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Finite groups with some $\mathcal{H}$-subgroups☆

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Abstract

A subgroup H is said to be an $\mathcal{H}$-subgroup of a finite group G if $H^g \cap N_G(H) \leq H$ for all $g \in G$. For every prime p dividing the order of G, let P be a Sylow p-subgroup of G and D a subgroup of P with $1 < |D| < |P|$. We investigate the structure of G under the assumption that each subgroup H of P with $|H| = |D|$ is an $\mathcal{H}$-subgroup of G. Some earlier results are generalized. Some results about formation are obtained.

MSC: 20D10; 20D20

Keywords: $\mathcal{H}$-subgroup; p-nilpotent group; Supersolvable group; Formation

1. Introduction

All groups considered in this paper are finite. $\mathcal{F}$ denotes a formation, a normal subgroup N of a group G is said to be $\mathcal{F}$-hypercentral in G provided N has a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_r = N$ such that each N_{i+1}/N_i is an $\mathcal{F}$-central chief factor of G, the product of all $\mathcal{F}$-hypercentral subgroups of G is again an $\mathcal{F}$-hypercentral subgroup of G. It is denoted by $Z_{\mathcal{F}}(G)$ and called the $\mathcal{F}$-hypercenter of G. $\mathcal{U}$ and $\mathcal{N}$ denote the classes of all supersolvable groups and nilpotent groups respectively. Other terminology and notations are standard, as in [11].

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In [3], Bianchi et al. introduced the concept of an $\mathcal{H}$-subgroup and investigated the influence of $\mathcal{H}$-subgroups on the structure of a group G. A subgroup H is called an $\mathcal{H}$-subgroup of G if $H^g \cap N_G(H) \leq H$ for all $g \in G$. We use $\mathcal{H}(G)$ to denote the set of all $\mathcal{H}$-subgroups of a group G. Bianchi et al. in [3] proved that a group G is a supersolvable T-group (a group in which normality is transitive) if and only if every subgroup of G is an $\mathcal{H}$-subgroup. Later on, Csörgö and Herzog [4] obtained that a group G is supersolvable if every cyclic subgroup of G of prime order or order 4 is an $\mathcal{H}$-subgroup. Asaad [1] proved that a group G is supersolvable if every maximal subgroup of every Sylow subgroup of G is an $\mathcal{H}$-subgroup. Recently, Guo and Wei [7] provided a unified version of the results mentioned above if the order of G is odd. By assuming some subgroups of G with the same order all belong to $\mathcal{H}(G)$, they get some sufficient conditions about a group G to be p-nilpotent or supersolvable. But the case of groups of even order remained open. In this paper, we continue to study this problem further and discuss the case when G is of even order. Moreover, the results were extended to saturated formations.

2. Preliminaries

For convenience, we list here some known results which are crucial in proving our main results.

Lemma 2.1. *Let K and H be subgroups of a group G. Then*

(1) *If $H \leq K$ and $H \in \mathcal{H}(G)$, then $H \in \mathcal{H}(K)$.*
(2) *If $N \trianglelefteq G$ and $N \leq H$, then $H \in \mathcal{H}(G)$ if and only if $H/N \in \mathcal{H}(G/N)$.*
(3) *Suppose that $N \trianglelefteq G$, P is a p-subgroup of G which belongs to $\mathcal{H}(G)$ and $(|N|, |P|) = 1$. Then $PN \in \mathcal{H}(G)$ and $PN/N \in \mathcal{H}(G/N)$.*

Proof. (1) is from [3, Lemma 7 (2)]; (2) is from [3, Lemma 2 (1)]; (3) is from [4, Lemma 6]. □

Lemma 2.2 (*[3, Theorem 6 (2)]*)**.** *Let G be a group and $H \in \mathcal{H}(G)$. If H is subnormal in G, then H is normal in G.*

Lemma 2.3 (*[8, Theorem 1]*)**.** *Let p be a fixed prime and P a Sylow p-subgroup of a group G. Then the following two statements are true:*

(1) *If p is odd and every minimal subgroup of P lies in the center of $N_G(P)$, then G is p-nilpotent.*
(2) *If $p = 2$ and every cyclic subgroup of P of order 2 and 4 is quasinormal in $N_G(P)$, then G is 2-nilpotent.*

Lemma 2.4 (*[1, Corollary 1.2]*)**.** *Let P be a Sylow p-subgroup of G and let p be the smallest prime dividing $|G|$. Then G is p-nilpotent if and only if every maximal subgroup of P belongs to $\mathcal{H}(G)$.*

Lemma 2.5 (*[7, Theorem 3.4]*)**.** *Let G be a group with odd order, p the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If P is cyclic or there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ belongs to $\mathcal{H}(G)$, then G is p-nilpotent.*

Lemma 2.6 (*[12, Lemma 2.16]*)**.** *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

Lemma 2.7. *Suppose that P is a normal 2-subgroup of a group G. If every cyclic subgroup of P of order 2 and 4 is normal in G, then $P \leq Z_\infty(G)$.*

Proof. Pick an arbitrary Sylow q-subgroup G_q of G, where $q \neq 2$. Consider the subgroup $M = PG_q$. Since every cyclic subgroup of P of order 2 and 4 is normal in G, they are normal in M. Thus it is easy to see that M is 2-nilpotent and then $M = P \times G_q$. This implies that $O^p(G) \leq C_G(P)$. Therefore $P \leq Z_\infty(G)$. □

Lemma 2.8. *Let P be a nontrivial normal p-subgroup of a group G, where p is an odd prime. If every minimal subgroup of P is normal in G, then $P \leq Z_{\mathcal{U}}(G)$.*

Proof. By [2, Theorem 1.1], it is obvious. □

3. Main results

Theorem 3.1. *Let P be a Sylow 2-subgroup of a group G. Suppose that there exists a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ belongs to $\mathcal{H}(G)$. When $|P : D| > 2$, suppose in addition that, H belongs to $\mathcal{H}(G)$ if there exists $D_1 \trianglelefteq H \leq P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4. Then G is 2-nilpotent.*

Proof. Suppose that the result is false and let G be a counterexample of minimal order. First, by Lemma 2.1 we know if M is a proper subgroup of G with $|M|_2 > |D|$, then M is 2-nilpotent. Furthermore, we have:

(1) $|D| > 2$ and $|P : D| > 2$.

If $|D| = 2$, then by hypothesis every cyclic subgroup of P with order 2 and order 4 belongs to $\mathcal{H}(G)$ and so it belongs to $\mathcal{H}(N_G(P))$. Hence Lemma 2.2 shows that every cyclic subgroup of P of order 2 and order 4 is normal in $N_G(P)$, then G is 2-nilpotent by Lemma 2.3, a contradiction. By Lemma 2.4, we have $|P : D| > 2$.

(2) G is not a non-abelian simple group.

Assume that G is non-abelian simple. Let H be a subgroup of P such that $|H| = |D|$. Then H is an $\mathcal{H}$-subgroup of G and $N_G(H)$ is a proper subgroup of G and so it is 2-nilpotent by inductive hypothesis. It follows that $N_G(H)/C_G(H)$ is a 2-group. By [6, Corollary B3], H is a Sylow 2-subgroup of the normal closure H^G of H in G which is G, as G is simple, and hence H is a Sylow 2-subgroup of G, but this contradicts to the fact that $1 < |D| = |H| < |P|$. Hence G is not a non-abelian simple group.

(3) $O_{2'}(G) = 1$.

If not, then by Lemma 2.1, the hypothesis hold for $G/O_{2'}(G)$. Therefore, $G/O_{2'}(G)$ is 2-nilpotent by our minimal choice of G and so G is 2-nilpotent, a contradiction.

(4) $O_2(G) \neq 1$.

By (2) and (3), G has a proper normal subgroup N of even order. Let $P_1 = P \cap N$ be a Sylow 2-subgroup of N. If $|P_1| > |D|$, then by Lemma 2.1, every subgroup H of P_1 with $|H| = |D|$ belongs to $\mathcal{H}(N)$. Moreover, H belongs to $\mathcal{H}(N)$ if there exists $D_1 \trianglelefteq H \leq P_1$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4. Thus N is 2-nilpotent by our minimal choice of G, and so N is a 2-group. If $|P_1| \leq |D|$, then there exists a subgroup H of P such that $P_1 \leq H$ and $|H| = 2|D|$. By (1), $HN < G$. Clearly H is a Sylow 2-subgroup of HN and every maximal subgroup of H belongs to $\mathcal{H}(HN)$. Then HN is 2-nilpotent by Lemma 2.4. Hence N is 2-nilpotent and so N is a 2-group.

(5) Let N be a minimal normal subgroup of G contained in P, then $|N| \leq |D|$ and so $N < P$.

If $|D| < |N|$, then there exists a subgroup H of N with $|H| = |D|$ such that $H \in \mathcal{H}(G)$. Since $H \trianglelefteq\trianglelefteq N \trianglelefteq G$, $H \trianglelefteq G$ by Lemma 2.2, which contradicts to the minimal normality of N in G. Thus we have $|N| \le |D|$ and so $N < P$.

(6) G has the unique minimal normal subgroup N contained in P and G/N is 2-nilpotent.

Let $N \le P$ be a minimal normal subgroup of G. If $|N| < |D|$, then G/N satisfies the hypothesis of the theorem by Lemma 2.1. Thus G/N is 2-nilpotent by the minimal choice of G. So we may suppose $|N| = |D|$ by (5). Let K be a subgroup of P containing N and $|K/N| = 2$. By (1), $|N| > 2$ and then N is noncyclic, so are all subgroups of P containing N. Hence there exists a maximal subgroup L of K such that $K = NL$. Clearly, $|N| = |L| = |D|$. Since $L \in \mathcal{H}(G)$, $L \in \mathcal{H}(N_G(P))$ by Lemma 2.1, so $L \trianglelefteq N_G(P)$ by Lemma 2.2. Then $K/N = LN/N \trianglelefteq N_G(P)N/N = N_{G/N}(P/N)$. Since $|P : D| > 2$, let $X/N \le P/N$ be cyclic of order 4. Since X is noncyclic and X/N is cyclic, there is a maximal subgroup L of X such that N is not contained in L. Thus $X = LN$ and $|L| = 2|D|$. Since $L/L \cap N \cong LN/N = X/N$ is cyclic of order 4, by hypothesis, $L \in \mathcal{H}(G)$ and so $L \in \mathcal{H}(N_G(P))$. Thus $L \trianglelefteq N_G(P)$ by Lemma 2.2. Then $X/N = LN/N \trianglelefteq N_G(P)N/N = N_{G/N}(P/N)$.

Now we have proved that every cyclic subgroup of P/N of order 2 and order 4 is normal in $N_{G/N}(P/N)$, so G/N is 2-nilpotent by Lemma 2.3. The uniqueness of N is obvious.

(7) The final contradiction.

By (6), G/N is 2-nilpotent. Then there exists a normal subgroup M of G such that $|G : M| = 2$. Let $P_1 = P \cap M$ be a Sylow 2-subgroup of M. By (1), $|P_1| > |D|$. Lemma 2.1 shows that every subgroup H of P_1 with order $|D|$ belongs to $\mathcal{H}(M)$. When $|P_1 : D| > 2$, H belongs to $\mathcal{H}(M)$ if there exists $D_1 \trianglelefteq H \le P_1$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4. Thus M is 2-nilpotent by our minimal choice of G. Then G is 2-nilpotent, the final contradiction. □

Remark. (1) By Lemma 2.4, we know that the converse of Theorem 3.1 is also true.
(2) The hypothesis that "When $|P : D| > 2$, suppose in addition that, H belongs to $\mathcal{H}(G)$ if there exists $D_1 \trianglelefteq H \le P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4" cannot be removed, for the semidirect product $[Q_8]Z_3$ is a counterexample.

By Theorem 3.1 and Lemma 2.5, we have:

Corollary 3.1. *Let G be a group. Suppose that for every prime p dividing $|G|$, there exists a Sylow p-subgroup P of G such that P is cyclic or P has a subgroup D satisfying $1 < |D| < |P|$ and every subgroup H of P with order $|D|$ belongs to $\mathcal{H}(G)$. When P is a 2-group and $|P : D| > 2$, suppose in addition that, H belongs to $\mathcal{H}(G)$ if there exists $D_1 \trianglelefteq H \le P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4. Then G has a Sylow tower of supersolvable type.*

Theorem 3.2. *Let E be a normal subgroup of G such that $G/E \in \mathcal{F}$, a saturated formation containing the class of all supersolvable groups $\mathcal{U}$. Suppose that for every prime p dividing $|E|$, there exists a Sylow p-subgroup P of E such that P is cyclic or P has a subgroup D with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ belongs to $\mathcal{H}(G)$. When P is a 2-group and $|P : D| > 2$, suppose in addition that, H belongs to $\mathcal{H}(G)$ if there exists $D_1 \trianglelefteq H \le P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4. Then $G \in \mathcal{F}$.*

Proof. Suppose that the result is false and let G be a counterexample such that $|G||E|$ is minimal.

By Corollary 3.1, G possesses a Sylow tower of supersolvable type. Let p be the largest prime dividing $|E|$ and P a Sylow p-subgroup of E, then P is normal in G. By Lemma 2.1,

G/P with its normal subgroup E/P satisfies the hypothesis of our theorem and hence $G/P \in \mathcal{F}$ by induction. Therefore we have $E = P$. If P is cyclic, then $G \in \mathcal{F}$ by Lemma 2.6. So we may assume that P is not cyclic. Since every subgroup of P with order $|D|$ belongs to $\mathcal{H}(G)$ and it is subnormal in G. By Lemma 2.2, we know that every subgroup of P with order $|D|$ is normal in G. When P is a 2-group and $|P : D| > 2$, H is also normal in G if there exists $D_1 \trianglelefteq H \leq P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4. Let N be a minimal normal subgroup of G contained in P, then clearly $|N| \leq |D|$.

If $|N| = |D|$, then every subgroup H of P with order $|N|$ is normal in G. In this case, we claim that $|N| = p$. If $|N| > p$, then N is not cyclic. Let $\overline{H} = H/N$ be a subgroup of $\overline{P} = P/N$ of order p. Then there exists a maximal subgroup L of H such that $H = LN$ and $L \trianglelefteq G$. However, $1 \neq L \cap N \trianglelefteq G$, which contradicts to the minimal normality of N. Thus $|N| = p$, which implies that every minimal subgroup of P is normal in G. If $p > 2$, then by Lemma 2.8, $P \leq Z_{\mathcal{U}}(G)$. Since $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ by [5, IV, Proposition 3.11], $P \leq Z_{\mathcal{F}}(G)$ and so $G \in \mathcal{F}$, as desired. If $p = 2$, then by hypothesis and the above discussion, we have every cyclic subgroup of P of order 2 and order 4 is normal in G. By Lemma 2.7, $P \leq Z_{\infty}(G)$. Thus $P \leq Z_{\mathcal{F}}(G)$ and so $G \in \mathcal{F}$, a contradiction.

Now we assume that $|N| < |D|$. By Lemma 2.1, G/N with its normal subgroup P/N satisfies the hypothesis of our theorem and hence $G/N \in \mathcal{F}$ by the minimal choice of G. Since $\mathcal{F}$ is a saturated formation, we may assume that N is the unique minimal normal subgroup of G contained in P. Let K be a subgroup of P of order $|D|$. Since K is normal in G by the above discussion, the unique minimality of N implies that N is contained in K. Noting that $|K| = |D| < |P|$, it follows that N is contained in every maximal subgroup of P, that is, $N \leq \Phi(P) \leq \Phi(G)$. So we have $G \in \mathcal{F}$, as required. □

Theorem 3.3. *Let E be a normal subgroup of G such that $G/E \in \mathcal{F}$, a saturated formation containing $\mathcal{U}$. Suppose that for every prime p dividing $|F^*(E)|$, there exists a Sylow p-subgroup P of $F^*(E)$ such that P is cyclic or P has a subgroup D with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ belongs to $\mathcal{H}(G)$. When P is a 2-group and $|P : D| > 2$, suppose in addition that, H belongs to $\mathcal{H}(G)$ if there exists $D_1 \trianglelefteq H \leq P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4. Then $G \in \mathcal{F}$.*

Proof. By Lemma 2.1, we have every subgroup H of P with $|H| = |D|$ belongs to $\mathcal{H}(F^*(E))$. When P is a 2-group and $|P : D| > 2$, H belongs to $\mathcal{H}(F^*(E))$ if there exists $D_1 \trianglelefteq H \leq P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4. By Corollary 3.1, $F^*(E)$ possesses a Sylow tower of supersolvable type. In particular, $F^*(E)$ is solvable and so $F^*(E) = F(E)$. Since every subgroup of $F(E)$ is subnormal in G, by hypothesis and Lemma 2.2, we know that for every prime p dividing $|F^*(E)|$, there exists a Sylow p-subgroup P of $F^*(E)$ such that P is cyclic or P has a subgroup D with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ is normal in G. When P is a 2-group and $|P : D| > 2$, H is also normal in G if there exists $D_1 \trianglelefteq H \leq P$ with $2|D_1| = |D|$ and H/D_1 is cyclic of order 4. By [12, Theorem 1.3], we have $G \in \mathcal{F}$, as required. □

Remark. Following Li [9], a subgroup H of a group G is called an NE-subgroup if it satisfies $N_G(H) \cap H^G = H$, where H^G denotes the normal closure of H in G. In [10], the author proved that: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of every Sylow subgroup of $F(H)$ are NE-subgroups of G, then $G \in \mathcal{F}$ (Theorem 4.4).

After this theorem, the author said: "we do not know whether Theorem 4.4 remains true if we replace $F(H)$ by $F^*(H)$ and drop the condition that H is solvable in Theorem 4.4".

Since an NE-subgroup is an $\mathcal{H}$-subgroup, obviously, the answers to the above question is positive by our Theorem 3.3.

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