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FUNDAMENTAL POLYHEDRA FOR MARGULIS SPACE-TIMES

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1 INTRODUCTION

COMPLETE affinely flat manifolds correspond to subgroups $\Gamma \subset \mathrm{Aff}(\mathbb{R}^n)$ which act properly discontinuously on $\mathbb{R}^n$, and $\pi_1(M) \cong \Gamma$ for $M = \mathbb{R}^n/\Gamma$. Milnor [6] has shown that if G is virtually polycyclic then there is some complete affinely flat manifold M such that $\pi_1(M) \cong G$, and he asked if the converse was true.

Margulis [4, 5] demonstrated that there exist free subgroups $\Gamma \subset \mathrm{Aff}(\mathbb{R}^3)$ acting properly discontinuously on $\mathbb{R}^3$, thus answering Milnor's question negatively. By Fried and Goldman [2], the underlying linear group of Γ must be conjugate to a subgroup of $\mathrm{SO}^\circ(2, 1)$. The corresponding quotient manifolds will thus be called Margulis space-times. Drumm and Goldman [1] have given geometric conditions for the group Γ to act properly discontinuously on $\mathbb{R}^3$.

0.2 Geometric conditions, similar to [1], for the group Γ to act properly discontinuously on $\mathbb{R}^3$, are obtained through the construction of a fundamental polyhedra for the action of Γ acting on $\mathbb{R}^3$. These fundamental polyhedra are noncompact [2, 3]. With the identifications, the manifolds are seen to have the topological type of a solid handlebody.

§1. GEOMETRY OF $\mathbb{R}^{2,1}$

1.1 Consider $\Gamma = \langle h_1, h_2, \ldots, h_n \rangle \subset V \rtimes \mathbb{G} = \mathbb{H}$ where $\mathbb{G} = \mathrm{SO}^\circ(2, 1)$ and $\mathbf{V}$ is the group of parallel translations in 3-space. Γ will act on $\mathbf{E} = \mathbb{R}^{2,1}$ with the Lorentzian inner product $\mathbb{B}(\mathbf{u}, \mathbf{v}) = u_1 v_1 + u_2 v_2 - u_3 v_3$ invariant under the action of $\mathbb{G}$. $\mathbf{C} = \{\mathbf{u} \in \mathbf{E} \mid \mathbb{B}(\mathbf{u}, \mathbf{u}) = 0\}$ is the null cone and $\mathbf{W} = \{\mathbf{u} \in \mathbf{C} \mid u_3 > 0\}$ is its upper nappe. For linearly independent $\mathbf{u}, \mathbf{v} \in \mathbf{E}$, $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes the plane spanned by $\mathbf{u}$ and $\mathbf{v}$. $\rho(\mathbf{A}, \mathbf{B})$ denotes the Euclidean distance between the sets $\mathbf{A}$ and $\mathbf{B}$, $cl(\mathbf{A})$ is the closure of $\mathbf{A}$ in the usual topology, and $\|\mathbf{u}\|$ is the Euclidean length of the vector $\mathbf{u}$.

The "Lorentzian cross product" is defined to be

$$\mathbf{u} \boxtimes \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_2 v_1 - u_1 v_2 \end{bmatrix}.$$

It can easily be checked that:

(i) $\mathbb{B}(\mathbf{u}, \mathbf{u} \boxtimes \mathbf{v}) = \mathbb{B}(\mathbf{v}, \mathbf{v} \boxtimes \mathbf{u}) = 0$;

(ii) $\mathbf{u} \boxtimes \mathbf{v} = -\mathbf{v} \boxtimes \mathbf{u}$;

(iii) $\mathbb{B}(\mathbf{u} \boxtimes \mathbf{v}, \mathbf{u} \boxtimes \mathbf{v}) = \mathbb{B}(\mathbf{u}, \mathbf{v})^2 - \mathbb{B}(\mathbf{u}, \mathbf{u})\mathbb{B}(\mathbf{v}, \mathbf{v})$.

1.2 Let $l\colon \mathbb{H} \to \mathbb{G}$ be the projection onto the linear part of elements of $\mathbb{H}$. l is injective [2] and if Γ acts properly discontinuously on E then $l(\Gamma)$ is also free, otherwise the kernel would be an abelian subgroup. The discussion will concern only purely hyperbolic $l(\Gamma)$, i.e. every nonidentity element of $l(\Gamma)$ is hyperbolic.

$g \in \mathbb{G}$ is hyperbolic if it has three distinct positive real eigenvalues, $\lambda(g) < 1 < \lambda(g)^{-1}$. Corresponding repelling, fixed, and attracting eigenvectors $\mathbf{x}_g^-$, $\mathbf{x}_g^0$, and $\mathbf{x}_g^+$, respectively, are defined so that $\mathbf{x}_g^-, \mathbf{x}_g^+ \in \mathbf{W} \cap \mathbf{S}^2$, where $\mathbf{S}^2$ is the Euclidean unit sphere, and $\mathbb{B}(\mathbf{x}_g^0, \mathbf{x}_g^0) = 1$ such that $\mathbb{B}(\mathbf{x}_g^0, \mathbf{x}_g^- \boxtimes \mathbf{x}_g^+) > 0$, i.e. $\{\mathbf{x}_g^0, \mathbf{x}_g^-, \mathbf{x}_g^+\}$ is a right handed basis for $\mathbf{E}$. For $\delta > 0$, $g \in \mathbb{G}$ is δ-hyperbolic if g is hyperbolic and $\rho(\mathbf{x}_g^+, \mathbf{x}_g^-) > \delta$.

$\mathbf{v} \in \mathbf{E}$ is called spacelike if $\mathbb{B}(\mathbf{v}, \mathbf{v}) > 0$, timelike if $\mathbb{B}(\mathbf{v}, \mathbf{v}) < 0$, and lightlike if $\mathbb{B}(\mathbf{v}, \mathbf{v}) = 0$. The set of spacelike vectors, the set of timelike vectors, and the set of lightlike vectors are each invariant under the action of $\mathbb{G}$. For spacelike $\mathbf{v}$, define unique vectors $\mathbf{x}_v^+$, $\mathbf{x}_v^- \in \mathbf{W} \cap \mathbf{S}^2$ such that $\mathbb{B}(\mathbf{v}, \mathbf{x}_v^\pm) = 0$ and $\mathbb{B}(\mathbf{v}, \mathbf{x}_v^- \boxtimes \mathbf{x}_v^+) > 0$. In fact, $\mathbf{x}_v^- \boxtimes \mathbf{x}_v^+$ is a positive scalar multiple of $\mathbf{v}$. Note that if $\mathbf{v} = \mathbf{x}_g^0$ then $\mathbf{x}_v^\pm = \mathbf{x}_g^\pm$. Also, if $\mathbf{u}, \mathbf{v} \in \mathbf{W} \cap \mathbf{S}^2$ are linearly independent then $\mathbf{u} \boxtimes \mathbf{v}$ is spacelike, $\mathbf{x}^-(\mathbf{u} \boxtimes \mathbf{v}) = \mathbf{u}$, and $\mathbf{x}^+(\mathbf{u} \boxtimes \mathbf{v}) = \mathbf{v}$.

1.3 Define a "conical neighborhood" $\mathbf{A} \subset \mathbf{W}$ of $\mathbf{v} \in \mathbf{W}$ to be an open connected subset of $\mathbf{W}$ containing $\mathbf{v}$, such that if $\mathbf{w} \in \mathbf{W}$ then $k\mathbf{w} \in \mathbf{W}$ for all $k > 0$.

A free subgroup G, with generators g_i, of $\mathbb{G}$ "acts as a Schottky group on $\mathbf{W}$" if there are conical neighborhoods $\mathbf{A}_i^\pm$ of $\mathbf{x}_{g_i}^\pm$ such that $cl(\mathbf{A}_i^\pm) \cap cl\left(\mathbf{A}_i^\mp \bigcup_{j \neq 1} (\mathbf{A}_j^+ \cup \mathbf{A}_j^-)\right) = \varnothing$ and $cl(g_i(\mathbf{A}_i^-)) = \mathbf{W} - \mathbf{A}_i^+$. For disjoint conical open sets $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{T}(\mathbf{A}, \mathbf{B}) = \{\mathbf{v} \in \mathbf{E} \mid \mathbb{B}(\mathbf{v}, \mathbf{a} \boxtimes \mathbf{b}) > 0 \ \forall \mathbf{a} \in cl(\mathbf{A}) \text{ and } \mathbf{b} \in cl(\mathbf{B})\}.$$

In particular, if $\mathbf{A}$ and $\mathbf{B}$ are connected $\mathbf{T}(\mathbf{A}, \mathbf{B})$ is an open infinite pyramid whose vertex is the origin and whose 4 corners are parallel to vectors in the boundary of $\mathbf{A}$ and $\mathbf{B}$.

§2. FUNDAMENTAL POLYHEDRA FOR THE LINEAR PART

2.1 First, fundamental domains for the action of $G_i = \langle g_i \rangle$ on $\mathbf{E}$-{2 half planes} are constructed. Only then will the translational part of the affine action be considered.

Let $\Gamma = \langle h_1, h_2, \ldots, h_n \rangle$, where $h_i(\mathbf{x}) = g_i\mathbf{x} + \mathbf{v}_i$ and $l(h_i) = g_i$ and $l(\Gamma)$ is a Schottky subgroup of $\mathbb{G}$. Let $\mathbf{A}_i^\pm$ be the conical (but not canonical) neighborhoods of $\mathbf{x}_{h_i}^\pm$ whose closures are distinct and $cl(g_i(\mathbf{A}_i^-)) = \mathbf{W} - \mathbf{A}_i^+$.

2.2 On $\mathscr{C} = C - \{\mathbf{v} \in \mathbf{C} \mid \mathbf{v} = k\mathbf{x}_{g_i}^\pm \text{ for some } k \in \mathbb{R}\}$, a fundamental domain for the action of G_i is the complement of $\mathbf{A}_i^+ \cup (-\mathbf{A}_i^+)$ and $\mathbf{A}_i^- \cup (-\mathbf{A}_i^-)$. Define $\mathbf{v}_{ij}^\pm, j \in \{1, 2, \ldots, n\}$, to be of norm 1 and in the boundary of $\mathbf{A}_i^\pm$ where $\mathbf{v}_{i1}^\pm \boxtimes \mathbf{v}_{i2}^\pm \neq 0$ and $g_i(\mathbf{v}_{ij}^-)/\|g_i(\mathbf{v}_{ij}^-)\| = \mathbf{v}_{ij}^+$.

The action under consideration is linear and it is easy to see that a fundamental domain for $\mathscr{C}$ and the timelike vectors is the region between $\mathbf{F}_i^+ = \{\mathbf{v} \in \langle \mathbf{v}_{i1}^+, \mathbf{v}_{i2}^+ \rangle \mid \mathbb{B}(\mathbf{v}, \mathbf{v}) \leq 0\}$ and $\mathbf{F}_i^- = \{\mathbf{v} \in \langle \mathbf{v}_{i1}^-, \mathbf{v}_{i2}^- \rangle \mid \mathbb{B}(\mathbf{v}, \mathbf{v}) \leq 0\}$.

For the spacelike vectors, as with the timelike vectors, the fundamental domain is bounded by planar objects that include the $\mathbf{v}_{ij}^\pm$'s. Consider the half planes that are $\mathbb{B}$-perpendicular, i.e. Euclidean tangent, to the cone.

Denote the "positive" half planes perpendicular to $\mathbf{C}$ at $\mathbf{v} \in \mathbf{C}$ as

$$\mathbf{P}(\mathbf{v}) = \{\mathbf{w} \in \mathbf{E} \mid \mathbf{x}_w^+ = \mathbf{v} \text{ or } k\mathbf{v} = \mathbf{w}, \quad k \in \mathbb{R}\},$$

and the "negative" half planes perpendicular to $\mathbf{C}$ at $\mathbf{v} \in \mathbf{C}$ as

$$\mathbf{N}(\mathbf{v}) = \{\mathbf{w} \in \mathbf{E} \mid \mathbf{x}_w^- = \mathbf{v} \quad \text{or } k\mathbf{v} = \mathbf{w}, k \in \mathbb{R}\}.$$

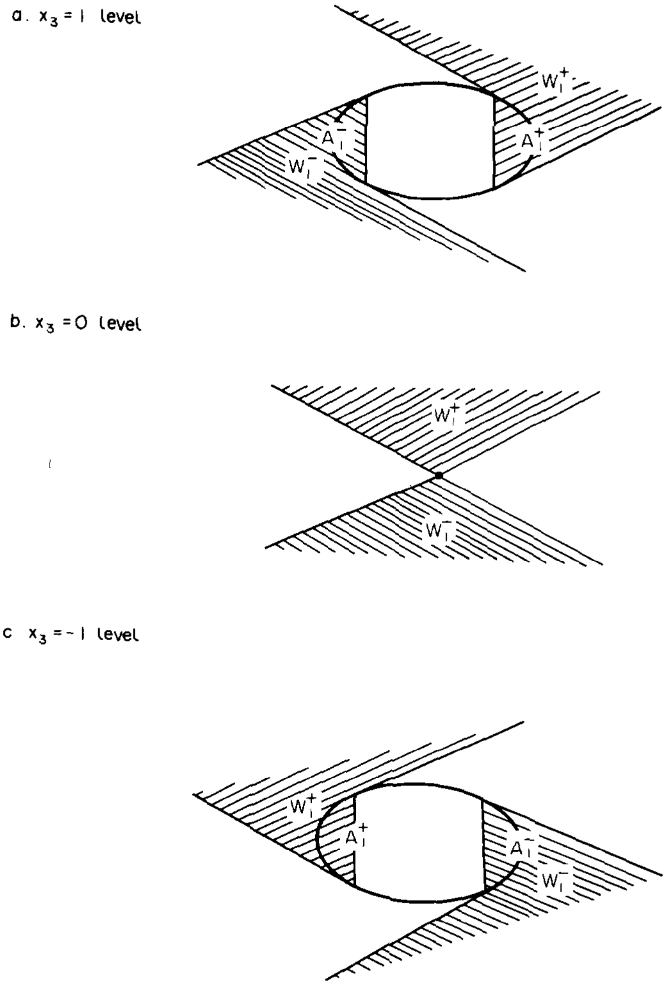

Fig. 1. Horizontal view of a pair of wedges.

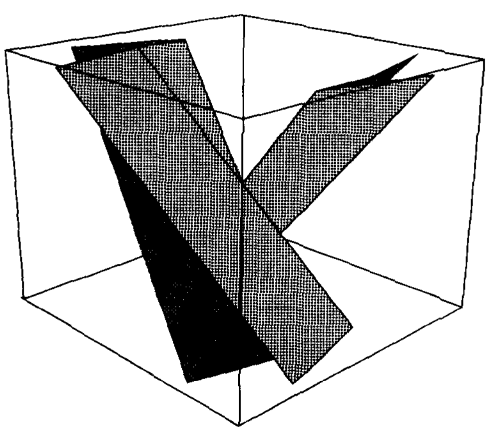

Fig. 2. 3D view of a pair of wedges.

Note that $g_i(\mathbf{P}(\mathbf{v}_{ij}^-)) = \mathbf{P}(\mathbf{v}_{ij}^+)$ and $g_i(\mathbf{N}(\mathbf{v}_{ij}^-)) = \mathbf{N}(\mathbf{v}_{ij}^+)$, since $g(\mathbf{x}_v^\pm)/\|g(\mathbf{x}_v^\pm)\| = \mathbf{x}_{g(v)}^\pm$ for hyperbolic $g \in \mathbb{G}$ and $\mathbf{v} \in \mathbf{E}$. Define the "wedges" $\mathbf{W}_i^\pm$ to be the open region bounded by $\mathbf{F}_i^\pm \cup \mathbf{P}(\mathbf{v}_{i1}^\pm) \cup \mathbf{P}(\mathbf{v}_{i2}^\pm)$ not containing $\mathbf{F}_i^\mp \cup \mathbf{P}(\mathbf{v}_{i1}^\mp) \cup \mathbf{P}(\mathbf{v}_{i2}^\mp)$ and the "wedges" $\mathbf{M}_i^\pm$ to be the half-space bounded by $\mathbf{F}_i^\pm \cup \mathbf{N}(\mathbf{v}_{i1}^\pm) \cup \mathbf{N}(\mathbf{v}_{i2}^\pm)$ and not containing $\mathbf{F}_i^\mp \cup \mathbf{N}(\mathbf{v}_{i1}^\mp) \cup \mathbf{N}(\mathbf{v}_{i2}^\mp)$.

A fundamental domain for the action of G_i

(i) on $\mathbf{E} - \mathbf{P}(\mathbf{x}_{g_i}^-) - \mathbf{P}(\mathbf{x}_{g_i}^+)$ is $\mathbf{E} - \mathbf{W}_1^+ - \mathbf{W}_i^-$ and
(ii) on $\mathbf{E} - \mathbf{N}(\mathbf{x}_{g_i}^-) - \mathbf{N}(\mathbf{x}_{g_i}^+)$ is $\mathbf{E} - \mathbf{M}_1^+ - \mathbf{M}_i^-$.

3. AFFINE FUNDAMENTAL POLYHEDRA

3.1 Fundamental domains for the action of $H_i = \langle h_i \rangle$ on $\mathbf{E}$ are obtained from the above domains. The fundamental domain for the action of Γ on $\mathbf{E}$ is the intersection of the fundamental domains of the action of the H_i's on $\mathbf{E}$ provided the fundamental domain for each H_i completely contains the complement of the fundamental domain for the other H_i. The fundamental domains involving the $\mathbf{P}(\mathbf{v})$'s will be discussed and the argument for the $\mathbf{N}(\mathbf{v})$'s is completely analogous.

$\mathbf{x}, \mathbf{y} \in \mathbf{E}$ are Γ-equivalent if there exists a $\gamma \in \Gamma$ such that $\gamma(\mathbf{x}) = \mathbf{y}$, and the relation between constructing a fundamental domain and showing proper discontinuity is given by:

3.2 LEMMA. *Γ, a group of affine homeomorphisms of $\mathbf{E}$, acts properly discontinuously and freely on $\mathbf{E}$ if there exists a 3-dimensional submanifold $\mathbf{X}$ with boundary (a fundamental domain), such that:*

(i) *no 2 elements of the interior of $\mathbf{X}$ are Γ-equivalent,*
(ii) *and every element of $\mathbf{E}$ is Γ-equivalent to an element of $\mathbf{X}$.*

The fundamental domain of a cyclic affine group is bounded by translates of components of the boundary of a fundamental domain for the corresponding cyclic linear group. $cl(h_i(\mathbf{W}_i^-)) = cl(g_i(\mathbf{W}_i^-) + \mathbf{v}_i) = (\mathbf{E} - \mathbf{W}_i^+) + \mathbf{v}_i$ and a fundamental domain for the action of $H_i = \langle h_i \rangle$ on $\mathbf{E}$ is the complement of $\mathbf{W}_i^-$ and $\mathbf{W}_i^+ + \mathbf{v}_i$ if these 2 sets have disjoint closures.

If $\rho(\mathbf{y}, \mathbf{z} + \mathbf{v}_i) > 0$ for all choices of $\mathbf{y} \in cl(\mathbf{W}_i^-)$ and $\mathbf{z} \in cl(W_i^+)$ then $cl(\mathbf{W}_i^-) \cap cl(\mathbf{W}_i^+ + \mathbf{v}_i) = \varnothing$. In particular, if for each pair of vectors $\mathbf{y} \in cl(\mathbf{W}_i^-)$ and $\mathbf{z} \in cl(\mathbf{W}_i^+)$ there is a vector $\mathbf{u} \in \mathbf{E}$ such that $\mathbb{B}(\mathbf{y}, \mathbf{u}) \neq \mathbb{B}(\mathbf{z} + \mathbf{v}_i, \mathbf{u})$ then $cl(\mathbf{W}_i^-) \cap cl(\mathbf{W}_i^+ + \mathbf{v}_i) = \varnothing$.

To construct $\mathbf{u}$ given the vectors $\mathbf{y} \in \mathbf{W}_i^-$ and $\mathbf{z} \in \mathbf{W}_i^+$, first examine the case in which $\mathbf{y}$ and $\mathbf{z}$ are both spacelike. In this case, $\mathbf{x}_y^+ \in cl(\mathbf{A}_i^-)$, $\mathbf{x}_z^+ \in cl(\mathbf{A}_i^+)$, and $\mathbf{u} = \mathbf{x}_y^+ \boxtimes \mathbf{x}_z^+$. If $\mathbf{y}$ is not spacelike and $\mathbf{z}$ is spacelike, let $\mathbf{u} = (\mathbf{y}/\|\mathbf{y}\|) \boxtimes \mathbf{x}_z^+$. If $\mathbf{z}$ is not spacelike and $\mathbf{y}$ is spacelike, let $\mathbf{u} = \mathbf{x}_y^+ \boxtimes (\mathbf{z}/\|\mathbf{z}\|)$. If neither $\mathbf{y}$ nor $\mathbf{z}$ are spacelike, let $\mathbf{u} = (\mathbf{y}/\|\mathbf{y}\|) \boxtimes (\mathbf{z}/\|\mathbf{z}\|)$. Note that $\mathbb{B}(\mathbf{y}, \mathbf{u}) = 0$ if $\mathbf{y}$ is not spacelike. If $\mathbf{y}$ is spacelike then $\mathbf{x}_y^+ = \mathbf{x}_u^-$ and $\mathbb{B}(\mathbf{y}, \mathbf{u}) < 0$. Similarly, $\mathbb{B}(\mathbf{z}, \mathbf{u}) \geq 0$.

If $\mathbb{B}(\mathbf{v}_i, \mathbf{u}) > 0$ for all possible vectors $\mathbf{u}$ described above then $cl(\mathbf{W}_i^-) \cap cl(\mathbf{W}_i^+ + \mathbf{v}_i) = \varnothing$ and a fundamental domain for H_i is the complement of $\mathbf{W}_i^-$ and $\mathbf{W}_i^+ + \mathbf{v}_i$. For $\mathbf{y} \in cl(\mathbf{W}_i^-)$ and $\mathbf{z} \in cl(\mathbf{W}_i^+)$ one can construct a vector $\mathbf{u}$ as above such that $\mathbb{B}(\mathbf{z} + \mathbf{v}_i, \mathbf{u}) = \mathbb{B}(\mathbf{z}, \mathbf{u}) + \mathbb{B}(\mathbf{v}_i, \mathbf{u}) > k$ and $\mathbb{B}(\mathbf{y}, \mathbf{u}) \leq 0$.

3.3 The set of $\mathbf{v}_i$'s such that $\mathbb{B}(\mathbf{v}_i, \mathbf{u}) > 0$ for a fixed $\mathbf{u}$ is a half space bounded by $\langle \mathbf{x}_u^-, \mathbf{x}_u^+ \rangle$, the null plane of $\mathbf{u}$. The set of translations giving rise to a fundamental domain for the given $\mathbf{A}_i^\pm$ in this construction is $\mathbf{T}(\mathbf{A}_i^-, \mathbf{A}_i^+)$, the set of "allowable translations."

Another set of allowable translations is obtained by noting that $cl(h_i(\mathbf{W}_i^- - g_i^{-1}(\mathbf{v}_i))) = \mathbf{E} - \mathbf{W}_i^+$. In this case the wedges separate if $\mathbb{B}(-g_i^{-1}(\mathbf{v}_i), \mathbf{u}) < 0$ for all $\mathbf{u} = \mathbf{z} \boxtimes \mathbf{w}$ where $\mathbf{w} \in cl(\mathbf{A}_i^+)$ and $\mathbf{z} \in cl(\mathbf{A}_i^-)$. Equivalently, if $u = g_i(\mathbf{u})$ then $\mathbb{B}(\mathbf{v}_i, \mathbf{u}) > 0$ for $u = \mathbf{z} \boxtimes \mathbf{w}$ where $\mathbf{w} \in cl(g_i(\mathbf{A}_i^+))$ and $\mathbf{z} \in cl(g_i(\mathbf{A}_i^-))$ and the set of allowable translations is $\mathbf{T}(g_i(\mathbf{A}_i^-), g_i(\mathbf{A}_i^+))$.

These two sets of allowable translations can be combined to make a larger third set of allowable translations. If $\mathbf{v}_i = \mathbf{v}_{i1} + \mathbf{v}_{i2}$ where $\mathbb{B}(\mathbf{v}_{i1}, \mathbf{u}) > 0$ and $\mathbb{B}(\mathbf{v}_{i2}, \mathbf{u}) > 0$ for all $\mathbf{u}$ and $\mathbf{u}$ described above, then the complement of $\mathcal{W}_i^- = \mathbf{W}_i^- - g_i^{-1}(\mathbf{v}_{i2})$ and $\mathcal{W}_i^+ = \mathbf{W}_i^+ + \mathbf{v}_{i1}$ is a fundamental domain for the action of H_i on $\mathbf{E}$ since the closures of the wedges are distinct.

A fundamental domain for the action of H_i on $\mathbf{E}$ is $\mathbf{E} - \mathcal{W}_i^- - \mathcal{W}_i^+$, if $\mathbf{v}_i \in \mathbf{T}(\mathbf{A}_i^-, g_i(\mathbf{A}_i^+))$ where $\mathbf{v}_{i1} \in \mathbf{T}(\mathbf{A}_i^-, \mathbf{A}_i^+)$ and $\mathbf{v}_{i2} \in \mathbf{T}(g_i(\mathbf{A}_i^-), g_i(\mathbf{A}_i^+))$ are such that $\mathbf{v}_i = \mathbf{v}_{i1} + \mathbf{v}_{i2}$.

3.4 In passing from the fundamental domain of H_i to the fundamental domain of Γ, it is useful to demand both wedges be translated away from the origin. The fundamental domain of Γ will be the complement of 2 pairs of ε-separated wedges. The pairs of wedges were constructed so that $h_i(cl(\mathbf{W}_i^- - g_i^{-1}(\mathbf{v}_{i2}))) = \mathbf{E} - (\mathbf{W}_i^+ + \mathbf{v}_{i1})$ and the quotient is obtained by identifying the boundaries of each pair of wedges.

In order to guarantee that the closures of the translated wedges are distinct, it is useful to consider each wedge paired with the other wedges. Let $\mathcal{A}_i^\pm = \mathbf{A}_i^\pm \bigcup_{j \neq i} (\mathbf{A}_j^+ \cup \mathbf{A}_j^-)$. If $\mathbf{v}_i \in \mathbf{T}(g_i(\mathcal{A}_i^+), \mathcal{A}_i^-)$ then $\mathbf{v}_i = \mathbf{v}_{i1} + \mathbf{v}_{i2}$ for some $\mathbf{v}_{i1} \in \mathbf{T}(\mathbf{A}_i^+, \mathcal{A}_i^-)$ and $\mathbf{v}_{i2} \in \mathbf{T}(g_i(\mathcal{A}_i^+), g_i(\mathbf{A}_i^-))$, and if $-\mathbf{v}_i \in \mathbf{T}(g_i(\mathcal{A}_i^+), \mathcal{A}_i^-)$ then $\mathbf{v}_i = \mathbf{v}_{i1} + \mathbf{v}_{i2}$ for some $-\mathbf{v}_{i2} \in \mathbf{T}(g_i(\mathcal{A}_i^+), g_i(\mathbf{A}_i^-))$ and $-\mathbf{v}_{i1} \in \mathbf{T}(\mathbf{A}_i^+, \mathcal{A}_i^-)$. In this case let $\mathcal{W}_i^+ = \mathbf{W}_i^+ + \mathbf{v}_{i1}$, $\mathcal{W}_i^- = \mathbf{W}_i^- - g_i^{-1}(\mathbf{v}_{i2})$, $\mathcal{M}_i^+ = \mathbf{M}_i^+ + \mathbf{v}_{i1}$, and $\mathcal{M}_i^- = \mathbf{M}_i^- - g_i^{-1}(\mathbf{v}_{i2})$.

3.5 THEOREM. *Let* $h_i(\mathbf{x}) = g_i(\mathbf{x}) + \mathbf{v}_i$ *for* $i \in \{1, 2, \ldots, n\}$, *and* $\Gamma = \langle h_1, h_2, \ldots, h_n \rangle$. *If* $l(\Gamma)$ *acts as a Schottky subgroup on* $\mathbf{W}$ *and:*

(i) $\mathbf{v}_i \in \mathbf{T}(g_i(\mathcal{A}_i^+), \mathcal{A}_i^-)$, *then* Γ *acts properly discontinuously on* $\mathbf{E}$. $\mathbf{E} - \left(\bigcup_i (\mathcal{W}_i^+ \cup \mathcal{W}_i^-) \right)$ *is a fundamental polyhedron for the action of* Γ *on* $\mathbf{E}$.

(ii) $-\mathbf{v}_i \in \mathbf{T}(g_i(\mathcal{A}_i^+), \mathcal{A}_i^-)$, *then* Γ *acts properly discontinuously on* $\mathbf{E}$. $\mathbf{E} - \left(\bigcup_i (\mathcal{M}_i^+ \cup \mathcal{M}_i^-) \right)$ *is a fundamental polyhedron for the action of* Γ *on* $\mathbf{E}$.

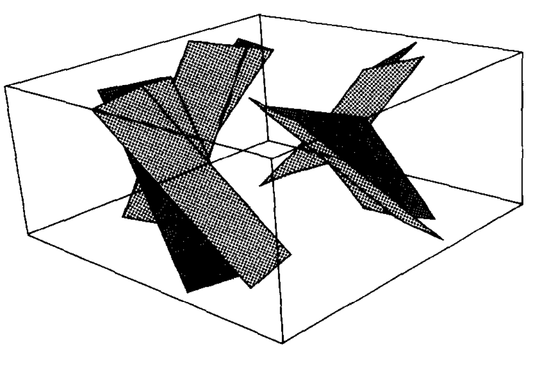

Fig. 3. Fundamental polyhedron for margulis space-time.

Proof. It suffices to prove the theorem for the case $\mathbf{v}_i \in \mathbf{T}(g_i(\mathscr{A}_i^+), \mathscr{A}_i^-)$. It is clear from the construction that no 2 elements of $\mathbf{X} = \mathbf{E} - \left(\bigcup_i (\mathscr{W}_i^+ \cup \mathscr{W}_i^-)\right)$ are Γ-equivalent.

3.6 Assume that there exists a $\mathbf{p} \in \mathbf{E}$ which is not Γ-equivalent to any point in $\mathbf{X}$. Thus, one may construct an infinite sequence of embedded images of the wedges all containing $\mathbf{p}$ in the following manner:

Let $\gamma_0 = e$ and $\mathscr{W}_{i_0}^{j_0} = \omega$. For integers $n > 1$ choose $\gamma_n \in \Gamma$, $i_n \in \{1, 2\}$, and $j_n \in \{-1, 1\}$, so that $\mathbf{p} \in \gamma_n(\mathscr{W}_{i_n}^{j_n})$, $\gamma_{n+1}(\mathscr{W}_{i_{n+1}}^{j_{n+1}}) \subset \gamma_n(\mathscr{W}_{i_n}^{j_n})$, and $\gamma_{n+1} = \gamma_n h_{i_n}^{j_n}$.

The leading term of γ_n is $h_{i_0}^{j_0}$ and by an application of the Brouwer fixed point theorem [1] it can be shown that $\mathbf{x}^+(\gamma_n) \in \mathbf{A}_{i_0}^{j_0}$, $\gamma_{n+1/2}$ is defined to be $h_k \gamma_n$, where $k \in \{1, -1\}$ is chosen so that $k \neq i_0$. From the previous discussion it is noted that $\mathbf{x}^+(\gamma_{n+1/2}) \in g_k(\mathbf{A}_{i_0}^{j_0})$. Define the plane $\mathbf{S}_m = \langle \mathbf{x}^+(\gamma_m), \mathbf{x}^0(\gamma_m) \rangle$ for all $m \in \{0, 1/2, 1, 3/2, 2 \ldots\}$.

Consider the intersection of the embedded images of the wedges and the plane $\mathbf{P} = \{\mathbf{x} \mid x_3 = p_3\}$. $\mathbf{p}$ is Γ-equivalent to elements in all of the wedges $\mathscr{W}_i^{\pm}$. One can assume that $\mathbf{p}$ is contained in a "small" wedge ω, where the angle between every pair of rays contained in $\omega \cap \mathbf{P}$ is $\leq \pi/2$. In particular, $\mathbf{S}_n \cap \mathbf{P}$ contains a ray lying completely within $\omega \cap \mathbf{P}$ for positive all positive integers n.

Choose $\mathbf{L}_0 \subset \mathbf{P}$ to be the line closest to $\mathbf{p}$ which bounds a half plane in $\mathbf{P}$ containing all of $\omega \cap \mathbf{P}$ and whose normal in $\mathbf{P}$ forms an angle of less than $\pi/4$ with all rays contained $\omega \cap \mathbf{P}$. Let $\mathbf{L}_n \subset \mathbf{P}$ be the closest line to $\mathbf{p}$ parallel to $\mathbf{L}_0$ and bounding a half plane in $\mathbf{P}$ containing $\gamma_n(\mathscr{W}_{i_n}^{j_n}) \cap \mathbf{P}$. $\{\mathbf{L}_0, \mathbf{L}_1, \mathbf{L}_2, \ldots\}$ is an infinite sequence of parallel lines in $\mathbf{P}$ constructed so that $\rho(\mathbf{p}, \mathbf{L}_{n+1}) \leq \rho(\mathbf{p}, \mathbf{L}_n)$. To arrive at a contradiction to the assumption, it is enough to show that $(\rho(\mathbf{p}, \mathbf{L}_n) - \rho(\mathbf{p}, \mathbf{L}_{n+1}))$ is bounded from below.

3.7 There exists an $\varepsilon > 0$ such that for any $\mathbf{x} \in \mathbf{X}$ the ε-ball centered at $\mathbf{x}$, $\mathbf{B}(\mathbf{x}, \varepsilon)$, is contained in $\mathbf{X} \cup (h_1^+ \mathbf{X}) \cup (h_1^- \mathbf{X}) \cup (h_2^+ \mathbf{X}) \cup (h_2^- \mathbf{X})$. Of course, $\rho(\mathbf{p}, \mathbf{L}_1) < (\rho(\mathbf{p}, \mathbf{L}_0) - \varepsilon)$.

If γ_m is δ-hyperbolic there is a minimum compression for any ε-ball in the plane $\mathbf{S}_m$. Quantitatively,

$$l(\gamma_m)\mathbf{B}(0, \varepsilon) \cap \mathbf{S}_m \supset \mathbf{B}(0, \varepsilon\delta/2) \cap \mathbf{S}_m. \tag{1}$$

Consider the case where γ_n is δ-hyperbolic for positive integer n. For every $y \in \gamma_n^{-1}(\mathbf{L}_n)$, $\mathbf{B}(y, \varepsilon)$ is contained in the complement of $h_{i_n}^{j_n}(\mathscr{W}_{i_{n+1}}^{j_{n+1}})$. Since the angle between $\mathbf{P} \cap \mathbf{S}_m$ and the normal to $\mathbf{L}_n$ in $\mathbf{P}$ was constructed to be less than $\pi/4$, $\mathbf{B}(x, \varepsilon\delta/2^{3/2})$ for all $x \in \mathbf{L}_n$ is contained in the complement of $\gamma_{n+1}(\mathscr{W}_{i_{n+1}}^{j_{n+1}})$ and

$$\rho(\mathbf{p}, \mathbf{L}_{n+1}) < \rho(\mathbf{p}, \mathbf{L}_n) - (\varepsilon\delta/2^{3/2}). \tag{2}$$

If γ_n is not δ-hyperbolic then $\gamma_{n+1/2}$ is δ-hyperbolic. (1) holds for $\gamma_{n+1/2}$ and the action of g_k^{-1} does not contract any vector by more than a factor of $\lambda(g_k)$ so that

$$l(\gamma_n)\mathbf{B}(0, \varepsilon) \cap (g_k^{-1}(\mathbf{S}_{n+1/2})) \subset \mathbf{B}(0, \lambda(g_k)\varepsilon\delta/2^{1/2}) \cap (g_k^{-1}(\mathbf{S}_{n+1/2})). \tag{3}$$

There exists a ray in $g_k^{-1}(\mathbf{S}_{n+1/2}) \cap \mathbf{P}$ lying in $\omega \cap \mathbf{P}$ and

$$\rho(\mathbf{p}, \mathbf{L}_{n+1}) < \rho(\mathbf{p}, \mathbf{L}_n) - (\lambda(g_k)\varepsilon\delta/2^{3/2}). \tag{4}$$

This is a contradiction since (2) and (4) bound $(\rho(\mathbf{p}, \mathbf{L}_n) - \rho(\mathbf{p}, \mathbf{L}_{n+1}))$ from below. Thus, there is no $\mathbf{p} \in \mathbf{E}$ which is not Γ-equivalent to an element of $\mathbf{X}$ and $\mathbf{X}$ is a fundamental domain for the action of Γ on $\mathbf{E}$. (i) follows. ■

3.8 Let $\partial\mathbf{S}$ denote the boundary of the set $\mathbf{S}$. In the case where the fundamental polyhedron is $\mathbf{E} - (\mathscr{W}_1^+ \cup \mathscr{W}_1^- \cup \mathscr{W}_2^+ \cup \mathscr{W}_2^-)$, Γ identifies $\partial\mathscr{W}_1^+$ with $\partial\mathscr{W}_1^-$ and $\partial\mathscr{W}_2^+$ with $\partial\mathscr{W}_2^-$. Similarly, Γ identifies $\partial\mathscr{M}_1^+$ with $\partial\mathscr{M}_1^-$ and $\partial\mathscr{M}_2^+$ with $\partial\mathscr{M}_2^-$ when the fundamental polyhedron is $\mathbf{E} - (\mathscr{M}_1^+ \cup \mathscr{M}_1^- \cup \mathscr{M}_2^+ \cup \mathscr{M}_2^-)$.

COROLLARY. *The Margulis space-times arising from Theorem* 2.6 *are homeomorphic to solid handlebodies of genus n.*

It seems natural to conjecture that all Margulis space-times are homeomorphic to solid handlebodies of genus n. This is certainly true for $n = 1$.

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