Annals of Pure and Applied Logic 55 (1992) 285–303
North-Holland

Woodin cardinals and presaturated ideals

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Communicated by A. Nerode
Received 21 March 1990

Abstract

Goldring, N., Woodin cardinals and presaturated ideals, Annals of Pure and Applied Logic 55 (1992) 285–303.

Models of set theory are constructed where the non-stationary ideal on $\mathcal{P}_{\omega_1}\lambda$ (λ an uncountable regular cardinal) is presaturated. The initial model has a Woodin cardinal. Using the Lévy collapse the Woodin cardinal becomes λ^+ in the final model. These models provide new information about the consistency strength of a presaturated ideal on $\mathcal{P}_{\omega_1}\lambda$ for λ greater than ω_1.

1. Introduction

In this paper we describe a generic extension of V where the non-stationary (henceforth NS) ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated. More specifically, we show that in the model obtained by collapsing all ordinals less than a Woodin cardinal to some regular uncountable cardinal λ, the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated (we define the notion of a presaturated ideal below).

This result improves on an unpublished result of Foreman, Magidor and Shelah, which says that in the model obtained by collapsing all ordinals less than a *supercompact* cardinal to a regular uncountable cardinal λ, the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated. The proof of this unpublished result is very much in the spirit of some of the proofs in [2], e.g., the proof that after collapsing all ordinals less than a supercompact to some regular uncountable cardinal λ, the NS ideal on λ is precipitous. Our proof relies heavily on their work.

Finally, as far as consistency results go, our result shows that, for example, the consistency strength of a presaturated ideal on $\mathcal{P}_{\omega_1}\omega_n$ is no greater than that of a Woodin cardinal. For $n > 1$, this reduces the known consistency strength of the above claim from that of a supercompact cardinal to that of a Woodin cardinal. For $n = 1$, our result gives no new consistency strength information since in that case, Woodin and Shelah have shown that, starting with a model with a Woodin cardinal, one can force to get a model where the NS ideal on $\aleph_1$ is saturated, let alone presaturated (their result improves on a result in [2], where the authors used a supercompact in the ground model). However, even in this case we get new information about the particular model where we get the NS ideal on $\mathcal{P}_{\omega_1}\omega_1$ to be presaturated.

2.

We start by introducing some definitions. We want to show that if δ is Woodin in V, then there is a forcing extension of V where the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated. In general, for any ideal on $\mathcal{P}_{\omega_1}\lambda$, we have the following definition (where I^+ stands for the positive sets with respect to I, i.e., the complement of I in $\mathcal{P}_{\omega_1}\lambda$).

Definition 2.1. An ideal I on $\mathcal{P}_{\omega_1}\lambda$ is *presaturated* iff whenever G is generic over I^+, λ^+ is not collapsed in $V[G]$.

The notion of a presaturated ideal was introduced by Baumgartner and Taylor in [1]. They also formulated there an equivalent definition, which will be more useful for our purposes. Before we state the equivalent definition (Lemma 2.3), we need one more definition:

Definition 2.2. Let $\kappa \leqslant \lambda$ be uncountable cardinals, κ regular. Let A be an antichain of stationary sets with respect to some ideal I in $P_\kappa\lambda$. Let B be any positive set in $P_\kappa\lambda$. Then:

$$A \restriction B = \{D \in A \mid D \cap B \notin I\}.$$

Loosely speaking, $A \restriction B$ is the part of the antichain which is 'below' B when forcing with I^+.

Note. In the above definition, we referred to 'antichains'. In general, A is an antichain in $\mathcal{P}_\kappa\lambda$ with respect to an ideal I if all members B, C $(B \neq C)$ of A are positive with respect to I and their intersection belongs to I. We will often just say 'A is an antichain' when the space and the ideal are clear from the context.

We now have the following lemma:

Lemma 2.3. *Assume $2^\lambda = \lambda^+$ and $\lambda^\omega = \lambda$. Then an ideal I on $\mathcal{P}_{\omega_1}\lambda$ is presaturated iff for any ω antichains $\langle A^i \mid i \in \omega\rangle$ and any positive set B, there is a positive set $D \subset B$ such that each $|A^i \restriction D| \leqslant \lambda$.*

For a proof see [1].

Remark. In the model where we will be proving that the NS ideal on $\mathcal{P}_{\omega_1}\lambda$ is presaturated, $2^\lambda = \lambda^+$ and $\lambda^\omega = \lambda$. Thus the above equivalence holds in that model. Also, we will only use the right-to-left implication ('new' definition $\Rightarrow$ 'old' definition) which does not depend on any cardinal arithmetic.

We will be interested only in the NS ideal on $\mathcal{P}_{\omega_1}\lambda$: A subset of $\mathcal{P}_\kappa\lambda$ is NS iff it is disjoint from a club subset of $\mathcal{P}_\kappa\lambda$. In general, there are two distinct notions of 'club' subsets of $\mathcal{P}_\kappa\lambda$, club and strongly club; (a set $C \subset \mathcal{P}_\kappa\lambda$ is club in $\mathcal{P}_\kappa\lambda$ iff it is unbounded in $\mathcal{P}_\kappa\lambda$ — i.e. every set in $\mathcal{P}_\kappa\lambda$ is covered by a set in C and it is closed — i.e. the union of any increasing sequence of length less than κ of sets in C is itself in C; C is strongly club in $\mathcal{P}_\kappa\lambda$ iff there is a structure $\mathcal{A}$ of the form $\mathcal{A} = \langle \lambda, f_i \rangle_{i\in\omega}$, where $f_i : \lambda^{<\omega} \to \lambda$ and $C = \{N \prec \mathcal{A} \mid |N| < \kappa\}$). However, in the case we are interested in, i.e., $\kappa = \omega_1$, the two notions coincide (see e.g., [2] — the result itself is due to Kueker, see [5]). The notions of club and NS for subsets of $\mathcal{P}_\kappa X$ are defined in a similar way.

The proof of the theorem relies heavily on the relationship between stationary sets of $\mathcal{P}_{\omega_1}\lambda$ and $\mathcal{P}_{\omega_1}X$, for X such that $\lambda \subset X$. We state here the facts we use. If S is a stationary (club) subset of $\mathcal{P}_{\omega_1}X$ then

$$S^* = \{Y \cap \lambda \mid Y \in S\}$$

is stationary (club) in $\mathcal{P}_{\omega_1}\lambda$.

Conversely, if S is a stationary (club) subset of $\mathcal{P}_{\omega_1}\lambda$ then

$$\bar{S} = \{Y \in \mathcal{P}_{\omega_1}X \mid Y \cap \lambda \in S\}$$

is a stationary (club) subset of $\mathcal{P}_{\omega_1}X$.

A proof of these facts can be found in [2].

Finally, the partial order we will force with is the Lévy partial order, which for cardinals $\lambda < \delta$, collapses all α, $\lambda < \alpha < \delta$ to λ, i.e., its domain is $\{p \mid |p| < \lambda \wedge p \text{ is a function} \wedge \mathrm{dom}(p) \subset \delta \times \lambda \wedge \forall \langle \alpha, \beta\rangle \in \mathrm{dom}(p)\, (p(\alpha, \beta) < \alpha)\}$ and it is ordered by reverse inclusion, i.e., $p \leq q$ iff $q \subset p$. We denote this partial order by $\mathrm{Lv}(<\delta, \lambda)$.

If λ is an infinite regular cardinal and δ an inaccessible cardinal greater than λ, then $\mathrm{Lv}(<\delta, \lambda)$ is $<\lambda$ closed and has the δ-c.c. (see [3, p. 191, Lemma 20.4]). Thus if G is generic over $\mathrm{Lv}(<\delta, \lambda)$ then $\delta = \lambda^+$ in $V[G]$ and no cardinals less than or equal to λ or greater than or equal to δ were collapsed.

We will use the following factorization of the Lévy collapse:

$$\mathrm{Lv}(<\delta, \lambda) = \mathrm{Lv}(<\gamma, \lambda) \times Q^\delta_\gamma,$$

where

$$Q^\delta_\gamma = \{p \in \mathrm{Lv}(<\delta, \lambda) \mid \forall \langle \alpha, \beta \rangle \in \mathrm{dom}(p)\ \alpha \geq \gamma\}.$$

In the proof we will often look at restrictions of a generic filter on $\mathrm{Lv}(<\delta, \lambda)$. We denote the filter on $\mathrm{Lv}(<\delta, \lambda)$ by G_δ and its restriction to $\mathrm{Lv}(<\alpha, \lambda)$ $(\lambda < \alpha < \delta)$ by G_α (G_α is generic over $\mathrm{Lv}(<\alpha, \lambda)$).

3.

We now state our theorem.

Theorem 3.1. *Let δ be a Woodin cardinal, let λ be a regular uncountable cardinal, $\lambda<\delta$. Let G be a generic filter over* $\mathrm{Lv}(<\delta, \lambda)$. *Then in $V[G]$ $(=V[G_\delta])$, the NS ideal over $\mathcal{P}_{\omega_1}\lambda$ is presaturated.*

Proof. By Lemma 2.3, we want to show that given any ω maximal antichains in $\mathcal{P}_{\omega_1}\lambda$ (in $V[G]$) $\langle A^i \mid i\in\omega\rangle$ with $A^i=\langle A^i_\beta \mid \beta<\gamma_i\rangle$ and a stationary set $B\subset \mathcal{P}_{\omega_1}\lambda$, there is a stationary subset D of B such that $|A^i \restriction D|\leqslant\lambda$ for all i. Since $2^\lambda=\lambda^+=\delta$ holds in $V[G]$, any maximal antichain A in $\mathcal{P}_{\omega_1}\lambda$ has size at most δ. Thus $\gamma_i\leqslant\delta$, for all $i\in\omega$. If $\gamma_i<\delta$, then $\gamma_i\leqslant\lambda$, and there is nothing to show regarding A^i. Hence we may assume $\gamma_i=\delta$ for all $i\in\omega$. From now until the end of the proof, we fix $A=\langle A^i \mid i\in\omega\}$ and B. Since their choice was arbitrary, we will be done if we can show that there is a stationary subset D of B such that $|A^i\restriction D|\leqslant\lambda$ for all $i\in\omega$.

It is difficult to prove our claim directly for δ, since δ is not (necessarily) the critical point of an elementary embedding. However, there are many ordinals below δ which *are* the critical points of elementary embeddings, and we will show that there is enough resemblance between these and δ for the proof to go through. This will be made clearer below.

First, we introduce the following definitions:

Definition 3.2. Assume $A=\langle A^i \mid i\in\omega\rangle$ is a sequence of maximal antichains in $\mathcal{P}_{\omega_1}\lambda$ and that $\alpha<\delta$. We say that *α reflects δ* for $\langle A^i \mid i\in\omega\rangle$ if $\langle A^i\restriction\alpha \mid i\in\omega\rangle\in V[G_\alpha]$ and, further, $A^i\restriction\alpha$ is a maximal antichain in $\mathcal{P}_{\omega_1}\lambda$ in $V[G_\alpha]$ for all $i\in\omega$.

Note. Since we have fixed $\langle A^i \mid i\in\omega\rangle$ at the beginning of the proof, we will just say 'α reflects δ', when referring to that sequence of antichains.

The proof will use extensively the relationship between stationary sets in $\mathcal{P}_{\omega_1}\lambda$ and those in $\mathcal{P}_{\omega_1}H_\theta$, for some large regular θ. For $N\subset H_\theta$ and A an antichain in $\mathcal{P}_{\omega_1}\lambda$, we ask about $N\cap\lambda$ belonging to a member of A. Here and elsewhere, H_α will stand for the set of all sets of hereditary cardinality less than α. The following definition appears in [2].

Definition 3.3. Let λ be a regular uncountable cardinal and let θ be a regular cardinal $>2^{\lambda^\omega}$. Let $\mathcal{A}=\langle H_\theta, \in, \Delta_\theta\rangle$. Let N be a countable elementary substructure of $\mathcal{A}$ and let $A=\langle A_\alpha \mid \alpha<\beta\rangle$, $\beta\leqslant 2^{\lambda^\omega}$ be an antichain of stationary sets in $\mathcal{P}_{\omega_1}\lambda$.

- We say *N contains an index for A* iff $\exists\gamma\in N$, $\gamma<\beta$ such that $N\cap\lambda\in A_\gamma$.
- We say that *N can be extended to contain an index for A* (relative to $\mathcal{A}$) iff $\exists M\prec\mathcal{A}$ such that $N\subseteq M$, $N\cap\lambda=M\cap\lambda$, $M\cap\lambda\in A_\gamma$ and $\gamma\in M$. We call such an M a *good extension* of N for A (relative to $\mathcal{A}$).

Remark. The definition of 'containing an index of A' does not depend on the structure $\mathscr{A}$; however, since the N's we will typically look at will be elementary substructures of some structure $\mathscr{A}$, we mentioned it for both parts of the definition, anyway. Also, we will sometimes omit the clause 'relative to $\mathscr{A}$' when the $\mathscr{A}$ in question will be clear from the context.

Let θ_α be $(2^{2^\alpha})^+$. Let Δ_{θ_α} be a well-ordering of $H_{\theta_\alpha}^{V[G_\alpha]}$. (From now on, unless otherwise stated, H_{θ_α} will always mean $H_{\theta_\alpha}^{V[G_\alpha]}$.) Let $S_{B,\alpha}$ be the set of all countable elementary substructures N of $\langle H_{\theta_\alpha}, \in, \Delta_{\theta_\alpha}\rangle$ such that for all i in ω, N contains an index for $A^i \upharpoonright \alpha$ and $N \cap \lambda \in B$. We can then show the following:

Lemma 3.4. *Let δ, λ, G be as in the statement of the theorem. Let A, B be as fixed above. Assume $B \in V[G_\alpha]$, α reflects δ and $S_{B,\alpha}$ is stationary in $(\mathscr{P}_{\omega_1} H_{\theta_\alpha})^{V[G_\alpha]}$. Then (in $V[G_\delta]$) there is a subset D of B such that $|A^i \upharpoonright D| \leq \lambda$ for all $i \in \omega$.*

Proof. We work in $V[G_\delta]$. $S_{B,\alpha}$ is still stationary in $V[G_\delta]$ since

$$\mathrm{Lv}(<\delta, \lambda)/\mathrm{Lv}(<\alpha, \lambda)$$

is proper. (It is Q_α^δ, which is ω-closed.) In $V[G_\delta]$ we have $f: \lambda \to H_{\theta_\alpha}$ which is one-one and onto. Since in $V[G_\delta]$, the set of all subsets of H_{θ_α} which are closed under f, f^{-1} is closed unbounded, its intersection with $S_{B,\alpha}$ is stationary. Let $T_{B,\alpha}$ be the (stationary) set of all N's in $S_{B,\alpha}$ which are closed under f, f^{-1}. Let $T^*_{B,\alpha} = \{f^{-1}{}''N \mid N \in T_{B,\alpha}\}$. We then have that

$$T^*_{B,\alpha} = \{N \cap \lambda \mid N \in T_{B,\alpha}\}.$$

($f^{-1}{}''N \subset N \cap \lambda$ follows from the closure of N under f^{-1}. On the other hand, $N \cap \lambda \subset f^{-1}{}''N$ follows from the closure of N under f, since $\alpha = f^{-1}(f(\alpha))$.)

Since $T^*_{B,\alpha}$ is stationary, it must meet every antichain in $\mathscr{P}_{\omega_1}\lambda$. In particular, it must meet each A^i. For the rest of the proof, fix i. Assume $T^*_{B,\alpha} \cap A^i_\gamma$ is stationary. Let $R = T^*_{B,\alpha} \cap A^i_\gamma$. Then $\hat{R} = \{f''N \mid N \in R\}$ is stationary in $\mathscr{P}_{\omega_1} H_{\theta_\alpha}$. Since $\hat{R} \subset S_{B,\alpha}$, every N in $\hat{R}$ contains an index for $A^i \upharpoonright \alpha$.

Let

$$g(N) = \text{the least } \beta \in N \text{ such that } N \cap \lambda \in A^i_\beta.$$

Then g is regressive on $\hat{R}$, so there is a stationary set $Q \subset \hat{R}$ such that g is constant on Q, say $g(N) = \eta$ for all N in Q. Since every N in R contains an index for $A^i \upharpoonright \alpha$, we must have that $\eta < \alpha$.

Now look at

$$Q^* = \{N \cap \lambda \mid N \in Q\} = \{f^{-1}{}''N \mid N \in Q\}.$$

We have that $Q^* \subset A^i_\eta$, since $N \cap \lambda \in A_\eta$ for all N in Q. Also, $Q^* \subset A^i_\gamma$ since $Q^* \subset R \subset A^i_\gamma$. Since Q^* is stationary, we have that $\gamma = \eta$.

So we have shown that if $T^*_{B,\alpha} \cap A^i_\gamma$ is stationary then $\gamma < \alpha$. Hence $|A^i \upharpoonright T^*_{B,\alpha}| \leq \lambda$. Since the choice of i was arbitrary, $|A^i \upharpoonright T^*_{B,\alpha}| \leq \lambda$ for all i. Also $T^*_{B,\alpha} \subset B$. Hence $T^*_{B,\alpha}$ is the set D we were looking for. $\square$

Thus to prove the theorem, it is sufficient to find an α such that B belongs to $V[G_\alpha]$, α reflects δ and $S_{B,\alpha}$ is stationary in $(P_{\omega_1}H_{\theta_\alpha})^{V[G_\alpha]}$. The rest of the proof is devoted to showing such an α exists.

We first reduce the requirement that $S_{B,\alpha}$ be stationary in $(P_{\omega_1}H_{\theta_\alpha})^{V[G_\alpha]}$ to an alternative requirement (this reduction is due to Foreman, Magidor and Shelah):

Lemma 3.5. *Assume α is such that B belongs to $V[G_\alpha]$, α reflects δ and*

$$C_\alpha = \{N \prec \langle H_{\theta_\alpha}, \in, \Delta_{\theta_\alpha}\rangle \mid |N| = \aleph_0 \text{ and for all } i,\ A^i \upharpoonright \alpha \in N \text{ and } N \text{ has a good extension for } A^i \upharpoonright \alpha\}$$

contains a club in $((\mathcal{P}_{\omega_1}H_{\theta_\alpha}))^{V[G_\alpha]}$. Then $S_{B,\alpha}$ is stationary in $(\mathcal{P}_{\omega_1}H_{\theta_\alpha})^{V[G_\alpha]}$.

Proof. We work in $V[G_\alpha]$. Assume C_α contains a club (in $(\mathcal{P}_{\omega_1}H_{\theta_\alpha})$) but that $S_{B,\alpha}$ is *not* stationary (in $(\mathcal{P}_{\omega_1}H_{\theta_\alpha})^{V[G_\alpha]}$). Then the complement of $S_{B,\alpha}$ contains a club. Thus there is a club set $T_{B,\alpha}$ of countable elementary substructures N of $\langle H_{\theta_\alpha}, \in, \Delta_{\theta_\alpha}\rangle$ which *either* lack an index for some $A^i \upharpoonright \alpha$ $(i \in \omega)$ *or* are such that $N \cap \lambda \notin B$. Now the set of countable elementary substructures N of $\langle H_{\theta_\alpha}, \in, \Delta_{\theta_\alpha}\rangle$ which satisfy $N \cap \lambda \in B$ is stationary, so that we can find an N in $T_{B,\alpha}$ such that for all $i \in \omega$, N has a good extension for $A^i \upharpoonright \alpha$ but there is an i such that N does not contain an index for $A^i \upharpoonright \alpha$.

Let N be such a structure: So N lacks an index for $A^i \upharpoonright \alpha$ but it may be extended to contain one; say M is such an extension. Thus M *contains* an index for $A^i \upharpoonright \alpha$. Have we reached a contradiction? No. First, M has an index for $A^i \upharpoonright \alpha$ but may lack one for $A^j \upharpoonright \alpha$, $j \neq i$. Second, even if M contained indices for $A^i \upharpoonright \alpha$ for all $i \in \omega$, M may no longer belong to $T_{B,\alpha}$.

Consider the first problem, that of adding indices to N for *all* antichains $A^i \upharpoonright \alpha$, $i \in \omega$. We may try to add indices one by one. Say $N_0 = N$ and N_1 is a good extension of N_0 for $A^0 \upharpoonright \alpha$. We now wish to add an index to N_1 for $A^1 \upharpoonright \alpha$. The problem is that N_1 may no longer belong to C_α, in which case we do not know that we can add an index for $A^1 \upharpoonright \alpha$ to it.

Thus it turns out that both problems arise out of the fact that a good extension of a member of C_α $(T_{B,\alpha})$ may no longer belong to C_α $(T_{B,\alpha})$. To solve this problem, we look at the sets

$$C'_\alpha = \{N \prec \langle H^{V[G_\alpha]}_{\theta^+_\alpha}, \in, \Delta_{\theta^+_\alpha}\rangle \mid |N| = \aleph_0 \text{ and } C_\alpha \in N\}$$

and

$$T'_{B,\alpha} = \{N \prec \langle H^{V[G_\alpha]}_{\theta^+_\alpha}, \in, \Delta_{\theta^+_\alpha}\rangle \mid |N| = \aleph_0 \text{ and } T_{B,\alpha} \in N\}.$$

C'_α and $T'_{B,\alpha}$ are both clubs in $(\mathcal{P}_{\omega_1}H_{\theta_\alpha})^{V[G_\alpha]}$, and they both have the property that if M is a good extension of one of their members, then M itself is a member of them. We only have to verify that they still have the original properties that C_α, $T_{B,\alpha}$ (respectively) had.

We start with the easier case, $T_{B,\alpha}$. Let $N \in T'_{B,\alpha}$. We have that $N \cap H_{\theta_\alpha} \in T_{B,\alpha}$, since $T_{B,\alpha}$ is club. Thus either $N \cap H_{\theta_\alpha} \cap \lambda \notin B$ or there is an i in ω such that

$N \cap H_{\theta_\alpha}$ does not contain an index for $A^i \restriction \alpha$. If $N \cap H_{\theta_\alpha} \cap \lambda \notin B$, then $N \cap \lambda \notin B$ ($\lambda \subset H_{\theta_\alpha}$). If $N \cap H_{\theta_\alpha}$ does not contain an index for $A^i \restriction \alpha$, neither does N, since an index for $A^i \restriction \alpha$ is a member of H_{θ_α}.

Next we show that C'_α has the property that, for any $i \in \omega$, every member of it can be extended to contain an index for $A^i \restriction \alpha$. We will rely on a technical lemma, whose proof we postpone until after the proof of the main lemma. The lemma we need is the following (this lemma is part of the Foreman, Magidor and Shelah proof. A version of it appears in [2]):

Lemma 3.6. *Let $\lambda < \theta < \sigma$ be regular uncountable cardinals such that $2^{2^\lambda} < \theta$. Let Δ_σ be a well-ordering of H_σ. Let $N \prec \langle H_\sigma, \in, \Delta_\sigma \rangle$, $|N| = \aleph_0$ and $\theta \in N$. Let $A \in N \cap H_\theta$ be a maximal antichain of $\mathcal{P}_{\omega_1}\lambda$ (with respect to the NS ideal). Then: If $N \cap H_\theta$ has a good extension for A (with respect to $\langle H_\theta, \in, \Delta_\sigma \restriction H_\theta \times H_\theta \rangle$), then so does N (with respect to $\langle H_\sigma, \in, \Delta_\sigma \rangle$).*

Now assume the lemma. Recall that

$$C'_\alpha = \{N \prec \langle H_{\theta_\alpha^+}^{V[G_\alpha]}, \in, \Delta_{\theta_\alpha^+} \rangle \mid |N| = \aleph_0 \text{ and } C_\alpha \in N\}$$

and we want to show that if N is in C'_α, then for all $i \in \omega$, N has a good extension for $A^i \restriction \alpha$ in C'_α.

Fix $N \in C'_\alpha$. Since $N \in C'_\alpha$, $C_\alpha \in N$, so that $N \cap H_\theta^{V[G_\alpha]} \in C_\alpha$ (since C_α is club in $\mathcal{P}_{\omega_1} H_\theta^{V[G_\alpha]}$). So $N \cap H_{\theta_\alpha}$ has a good extension for $A^i \restriction \alpha$. But then so does N, by Lemma 3.6. (Note that we defined $\Delta_{\theta_\alpha^+}$ to be an extension of Δ_{θ_α}; we also have that $\theta \in N$ and that $2^{2^\lambda} < \theta_\alpha$, so that we are justified in applying Lemma 3.6.)

So both C'_α and $T'_{B,\alpha}$ have the property that we wanted them to have. We can now proceed to get a contradiction to our assumption (that $T_{B,\alpha}$ was club). Let

$$\bar{B} = \{N \prec \langle H_{\theta_\alpha^+}, \in, \Delta_{\theta_\alpha^+} \rangle \mid |N| = \aleph_0 \text{ and } N \cap \lambda \in B\}.$$

$\bar{B}$ is stationary.

Let $N_0 \in T'_{B,\alpha} \cap C'_\alpha \cap \bar{B}$. Then $N_0 \cap \lambda \in B$, so $N_0 \in T'_{B,\alpha}$ implies that there is an i such that N_0 does not contain an index for $A^i \restriction \alpha$. $N_0 \in C'_\alpha$, so N_0 has a good extension for (the least such) $A^i \restriction \alpha$, say N_1.

Construct a sequence of models $N_0 \prec N_1 \prec N_2 \prec \cdots$ such that N_{k+1} is a good extension of N_k for A_k, i.e., N_{k+1} contains indices for (at least) $A^0 \restriction \alpha, \ldots, A^k \restriction \alpha$. Consider $M = \bigcup_{i \in \omega} N_i$. We have that $|M| = \aleph_0$, $M \prec \langle H_{\theta_\alpha^+}, \in, \Delta_{\theta_\alpha^+} \rangle$, M has indices for $\langle A^i \restriction \alpha \mid i \in \omega \rangle$ but $M \in C'_\alpha$, contradiction (since $N \in C'_\alpha$ implies that there is an i such that N does not contain an index for $A^i \restriction \alpha$). $\square$

Thus, in order to finish the proof of the theorem, it is enough to show that there is an $\alpha < \delta$ such that α reflects δ (for A), $B \in V[G_\alpha]$ and the set C_α defined above is club in $(\mathcal{P}_{\omega_1} H_{\theta_\alpha})^{V[G_\alpha]}$. We call such an α *good* for A, B.

We conclude this section by proving Lemma 3.6:

Proof of Lemma 3.6. First note that if $N \prec \langle H_\sigma, \in, \Delta_\sigma \rangle$ then $N \cap H_\theta \prec \langle H_\theta, \in, \Delta_\sigma \restriction H_\theta \times H_\theta \rangle$ so that the statement of the lemma makes sense.

Let $\mathcal{A} = \langle H_\sigma, \in, \Delta_\sigma \rangle$, $\mathcal{B} = \langle H_\theta, \in, \Delta_\sigma \restriction H_\theta \times H_\theta \rangle$. We will prove this lemma by describing a canonical way for constructing good extensions (when they exist). We will then argue that since this process works for $N \cap H_\theta$, it works for N.

We set out this process in a separate lemma. We use here and later on the notation SH(N) for the Skolem Hull of N. Hence we use $\mathrm{SH}^{\mathcal{A}}(N)$ for the Skolem Hull of N in $\mathcal{A}$.

Lemma 3.7. *Let $N \prec \langle H_\theta, \in, \Delta_\theta \rangle$ and let A be a maximal antichain of $\mathcal{P}_{\omega_1}\lambda$ with $2^{(\lambda^\omega)} < \theta$, $\lambda < \theta$ regular uncountable cardinals. If N has a good extension for A, then there exists a $\beta < 2^{(\lambda^\omega)}$ such that $\mathrm{SH}(N \cup \{\beta\})$ in $\langle H_\theta, \in, \Delta_\theta \rangle$ is such an extension.*

Proof. Let $\mathcal{A} = \langle H_\theta, \in, \Delta_\theta \rangle$. Let $M \prec \mathcal{A}$ be a good extension of N, say $M \cap \lambda \in A_\beta$. Then $\mathrm{SH}^{\mathcal{A}}(N \cup \{\beta\}) \subset M$, since $N \subset M$, $\beta \in M$, and $M \prec \mathcal{A}$. But then $\mathrm{SH}^{\mathcal{A}}(N \cup \{\beta\})$ must be a good extension of N for A. (Since $N \cap \lambda = M \cap \lambda$, $N \cap \lambda = (\mathrm{SH}^{\mathcal{A}}(N \cup \{\beta\})) \cap \lambda$.) □

We now go back to the 'main' lemma:

$N \cap H_\theta$ has a good extension for A. By the lemma just proved, we may assume this extension is $M = \mathrm{SH}^{\mathcal{B}}((N \cap H_\theta) \cup \{\beta\})$ (for some $\beta < 2^{(\lambda^\omega)}$).

We want to show that $K = \mathrm{SH}^{\mathcal{A}}(N \cup \{\beta\})$ is a good extension of N. It is enough to show that $K \cap \lambda = M \cap \lambda$: We know that $(N \cap H_\theta) \cap \lambda = M \cap \lambda$, so that $N \cap \lambda = M \cap \lambda$. So if $K \cap \lambda = M \cap \lambda$, then $K \cap \lambda = N \cap \lambda$ and K is a good extension of N. (K contains an index for A, namely, β.)

So we want to show that

$$(\mathrm{SH}^{\mathcal{B}}(N \cap H_\theta) \cup \{\beta\}) \cap \lambda = (\mathrm{SH}^{\mathcal{A}}(N \cup \{\beta\})) \cap \lambda.$$

If $\alpha \in (\mathrm{SH}^{\mathcal{A}}(N \cup \{\beta\})) \cap \lambda$, then $\alpha = \tau(a, \beta)$ for some Skolem term τ and some $a \in N$.

Let

$$f(\gamma) = \begin{cases} \tau(a, \gamma) & \text{if } \tau(a, \gamma) \text{ is an ordinal } <\lambda, \\ \emptyset & \text{otherwise.} \end{cases}$$

f is a function from $2^{(\lambda^\omega)}$ into λ. Hence $f \in H_\theta$. Since $a \in N, f \in N$, as well. Hence $f \in N \cap H_\theta$. But then $f(\beta) \in M$; thus $\alpha \in M$, since $\alpha = \tau(a, \beta) = f(\beta)$. So $K \cap \lambda \subset M \cap \lambda$ and hence $K \cap \lambda = M \cap \lambda$. □

4.

Thus far, we have not used the fact that δ was a Woodin cardinal. We will use the Woodin property of δ to show that there is an $\alpha < \delta$ which is good for A and B.

We will use the following definition of a Woodin cardinal:

Definition 4.1. δ is a *Woodin cardinal* iff for all $A \subset V_\delta$ there are arbitrarily large $\kappa < \delta$ such that for all $\alpha < \delta$ there is a $j : V \to M$ with j elementary, $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$, M transitive, $V_\alpha \subset M$ and $A \cap V_\alpha = j(A) \cap V_\alpha$.

Remarks. (1) In the definition above we can replace A by $\langle A_\beta \mid \beta < \delta \rangle$, $A_\beta \subset V_\delta$, demanding that $A_\beta \cap V_\alpha = j(A_\beta) \cap V_\alpha$ for all $\beta \in \delta$ since we can code δ subsets of V_δ by a single subset of V_δ. We will often use this version of Woodinness.

(2) In the definition of a Woodin cardinal, j and M can be chosen such that M is ω-closed in V. See [6, p. 104, Lemma 4.2], for a proof.

Informally, we would like to apply the Woodin property of δ to the sequence of antichains $A = \langle A^i \mid i \in \omega \rangle$. Loosely speaking, we would like to be able to find a $\kappa < \delta$ such that for any $\beta < \delta$ there is always an elementary embedding with critical point κ from one model to another where the two models 'agree' on the antichains up to β, i.e., $j(A^i) \restriction \beta = A^i \restriction \beta$.

The problem with getting the above situation to hold is that A can be thought of as a subset of V_δ in $V[G]$, but δ is no longer Woodin in $V[G]$ (it was made to be λ^+). Thus, we cannot apply the Woodin property of δ directly to A.

Also, even if we find a way to get around this problem, there is a question of what kind of elementary embeddings we will now have. Since δ is λ^+ in $V[G]$, we will have to make use of generic elementary embeddings, i.e., extend an elementary embedding $j : V \to M$, originally in V, to be defined on a generic extension of V.

The solution taken in this paper is the following: We begin with the model $V[G] = V[G_\delta]$. We will use the Woodin property of δ to show that for any $\alpha < \delta$ there exists an elementary embedding $j : V \to M$, $\text{crit}(j) = \kappa$, $j(\kappa) > \alpha$ such that j has the following property: We can find a V-generic filter over $j(\text{Lv}(<\kappa, \lambda))$, $H_{j(\kappa)}$, such that j can be extended to a j^*, $j^* : V[G_\kappa] \to M[H_{j(\kappa)}]$ (j^* an elementary embedding), with the property that $M[H_{j(\kappa)}]$ is ω-closed in $V[H_{j(\kappa)}]$. Further, we may choose $H_{j(\kappa)}$ such that in addition:

(a) $H_\alpha = G_\alpha$.

(b) $j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha$ for all $i \in \omega$.

In fact, since our aim is to find a κ which is good for A, B we will want the κ mentioned above to reflect δ and be such that $B \in V[G_\kappa]$. We will postpone the proof that a κ satisfying all the above mentioned requirements exists to the next section. We will now show how, assuming such a κ exists, we can complete the proof of the theorem.

Lemma 4.2. *Let $\lambda < \kappa < \delta$ be regular uncountable cardinals. Let G_δ be generic over* $\text{Lv}(<\delta, \lambda)$. *Let $A = \langle A^i \mid i \in \omega \rangle$ be a sequence of maximal antichains in*

$(\mathcal{P}_{\omega_1}\lambda)^{V[G_\delta]}$ *(with* $A^i = \langle A^i_\beta \mid \beta < \delta \rangle$*). Let* B *be a stationary set in* $\mathcal{P}_{\omega_1}\lambda$ *(in* $V[G_\delta]$*). Assume* κ *satisfies the following:*

1. κ *reflects* δ.
2. $B \in V[G_\kappa]$.
3. *For any* $\alpha < \delta$ *there exists an elementary embedding* $j : V \to M$, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$ *such that* j *has the following property*:

There is a V*-generic filter over* $j(\mathrm{Lv}(<\kappa, \lambda))$, $H_{j(\kappa)}$, *such that* j *can be extended to a* j^*, $j^* : V[G_\kappa] \to M[H_{j(\kappa)}]$ *(*j^* *an elementary embedding), with the property that* $M[H_{j(\kappa)}]$ *is* ω*-closed in* $V[H_{j(\kappa)}]$*. Further, we may choose* $H_{j(\kappa)}$ *such that in addition*:

(a) $H_\alpha = G_\alpha$.

(b) $j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha$ *for all* $i \in \omega$.

Then κ *is good for* A, B.

Proof. Our convention above regarding H_θ's still holds, i.e., H_{θ_κ} is $H^{V[G_\kappa]}_{\theta_\kappa}$, unless otherwise indicated. By the definition of κ being good (after end of proof of Lemma 3.5), it is enough to show that the set

$$C_\kappa = \{N < \langle H_{\theta_\kappa}, \in, \Delta_{\theta_\kappa}\rangle \mid |N| = \aleph_0 \text{ and for all } i,\ A^i \restriction \kappa \in N \text{ and } N \text{ has a good extension for } A^i \restriction \kappa\}$$

contains a club in $(\mathcal{P}_{\omega_1}H_{\theta_\kappa})$ (in $V[G_\kappa]$). Assume not. Let i be the least such that the set $S_{\kappa,i}$ of all $N < \langle H_{\theta_\kappa}, \in, \Delta_{\theta_\kappa}\rangle$ such that $|N| = \aleph_0$ and N does *not* have a good extension for A^i is stationary in $\mathcal{P}_{\omega_1}H_{\theta_\kappa}$.

We first work in $V[G_\delta]$. Let $g : \lambda \to H^{V[G_\kappa]}_{\theta_\kappa}$ be one-one and onto ($g \in V[G_\gamma]$, where $\gamma = |H_{\theta_\alpha}|^+$). Let $S^*_{\kappa,i} = \{g^{-1}{}''N \mid N \in S_{\kappa,i}\}$. Then $S^*_{\kappa,i}$ is stationary in $\mathcal{P}_{\omega_1}\lambda$. Let β be the least ordinal such that $S^*_{\kappa,i} \cap A^i_\beta$ is stationary. Let $\alpha > \beta$, $|H_{\theta_\kappa}|^+$ be such that $\langle A^i \restriction \alpha \mid i \in \omega\rangle \in V[G_\alpha]$ (we will show in Section 5, Corollary 5.5, that the set of all α's such that $\langle A^i \restriction \alpha \mid i \in \omega\rangle \in V[G_\alpha]$ is club in δ). Use hypothesis 3(a–b) in the statement of the lemma to find a j, $j : V[G_\kappa] \to M[H_{j(\kappa)}]$ (with critical point κ) where $H_\alpha = G_\alpha$, $M[H_{j(\kappa)}]$ is ω-closed in $V[H_{j(\kappa)}]$ and $j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha$, for all $i \in \omega$.

Now work in $V[H_{j(\kappa)}]$:

Since $A^i \restriction \alpha \in V[G_\alpha]$, we have that $A^i_\beta \in V[G_\alpha]$. Since $\alpha > |H_{\theta_\kappa}|^+$, $S^*_{\kappa,i} \in V[G_\alpha]$. Thus $S^*_{\kappa,i} \cap A^i_\beta \in V[G_\alpha]$ and is stationary in $V[G_\alpha]$ (since it is stationary in a bigger model, $V[G_\delta]$). Thus $S^*_{\kappa,i} \cap A^i_\beta$ is stationary in $V[H_{j(\kappa)}]$: The forcing $j(\mathrm{Lv}(<\kappa, \lambda))/\mathrm{Lv}^M(<\alpha, \lambda)$ is ω-closed in M; but since M is ω-closed in V, this means that the quotient forcing is actually ω-closed, and thus proper, in V. So the fact that $S^*_{\kappa,i} \cap A^i_\beta$ is stationary in $V[G_\alpha]$ implies that $S^*_{\kappa,i} \cap A^i_\beta$ is stationary in $V[H_{j(\kappa)}]$.

Let

$$C_1 = \{N < H^{M[H_{j(\kappa)}]}_{\theta_{j(\kappa)}} \mid |N| = \aleph_0 \text{ and } \beta, g, g^{-1}, j \restriction H_{\theta\kappa} \in N\}.$$

(j is defined in $V[H_{j(\kappa)}]$, so there is no problem in closing under $j \restriction H_{\theta_\kappa}$.) C_1 is club (in $\mathcal{P}_{\omega_1}(H^{M[H_{j(\kappa)}]}_{j(\theta)})$). So there exists an N in C_1, $N \cap \lambda \in S^*_{\kappa,i} \cap A^i_\beta$. Look at $g''(N \cap \lambda) = K$. Since $N \cap \lambda \in S^*_{\kappa,i}$, $K \in S_{\kappa,i}$. So $j(K) \in J(S_{\kappa,i})$. (Note that $S_{\kappa,i} \in V[G_\kappa]$, so that $K \in S_{\kappa,i} \Rightarrow K \in V[G_\kappa]$, and thus j is well defined on these sets).

Now move back to $M[H_{j(\kappa)}]$: In $M[H_{j(\kappa)}]$, $j(K) \in j(S_{\kappa,i})$ means $j(K)$ has no good extension for $j(A^i \restriction \kappa)$; but it does, since N is such an extension.

We need to argue:

(1) $N \in M[H_{j(\kappa)}]$.
(2) N is a good extension of $j(K)$ for $j(A^i \restriction \kappa)$.

(1) follows from the fact that $M[H_{j(\kappa)}]$ is ω-closed in $V[H_{j(\kappa)}]$; N is a countable object which is a subset of a set of $M[H_{j(\kappa)}]$ (namely, $H^{M[H_{j(\kappa)}]}_{\theta_{j(\kappa)}}$) and thus belongs to $M[H_{j(\kappa)}]$.

To show (2) holds, we need to show:

(a) N contains an index for $j(A^i \restriction \kappa)$.
(b) $N \cap \lambda = j(K) \cap \lambda$.
(c) $j(K) \subset N$.

Proof. (a) We chose N so that $N \cap \lambda \in A^i_\beta$, and $\beta \in N$. We also have that $j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha$ so that $A^i_\beta \in j(A^i \restriction \kappa)$; thus N contains an index for $j(A^i \restriction \kappa)$.

(b) and (c). This is the same argument as in the [2] proof: $N \cap \lambda = K \cap \lambda$ since N is closed under g, g^{-1}. $K \cap \lambda = j(K) \cap \lambda$ since the critical point of j is greater than λ. Finally, $K \subset N$ since N is closed under g, and $j(K) \subset N$ since N is closed under $j \restriction H_{\theta_\kappa}$.

So in $M[H_{j(\kappa)}]$, $j(K) \in j(S_{\kappa,i})$. But $j(K)$ *does* have a good extension for $j(A^i \restriction \kappa)$; this is a contradiction. Thus C_κ is club in $(\mathcal{P}_{\omega_1} H_{\theta_\kappa})$ and κ is good for A, B. □

Assuming a κ as in the hypotheses of Lemma 4.2 can always be found (in our circumstances), we can find a κ such that these hypotheses hold with respect to the particular sequence of antichains $\langle A^i \mid i \in \omega \rangle$ and the particular stationary set B that we have fixed at the beginning of the proof of the theorem. Then Lemma 4.2 guarantees that κ is good for A, B and thus the proof of Theorem 3.1 is completed.

5.

This section is devoted to showing that a κ satisfying the hypotheses of Lemma 4.2 exists. We will assume throughout this section that δ is a Woodin cardinal and that G_δ is a generic filter over $\mathrm{Lv}(<\delta, \lambda)$ ($\lambda < \delta$ regular). Also, since the hypotheses of Lemma 4.2 refer to one sequence of antichains in $\mathcal{P}_{\omega_1}\lambda$ in $V[G_\delta]$ and one stationary set in $\mathcal{P}_{\omega_1}\lambda$, we will assume, as before, that $\langle A^i \mid i \in \omega \rangle$ is a sequence of maximal antichains in $\mathcal{P}_{\omega_1}\lambda$ in $V[G_\delta]$ and that B is a stationary set in $\mathcal{P}_{\omega_1}\lambda$ in $V[G_\delta]$.

We first discuss the issue of 'extending elementary embeddings'. In our situation, δ is a Woodin cardinal and we have a $\kappa < \delta$ such that $j: V \to M$ has critical point κ (j an elementary embedding). For any V generic filter G_κ over $\mathrm{Lv}(<\kappa, \lambda)$ we can always find a V-generic filter over $j(\mathrm{Lv}(<\kappa, \lambda))$, $H_{j(\kappa)}$, such that $G_\kappa \subset H_{j(\kappa)}$. This is true because

(a) $j(\mathrm{Lv}(<\kappa, \lambda)) = \mathrm{Lv}^M(<\kappa, \lambda) \times (Q_\kappa^{j(\kappa)})^M$.
(b) $\mathrm{Lv}^M(<\kappa, \lambda) = \mathrm{Lv}^V(<\kappa, \lambda)$.

(a) follows by the factoring of the Lévy collapse remarked on earlier and the fact that $j(\mathrm{Lv}(<\kappa, \lambda)) = \mathrm{Lv}^M(<j(\kappa), \lambda)$.

(b) is true because the critical point of j is κ and so $V_\kappa \in M$ and thus $\mathrm{Lv}(<\kappa, \lambda)$ computed in M is the same as when computed in V.

From (a) and (b) together we get that

(c) $j(\mathrm{Lv}(<\kappa, \lambda)) = \mathrm{Lv}^V(<\kappa, \lambda) \times (Q_\kappa^{j(\kappa)})^M$.

This allows us to pick a V-generic filter over $j(\mathrm{Lv}(<\kappa, \lambda))$ extending G_κ.

This, in turn, implies that j can be extended to an elementary j^*, $j^*: V[G_\kappa] \to M[H_{j(\kappa)}]$. See e.g., [4]. From now on we will denote both j and j^* by j.

In our case, we start with the model $V[G_\delta]$, where G_δ is generic over $\mathrm{Lv}(<\delta, \lambda)$. Starting with $j: V \to M$, $\mathrm{crit}(j) = \kappa$, we not only want to be able to pick a V-generic filter $H_{j(\kappa)}$ extending G_κ as described above, but also to be able to guarantee that $H_{j(\kappa)}$ 'agrees to a large extent' with G_δ. More precisely, we want κ to be such that for any $\alpha < \delta$, $\exists j: V \to M$, $\mathrm{crit}(j) = \kappa$ such that j can be extended in the way described above but with $H_\alpha = G_\alpha$ (where $H_\alpha = H_{j(\kappa)} \restriction \mathrm{Lv}^M(<\alpha, \lambda)$). Here it is the Woodinness of δ which guarantees this can be done.

Claim 5.1. *Let δ be a Woodin cardinal and let $\kappa < \delta$ be such that for all $\alpha < \delta$ there is an elementary embedding j, $j: V \to M$, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and $j(\mathrm{Lv}(<\delta, \lambda)) \cap V_\alpha = \mathrm{Lv}(<\delta, \lambda) \cap V_\alpha$. Let G_δ be V generic over $\mathrm{Lv}(<\delta, \lambda)$. Then $\mathrm{Lv}^V(<\alpha, \lambda) = \mathrm{Lv}^M(<\alpha, \lambda)$ and there exists a V-generic filter over $j(\mathrm{Lv}(<\kappa, \lambda))$, $H_{j(\kappa)}$, such that $G_\alpha = H_\alpha$.*

Proof. Let κ be as in the claim. Since $\mathrm{Lv}(<\alpha, \lambda) \subset V_\alpha$, the above equality will guarantee that $j(\mathrm{Lv}(<\delta, \lambda)) \cap V_\alpha = \mathrm{Lv}(<\alpha, \lambda)$; or that $\mathrm{Lv}^M(<\alpha, \lambda) = \mathrm{Lv}^V(<\alpha, \lambda)$. Hence G_α will be (V-) generic over the initial part of the forcing $(j(\mathrm{Lv}(<\kappa, \lambda)))$ in M (note that $j(\mathrm{Lv}(<\kappa, \lambda)) = j(\mathrm{Lv}(<\delta, \lambda)) \restriction j(\kappa)$, since $j(\kappa) > \alpha$); we can then choose any $V[G_\alpha]$-generic filter over the rest of the forcing $(j(\mathrm{Lv}(<\kappa, \lambda))/\mathrm{Lv}(<\alpha, \lambda))$ to get our filter $H_{j(\kappa)}$. □

We also want that $j: V[G_\kappa] \to M[H_{j(\kappa)}]$ will be such that $M[H_{j(\kappa)}]$ is ω-closed in $V[H_{j(\kappa)}]$. This will follow if we choose $j: V \to M$ so that M is ω-closed in V:

Claim 5.2. *Let $j: V \to M$ be an elementary embedding from V to M, with* $\operatorname{crit}(j) = \kappa$. *Let G be a generic filter over* $\mathrm{Lv}(<\kappa, \lambda)$. *Let H be a V-generic filter over $j(\mathrm{Lv}(<\kappa, \lambda))$ extending G. Let $\eta < \lambda$. Then if M is η closed in V (i.e. ${}^\eta M \subset M$, where ${}^\eta M$ is computed in V) then $M[H]$ is η closed in $V[H]$.*

Proof. Let $\langle x_\alpha \mid \alpha \in \eta \rangle$ be an η sequence of elements of $M[H]$ in $V[H]$. Let τ be a $j(\mathrm{Lv}(<\kappa, \lambda))$ name for $\langle x_\alpha \mid \alpha \in \eta \rangle$. We may assume

$$1 \Vdash_{j(\mathrm{Lv}(<\kappa,\lambda)),V} \text{'}\tau \text{ is an } \eta \text{ sequence of elements each of which is in } M[H]\text{'}.$$

It is enough to show that

$$D = \{p \mid p \Vdash_{j(\mathrm{Lv}(<\kappa,\lambda)),V} \tau \in M[H]\}$$

is dense in $j(\mathrm{Lv}(<\kappa, \lambda))$ (since then the generic filter H must meet this set).

Let $p \in j(\mathrm{Lv}(<\kappa, \lambda))$. Construct a descending sequence of conditions (in V) $\langle p_\alpha | \alpha < \eta \rangle$ with $p_0 = p$ and such that $p_\alpha \Vdash \tau(\check{\alpha}) = \sigma_\alpha$, for some M name σ_α (this is possible since $p_\alpha \Vdash \tau(\check{\alpha}) \in M[H]$). $j(\mathrm{Lv}(<\kappa, \lambda))$ is $<\lambda$ closed in M, so, in particular, it is η closed in M. M is η closed in V (this was our assumption) so that $j(\mathrm{Lv}(<\kappa, \lambda))$ is η closed in V, as well. Hence there is a lower bound for $\langle p_\alpha \mid \alpha < \eta \rangle$ in $j(\mathrm{Lv}(<\kappa, \lambda))$. Let q be a lower bound for $\langle p_\alpha \mid \alpha < \eta \rangle$. Then $q \Vdash \tau \in M[H]$. $\square$

We now want to explain how we can make sure that (we can find a κ such that) we can find an extended j with $j: V[G_\kappa] \to M[H_{j(\kappa)}]$ such that $j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha$. The idea is to use the Woodin property in V for *names* of the antichains. It turns out that it is simpler to work with names for subsets of δ, rather than names for the antichains themselves. We thus proceed to code $\langle A^i \mid i \in \omega \rangle$ as a subset of δ.

Coding of $\langle A^i \mid i \in \omega \rangle$ as a subset of δ

We will use the following one-one and onto functions for our codings:

$$f_1: \lambda \to (\mathcal{P}_{\omega_1}\lambda)^V = (\mathcal{P}_{\omega_1}\lambda)^{V[G_\delta]},$$

$$f_2: \lambda \cdot \delta \to \delta,$$

$$f_3: \delta \cdot \omega \to \delta,$$

where f_2 and f_3 belong to V and f_1 belongs to $V[G_\alpha]$ for all $\alpha > \beta$, where $\beta = (|\mathcal{P}_{\omega_1}\lambda|)^V$. $\beta < \delta$ since δ is inaccessible. Note that $(\mathcal{P}_{\omega_1}\lambda)^V = (\mathcal{P}_{\omega_1}\lambda)^{V[G_\delta]}$ because $\mathrm{Lv}(<\delta, \lambda)$ is ω-closed.

1. Coding subsets of $\mathcal{P}_{\omega_1}\lambda$: If $X \subset \mathcal{P}_{\omega_1}\lambda$, then $f^{-1}{}''X$ codes X. $f^{-1}{}''X \subset \lambda$ (thus subsets of $\mathcal{P}_{\omega_1}\lambda$ can be thought of as subsets of λ, on a tail-end of the forcing).

2. Coding $A^i = \langle A^i_\alpha \mid \alpha < \delta \rangle$: Let $X^i_\alpha = f_1^{-1}{}''A^i_\alpha$. Thus we can think of our antichain as a sequence of subsets of λ, $\langle X^i_\alpha \mid \alpha < \delta \rangle$. Define $\langle g_\alpha \mid \alpha < \delta \rangle$, $g_\alpha : \lambda \to \delta$ as follows:

   $$g_\alpha(\beta) = f_2(\lambda \cdot \alpha + \beta) \quad \text{for } \beta < \lambda.$$

   Since the ranges of the g_α's are disjoint, $X^i = \bigcup_{\alpha<\delta} g''_\alpha X^i_\alpha$ codes $\langle X^i_\alpha \mid \alpha < \delta \rangle$ and $X^i \subset \delta$.
3. Coding $\langle A^i \mid i \in \omega \rangle$: Let X^i code A^i $(i \in \omega)$ as in 2. Define $\langle h_i \mid i \in \omega \rangle$, $h_i : \delta \to \delta$ as follows: $h_i(\beta) = f_3(\delta \cdot i + \beta)$ for $\beta < \delta$. Then $\bigcup_{i\in\omega} h''_i X^i$ codes $\langle A^i \mid i \in \omega \rangle$ as a subset of δ.

From now until the end of this section, f_1–f_3 will refer to the coding functions mentioned above and X, X^i will code $\langle A^i \mid i \in \omega \rangle$, A^i (resp.). The coding will be done in the way it was set up above.

When can we code and decode $\langle A^i \mid i \in \omega \rangle$ correctly? More specifically, we will be working in some $V[G_\alpha]$. For which α's does $X \cap \alpha$ belong to $V[G_\alpha]$ and code there the sequence $\langle A^i \restriction \alpha \mid i \in \omega \rangle$?

We answer this question in two stages. First, we have:

Lemma 5.3. *Let δ be an inaccessible cardinal. Let $E \subset \delta$ in $V[G_\delta]$, where G_δ is generic over* Lv$(<\delta, \lambda)$, *$\lambda < \delta$ regular cardinal. Then there is a club $C \subset \delta$ (C in V) such that $\alpha \in C \Rightarrow E \cap \alpha \in V[G_\alpha]$.*

Proof. Let $E \subset \delta$ in $V[G_\delta]$. Let $\underset{\sim}{E}$ be a name for E. For $\beta < \delta$, let F_β be a maximal antichain in Lv$(<\delta, \lambda)$ that decides whether β belongs to E (i.e., $p \in F_\beta \Rightarrow p \Vdash \check\beta \in \underset{\sim}{E}$ or $p \Vdash \check\beta \notin \underset{\sim}{E}$). Note that $|F_\beta| < \delta$ (because Lv$(<\delta, \lambda)$ has the δ-c.c.) and $p \in$ Lv$(<\delta, \lambda) \Rightarrow p \in$ Lv$(<\alpha, \lambda)$ for some $\alpha < \delta$. Thus we can define $f : \delta \to \delta$ as follows:

$$f(\gamma) = \text{least } \alpha \text{ such that } \{F_\beta \mid \beta < \gamma\} \subset \mathrm{Lv}(<\alpha, \lambda).$$

Let C be the set of closure points of f. Then $\alpha \in C \Rightarrow E \cap \alpha \in V[G_\alpha]$ since $E \cap \alpha = \{\beta < \alpha \mid \exists p \in G_\alpha\ p \Vdash \check\beta \in \underset{\sim}{E}\}$. $\square$

Corollary 5.4. *There is a club set $C_5 \subset \delta$ such that $\alpha \in C_5 \Rightarrow X \cap \alpha \in C_5$ and $X \cap \alpha$ codes $\langle A^i \restriction \alpha \mid i \in \omega \rangle$ (the coding and decoding can be done in $V[G_\alpha]$).*

Proof. Our coding functions f_1, f_2, f_3 all belong to $V[G_\beta]$, for all $\beta > |\mathcal{P}_{\omega_1}\lambda|^V$, so that the coding and decoding can be done in almost all $V[G_\alpha]$'s. The main problem is when does $X \cap \alpha$ code $\langle A^i \restriction \alpha \mid i \in \omega \rangle$, and the answer is that α has to be a closure point of the coding functions — and there are club many such α's. More specifically, let

$$C_1 = \{\alpha < \delta \mid \alpha > (|\mathcal{P}_{\omega_1}\lambda|)^V\}.$$
$$C_2 = \{\alpha < \delta \mid f_2 \restriction \lambda \cdot \alpha : \lambda \cdot \alpha \to \alpha \text{ and is onto } \alpha\}.$$
$$C_3 = \{\alpha < \delta \mid f_3 \restriction \{\delta \cdot i + \beta \mid i \in \omega, \beta < \alpha\} : \{\delta \cdot i + \beta \mid i \in \omega, \beta < \alpha\} \to \alpha \text{ and is onto } \alpha\}.$$

C_1, C_2 and C_3 are all clubs in δ. Let $X^i \subset \delta$ code A^i for $i \in \omega$ as explained above. Let $X \subset \delta$ code $\langle A^i \mid i \in \omega \rangle$. By Lemma 5.3, there exists a club $C_4 \subset \delta$ such that $\alpha \in C_4 \Rightarrow X \cap \alpha \in V[G_\alpha]$. Let

$$C_5 = C_1 \cap C_2 \cap C_3 \cap C_4.$$

Let $\alpha \in C_5$. Since $\alpha \in C_4$, $X \cap \alpha \in V[G_\alpha]$. Since $\alpha \in C_3$, $X \cap \alpha$ codes (via f_3) $\langle X^i \cap \alpha \mid i \in \omega \rangle$. Since $\alpha \in C_2 \cap C_1$, each $X^i \cap \alpha$ codes $A^i \restriction \alpha$ (via f_2 and f_1). (*Note.* We need $\alpha \in C_1$ so that $f_1 \in V[G_\alpha]$.) Thus $\langle A^i \restriction \alpha \mid i \in \omega \rangle \in V[G_\alpha]$ (and hence also $A^i \restriction \alpha \in V[G_\alpha]$ for each $i \in \omega$). $\square$

Note, in particular, that we have the following:

Corollary 5.5. *There is a club set in δ of points α such that $\langle A^i \restriction \alpha \mid i \in \omega \rangle \in V[G_\alpha]$.*

(We used this fact in the proof of Lemma 4.2.)

We now continue in trying to get a $j: V[G_\kappa] \to M[H_{j(\kappa)}]$ such that $j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha$ by starting with a $j: V \to M$ for which there is agreement between the names for these sets and their images under j.

Let $\underset{\sim}{A}^i$ be an $\mathrm{Lv}(<\delta, \lambda)$ name for A^i ($i \in \omega$). We know we can find a κ such that for any $\alpha < \delta$ $\exists j, j: V \to M$, such that $j(\underset{\sim}{A}^i) \cap V_\alpha = \underset{\sim}{A}^i \cap V_\alpha$. We want to show that we can get agreement on the A^i's themselves from the agreement on the names. To work out all the details for this, we will have to choose 'nice' names for A^i ($i \in \omega$). In fact, it will be easier to work with the codes X^i for A^i (discussed above) and names for them.

Fix α. Suppose we want a $j, j: V[G_\kappa] \to M[H_{j(\kappa)}]$ with $j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha$. Assume that (as before) X^i is a code for A^i, and assume we have $j: V[G_\kappa] \to M[H_{j(\kappa)}]$ with $j(X^i \cap \kappa) \cap \alpha = X^i \cap \alpha$. Also assume that κ, $\alpha \in C_5$. Since $\kappa \in C_5$, $X^i \cap \kappa$ codes $A^i \restriction \kappa$ (via f_1, f_2). So $j(X^i \cap \kappa)$ codes $j(A^i \restriction \kappa)$ (via $j(f_1), j(f_2)$). Since $\alpha \in C_5$, $j(X^i \cap \kappa) \cap \alpha$ codes $j(A^i \restriction \kappa) \restriction \alpha$, while $X^i \cap \alpha$ codes $A^i \restriction \alpha$. We want to argue that if the codes $(X^i \cap \alpha, j(X^i \cap \kappa) \cap \alpha)$ are the same the sets they code $(A^i \restriction \alpha, j(A^i \restriction \kappa) \restriction \alpha)$ are the same. This will be true as long as the coding functions we use in $V[G_\kappa]$ and $V[H_{j(\kappa)}]$ are the same, or at least agree on the relevant domains.

Now $f_1: \lambda \to \mathcal{P}_{\omega_1}\lambda$ (since $\kappa \in C_5$, $f_1 \in V[G_\kappa]$). So $j(f_1) = f_1$. Also, $f_2: \lambda \cdot \delta \to \delta$; and $\alpha \in C_5 \Rightarrow f_2: \lambda \cdot \alpha \to \alpha$ is one-one and onto; so if we choose $\alpha \in C_5$ and κ so that $j(f_2) \cap V_\alpha = f_2 \cap V_\alpha$, we will get that $f_2 \restriction \lambda \cdot \alpha = j(f_2) \restriction \lambda \cdot \alpha$, and hence f_2 and $j(f_2)$ will agree when decoding $X^i \cap \alpha$ and $j(X^i \cap \kappa) \cap \alpha$, respectively.

We summarize the last few paragraphs in the following claim (the notation remains fixed).

Claim 5.6. *Let κ, $\alpha < \delta$ be such that κ, $\alpha \in C$ and let $j: V[G_\kappa] \to M[H_{j(\kappa)}]$ be an elementary embedding such that $j(f_2) \cap V_\alpha = f_2 \cap V_\alpha$ and $j(X^i \cap \kappa) \cap \alpha = X^i \cap \alpha$. Then $j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha$.*

We will now choose 'nice' names for X^i. Let $\langle \bar{D}^i_\beta \mid \beta < \delta \rangle$ be a name for X^i, where D^i_β is a maximal antichain in $\mathrm{Lv}(<\kappa, \lambda)$ deciding whether β belongs to X^i and $\bar{D}^i_\beta$ is the antichain split into two: those conditions deciding that β belongs to X^i and those that decide that β does not belong to X^i. (Let $\underset{\sim}{X}^i$ be a name for X^i. Then $p \in D^i_\beta \Rightarrow p \Vdash \check{\beta} \in \underset{\sim}{X}^i$ or $p \Vdash \check{\beta} \notin \underset{\sim}{X}^i$; $\bar{D}^i_\beta = \{\langle p, i\rangle \mid p \in D^i_\beta, i \in \{0, 1\}$ and $p \Vdash \check{\beta} \in \underset{\sim}{X}^i$ and $i = 0$ or $p \Vdash \check{\beta} \notin \underset{\sim}{X}^i$ and $i = 1\}$.)

Thus $\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle$ is an $\mathrm{Lv}(<\kappa, \lambda)$ name for $X^i \cap \kappa$; $\langle \bar{D}^i_\beta \mid \beta < \alpha \rangle$ is a name for $X^i \cap \alpha$. Suppose we arrange that $(j(\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle)) \restriction \alpha = \langle \bar{D}^i_\beta \mid \beta < \alpha \rangle$. (Let $j(\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle) = \langle \bar{E}^i_\beta \mid \beta < j(\kappa) \rangle$. Then by $j(\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle) \restriction \alpha$ I mean $\langle \bar{E}^i_\beta \mid \beta < \alpha \rangle$.)

By elementarity of j, $j(\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle)$ is a name for $j(X^i \cap \kappa)$, and so $j(\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle) \restriction \alpha$ is a name for $j(X^i \cap \kappa) \cap \alpha$. So the *names* for $X^i \cap \alpha$, $j(X^i \cap \kappa) \cap \alpha$ are the same; the *sets* will be the same if the generic filters interpreting the names are the same, at least as far as the conditions mentioned in the names are concerned.

So let D be a club set in δ ($D \in V$) such that $\gamma \in D \Rightarrow \langle \bar{D}^i_\beta \mid \beta < \gamma \rangle \subset V_\gamma$ for all i. We then have:

Lemma 5.7. *Let X^i $(i \in \omega)$, $\langle \bar{D}^i_\beta \mid \beta < \delta \rangle$ and D be as described above. Let $\alpha \in D$ and let $\kappa < \delta$ be the critical point of an elementary embedding $j : V \to M$ such that $j(\kappa) > \alpha$ and $j(\langle \bar{D}^i_\beta \mid \beta < \delta \rangle) \cap V_\alpha = \langle \bar{D}^i_\beta \mid \beta < \delta \rangle \cap V_\alpha$. Let $H_{j(\kappa)}$ be V-generic over $j(\mathrm{Lv}(<\kappa, \lambda))$ such that $H_\alpha = G_\alpha$. Then*:

(1) $(j(\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle)) \restriction \alpha = \langle \bar{D}^i_\beta \mid \beta < \alpha \rangle$,

(2) $j(X^i \cap \kappa) \cap \alpha = X^i \cap \alpha$.

Proof. (1) Since $\alpha \in D$, $\langle \bar{D}^i_\beta \mid \beta < \alpha \rangle$ '$\subset$' $\langle \bar{D}^i_\beta \mid \beta < \delta \rangle \cap V_\alpha$. Also, if $j(\langle \bar{D}^i_\beta \mid \beta < \delta \rangle) = \langle \bar{E}^i_\beta \mid \beta < \delta \rangle$, then $\langle \bar{E}^i_\beta \mid \beta < \delta \rangle$ is determined by $\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle$, since $j(\kappa) > \alpha$.

(2) Since $\langle \bar{D}^i_\beta \mid \beta < \alpha \rangle \subset V_\alpha$, all we need to interpret this name in $V[G_\delta]$ is G_α. Since $(j(\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle)) \restriction \alpha = \langle \bar{D}^i_\beta \mid \beta < \alpha \rangle$, all we need to interpret $(j(\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle)) \restriction \alpha$ in $M[H_{j(\kappa)}]$ is H_α. Since $G_\alpha = H_\alpha$, these interpretations will be the same in both models. Thus, as argued in the paragraphs before the statement of this lemma, $X^i \cap \alpha$ (named by $\langle \bar{D}^i_\beta \mid \beta < \alpha \rangle$) is the same as $j(X^i \cap \kappa) \cap \alpha$ (named by $j(\langle \bar{D}^i_\beta \mid \beta < \kappa \rangle) \restriction \alpha$). $\square$

We are now almost done showing that there exists a κ satisfying the hypotheses of Lemma 4.2. What we still need to show is that we can make sure that κ, the critical point of our embedding, reflects δ and that $B \in V[G_\kappa]$. Note first that we have the following general fact about Woodin cardinals:

Lemma 5.8. *Let δ be a Woodin cardinal, $C \subset \delta$ club in δ. Let $\kappa < \delta$ be such that for all $\alpha < \delta$ $\exists j : V \to M$ such that* $\mathrm{crit}(j) = \kappa$ *and $j(C) \cap \alpha = C \cap \alpha$. Then $\kappa \in C$.*

Proof. Let κ be as in the claim, and assume $\kappa \notin C$. Then κ is not a limit point of C. Let α be the largest ordinal in C that is smaller than κ. Let β be the least ordinal in C that is bigger than κ. Let $\gamma > \beta$ and let $j : V \to M$, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \gamma$ and $C \cap \gamma = j(C) \cap \gamma$. Then we have:

$$V \models \alpha \text{ is the largest ordinal in } C \text{ which is less than } \kappa.$$

By elementarity of j,

$$M \models j(\alpha) \text{ is the largest ordinal in } j(C) \text{ which is less than } j(\kappa).$$

So

$$M \models \alpha \text{ is the largest ordinal in } j(C) \text{ which is less than } j(\kappa).$$

(Note that since $\alpha < \kappa, j(\alpha) = \alpha$.) But $\beta \in C$, $\beta > \kappa$, $\beta < \gamma$ so $\beta \in j(C)$ and $\alpha < \beta < j(\kappa)$, contradiction. $\square$

Recall the club set C_5 defined above of points α such that (among other things) $\langle A^i \restriction \alpha \mid i \in \omega \rangle$ belongs to $V[G_\alpha]$. The set B is a subset of $\mathcal{P}_{\omega_1}\lambda$ and thus belongs to some $V[G_\beta]$ for some β less than δ. Let C be $C_5 \backslash \beta$. We immediately get the following:

Corollary 5.9. *Let δ be a Woodin cardinal. Let $\kappa < \delta$ be such that for all $\alpha < \delta$ there is a $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $j(C) \cap \alpha = C \cap \alpha$. Then $B \in V[G_\kappa]$ and $\langle A^i \restriction \kappa \mid i \in \omega \rangle \in V[G_\kappa]$.*

Proof. Immediate from lemma, since it implies that $\kappa \in C$. $\square$

Thus it only remains to show that we can make sure $A^i \restriction \kappa$ is a maximal antichain in $V[G_\kappa]$ for all $i \in \omega$. In fact, as we will argue below, no extra conditions have to be put on κ in order to guarantee that:

Lemma 5.10. *Let δ be a Woodin cardinal, G_δ a generic filter over $\mathrm{Lv}(<\delta, \lambda)$ and let $\kappa < \delta$ be such that for all α there is an elementary embedding $j : V \to M$ with critical point κ, $j(\kappa) > \alpha$ and $V_\alpha \subset M$ and such that:*

$$j(C) \cap \alpha = C \cap \alpha,$$

$$j(\mathrm{Lv}(<\delta, \lambda)) \cap V_\alpha = \mathrm{Lv}(<\delta, \lambda) \cap V_\alpha,$$

$$j(\langle \bar{D}^i_\beta \mid \beta < \delta \rangle) \cap V_\alpha = \langle \bar{D}^i_\beta \mid \beta < \delta \rangle \cap V_\alpha \quad (\text{for all } i \in \omega)$$

$$j(f_2) \cap V_\alpha = f_2 \cap V_\alpha.$$

Then κ satisfies the hypotheses of Lemma 4.2.

Proof. From the claims and lemmas established thus far, we have that for a κ as above:

1. $B \in V[G_\kappa]$ (this follows from Corollary 5.9).
2. $\langle A^i \restriction \kappa \mid i \in \omega \rangle \in V[G_\kappa]$ (this follows from Corollary 5.9).
3. For any $\alpha < \delta\ \exists j : V \to M$, $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$ such that j has the following property:
   There is a V-generic filter over $j(\mathrm{Lv}(<\kappa, \lambda))$, $H_{j(\kappa)}$, such that j can be extended to a j^*, $j^* : V[G_\kappa] \to M[H_{j(\kappa)}]$ (j^* an elementary embedding), with the property that $M[H_{j(\kappa)}]$ is ω-closed in $V[H_{j(\kappa)}]$. Further, we may choose $H_{j(\kappa)}$ such that in addition:
   (a) $H_\alpha = G_\alpha$. (That we may have $H_\alpha = G_\alpha$ follows from Claim 5.1. That j may be chosen (in advance) so that $M[H_{j(\kappa)}]$ is ω-closed in $V[H_{j(\kappa)}]$ follows from Remark (2) after Definition 4.1 and Claim 5.2.)
   (b) $j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha$ for all $i \in \omega$. (This follows from Claim 5.6 and Lemma 5.7.)

Thus we only have to show that $\langle A^i \restriction \kappa \mid i \in \omega \rangle$ is a maximal antichain in $\mathcal{P}_{\omega_1}\lambda$ in $V[G_\kappa]$, thus completing the proof that κ reflects δ. So assume that for some $i \in \omega$, $A^i \restriction \kappa$ is *not* maximal in $V[G_\kappa]$. Then $\exists S \in V[G_\kappa]$ such that S is stationary in $\mathcal{P}_{\omega_1}\lambda$ but $S \cap A^i_\alpha$ is *not* stationary in $V[G_\kappa]$ for all $\alpha < \kappa$. Let β be such that $S \cap A^i_\beta$ is stationary. Let $\alpha > \beta$, $\alpha \in C$, α a limit point of C, and use property 3 with respect to α, i.e., let j be an elementary embedding, $j : V[G_\kappa] \to M[H_{j(\kappa)}]$, where $H_\alpha = G_\alpha$ and

$$j(A^i \restriction \kappa) \restriction \alpha = A^i \restriction \alpha.$$

Since j is elementary,

$$M[H_{j(\kappa)}] \models j(S) \cap j(A^i_\alpha) \text{ is not stationary for all } \alpha \text{ less than } j(\kappa).$$

Since $S \subset \mathcal{P}_{\omega_1}\lambda$ (and $\kappa = \lambda^+$), $j(S) = S$. Thus

$$M[H_{j(\kappa)}] \models S \cap A^i_\alpha \text{ is not stationary for all } \alpha \text{ less than } j(\kappa).$$

But $S \cap A^i_\beta$ is stationary in $V[G_\alpha]$ ($\alpha \in C$ so $A^i \restriction \alpha \in V[G_\alpha]$; $S \in V[G_\kappa]$). So $S \cap A^i_\beta$ must be stationary in $M[G_\alpha]$ as well ($M[G_\alpha] \subset V[G_\alpha]$), provided $S \cap A^i_\beta$ belongs to $M[G_\alpha]$. But $\alpha \in j(C)$ (since $\alpha \in C$ and $C \cap \alpha = j(C) \cap \alpha$ and α is a limit point of C). So $j(A^i) \restriction \alpha \in M[G_\alpha]$, and hence $A^i_\beta \in M[G_\alpha]$. Also, since $S \subset \mathcal{P}_{\omega_1}\lambda$, $S \in V[G_\kappa]$, and $\mathrm{Lv}(<\kappa, \lambda)$ has the κ chain condition, it follows that there is a $\gamma < \kappa$ such that $S \in V[G_\gamma]$. But $S \in V[G_\gamma] \Rightarrow j(S) \in M[H_{j(\gamma)}] = M[H_\gamma] = M[G_\gamma]$ since $\gamma < \kappa$. Hence S, and thus $S \cap A^i_\beta$, belong to $M[G_\alpha]$ and $S \cap A^i_\beta$ is stationary in $M[G_\alpha]$. Thus $S \cap A^i_\beta$ is stationary in $M[H_{j(\kappa)}]$ (since $M \models \mathrm{Lv}(< j(\kappa), \lambda)/\mathrm{Lv}(<\alpha, \lambda)$ is proper). But $\beta < \alpha < j(\kappa)$, contradiction. □

The proof of this lemma completes the proof of Theorem 3.1. □

Acknowledgements

I owe much thanks to D.A. Martin, M. Magidor and Y. Matsubara.

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