The Limit Sets of Uniformly Asymptotically 
Zhukovskij Stable Orbits

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Abstract—In this article, we prove that the omega limit set of a uniformly asymptotically 
Zhukovskij stable orbit of a differential system in $\mathbb{R}^n$ is a closed orbit or a fixed point and also 
it is a uniform attractor. Further, if the system is defined on a compact subset of $\mathbb{R}^n$ and each orbit 
is uniformly asymptotically Zhukovskij stable, then the set of fixed points and closed orbits is finite.

Keywords—Zhukovskij stability, Minimal set, Closed orbit.

1. INTRODUCTION

Lyapunov stability, Poincaré (orbital) stability, and Zhukovskij stability are three different, and perhaps the most important, stabilities of solutions of differential equations. Paper [1] provides excellent comparisons and analyses for these kinds of stabilities. In a recent paper [2], Yang discussed the periodicity of limit sets of uniformly asymptotically Zhukovskij stable orbits. However, we think that the proof of his result [2, Theorem 2.1] is flawed. For example, in his proof, the Poincaré map $\Phi_t$ [2, p. 48] should not be defined by the same $t$ for all $y \in D_\sigma(x_0)$. Otherwise, $\text{diam}(\Phi_t(D_\sigma(x))) \to 0$ does not hold as $t \to +\infty$. Also the point $\omega$ [2, p. 48] in the omega limit set of $x(t,x_0)$ should be excluded from being a fixed point, since it is impossible to define a transversal $D_\sigma(\omega)$ at a fixed-point $\omega$. In this article, we shall prove a generalized conclusion: the omega limit set of a uniformly asymptotically Zhukovskij stable orbit of the flow defined by a system in $\mathbb{R}^n$ is a closed orbit or a fixed point and also it is a uniform attractor. Moreover, we obtain a result about the number of fixed points and closed orbits. In a subsequent paper, we shall extend those results to flows defined on a locally compact metric space.

First, we fix some terminologies. Consider the system in $\mathbb{R}^n$ defined by a differential equation $\dot{x} = F(x)$, where $x \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Assume that solutions of arbitrary initial value problems are unique so that the vector field $V = F(x)$ defines a continuous flow $f(x,t)$. Write $x \cdot t = f(x,t)$ and let $A \cdot J = \{x \cdot t \mid x \in A, t \in J\}$ for $A \subset \mathbb{R}^n$ and $J \subset \mathbb{R}$. Thus, $x \cdot R$ and $x \cdot R^+$ are the orbit and the positive semiorbit, respectively, of $x \in \mathbb{R}^n$. The omega limit set of $x$ is the set $\omega(x) = \{y \in \mathbb{R}^n \mid$ there is a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $t_n \to +\infty$ and $x(t_{n+1}) \to y\}$. The author wishes to thank the referee for many useful suggestions.

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A set $Y \subset \mathbb{R}^n$ is invariant under the flow $f$ if $Y \cdot R = Y$, and an invariant set $Y$ is a minimal set, provided:

(i) $Y$ is a closed, nonempty set; and,
(ii) if $Z$ is a closed, nonempty, invariant subset of $Y$, then $Z = Y$.

We denote by $d$, the ordinary metric on $\mathbb{R}^n$, and let the open ball $B_r(x) = \{ z \in \mathbb{R}^n \mid d(x, z) < r \}$ with center $x$ and radius $r > 0$. The closed ball $\overline{B}_r(x) = \{ z \in \mathbb{R}^n \mid d(x, z) \leq r \}$ is the closure of $B_r(x)$. For $p \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, let $d(p, A) = \inf \{ d(p, z) \mid z \in A \}$, and then we define $N_r(A) = \{ z \in \mathbb{R}^n \mid d(z, A) < r \}$ for $r > 0$, which is called the generalized open $r$-ball about $A$ of radius $r$.

**Zhukovskij Stability.** (See [1].) The orbit $x \cdot R$ of a point $x$ in $\mathbb{R}^n$ is Zhukovskij stable provided that given any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that for any $y \in B_1(x)$, one can find a time parameterization $\tau_y$ such that $d(x \cdot t, y \cdot \tau_y(t)) < \epsilon$ holds for $t \geq 0$, where $\tau_y$ is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_y(0) = 0$. Moreover, if $d(x \cdot t, y \cdot \tau_y(t)) \to 0$ as $t \to +\infty$ also holds, the orbit $x \cdot R$ is said to be asymptotically Zhukovskij stable.

**Definition 1.** The orbit $x \cdot R$ of a point $x$ is uniformly asymptotically Zhukovskij stable provided that given any $\epsilon > 0$, there is a $\delta > 0$ such that for each $t' \geq 0$ and $y \in B_1(x \cdot t')$, one can find a time parameterization $\tau_y$ such that $d(x \cdot (t + t'), y \cdot \tau_y(t)) < \epsilon$ holds for $t \geq 0$, and also

$$d(x \cdot (t + t'), y \cdot \tau_y(t)) \to 0, \quad \text{as} \quad t \to +\infty,$$

where $\tau_y$ is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_y(0) = 0$.

Geometrically, the semiorbit of $y \cdot R^+$ will stay in a long tube of $x \cdot R^+$ with different time scales, and the diameter of the tube goes to zero as time tends to infinity.

Consider the system in $\mathbb{R}^2$ defined by differential equations in polar coordinates

$$\frac{dr}{dt} = r(1 - r), \quad \frac{d\theta}{dt} = 1 - r.$$  

It is easy to see that every orbit outside the unit circle is asymptotically Zhukovskij stable, but not uniformly asymptotically Zhukovskij stable. Furthermore, the unit circle is composed of fixed points of the system, and so is not a closed orbit. But it is the omega limit set of every point outside the unit circle. This shows that the last remark in [2] is not true.

## 2. The Main Results

**Lemma 2.1.** If an orbit $x \cdot R$ of a point $x$ is uniformly asymptotically Zhukovskij stable with nonempty omega limit set, then its omega limit set $\omega(x)$ is minimal.

**Proof.** Otherwise, $\omega(x)$ has a proper closed invariant subset $A \subset \omega(x)$ with $A \neq \emptyset$. Choose a point $p \in \omega(x) \setminus A$, then $\lambda = d(p, A) > 0$. Now, for a sufficiently large $t'$, we can find a point $q \in A$ satisfying $d(x \cdot t', q) < \delta$, where $\delta$ is the number defined in Definition 1. Also there exists a sequence $t_i \geq t'$ such that $t_i \to +\infty$ and $x \cdot t_i \to p$. Since $A$ is invariant, it follows that $q \cdot R \subset A$. However, for large $t_i$ we have $d(x \cdot t_i, p) < \lambda/2$, so it follows that $d(x \cdot t_i, q) \cdot R) \geq d(p, A) - d(x \cdot t_i, p) \geq \lambda/2$ for large $t_i$. This contradicts (1) in Definition 1, since $d(x \cdot t', q) < \delta$. Thus, $\omega(x)$ is minimal.

**Remark.** If $\omega(x)$ has at least two points, it follows from Lemma 2.1 that there are no fixed points in $\omega(x)$. Furthermore, if there is a closed orbit $\gamma$ in $\omega(x)$, then $\omega(x) = \gamma$. From the proof of Lemma 2.1, it is easy to see that any closed nonempty invariant set $A$ must be at least a distance $\delta$ from a uniformly asymptotically Zhukovskij stable semiorbit $x : R^+ \to \emptyset$.

**Theorem 2.2.** If an orbit $x \cdot R$ is uniformly asymptotically Zhukovskij stable with nonempty omega limit set, then its omega limit set $\omega(x)$ is a fixed point or a closed orbit.

**Proof.** If $\omega(x)$ is a singleton, certainly it is just a fixed point of the flow $f$. In the sequel, we suppose that $\omega(x)$ is not a singleton and prove that $\omega(x)$ is a closed orbit. Choose a point $p \in \omega(x)$,
which cannot be a fixed point, owing to the above remark. Now, choose a sequence \( \{ t_i \}_{i=1}^{\infty} \subset \mathbb{R}^+ \) such that \( t_i \to +\infty \) and \( x \cdot t_i \to p \). Then, there is a positive \( \sigma (\sigma < \delta) \) such that the closed ball \( \overline{B}_\sigma(p) \) lies in the open ball \( B_\delta(x \cdot t_k) \) for some \( t_k \in \{ t_i \}_{i=1}^{\infty} \) and so does the set \( \overline{B}_\sigma(p) \cdot [-\theta, \theta] \) for a sufficiently small \( \theta > 0 \). Since \( p \) is not stationary, by the tubular flow theorem (see [3, Chapter 5, Section 2]), there is a transversal \( \Sigma \subset \overline{B}_\sigma(p) \cdot [-\theta, \theta] \) such that for each \( y \in \overline{B}_\sigma(p) \cdot [-\theta, \theta] \), the arc of \( y \cdot R \) in \( \overline{B}_\sigma(p) \cdot [-\theta, \theta] \) crosses \( \Sigma \) at a unique \( t = \phi(y) \), where \( \phi(y) \) is continuous on \( y \in \overline{B}_\sigma(p) \cdot [-\theta, \theta] \). Because \( \overline{B}_\sigma(p) \cdot [-\theta, \theta] \subset B_\delta(x \cdot t_k) \), it follows from (1) in Definition 1 that for each \( y \in \overline{B}_\sigma(p) \cdot [-\theta, \theta] \) there is a \( T(y) > 0 \) such that \( d(x \cdot (t + t_k), y \cdot \tau_t(t)) < \delta/2 \) for \( t \geq T(y) \). Thus, from the compactness of \( \overline{B}_\sigma(p) \cdot [-\theta, \theta] \) and the continuity of the flow \( f \), one can find a positive \( M = \sup \{ T(y) \mid y \in \overline{B}_\sigma(p) \cdot [-\theta, \theta] \} < +\infty \) such that for each \( y \in \overline{B}_\sigma(p) \cdot [-\theta, \theta] \), \( d(x \cdot (t + t_k), y \cdot \tau_t(t)) < \delta/2 \) holds for \( t \geq M \). Fix a \( t_1 > t_k \) and \( t_1 - t_k \geq M \) with \( d(x \cdot t_1, p) < \delta/2 \). Now, we define a Poincaré map \( \Phi : \Sigma \to \Sigma \) as follows. If \( y \in \Sigma \) \( (\subset \overline{B}_\sigma(p) \cdot [-\theta, \theta]) \), then we have \( d(x \cdot (t + t_k), y \cdot \tau_t(t)) < \delta/2 \) for \( t \geq M \), which implies that \( d(p, x \cdot t_k) \leq d(p, y \cdot \tau_t(t_k)) \leq d(p, x \cdot t_k) + d(x \cdot t_k, y \cdot \tau_t(t_k)) < \sigma/2 + \sigma/2 = \sigma \). So it follows that \( y \cdot \tau_t(t_k) \in \overline{B}_\sigma(p) \) and then \( y \cdot \tau_t(t_k) + \phi(y) \in \Sigma \) for a \( \phi(y) \in [-\theta, \theta] \). Hence, we define \( \Phi(y) = y \cdot (\tau_t(t_k) + \phi(y)) \). The continuity of \( \phi \) comes from the continuity of the flow \( f, \tau_t, \) and \( \phi \). Note that \( \Phi \) may not be the first return map. Next, by the local structure of the flow at \( p \) (see [3, Chapter 5, Section 2]), \( [\Sigma \cdot [-e, e] \) for a positive \( e < \theta \) is homeomorphic to a compact convex set in \( \mathbb{R}^n \), so it has the fixed-point property from the Schauder fixed-point theorem. Since \( \Sigma \) is a retract of \( \Sigma \cdot [-e, e] \) by the flow, it also has the fixed-point property. It follows that \( \Phi \) has a fixed point \( q \in \Sigma \). Obviously, \( q \cdot R \) is a closed orbit, and from the Remark we immediately obtain \( \omega(x) = q \cdot R \). This completes the proof.

For the next result, we recall the definition of a uniform attractor. The first prolongational limit set of \( x \) is the set \( J^+(x) = \{ y \in \mathbb{R}^n \mid \text{there are a sequence } x_n \in \mathbb{R}^n \text{ and a sequence } t_n \text{ such that } x_n \to x, t_n \to +\infty \text{ and } x_n \cdot t_n \to y \} \). If \( K \) is a nonempty compact subset of \( \mathbb{R}^n \), the region of uniform attraction of \( K \) is the set \( A_u(K) = \{ x \in \mathbb{R}^n \mid J^+(x) \not\subset K \} \). \( K \) is said to be a uniform attractor if \( A_u(K) \) is a neighborhood of \( K \).

**Lemma 2.3.** (See [4, Chapter 5, Proposition 1.2].) For each neighborhood \( V \) of \( K \), if there exists a neighborhood \( U \) of \( x \) and a \( T > 0 \) such that \( U \cdot t \subset V \) holds for \( t \geq T \), then \( x \in A_u(K) \).

**Theorem 2.4.** If the orbit \( x \cdot R \) of \( x \in \mathbb{R}^n \) is uniformly asymptotically Zhukovskij stable with nonempty omega limit set, then \( w(x) \) is a uniform attractor.

**Proof.** With the number \( \delta \) defined as in Definition 1, we only need to prove \( N_{\delta/2}(\omega(x)) \subset A_u(\omega(x)) \), i.e., for \( y \in N_{\delta/2}(\omega(x)) \) we shall show that \( y \in A_u(\omega(x)) \). Choose \( \sigma > 0 \) such that \( B_\sigma(y) \subset N_{\delta/2}(\omega(x)) \). Given \( \epsilon > 0 \) \( (\epsilon < \delta/2) \), let \( t \cdot x \in N_t(\omega(x)) \) for \( t \geq T > 0 \); then it follows that for every \( z \in B_\sigma(y) \subset N_{\delta/2}(\omega(x)) \), there exists a \( p \in \omega(x) \) such that \( d(x, p) < \delta/2 \) and \( d(z \cdot t_0, p) < \delta/2 \) for some \( t_0 \geq T \). Thus, \( d(z, x \cdot t_0) \leq d(z, p) + d(p, x \cdot t_0) < \delta \) holds, so by Definition 1 it follows that for \( t \geq T_i \geq T \), \( d(z \cdot \tau_t(t), x \cdot (t_0 + t)) < \epsilon \). Now, it is easy to see that \( z \cdot \tau_t(t) \in N_{\delta/2}(\omega(x)) \) for \( t \geq T_i \), since \( x \cdot t \in N_t(\omega(x)) \) for \( t \geq T \). By the compactness of \( B_\sigma(y) \), we can take \( T' = \sup \{ \tau_t(T) \mid z \in B_\sigma(y) \} \). Then, for all \( z \in B_\sigma(y) \), \( z \cdot t \in N_{\delta/2}(\omega(x)) \) for \( t \geq T' \), hence, \( B_\sigma(y) \cdot t \subset N_{\delta/2}(\omega(x)) \) for \( t \geq T' \). So by the Lemma 2.3, we have \( y \in A_u(\omega(x)) \). The proof is therefore complete.

Obviously, similar results about alpha limit sets are also true. In the following, we shall consider a subsystem defined by the flow on a compact invariant subset \( W \subset \mathbb{R}^n \).

**Theorem 2.5.** If each orbit \( x \cdot R \) of \( x \in W \) is uniformly asymptotically Zhukovskij stable, then the set of fixed points and closed orbits in \( W \) is finite.

**Proof.** Let \( 2^W = \{ C \subset W \mid C \text{ is nonempty and closed} \} \) be the hyperspace of compact subspaces of \( W \). \( 2^W \) is a compact metric space under the Hausdorff metric \( H_d \) (see [5]). Since each orbit \( x \cdot R \) of \( x \in W \) is uniformly asymptotically Zhukovskij stable, by the compactness of \( W \), one can
take a common $\delta > 0$ that is suitable for every orbit $x \cdot R$ in $W$ as in Definition 1. Denote by $C(W)$, the set of all minimal subsets of the flow $f$ on $W$. Thus, for $A$ and $B$ in $C(W)$, if $A \neq B$, from the Remark we have $A \cap B = \emptyset$. Also $d(A,B) = \inf\{d(a,b) \mid a \in A$ and $b \in B\} > \delta$. So $H_d(A,B) > \delta$, and it follows that $C(W)$ is a discrete subset of $2^W$ with each pair of its members at least $\delta$-apart. By the compactness of $2^W$, we conclude that $C(W)$ is finite, and Theorem 2.5 follows.

REMARKS.

(1) In Theorem 2.5, the fixed points and closed orbits in $W$ are uniform attractors, moreover they are also uniformly asymptotically Zhukovskij stable.

(2) It is an interesting problem to consider the converse of Theorem 2.2. We think that if a closed orbit is a uniform attractor, then all the orbits in its neighborhood are uniformly asymptotically Zhukovskij stable.

(3) In a forthcoming paper [6], most results of this article are extended to the case in which $f(x,t)$ is a flow defined on a locally compact metric space.

REFERENCES


