sh$ in the definition of $a_j(\theta)$ and M. We obtain for $\theta \notin \operatorname{supp}(h)$

$$a_0(\theta, s) = a_0(\theta) + s \int_{\mathbb{R}^3} h_\theta(x)|x|^{-6}, \qquad a_1(\theta, s) = a_1(\theta), \qquad a_2(\theta, s) = a_2(\theta).$$

From (1.4) there is $s_0 > 0$ such that for $0 \leqslant s \leqslant s_0$:

- $\operatorname{Crit}(k) = \operatorname{Crit}(k + sh)$,
- $k + sh$ is a Morse function,
- $a_0(\theta) \cdot a_0(\theta, s) > 0$, if $\nabla k(\theta) = 0$ and $a_0(\theta) \neq 0$.

The main reason for introducing the perturbation h is that the sets M_s are empty, because

$$a_0(\theta, s) \neq 0 \quad \text{if } \nabla k(\theta) = 0.$$

By standard elliptic regularity the operator L_s, defined by

$$L_s : \varphi \mapsto (-8\Delta_{S^3} + 6)^{-1}\big(6\big(1 + t_0\big(k(\theta) + sh(\theta)\big)\big)\varphi^5\big),$$

is compact from $C^2(S^3)$ into $C^2(S^3)$. From the a priori estimates in [21], as $t_0 \notin T$, there is $C_{t_0} > 0$ such that all positive solution to (1.5) with $s = 0$ lie in $\mathcal{B}_{C_{t_0}}$,

$$\mathcal{B}_C := \big\{\varphi \in C^2\big(S^3\big)\colon \|\varphi\|_{C^2(S^3)} < C \text{ and } C^{-1} < \varphi\big\}.$$

Moreover, as $\operatorname{Crit}(k + sh)$ does not change when s moves from 0 to s_0, we may apply Theorem 7.1 in [21]. Thus, for any $0 < \delta < s_0$ there is $C_\delta > 0$ such that all positive solution to (1.5) with $s \in [\delta, s_0]$ lie in $\mathcal{B}_{C_\delta}$. The Leray–Schauder degree $\deg(Id - L_s, \mathcal{B}_{C_\delta}, 0)$, which is well defined and independent of $s \in [\delta, s_0]$ by the a priori estimates, is computed in [20] and equals

$$\deg(Id - L_s, \mathcal{B}_{C_\delta}, 0) = -\Bigg(1 + \sum_{\theta \in \operatorname{Crit}_-(k+sh)} (-1)^{\operatorname{ind}(k,\theta)}\Bigg), \tag{1.6}$$

where the set $\mathrm{Crit}_-(k+sh)$ is given by

$$\mathrm{Crit}_-(k+sh) := \Big\{\theta \in \mathrm{Crit}(k)\colon \lim_{\mu\to 0^+} \mathrm{sgn}\big(\Delta k(\theta) + a_0(\theta,s)\mu\big) = -1\Big\}.$$

As $h \geqslant 0$, we have for $\theta \in \mathrm{Crit}(k)$ that $a_0(\theta,s) < 0$ if and only if $a_0(\theta) < 0$. hence

$$\mathrm{Crit}_-(k+sh) = \big\{\theta \in \mathrm{Crit}(k)\colon\ \Delta k(\theta) < 0 \text{ or } \big(\Delta k(\theta) = 0 \text{ and } a_0(\theta) < 0\big)\big\}.$$

The constant C_δ in [21] depends on

$$\sup_{s\in[\delta,s_0]} \big\{\big|a_0(\theta,s)\big|^{-1}\colon\ \nabla k(\theta) = 0,\ \Delta k_\theta(0) = 0 \text{ and } a_0(\theta,s) \neq 0\big\}.$$

Consequently, we cannot assume that C_δ remains bounded as $\delta \to 0$. Indeed, we shall show that as s moves to 0 the family of solutions splits into solutions, that remain uniformly bounded as $s \to 0^+$ and converge to solutions of (1.5) with $s = 0$, and solutions that blow up as $s \to 0^+$. When s moves to 0^+ the total degree, which is computed in (1.6), is given by the sum of two degree's, the degree of the "bounded solutions", that we are interested in, and the degree of the "blow-up solutions". We will compute the degree of the solutions, that blow up when $s \to 0^+$, as a sum of local degree's. Subtracting the result from (1.6) leads to the formula in (1.3).

2. Preliminaries

For fixed $\theta \in S^3$ in stereographic coordinates $\mathcal{S}_\theta(\cdot)$ Eq. (1.5) is equivalent to

$$-\Delta u = \big(1 + t_0\big(k_\theta(x) + sh_\theta(x)\big)\big)u^5 \quad \text{in } \mathbb{R}^3,\ u > 0. \tag{2.1}$$

where $h_\theta = h \circ \mathcal{S}_\theta$ and

$$u(x) = \mathcal{R}_\theta(\varphi)(x) := 3^{\frac14}\big(1 + |x|^2\big)^{-\frac12}\varphi \circ \mathcal{S}_\theta(x). \tag{2.2}$$

The transformation (2.2) gives rise to a Hilbert space isomorphism between $H^{1,2}(S^3)$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)$, the closure of $C_c^\infty(\mathbb{R}^3)$ with respect to

$$\|u\|^2 := \int_{\mathbb{R}^3} |\nabla u|^2 = \langle u, u\rangle.$$

Due to elliptic regularity (see [8,19]) and Harnack's inequality it is enough to find a weak non-negative solution of (1.5) in $H^{1,2}(S^3)$, or of the equivalent equation in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Although we take advantage of both formulations, we mainly consider (2.1) to analyze the blow-up behavior and to compute local degrees. Weak solutions to (2.1) correspond to critical points of $f_{t_0,s} : \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$

$$f_{t_0,s}(u) := \int_{\mathbb{R}^3} \frac12|\nabla u|^2 - \frac16\big(1 + t_0\big(k_\theta(x) + sh_\theta(x)\big)\big)u^6\,dx.$$

We denote by f_0 the unperturbed functional with $t_0 = s = 0$. The positive solutions of (2.1) for $t_0 = s = 0$, i.e. the positive critical points of f_0, are completely known (see [9,13,15]) and given by a noncompact manifold

$$Z := \left\{ z_{\mu,y}(x) := \mu^{-\frac{1}{2}} 3^{\frac{1}{4}} \left(1 + \left|\frac{x-y}{\mu}\right|^2\right)^{-\frac{1}{2}}:\ y \in \mathbb{R}^3,\ \mu > 0 \right\},$$

We state some properties of the critical manifold Z and f_0 (see [2,21] for details). We define for $\mu > 0$ and $y \in \mathbb{R}^3$ the maps $\mathcal{U}_\mu, \mathcal{T}_y : \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathcal{D}^{1,2}(\mathbb{R}^3)$ by

$$\mathcal{U}_\mu(u) := \mu^{-\frac{1}{2}} u\left(\frac{\cdot}{\mu}\right) \quad \text{and} \quad \mathcal{T}_y(u) := u(\cdot - y).$$

With this notation the critical manifold Z is given by

$$Z = \left\{ z_{\mu,y} = \mathcal{T}_y \circ \mathcal{U}_\mu(z_{1,0}):\ y \in \mathbb{R}^3,\ \mu > 0 \right\}.$$

The dilation $\mathcal{U}_\mu$ and the translation $\mathcal{T}_y$ are automorphisms of $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and for every $\mu > 0$, $y \in \mathbb{R}^3$, and $v \in \mathcal{D}^{1,2}(\mathbb{R}^3)$

$$\begin{gathered} (\mathcal{U}_\mu)^{-1} = (\mathcal{U}_\mu)^t = U_{\mu^{-1}}, \qquad (\mathcal{T}_y)^{-1} = (\mathcal{T}_y)^t = \mathcal{T}_{-y}, \\ f_0 = f_0 \circ \mathcal{U}_\mu = f_0 \circ \mathcal{T}_y \quad \text{and} \\ f_0''(v) = (\mathcal{T}_y \circ \mathcal{U}_\mu)^{-1} \circ f_0''\big(\mathcal{T}_y \circ \mathcal{U}_\mu(v)\big) \circ (\mathcal{T}_y \circ \mathcal{U}_\mu), \end{gathered} \tag{2.3}$$

where $(\cdot)^t$ denotes the adjoint. The tangent space $T_{z_{\mu,y}} Z$ at a point $z_{\mu,y} \in Z$ is spanned by 4 orthonormal functions,

$$\begin{gathered} T_{z_{\mu,y}} Z = \left\langle (\dot{\xi}_{\mu,y})_i:\ i = 0 \ldots 3 \right\rangle, \\ (\dot{\xi}_{\mu,y})_i := \begin{cases} \left\| \frac{d}{d\mu} z_{\mu,y} \right\|^{-1} \frac{d}{d\mu} z_{\mu,y} & \text{if } i = 0, \\ \left\| \frac{d}{dy_i} z_{\mu,y} \right\|^{-1} \frac{d}{dy_i} z_{\mu,y} & \text{if } 1 \leqslant i \leqslant 3. \end{cases} \end{gathered}$$

The maps $\mathcal{U}_\mu$ and $\mathcal{T}_y$ are isomorphism of the tangent spaces, and moreover

$$\begin{gathered} (\dot{\xi}_{\mu,y})_i = \mathcal{T}_y \circ \mathcal{U}_\mu\big((\dot{\xi}_{1,0})_i\big), \\ \mathcal{T}_y \circ \mathcal{U}_\mu : (T_z Z)^\perp \xrightarrow{\cong} (T_{\mathcal{T}_y \circ \mathcal{U}_\mu(z)} Z)^\perp. \end{gathered} \tag{2.4}$$

We consider $f'_{t_0,s}(u)$ as an element of $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $f''_{t_0,s}(u)$ as a map in $\mathcal{L}(\mathcal{D}^{1,2}(\mathbb{R}^3))$. With this identification $f''_{t_0,s}(u)$ is a self-adjoint, compact perturbation of the identity map in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. The spectrum $\sigma(f_0''(z_{\mu,y}))$ consists of point-spectrum accumulating at 1 and is computed together with the eigenspaces in [21]. Since Z is a manifold of critical points of f_0', the tangent space $T_z Z$ at a point $z \in Z$ is contained in the kernel $N(f_0''(z))$ of $f_0''(z)$, knowing the eigenspaces we see

$$T_z Z = N\big(f_0''(z)\big) \quad \text{for all } z \in Z. \tag{2.5}$$

If (2.5) holds the critical manifold Z is called nondegenerate (see [1]). The operator $f_0''(z)$ maps the space $\mathcal{D}^{1,2}(\mathbb{R}^3)$ into $T_z Z^\perp$ and is invertible in $\mathcal{L}(T_z Z^\perp)$. From (2.3) and (2.4), we obtain in this case

$$\big\|\big(f_0''(z_{1,0})\big)^{-1}\big\|_{\mathcal{L}(T_{z_{1,0}}Z^\perp)} = \big\|\big(f_0''(z)\big)^{-1}\big\|_{\mathcal{L}(T_z Z^\perp)} \quad \forall z \in Z. \tag{2.6}$$

Moreover, $T_{z_{\mu,y}} Z^\perp$ splits orthogonally into (see [21])

$$T_{z_{\mu,y}} Z^\perp = \langle z_{\mu,y}\rangle \oplus^\perp \big\langle \Phi_{i,j,l}^{\mu,y}\colon\ i, j \in \mathbb{N}_0,\ 2 \leqslant i + j \leqslant n,\ 1 \leqslant l \leqslant c_i\big\rangle, \tag{2.7}$$

where $\Phi_{i,j,l}^{\mu,y}$ are eigenfunctions of $f_0''(z_{\mu,y})$ with positive eigenvalue

$$\lambda_{i,j} = 1 - \frac{15}{(4 + 2(i + j - 1))^2 - 1}.$$

The dimension of the eigenspace corresponding to $\lambda_{i.j}$ is denoted by c_i. The functions $\Phi_{i,j,l}^{\mu,y}$ are smooth and given in terms of Jacobi polynomials and spherical harmonics. The operator $f_0''(z_{\mu,y})$ has precisely one negative eigenvalue -4 with one-dimensional eigenspace $\langle z_{\mu,y}\rangle$.

3. The blow-up analysis

Based on the results in [17,23] we have the following lemma (see [21, Corollary 3.2])

Lemma 3.1. *Suppose* $1 + t_0 k \in C^1(S^3)$ *is positive and* $h \in C^\infty(S^3)$ *is nonnegative. If* $(s_i, \varphi_i) \in [0, 1] \times C^2(S^3)$ *solve* (1.5) *with* $s = s_i$*, then after passing to a subsequence either* (φ_i) *is uniformly bounded in* $L^\infty(S^3)$ (*and hence in* $C^{2,\alpha}(S^3)$ *by standard elliptic regularity*) *or there exist* $\theta \in S^3$ *and sequences* $(\mu_i) \in (0,\infty)$, $(y_i) \in \mathbb{R}^3$ *satisfying* $\lim_{i\to\infty} \mu_i = 0$ *and* $\lim_{i\to\infty} y_i = 0$, *such that* (*in stereographic coordinates* $\mathcal{S}_\theta(\cdot)$)

$$\mathcal{R}_\theta(\varphi_i) - \big(1 + t_0\big(k_\theta(y_i) + s_i h_\theta(y_i)\big)\big)^{-\frac{1}{4}} z_{\mu_i,y_i} \quad \textit{is orthogonal to } T_{z_{\mu_i,y_i}} Z,$$

$$\big\|\mathcal{R}_\theta(\varphi_i) - \big(1 + t_0\big(k_\theta(y_i) + s_i h_\theta(y_i)\big)\big)^{-\frac{1}{4}} z_{\mu_i,y_i}\big\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = o(1).$$

4. The finite dimensional reduction

For the rest of the paper, unless otherwise indicated, integration extends over $\mathbb{R}^3$ and is done with respect to the variable x.

Lemma 4.1. *Suppose* $1 + k \in C^5(S^3)$ *is a positive Morse function,* $t_0 \in (0, 1] \setminus T$, $h \in C^\infty(S^3)$ *satisfies* (1.4), *and* $\theta \in S^3$. *Then there exist* $s_0 = s_0(t_0, k, h) > 0$, $\mu_0 = \mu_0(t_0, k, h) > 0$ *and two functions* $w : \Omega \to \mathcal{D}^{1,2}(\mathbb{R}^3)$ *and* $\vec{\alpha} : \Omega \to \mathbb{R}^4$ *depending on* t_0, k, h, *and* θ, *where*

$$\Omega := \big\{(s, \mu, y) \in (-s_0, +s_0) \times (0, \mu_0) \times \mathbb{R}^3\big\}$$

such that for any $(s, \mu, y) \in \Omega$,

$$w(s, \mu, y) \quad \textit{is orthogonal to } T_{z_{\mu,y}} Z, \tag{4.1}$$

$$f'_{t_0,s}\big(z_{\mu,y}+w(s,\mu,y)\big)=\vec{\alpha}(s,\mu,y)\cdot\dot{\xi}_{\mu,y}\in T_{z_{\mu,y}}Z, \tag{4.2}$$

$$\big\|w(s,\mu,y)-w_0(s,\mu,y)\big\|+\big\|\vec{\alpha}(s,\mu,y)\big\|<\rho_0, \tag{4.3}$$

where $\{(\dot{\xi}_{\mu,y})_i\colon\ i=0\dots3\}$ *denotes the basis of* $T_{z_{\mu,y}}Z$ *given in* (1.3) *and*

$$w_0(s,\mu,y):=\Big(\big(1+t_0\big(k_\theta(y)+sh_\theta(y)\big)\big)^{-\frac{1}{4}}-1\Big)z_{\mu,y}.$$

The functions w *and* $\vec{\alpha}$ *are of class* C^2 *and unique in the sense that if* $(v,\vec{\beta})$ *satisfies* (4.1)–(4.3) *for some* $(s,\mu,y)\in\Omega$ *then* $(v,\vec{\beta})$ *is given by* $(w(s,\mu,y),\vec{\alpha}(s,\mu,y))$.

Moreover, we have as $\mu\to0$

$$\left|\vec{\alpha}(s,\mu,y)-\sum_{j=1}^{4}\vec{\alpha}_j(s,\mu,y)\right|=O\big(\mu^{4+\frac{1}{4}}+\mu^2\big|\nabla k_\theta(y)+s\nabla h_\theta(y)\big|^2\big)$$
$$+O\big(\mu^3\big|\nabla k_\theta(y)+s\nabla h_\theta(y)\big|+\mu^4\big|\Delta k_\theta(y)+s\Delta h_\theta(y)\big|\big),$$

where α_1, α_2 *are given by*

$$\vec{\alpha}_1(s,\mu,y):=-\mu\big(1+t_0\big(k_\theta(y)+sh_\theta(y)\big)\big)^{-\frac{5}{4}}\frac{t_0\pi}{3^{\frac{1}{4}}\sqrt{5}}\begin{pmatrix}0\\ \nabla k_\theta(y)+s\nabla h_\theta(y)\end{pmatrix},$$
$$\vec{\alpha}_2(s,\mu,y):=-\mu^2\big(1+t_0\big(k_\theta(y)+sh_\theta(y)\big)\big)^{-\frac{5}{4}}\frac{t_0\pi}{3^{\frac{1}{4}}\sqrt{5}}\begin{pmatrix}\Delta(k_\theta+sh_\theta)(y)\\ \vec{0}\end{pmatrix},$$

for $1\leqslant i\leqslant3$,

$$\vec{\alpha}_3(s,\mu,y)_i:=-\mu^3\big(1+t_0\big(k_\theta(y)+sh_\theta(y)\big)\big)^{-\frac{5}{4}}\frac{t_0\pi}{3^{\frac{1}{4}}2\sqrt{5}}\frac{\partial}{\partial x_i}\Delta(k_\theta+sh_\theta)(y),$$
$$\vec{\alpha}_4(s,\mu,y)_i:=-\mu^4\big(1+t_0(k_\theta+sh_\theta)(y)\big)^{-\frac{5}{4}}\frac{t_0 3^{\frac{3}{4}}8}{\pi\sqrt{5}}$$
$$\times\fint\big((k_\theta+sh_\theta)(x+y)-T^3_{(k_\theta+sh_\theta)(\cdot+y),0}(x)\big)\frac{x_i}{|x|^8},$$

and

$$\vec{\alpha}_3(s,\mu,y)_0:=-\mu^3\big(1+t_0(k_\theta+sh_\theta)(y)\big)^{-\frac{5}{4}}\frac{t_0 3^{\frac{3}{4}}4}{\pi\sqrt{5}}$$
$$\times\fint\big((k_\theta+sh_\theta)(x+y)-T^2_{(k_\theta+sh_\theta)(\cdot+y),0}(x)\big)\frac{1}{|x|^6},$$

$$\vec{\alpha}_4(s,\mu,y)_0:=\mu^4\big(1+t_0\big(k_\theta+sh_\theta\big)(y)\big)^{-\frac{5}{4}}\frac{t_0 3^{\frac{3}{4}}\pi\sqrt{5}}{30}\Delta^2(k_\theta+sh_\theta)(y)$$
$$+\frac{-t_0^2\mu^4 3^{\frac{3}{4}}\sqrt{5}}{16(1+t_0(k_\theta+sh_\theta)(y))^{\frac{9}{4}}}\int_{\partial B_1(0)}\big|D^2(k_\theta+sh_\theta)(y)(x)^2\big|^2\,dSx.$$

Replacing k by $k + sh$ the existence part, uniqueness, and the asymptotic estimates as $\mu \to 0$ follow directly from Lemmas 4.2–4.7 in [21]. It only remains to show the C^2-dependence on s, which we omit, since it is analogous to the proof given in [21].

Concerning the derivatives of $\vec{\alpha}$ with respect to μ and y we may apply the results in [21, Lemma 5.1] and [20, Lemma A.4–A.5] to obtain the following two lemmas.

Lemma 4.2. *Under the assumptions of Lemma* 4.1 *we have for all* $(s, \mu, y) \in \Omega$ *and* $1 \leqslant i, j \leqslant 3$

$$\begin{aligned}\frac{\partial \alpha(s,\mu,y)_i}{\partial y_j} &= -\frac{t_0 \mu \pi}{3^{\frac{1}{4}}\sqrt{5}}\big(1 + t_0(k_\theta + sh_\theta)(y)\big)^{-\frac{5}{4}} \frac{\partial^2 (k_\theta + sh_\theta)(y)}{\partial x_i \partial x_j} \\ &\quad + O\big(\big|\nabla(k_\theta + sh_\theta)(y)\big|^2 \mu + \mu^{2+\frac{1}{4}}\big), \\ \frac{\partial \alpha(s,\mu,y)_0}{\partial y_j} &= -\frac{t_0 \mu^2 \pi}{3^{\frac{1}{4}}\sqrt{5}}\big(1 + t_0(k_\theta + sh_\theta)(y)\big)^{-\frac{5}{4}} \frac{\partial}{\partial x_j}\Delta(k_\theta + sh_\theta)(y) \\ &\quad + O\big(\big|\nabla(k_\theta + sh_\theta)(y)\big|^2 \mu + \mu^{2+\frac{1}{4}}\big).\end{aligned}$$

Lemma 4.3. *Under the assumptions of Lemma* 4.1 *we have for all* $(s, \mu, y) \in \Omega$ *and* $1 \leqslant i \leqslant 3$

$$\begin{aligned}\frac{\partial \alpha(s,\mu,y)_i}{\partial \mu} &= \sum_{j=1}^{3} \frac{\partial \alpha_j(s,\mu,y)_i}{\partial \mu} + O\big(\big(\big|\nabla k_\theta(y)\big|^2 + \big|\nabla h_\theta(y)\big|^2\big)\mu + \mu^3\big), \\ \frac{\partial \alpha(s,\mu,y)_0}{\partial \mu} &= \sum_{j=2}^{4} \frac{\partial \alpha_j(s,\mu,y)_0}{\partial \mu} + O\big(\big(\big|\Delta k_\theta(y)\big| + \big|\Delta h_\theta(y)\big|\big)\mu^3 + \mu^{3+\frac{1}{4}}\big) \\ &\quad + O\big(\big(\big|\nabla k_\theta(y)\big|^2 + \big|\nabla h_\theta(y)\big|^2\big)\mu + \big(\big|\nabla k_\theta(y)\big| + \big|\nabla h_\theta(y)\big|\big)\mu^2\big).\end{aligned}$$

In order to compute the derivative of $\vec{\alpha}$ with respect to s one has to mimic the lengthy calculation of the t-derivative in [21, Lemma 5.2–5.3]. We will again just state the result and refer to [22] for details. This will be the last point where we are less precise concerning the s-dependence.

Lemma 4.4. *Under the assumptions of Lemma* 4.1 *we have for all* $(s, \mu, y) \in \Omega$ *and* $1 \leqslant i \leqslant 3$

$$\begin{aligned}\frac{\partial \alpha(s,\mu,y)_i}{\partial s} &= \sum_{j=1}^{3} \frac{\partial \alpha_j(s,\mu,y)_i}{\partial s} + O\big(\big(\big|\nabla k_\theta(y)\big|^2 + \big|\nabla h_\theta(y)\big|^2\big)\mu^2 + \mu^4\big), \\ \frac{\partial \alpha(s,\mu,y)_0}{\partial s} &= \sum_{j=2}^{4} \frac{\partial \alpha_j(s,\mu,y)_0}{\partial s} + O\big(\big(\big|\Delta k_\theta(y)\big| + \big|\Delta h_\theta(y)\big|\big)\mu^4 + \mu^{4+\frac{1}{4}}\big) \\ &\quad + O\big(\big(\big|\nabla k_\theta(y)\big|^2 + \big|\nabla h_\theta(y)\big|^2\big)\mu^2 + \big(\big|\nabla k_\theta(y)\big| + \big|\nabla h_\theta(y)\big|\big)\mu^3\big).\end{aligned}$$

Lemma 4.5. *Under the assumptions of Lemma* 4.1 *suppose*

$$\nabla k_\theta(0) = 0 \quad \textit{and} \quad \Delta k_\theta(0) = 0.$$

Consider the function $\hat{\alpha} : \Omega \to \mathbb{R}^3$, defined by

$$\hat{\alpha}(s,\mu,y) := \frac{3^{\frac{1}{4}}\sqrt{5}}{t_0\mu\pi}\big(1+t_0k(\theta)\big)^{\frac{5}{4}}\big(\vec{\alpha}(s,\mu,y)_1,\ldots,\vec{\alpha}(s,\mu,y)_3\big)^T.$$

Then there are $\mu_1 = \mu_1(t_0,k,h) > 0$ and a C^2-function $\beta : (-s_0,s_0)\times(0,\mu_1)\to\mathbb{R}^3$ depending on t_0, k, and h, such that

$$\beta(s,\mu) = -\mu^2\frac{1}{2}\big(D^2k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0) + O\big(\mu^3\big),$$

as $\mu\to 0$ and

$$\hat{\alpha}\big(s,\mu,\beta(s,\mu)\big) = 0 \quad \textit{for all } (s,\mu)\in(-s_0,s_0)\times(0,\mu_1).$$

Moreover, β is unique in the sense that, if $y\in B_{\mu_1}(0)$ satisfies $\hat{\alpha}(s,\mu,y)=0$ for some $s\in(-s_0,s_0)$ and $0<\mu<\mu_1$, then $y=\beta(t,\mu)$.

Proof. Lemma 4.2 suggests to apply the implicit function theorem, but unfortunately $\vec{\alpha}$ may not be differentiable for $\mu=0$. Instead we apply directly Banach's fixed-point theorem to the function

$$F(s,\mu,y) := y + \big(D^2k_\theta(0)\big)^{-1}\hat{\alpha}(s,\mu,y)$$

in $B_\delta(0)$, where $0<\delta<\operatorname{dist}(0,\operatorname{supp}(h_\theta))$ will be chosen later.

For $y\in B_\delta(0)$ we use the fact that $\nabla k_\theta(0)=0$ and get

$$\left(\frac{1+t_0k_\theta(0)}{1+t_0k_\theta(y)}\right)^{\frac{5}{4}} = 1 + O\big(\delta^2\big). \tag{4.4}$$

Fix $y_1, y_2\in B_\delta(0)$ and $(s,\mu)\in(-s_0,s_0)\times(0,\mu_0)$, then by Lemmas 4.2 and (4.4)

$$\begin{aligned}
&\big|F(s,\mu,y_1)-F(s,\mu,y_2)\big| \\
&\quad= \left|(y_1-y_2) + \big(D^2k_\theta(0)\big)^{-1}\int_0^1 \frac{\partial\hat{\alpha}}{\partial y}\big(s,\mu,y_2+t(y_1-y_2)\big)(y_1-y_2)\,dt\right| \\
&\quad\leqslant \left|(y_1-y_2) - \left(\int_0^1 \big(D^2k_\theta(0)\big)^{-1}D^2k_\theta\big(y_2+t(y_1-y_2)\big)\,dt\right)(y_1-y_2)\right| \\
&\qquad + O\Big(\delta^2 + \sup_{y\in B_\delta(0)}\big|\nabla k_\theta(y)\big| + \mu^{\frac{1}{4}}\Big)|y_1-y_2| \\
&\quad\leqslant O\big(\delta+\mu^{\frac{1}{4}}\big)|y_1-y_2|.
\end{aligned}$$

For $y\in B_\delta(0)$ we estimate using Lemma 4.1

$$\begin{aligned}\big|F(s,\mu,y)\big| &= \big|y+\big(D^2k_\theta(0)\big)^{-1}\big(\hat\alpha(s,\mu,y)\big)\big|\\ &\leqslant \big|y-\big(D^2k_\theta(0)\big)^{-1}\big(\nabla k_\theta(y)+O\big(\delta^2+\mu^2\big)\big)\big|\\ &\leqslant \big|y-\big(D^2k_\theta(0)\big)^{-1}\big(D^2k_\theta(0)y+O\big(\delta^2+\mu^2\big)\big)\big|\\ &\leqslant O\big(\delta^2+\mu^2\big).\end{aligned}$$

Consequently, there is $\mu_1>0$ such that $F(s,\mu,\cdot)$ is a contraction in $B_{\mu_1}(0)$ for any $0<\mu<\mu_1$ and $s\in[-s_0,s_0]$. From Banach's fixed-point theorem we may define $\beta(s,\mu)$ to be the unique fixed-point of $F(s,\mu,\cdot)$ in $B_{\mu_1}(0)$. After shrinking μ_1 if necessary we may apply Lemma 4.2 and the usual implicit function theorem to see that the function β is twice differentiable for $\mu>0$.

To deduce the expansion for small μ we fix $\rho>0$ and

$$y\in U_\rho := B_\rho\bigg(-\mu^2\frac{1}{2}\big(D^2k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0)\bigg).$$

Then, by Lemma 4.1 and (4.4)

$$\begin{aligned}&\bigg|F(s,\mu,y)+\mu^2\frac{1}{2}\big(D^2k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0)\bigg|\\ &\quad\leqslant \bigg|y+\big(D^2k_\theta(0)\big)^{-1}\bigg(\hat\alpha(s,\mu,y)+\mu^2\frac{1}{2}\nabla\Delta k_\theta(0)\bigg)\bigg|\\ &\quad\leqslant \bigg|y+\big(D^2k_\theta(0)\big)^{-1}\bigg(-\nabla k_\theta(y)-\mu^2\frac{1}{2}\big(\nabla\Delta k_\theta(y)-\nabla\Delta k_\theta(0)\big)\\ &\qquad+O\big(\mu|y|^2+\mu^2|y|+\mu^3\big)\bigg)\bigg|\\ &\quad\leqslant O\big(\rho^2+\mu^2\rho+\mu^3\big).\end{aligned}$$

Consequently, we may choose for small $0<\mu$ a radius $0<\rho=O(\mu^3)$ such that F maps $U_\rho\subset B_{\mu_1}(0)$ into itself. Consequently, the unique fixed-point $\beta(s,\mu)$ must lie in this ball. This ends the proof. □

Hence, to exclude or to construct blow-up sequences, which blow up at a nondegenerate critical point θ of k with $\Delta k_\theta(0)=0$ it suffices to study $\alpha(s,\mu,\beta(s,\mu))_0$.

Lemma 4.6. *Under the assumptions of Lemma* 4.5 *and* $k\in C^5(S^3)$ *we have*

$$\begin{aligned}\big(\alpha\big(s,\mu,\beta(s,\mu)\big)\big)_0 &= -t_0\mu^3\big(1+t_0k(\theta)\big)^{-\frac{5}{4}}\frac{3^{\frac{3}{4}}4}{\pi\sqrt5}\bigg(a_0(\theta)+s\int_{\mathbb{R}^3}h_\theta(x)|x|^{-6}\bigg)\\ &\quad+t_0\mu^4\frac{\pi 3^{\frac{3}{4}}\sqrt5}{30(1+t_0k(\theta))^{\frac{9}{4}}}\big(a_1(\theta)+t_0a_2(\theta)\big)+O\big(\mu^{4+\frac14}\big).\end{aligned}$$

Proof. In view of Lemma 4.5 and because $\nabla k_\theta(0) = 0$ we may estimate functions of $\beta(s,\mu)$ and of $k(\beta(s,\mu))$ as follows

$$F(\beta) = F(0) - \mu^2 F'(0)\frac{1}{2}\big(D^2 k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0) + O\big(\mu^3\big),$$
$$F\big(k(\beta)\big) = F\big(k_\theta(0)\big) + O\big(\mu^4\big). \tag{4.5}$$

To prove the claim of the lemma we expand $\alpha(s,\mu,\beta(s,\mu))_0$ according to Lemma 4.1 and use (4.5). □

Lemma 4.7. *Under the assumptions of Lemma* 4.5 *suppose*

$$a_0(\theta) = 0 \quad \textit{and} \quad a_1(\theta) + t_0 a_2(\theta) > 0,$$

and define

$$\gamma(s,\mu) := -\frac{1}{t_0\mu^3}\big(1 + tk(\theta)\big)^{\frac{5}{4}}\frac{\pi\sqrt{5}}{3^{\frac{3}{4}}4}\alpha\big(s,\mu,\beta(s,\mu)\big)_0.$$

Then as $\mu \to 0$

$$\frac{\partial\gamma(s,\mu)}{\partial s} = \int h_\theta(x)|x|^{-6} + O(\mu), \tag{4.6}$$
$$\frac{\partial\gamma(s,\mu)}{\partial\mu} = -\frac{\pi^2}{24}\big(1 + t_0 k(\theta)\big)^{-1}\big(a_1(\theta) + t_0 a_2(\theta)\big) + O\big(\mu^{\frac{1}{4}}\big). \tag{4.7}$$

Proof. As $\nabla k_\theta(0) = 0$ we get from (1.4) that $\operatorname{dist}(0, \operatorname{supp}(h_\theta)) > 0$. As $\beta(s,\mu) = O(\mu^2)$ as $\mu \to 0$ we get that any term which depends only locally on sh_θ is independent of s for small $\mu > 0$.

We have

$$\frac{d}{ds}\alpha\big(s,\mu,\beta(s,\mu)\big)_0 = \frac{\partial(\alpha)_0}{\partial s}\bigg|_{(s,\mu,\beta(s,\mu))} + \frac{\partial(\alpha)_0}{\partial y}\bigg|_{(s,\mu,\beta(s,\mu))}\frac{\partial\beta}{\partial s}\bigg|_{(s,\mu)}.$$

The derivatives of $\alpha(\cdot)_0$ are given in Lemmas 4.2–4.4. To compute the derivative of β we use the fact that $\hat{\alpha}(s,\mu,\beta(s,\mu)) \equiv 0$. By (4.5) and Lemmas 4.2–4.4 we have

$$\begin{aligned}\frac{\partial\beta}{\partial s}\bigg|_{(s,\mu)} &= -\bigg(\frac{\partial\hat{\alpha}}{\partial y}\bigg|_{(s,\mu,\beta(s,\mu))}\bigg)^{-1}\frac{\partial\hat{\alpha}}{\partial s}\bigg|_{(s,\mu,\beta(s,\mu))} \\ &= \Big(\big(D^2 k_\theta(0)\big)^{-1} + O\big(\mu^{1+\frac{1}{4}}\big)\Big) \\ &\quad\times\frac{3^{\frac{1}{4}}\sqrt{5}}{t_0\mu\pi}\big(1 + t_0 k(\theta)\big)^{\frac{5}{4}}\left[\sum_{j=1}^{3}\frac{\alpha_j(s,\mu,\beta)_i}{\partial s} + O\big(\mu^4\big)\right]_{i=1\ldots3} \\ &= O\big(\mu^3\big),\end{aligned}$$

where we used that $a_j(s,\mu,y)_i$ is independent of s for small $|y|$, $\mu > 0$. From Lemma 4.2 we get

$$\frac{\partial(\alpha)_0}{\partial y}\bigg|_{(s,\mu,\beta(s,\mu))}\frac{\partial\beta}{\partial s}\bigg|_{(s,\mu)} = O\big(\mu^5\big).$$

Furthermore, by Lemma 4.4

$$\begin{aligned}\frac{d\alpha(s,\mu,\beta)_0}{ds} &= \sum_{j=2}^{4}\frac{\partial\vec{\alpha}_j(s,\mu,\beta)_0}{\partial s} + O\big(\mu^{4+\frac{1}{4}}\big)\\ &= -t_0\mu^3\big(1+t_0k_\theta(0)\big)^{-\frac{5}{4}}\frac{3^{\frac{3}{4}}4}{\pi\sqrt{5}}\oint h_\theta(x)\frac{1}{|x|^6} + O\big(\mu^{4+\frac{1}{4}}\big).\end{aligned}$$

The definition of γ, (4.5), and Lemma 4.6 yield (4.6).

Concerning (4.7) we get

$$\frac{d}{d\mu}\alpha\big(s,\mu,\beta(s,\mu)\big)_0 = \frac{\partial(\alpha)_0}{\partial\mu}\bigg|_{(s,\mu,\beta(s,\mu))} + \frac{\partial(\alpha)_0}{\partial y}\bigg|_{(s,\mu,\beta(s,\mu))}\frac{\partial\beta}{\partial\mu}\bigg|_{(s,\mu)}.$$

By (4.5) and Lemmas 4.2–4.3 we have

$$\begin{aligned}\frac{\partial\beta}{\partial\mu}\bigg|_{(s,\mu)} &= -\left(\frac{\partial\hat{\alpha}}{\partial y}\bigg|_{(s,\mu,\beta(s,\mu))}\right)^{-1}\frac{\partial\hat{\alpha}}{\partial\mu}\bigg|_{(s,\mu,\beta(s,\mu))}\\ &= \Big(\big(D^2k_\theta(0)\big)^{-1} + O\big(\mu^{1+\frac{1}{4}}\big)\Big)\\ &\quad\times\frac{3^{\frac{1}{4}}\sqrt{5}}{t_0\mu\pi}\big(1+t_0k(\theta)\big)^{\frac{5}{4}}\left[\sum_{j=1}^{3}\frac{\alpha_j(s,\mu,\beta)_i}{\partial\mu} + O\big(\mu^3\big)\right]_{i=1\dots3}\\ &= \Big(\big(D^2k_\theta(0)\big)^{-1} + O\big(\mu^{1+\frac{1}{4}}\big)\Big)\\ &\quad\times\left(-\mu\nabla\Delta k_\theta(0) + \frac{3^{\frac{1}{4}}\sqrt{5}}{t_0\mu\pi}\left[\frac{1}{\mu}\sum_{j=1}^{3}\alpha_j(s,\mu,\beta)_i + O\big(\mu^3\big)\right]_{i=1\dots3}\right)\\ &= -\mu\big(D^2k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0) + O\big(\mu^2\big).\end{aligned}$$

Hence, by Lemmas 4.2 and 4.3

$$\begin{aligned}&\frac{\partial(\alpha)_0}{\partial y}\bigg|_{(s,\mu,\beta(s,\mu))}\frac{\partial\beta}{\partial\mu}\bigg|_{(s,\mu)}\\ &\quad= \frac{t_0\mu^3\pi}{3^{\frac{1}{4}}\sqrt{5}}\big(1+t_0k_\theta(0)\big)^{-\frac{5}{4}}\nabla\Delta k_\theta(0)\big(D^2k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0) + O\big(\mu^{3+\frac{1}{4}}\big),\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial\gamma(s,\mu)}{\partial\mu} &= \frac{3}{t_0\mu^4}\big(1+tk(\theta)\big)^{\frac{5}{4}}\frac{\pi\sqrt{5}}{3^{\frac{3}{4}}4}\alpha\big(s,\mu,\beta(s,\mu)\big)_0 \\
&\quad - \frac{1}{t_0\mu^3}\big(1+tk(\theta)\big)^{\frac{5}{4}}\frac{\pi\sqrt{5}}{3^{\frac{3}{4}}4}\sum_{j=2}^{4}\frac{\partial\alpha_j(s,\mu,\beta)_0}{\partial\mu} \\
&\quad - \frac{\pi^2}{12}\nabla\Delta k_\theta(0)\big(D^2k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0) + O\big(\mu^{3+\frac{1}{4}}\big) \\
&= \frac{3}{t_0\mu^4}\big(1+tk(\theta)\big)^{\frac{5}{4}}\frac{\pi\sqrt{5}}{3^{\frac{3}{4}}4}\sum_{j=2}^{4}\alpha_j(s,\mu,\beta)_0 - \frac{(1+tk(\theta))^{\frac{5}{4}}}{t_0\mu^4}\frac{\pi\sqrt{5}}{3^{\frac{3}{4}}4} \\
&\quad \times\left(3\sum_{j=2}^{4}\alpha_j(s,\mu,\beta)_0 - \alpha_2(s,\mu,\beta)_0 + \alpha_4(s,\mu,\beta)_0\right) \\
&\quad - \frac{\pi^2}{12}\nabla\Delta k_\theta(0)\big(D^2k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0) + O\big(\mu^{\frac{1}{4}}\big) \\
&= -\frac{1}{t_0\mu^4}\big(1+tk(\theta)\big)^{\frac{5}{4}}\frac{\pi\sqrt{5}}{3^{\frac{3}{4}}4}\big(-\alpha_2(s,\mu,\beta)_0 + \alpha_4(s,\mu,\beta)_0\big) \\
&\quad - \frac{\pi^2}{12}\nabla\Delta k_\theta(0)\big(D^2k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0) + O\big(\mu^{\frac{1}{4}}\big).
\end{aligned}
$$

If we use (4.5) and the expansion in Lemma 4.1 we find

$$
\begin{aligned}
&-\frac{1}{t_0\mu^4}\big(1+tk(\theta)\big)^{\frac{5}{4}}\frac{\pi\sqrt{5}}{3^{\frac{3}{4}}4}\big(-\alpha_2(s,\mu,\beta)_0 + \alpha_4(s,\mu,\beta)_0\big) \\
&\quad = \frac{\pi^2}{24}\nabla\Delta k_\theta(0)\big(D^2k_\theta(0)\big)^{-1}\nabla\Delta k_\theta(0) \\
&\qquad - \frac{\pi^2}{24}\Delta^2k_\theta(0) + \frac{t_0}{1+t_0k(\theta)}\frac{5\pi}{64}\int_{\partial B_1(0)}\big|D^2k_\theta(0)(x)^2\big|^2\,dSx.
\end{aligned}
$$

Summing up yields the claim of the lemma. □

Lemma 4.8. *Under the assumptions of Lemma* 4.1 *we define* $M_*\subset S^3$ *by*

$$
M_* := \big\{\theta\in \operatorname{Crit}(k)\colon\ \Delta k_\theta(0)=0=a_0(\theta),\ a_1(\theta)+t_0a_2(\theta)>0\big\}. \tag{4.8}
$$

Then there is $\delta>0$ *such that for any* $\theta\in M_*$ *there exists a unique* C^1*-curve*

$$
\{0<\mu<\delta\}\ni\mu\mapsto\big(s^\theta(\mu),\varphi^\theta(\mu,\cdot)\big)\in(0,\delta)\times C^{2,\alpha}\big(S^3\big),
$$

such that as $\mu\to 0$

$$s^\theta(\mu) = \mu\frac{\pi^2}{24}\left(\int h_\theta(x)|x|^{-6}\right)^{-1}\frac{a_1(\theta)+t_0a_2(\theta)}{1+t_0k(\theta)} + O\left(\mu^{1+\frac{1}{4}}\right),$$
$$\frac{\partial s^\theta}{\partial\mu}(\mu) = \frac{\pi^2}{24}\left(\int h_\theta(x)|x|^{-6}\right)^{-1}\frac{a_1(\theta)+t_0a_2(\theta)}{1+t_0k(\theta)} + O\left(\mu^{\frac{1}{4}}\right).$$

and $\varphi^\theta(\mu,\cdot)$ *solves* (1.1) *for* $s = s^\theta(\mu)$ *and blows up like*

$$\left\|\mathcal{R}_\theta\left(\varphi^\theta(\mu,x)\right) - \left(1+t_0k(\theta)\right)^{-\frac{1}{4}}z_{\mu,0}(x)\right\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)\cap C^2(B_1(0))} = O\left(\mu^2\right).$$

The curves are unique, in the sense that, if $(s_i,\varphi_i) \in (0,\delta)\times C^{2,\alpha}(S^3)$ *blow up at some* $\theta\in S^3$ *then* $\theta\in M_*$ *and there is a sequence of positive numbers* (μ_i) *converging to zero such that* $(s_i,\varphi_i) = (s^\theta(\mu_i),\varphi^\theta(\mu_i,\cdot))$ *for all but finitely many* $i\in\mathbb{N}$.

Proof. We fix $\theta\in M_*$. To construct $s^\theta(\mu)$ we proceed as in Lemma 4.5 and use Banach's fixed-point theorem applied to

$$F_2(s,\mu) := s - \left(\int h_\theta(x)|x|^{-6}\right)^{-1}\gamma(s,\mu).$$

Since we know the expansion of γ and $\frac{\partial\gamma}{\partial s}$ as $\mu\to 0$ it is easy to see that $F_2(\cdot,\mu)$ is a contraction in

$$B_r\left(\mu\frac{\pi^2}{24}\left(\int h_\theta(x)|x|^{-6}\right)^{-1}\frac{a_1(\theta)+t_0a_2(\theta)}{1+t_0k(\theta)}\right)$$

for any $0 < \text{const}\,\mu^{1+\frac{1}{4}} \leqslant r \leqslant r_1$ and the existence part of the claim follows from that. The differentiability of s with respect to μ follows from Lemma 4.7 and the usual implicit function theorem.

Assume (s_i,φ_i) blow up at some $\theta\in S^3$. Then we apply Lemma 3.1 and find in stereographic coordinates $\mathcal{S}_\theta$ sequences $y_i\to 0$, $\mu_i\to 0$ such that

$$\mathcal{R}_\theta(\varphi_i)(x) - \left(1+t_0\left(k(\theta)+s_ih(\theta)\right)\right)^{-\frac{1}{4}}z_{\mu_i,y_i}(x)$$

is orthogonal to $T_{\mu_i,y_i}Z$ and converges to 0 as $i\to\infty$. Consequently, if we set

$$w(i) := \mathcal{R}_\theta(\varphi_i) - z_{\mu_i,y_i}$$

we find as z_{μ_i,y_i} is orthogonal to $T_{\mu_i,y_i}Z$,

$$w(i)\quad\text{is orthogonal to } T_{\mu_i,y_i}Z \quad\text{and}\quad w(i) - w_0(s_i,\mu_i,y_i) \to_{i\to\infty} 0,$$

where w_0 is defined in Lemma 4.1. Moreover

$$0 = f'_{t_0,s}\left(\mathcal{R}_\theta(\varphi_i)\right) = f'_{t_0,s}\left(z_{\mu,y}+w(i)\right).$$

The uniqueness part of Lemma 4.1 shows for large i

$$w(i) = w(s_i, \mu_i, y_i) \text{ and } \vec{\alpha}(s_i, \mu_i, y_i) = 0.$$

As $\mu_i \to 0$ the expansion of $\vec{\alpha}$ of order μ and μ^2 in Lemma 4.1 shows

$$\nabla k_\theta(0) = 0 \quad \text{and} \quad \Delta k_\theta(0) = 0.$$

From Lemma 4.5 we infer that

$$y_i = \beta(s_i, \mu_i)$$

and the expansion in Lemma 4.6 gives

$$0 = -t_0 \mu_i^3 \big(1 + t_0 k(\theta)\big)^{-\frac{5}{4}} \frac{3^{\frac{3}{4}} 4}{\pi\sqrt{5}} \Big(a_0(\theta) + s_i \int\limits_{\mathbb{R}^3} h_\theta(x)|x|^{-6}\Big)$$
$$+ t_0 \mu_i^4 \big(1 + t_0 k(\theta)\big)^{-\frac{9}{4}} \frac{\pi 3^{\frac{3}{4}} \sqrt{5}}{30} \big(a_1(\theta) + t_0 a_2(\theta)\big) + O\big(\mu_i^{4+\frac{1}{4}}\big).$$

Consequently

$$\Big(a_0(\theta) + s_i \int\limits_{\mathbb{R}^3} h_\theta(x)|x|^{-6}\Big) \to 0 \quad \text{as } i \to \infty,$$

and from the choice of h, assuming $0 < \delta < s_0$, we deduce that $a_0(\theta) = 0$. Hence

$$s_i \int\limits_{\mathbb{R}^3} h_\theta(x)|x|^{-6} = \mu_i \big(1 + t_0 k(\theta)\big)^{-1} \frac{\pi^2}{24} \big(a_1(\theta) + t_0 a_2(\theta)\big) + O\big(\mu_i^{1+\frac{1}{4}}\big).$$

Thus, $a_1(\theta) + t_0 a_2(\theta)$ has to be positive, which shows $\theta \in M_*$, and for large i

$$s_i \in B_{r_1}\Big(\mu_i \frac{\pi^2}{24} \Big(\int h_\theta(x)|x|^{-6}\Big)^{-1} \frac{a_1(\theta) + t_0 a_2(\theta)}{1 + t_0 k(\theta)}\Big).$$

The uniqueness of the fixed point implies $s_i = s_i^\theta(\mu_i)$ and the claim follows. □

5. The Leray–Schauder degree

From Section 1 we know that the degree $\deg(Id - L_s, \mathcal{B}_{C_\delta}, 0)$ of the problem (1.5) is independent of $s \in [\delta, s_0]$ and equals

$$\deg(Id - L_s, \mathcal{B}_{C_\delta}, 0) = -\Big(1 + \sum_{\theta \in \mathrm{Crit}_-(k+sh)} (-1)^{\mathrm{ind}(k,\theta)}\Big),$$

where the set $\mathrm{Crit}_-(k+sh)$ is independent of s and given by

$$\mathrm{Crit}_-(k+sh)=\big\{\theta\in\mathrm{Crit}(k)\colon\ \Delta k(\theta)<0 \text{ or } \big(\Delta k(\theta)=0 \text{ and } a_0(\theta)<0\big)\big\}.$$

By Lemma 4.8 and the a priori estimate for $s=0$ the set of functions

$$L_b:=\Big\{\varphi \text{ solves (1.5) for some } s\in[0,s_0],\ \varphi\notin\bigcup_{\theta\in M_*}\big\{\varphi^\theta\big(s^\theta(\mu),\cdot\big)\colon\ 0<\mu<\delta\big\}\Big\}$$

is uniformly bounded from above and by standard elliptic regularity also in $C^{2,\alpha}(S^3)$. By Sobolev's and Harnack's inequality this gives a uniform lower bound, thus there is $C_1>0$ such that $L_b\subset\mathcal{B}_{C_1}$.

Again from Lemma 4.8 and since $\frac{\partial s^\theta}{\partial\mu}$ is uniformly positive, there is $s_1>0$ small, such that for any $0<s\leqslant s_1$ and any $\theta\in M_*$ there exists exactly one $\mu^\theta(s)\in(0,\delta)$ satisfying

$$s^\theta\big(\mu^\theta(s)\big)=s.$$

Moreover, we may assume, shrinking s_1

$$\big\|\varphi^\theta\big(\mu^\theta(s),\cdot\big)\big\|_\infty\geqslant 2C_1\quad\forall\theta\in M_*,$$
$$\big\|\varphi^{\theta_1}\big(\mu^{\theta_1}(s),\cdot\big)-\varphi^{\theta_2}\big(\mu^{\theta_1},\cdot\big)\big\|_\infty\geqslant C_1\quad\forall\theta_1\neq\theta_2 \text{ in } M_*.$$

Hence, there are two types of solutions to (1.5) as $s\to0^+$: the solutions in $L_b\subset\mathcal{B}_{C_1}$ remain uniformly bounded as $s\to0^+$ and the solutions $\{\varphi^\theta(\mu^\theta(s),\cdot)\colon\ \theta\in M_*\}$ that blow up as $s\to0^+$ and are uniformly isolated for each fixed small $s>0$. Consequently, using the additivity of the degree, we find for any $0<s\leqslant s_1$

$$\begin{aligned}
&\deg(Id-L_s,\mathcal{B}_{C_s},0)\\
&\quad=\deg(Id-L_s,\mathcal{B}_{C_1},0)+\sum_{\theta\in M_*}\deg_{loc}\big(Id-L_s,\varphi^\theta\big(\mu^\theta(s),\cdot\big)\big)\\
&\quad=\deg(Id-L_0,\mathcal{B}_{C_1},0)+\sum_{\theta\in M_*}\deg_{loc}\big(Id-L_s,\varphi^\theta\big(\mu^\theta(s),\cdot\big)\big).
\end{aligned}$$

Together with (1.6) we get for any $0<s\leqslant s_1$

$$\begin{aligned}
&\deg(Id-L_0,\mathcal{B}_{C_1},0)\\
&\quad=-\Big(1+\sum_{\theta\in\mathrm{Crit}_-(k+sh)}(-1)^{\mathrm{ind}(k,\theta)}\Big)-\sum_{\theta\in M_*}\deg_{loc}\big(Id-L_s,\varphi^\theta\big(\mu^\theta(s),\cdot\big)\big).
\end{aligned}$$

It remains to compute the local degree $\deg_{loc}(Id-L_s,\varphi^\theta(\mu^\theta(s),\cdot))$ for any $\theta\in M_*$. We use the transformation $\mathcal{R}_\theta$ in (2.2) to define the weighted space

$$C^2(\mathbb{R}^3, \mathcal{R}_\theta) := \{u \in C^2(\mathbb{R}^3): u \in \mathcal{R}_\theta(C^2(S^3))\},$$
$$\|u\|_{C^2(\mathbb{R}^3,\mathcal{R}_\theta)} := \|(\mathcal{R}_\theta)^{-1}(u)\|_{C^2(S^3)}.$$

Note that $C^2(\mathbb{R}^3, \mathcal{R}_\theta) \hookrightarrow \mathcal{D}^{1,2}(\mathbb{R}^3)$, because $\mathcal{R}_\theta$ is an isomorphism between $H^{1,2}(S^3)$ and $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Using $\mathcal{R}_\theta$ we obtain

$$\begin{aligned}\deg_{loc}(Id - L_s, \varphi^\theta(\mu^\theta(s), \cdot)) &= \deg_{loc}(Id - \mathcal{R}_\theta L_s(\mathcal{R}_\theta)^{-1}, u_{\theta,s}) \\ &= \deg_{loc}(f'_{t_0,s}, u_{\theta,s}),\end{aligned}$$

where $u_{\theta,s} = \mathcal{R}_\theta(\varphi^\theta(\mu^\theta(s), \cdot)) \in C^2(\mathbb{R}^3, \mathcal{R}_\theta)$. Note that by duality we consider $f'_{t_0,s}$ as a map from the Hilbert space $D^{1,2}(\mathbb{R}^3)$ into itself.

Lemma 5.1. *Under the assumptions of Lemma* 4.8 *there holds for* $0 < s \leqslant s_1$

$$\sum_{\theta \in M_*} \deg_{loc} l(f'_{t_0,s}, u_{\theta,s}) = \sum_{\theta \in M_*} (-1)^{\mathrm{ind}(k,\theta)}.$$

Proof. Fix $\theta \in M_*$. The solution $u_{\theta,s} \in C^2(\mathbb{R}^3, \mathcal{R}_\theta)$ is given in notation of Lemmas 4.1 and 4.5 by

$$u_{\theta,s} = z_{\mu^\theta(s), y^\theta(s)} + w(s, \mu^\theta(s), y^\theta(s)),$$

where $y^\theta(s) = \beta^\theta(s, \mu^\theta(s))$. $(\mu^\theta(s), y^\theta(s))$ is the only zero of $\vec{\alpha}(s, \cdot, \cdot)$ for μ and $|y|$ bounded above by a small fixed constant. As $y^\theta(s) = O(s^2)$ we may replace y^θ by 0 (in various expressions below) and get an addition $O(s^2)$-error.

We drop the s-dependence of μ^θ and y^θ in the notation when there is no possibility of confusion. Moreover by Lemma 4.8 we have $s^\theta(\mu) \sim \mu$ and we may estimate the errors in terms of s.

As seen above by Lemma 4.8 the solution $u_{\theta,s}$ remains uniform isolated in $C^2(\mathbb{R}^3, \mathcal{R}_\theta)$ as well as in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ for $s \in (0, s_1]$. From (4.2) and regularity results [8,19] we infer that $w(s, \mu, y) \in C^2(\mathbb{R}^3, \mathcal{R}_\theta)$ depends continuously on (s, μ, y).

To compute the local degree, we first show that $f''_{t_0,s}(u_{\theta,s})$ is nondegenerate. To this end we let

$$\varphi(s,\theta)_0 := \mu^\theta c_\xi^{-1} \frac{\partial}{\partial \mu}\left(z_{\mu,\beta(s,\mu)} + w(s, \mu, \beta(s,\mu))\right)\Big|_{\mu^\theta},$$
$$\varphi(s,\theta)_i := \mu^\theta c_\xi^{-1} \frac{\partial}{\partial y_i}\left(z_{\mu^\theta,y} + w(s, \mu^\theta, y)\right)\Big|_{y^\theta}, \quad i = 1 \ldots 3.$$

The derivatives of β and w with respect to μ are computed in [20, Appendix A] the derivatives of w with respect to y_i are given in [21, Lemma 5.1]. We have

$$\left|\frac{\partial \beta}{\partial \mu}(s, \mu^\theta)\right| = O(s),$$

$$\left\| \frac{\partial w}{\partial \mu}\big(s,\mu^\theta,y^\theta\big) - \frac{\partial w_0}{\partial \mu}\big(s,\mu^\theta,y^\theta\big) \right\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = O(s),$$
$$\left\| \frac{\partial w}{\partial y_i}\big(s,\mu^\theta,y^\theta\big) - \frac{\partial w_0}{\partial y_i}\big(s,\mu^\theta,y^\theta\big) \right\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = O(s).$$

Therefore we get

$$u_{\theta,s} = \big(1+t_0k_\theta(0)\big)^{-\frac{1}{4}} z_{\mu^\theta,y^\theta} + O\big(s^2\big)_{\mathcal{D}^{1,2}(\mathbb{R}^3)},$$
$$\varphi(s,\theta)_0 = \big(1+t_0k_\theta(0)\big)^{-\frac{1}{4}} (\dot{\xi}_{\mu^\theta,y^\theta})_0 + O\big(s^2\big)_{\mathcal{D}^{1,2}(\mathbb{R}^3)},$$
$$\varphi(s,\theta)_i = \big(1+t_0k_\theta(0)\big)^{-\frac{1}{4}} (\dot{\xi}_{\mu^\theta,y^\theta})_i + O\big(s^2\big)_{\mathcal{D}^{1,2}(\mathbb{R}^3)},$$

By Lemma 4.8 and Lemma 4.1 we find

$$f'_{t_0,s}\big(z_{\mu,\beta(s,\mu)} + w\big(s,\mu,\beta(s,\mu)\big)\big) = \alpha\big(s,\mu,\beta(s,\mu)\big)_0\big(\dot{\xi}_{\mu,\beta(s,\mu)}\big)_0.$$

Differentiating with respect to μ by Lemma 4.7 leads to

$$\begin{aligned} &f''_{t_0,s}(u_{\theta,s})\big(\mu^\theta\big)^{-1} c_\xi \varphi(s,\theta)_0 \\ &\quad = \left(t_0\big(\mu^\theta\big)^3\big(1+tk_\theta(0)\big)^{-\frac{9}{4}} \frac{3^{\frac{3}{4}}}{6\sqrt{5}}\big(a_1(\theta)+t_0a_2(\theta)\big) + O\big(s^{3+\frac{1}{4}}\big)\right)(\dot{\xi}_{\mu^\theta,y^\theta})_0. \end{aligned}$$

Moreover, differentiating

$$f'_{t_0,s}\big(z_{\mu,y} + w(s,\mu,y)\big) = \sum_{i=0}^{3} \alpha(s,\mu,y)_i(\dot{\xi}_{\mu,y})_i$$

with respect to y_j we get from Lemma 4.2

$$\begin{aligned} f''_{t_0,s}(u_{\theta,s})\frac{c_\xi}{\mu^\theta}\varphi(s,\theta)_j = {}& -\frac{t_0\mu^\theta\pi\sum_{i=1}^3\big(\frac{\partial^2 k_\theta}{\partial x_i\partial x_j}(0)+O(s)\big)(\dot{\xi}_{\mu^\theta,y^\theta})_i}{3^{\frac{1}{4}}\sqrt{5}(1+t_0k_\theta(0))^{\frac{5}{4}}} \\ & + O\big(s^2\big)(\dot{\xi}_{\mu^\theta,y^\theta})_0. \end{aligned}$$

Orthogonal to $T_{z_{\mu^\theta,y^\theta}}Z$ we use

$$\begin{aligned} f''_{t_0,s}(u_{s,\theta}) &= f''_0(z_{\mu^\theta,y^\theta}) + O\big(\big\|w\big(s,\mu^\theta,y^\theta\big) - w_0\big(s,\mu^\theta,y^\theta\big)\big\|\big)_{\mathcal{L}(\mathcal{D}^{1,2}(\mathbb{R}^3))} \\ &\quad - \frac{5t_0}{(1+t_0k_\theta(y^\theta))}\int_{\mathbb{R}^3}\big((k_\theta+sh_\theta)(x)-k_\theta(y)\big)(z_{\mu^\theta,y^\theta})^4\cdot\cdot\,dx \\ &= f''_0(z_{\mu^\theta,y^\theta}) + O\big(\mu^\theta\big)_{\mathcal{L}(\mathcal{D}^{1,2}(\mathbb{R}^3))}. \end{aligned} \tag{5.1}$$

The $O(\mu)$-estimates are given in [21, Lemma 4.1] or can be obtained using Hölder's and Sobolev's inequality and the fact that $k_\theta(x) - k_\theta(y)$ is bounded in $\mathbb{R}^3$ and of order $O(|x-y|)$ for $|x-y| \ll 1$.

To obtain a contradiction assume there is a function $v \in C^2(\mathbb{R}^3, \mathcal{R}_\theta) \setminus \{0\}$ with $f''_{t_0,s}(u_{\theta,s})v = 0$. We may assume $\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = 1$. Then by (5.1)

$$O(s) = \big\| f''_0(z_{\mu^\theta,y^\theta})v \big\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \geqslant c \|Proj_{T_{z_{\mu^\theta,y^\theta}} Z^\perp} v\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)},$$

because $f''_0(z_{\mu^\theta,y^\theta})$ is an isomorphism of $T_{z_{\mu^\theta,y^\theta}} Z^\perp$. Moreover,

$$\begin{aligned} 0 &= f''_{t_0,s}(u_{\theta,s}) c_\xi \varphi(s,\theta)_0 v \\ &= \left(\frac{3^{\frac{3}{4}} t_0 (\mu^\theta)^4}{6\sqrt{5}(1 + t k_\theta(0))^{\frac{9}{4}}} \big(a_1(\theta) + t_0 a_2(\theta)\big) + O\big(s^{3+\frac{1}{4}}\big) \right) \big\langle (\dot{\xi}_{\mu^\theta,y^\theta})_0, v \big\rangle_{\mathcal{D}^{1,2}(\mathbb{R}^3)}, \end{aligned}$$

and

$$\begin{aligned} \vec{0} &= \big(f''_{t_0,s}(u_{\theta,s}) c_\xi \varphi(s,\theta)_j v \big)_j \\ &= -\frac{t_0 (\mu^\theta)^2 \pi}{3^{\frac{1}{4}} \sqrt{5}(1 + t_0 k_\theta(0))^{-\frac{5}{4}}} \big(D^2 k_\theta(0) + O(s) \big) \big(\big\langle (\dot{\xi}_{\mu^\theta,y^\theta})_i, v \big\rangle_{\mathcal{D}^{1,2}(\mathbb{R}^3)} \big)_i \\ &\quad + O\big(s^3\big) \big\langle (\dot{\xi}_{\mu^\theta,y^\theta})_0, v \big\rangle_{\mathcal{D}^{1,2}(\mathbb{R}^3)}. \end{aligned}$$

Since $D^2 k_\theta(0)$ is invertible, we see that $Proj_{T_{z_{\mu^\theta,y^\theta}} Z} v = 0$, contradicting the fact that $\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^3)} = 1$. Since $f''_{t_0,s}(u_{\theta,s})$ is of the form *id-compact* in $C^2(\mathbb{R}^3, \mathcal{R}_\theta)$ (as well as in $\mathcal{D}^{1,2}(\mathbb{R}^3)$) we get

$$\big\| f''_{t_0,s}(u_{\theta,s}) v \big\|_{C^2(\mathbb{R}^3,\mathcal{R}_\theta)} \geqslant c \|v\|_{C^2(\mathbb{R}^3,\mathcal{R}_\theta)}.$$

For $f'_{t_0,s}(u) = f''_{t_0,s}(u_{\theta,s})(u - u_{\theta,s}) + O(\|u - u_{\theta,s}\|^2_{C^2(\mathbb{R}^3,\mathcal{R}_\theta)})$,

$$\deg_{loc}(f'_{t_0,s}, u_{\theta,s}) = \deg_{loc}\big(f''_{t_0,s}(u_{\theta,s}), 0\big).$$

To compute $\deg_{loc}(f''_{t_0,s}(u_{\theta,s}), 0)$ we consider the finite dimensional spaces (see (2.7))

$$\begin{aligned} X_{n,s} := \langle u_{\theta,s} \rangle &\oplus \big\langle \varphi(s,\theta)_0 \big\rangle \oplus \big\langle \varphi(s,\theta)_i \colon 1 \leqslant i \leqslant 3 \big\rangle \\ &\oplus \big\langle \Phi^{\mu^\theta,y^\theta}_{i,j,l} \colon i,j \in \mathbb{N}_0,\ 2 \leqslant i+j \leqslant n,\ 1 \leqslant l \leqslant c_i \big\rangle. \end{aligned}$$

The functions, spanning $X_{n,s}$, are a basis, as they are orthogonal in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ up to an $O(s^2)$-error. The linear operator $Proj_{X_{n,s}} f''_{t_0,s}(u_{\theta,s})$ restricted to $X_{n,s}$ is given by, up to a multiplication of the elements in the diagonal by positive constants

$$\begin{pmatrix} -4 & 0 & 0 & 0 \\ 0 & \mu^4(a_1(\theta)+t_0a_2(\theta)) & 0 & 0 \\ 0 & 0 & -\mu^2 D^2 k_\theta(0) & 0 \\ 0 & 0 & 0 & f_0''(z_{\mu^\theta,y^\theta})|_{\langle \Phi_{i,j,l}^{\mu^\theta,y^\theta}\rangle} \end{pmatrix} + \begin{pmatrix} O(\mu) & O(\mu) & O(\mu) & O(\mu) \\ O(\mu^6) & O(\mu^{4+\frac{1}{4}}) & O(\mu^6) & O(\mu^6) \\ O(\mu^4) & O(\mu^3) & O(\mu^3) & O(\mu^4) \\ O(\mu) & O(\mu) & O(\mu) & O(\mu) \end{pmatrix}.$$

Thus, we find for large n and small s

$$\begin{aligned} \deg_{loc}\big(f''_{t_0,s}(u_{\theta,s}),0\big) &= \operatorname{sgn}\det\big(Proj_{X_{n,s}} f''_{t_0,s}(u_{\theta,s})\big) \\ &= \operatorname{sgn}\det\big(D^2 k_\theta(0)\big) = (-1)^{\operatorname{ind}(k,\theta)}, \end{aligned}$$

which proofs the claim. □

Remark 5.2. From the proof of Lemma 5.1 we see that $f''_{t_0,s}(u_{\theta,s})$ is nondegenerate and the Morse-Index of $u_{\theta,s}$, i.e. the number of negative eigenvalues of $f''_{t_0,s}(u_{\theta,s})$, is given by

$$\operatorname{ind}(f_{t_0,s}, u_{\theta,s}) = 1 + \operatorname{ind}(-k,\theta) = 4 - \operatorname{ind}(k,\theta).$$

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