The Hardy Space of Analytic Functions Associated with Certain One-Parameter Families of Integral Operators

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Recently, H. M. Srivastava and S. Owa [*J. Math. Anal. Appl.* **118** (1986), 80–87] proved a number of results relevant to Komatu's conjectures (cf. Y. Komatu [*Bull. Fac. Sci. Engrg. Chuo Univ. Ser. I Math.* **22** (1979), 1–22]) for a certain one-parameter additive family of operators defined on analytic functions regular in the open unit disk. The object of the present paper is to investigate several one-parameter families of integral operators and to prove certain inclusion theorems involving the Hardy space $\mathscr{H}^p$ of analytic functions and the class of functions whose derivative has a positive real part. An application involving a generalized hypergeometric function is also considered.

1. INTRODUCTION AND DEFINITIONS

Let $\mathscr{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are analytic in the open unit disk

$$\mathscr{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Also let $\mathscr{S}$, $\mathscr{S}^*$, and $\mathscr{K}$ denote the subclasses of $\mathscr{A}$ consisting of functions which are univalent, starlike, and convex in $\mathscr{U}$, respectively.

A function $f(z)$ belonging to the class $\mathscr{A}$ is said to be in the class $\mathscr{R}$ if it satisfies the inequality:

$$\operatorname{Re}\{f'(z)\} > 0 \qquad (z \in \mathscr{U}). \tag{1.2}$$

The class $\mathscr{R}$ was studied systematically by MacGregor [7] who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

Finally, let $\mathscr{H}^p$ $(0 < p \leqslant \infty)$ denote the Hardy space of analytic functions $f(z)$ in $\mathscr{U}$, and define the integral means

$$M_p(r, f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} & (0 < p < \infty) \\ \max\limits_{|z| \leqslant r} |f(z)| & (p = \infty). \end{cases} \tag{1.3}$$

Then, by definition, an analytic function $f(z)$ in $\mathscr{U}$ belongs to the Hardy space $\mathscr{H}^p$ $(0 < p \leqslant \infty)$ if

$$\lim_{r \to 1-} \{M_p(r, f)\} < \infty \qquad (0 < p \leqslant \infty). \tag{1.4}$$

For $1 \leqslant p \leqslant \infty$, $\mathscr{H}^p$ is a Banach space with the norm defined by (cf. [3, p. 23; 6, p. 97])

$$\|f\|_p = \lim_{r \to 1-} \{M_p(r, f)\} \qquad (1 \leqslant p \leqslant \infty). \tag{1.5}$$

Furthermore, $\mathscr{H}^\infty$ is the class of bounded analytic functions in $\mathscr{U}$, while $\mathscr{H}^2$ is the class of power series $\sum a_n z^n$ with

$$\sum |a_n|^2 < \infty. \tag{1.6}$$

Motivated by Komatu's conjectures on the inclusion properties of a certain one-parameter additive family of operators (cf. [5]), involving the subclasses $\mathscr{S}$, $\mathscr{S}^*$, and $\mathscr{K}$ of the class $\mathscr{A}$, Srivastava and Owa [11] investigated two general classes of analytic and univalent functions defined in terms of some operators of fractional calculus (that is, fractional integral and fractional derivative). In the present paper we aim at proving a number of inclusion theorems involving the Hardy space $\mathscr{H}^p$, the class $\mathscr{R}$ of functions in $\mathscr{A}$ satisfying the inequality (1.2), and the following one-parameter families of integral operators:

$$\mathscr{P}^\alpha f = \mathscr{P}^\alpha f(z) = \frac{2^\alpha}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\,dt \qquad (\alpha > 0;\, f \in \mathscr{A}), \tag{1.7}$$

$$\mathscr{Q}_\beta^\alpha f = \mathscr{Q}_\beta^\alpha f(z) = \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^\beta} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\,dt$$
$$(\alpha > 0;\, \beta > -1;\, f \in \mathscr{A}), \tag{1.8}$$

and

$$\mathscr{I}_\alpha f = \mathscr{I}_\alpha f(z) = \frac{\alpha+1}{z^\alpha} \int_0^z t^{\alpha-1} f(t)\,dt \qquad (\alpha > -1;\, f \in \mathscr{A}), \tag{1.9}$$

where $\Gamma(\alpha)$ is the familiar Gamma function, and (in general)

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\,\Gamma(\beta+1)} = \binom{\alpha}{\alpha-\beta}. \tag{1.10}$$

We also consider an interesting application of one of our main results to a generalized hypergeometric function.

It should be remarked in passing that various forms of each of these integral operators have appeared in the mathematical literature. For

$$\alpha \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

the operators $\mathscr{P}^\alpha$, $\mathscr{Q}_1^\alpha$, and $\mathscr{I}_\alpha$ were considered by Bernardi [1, 2]. In fact, the operators $\mathscr{Q}_\beta^\alpha$ is related rather closely to the Beta or Euler transformation. Moreover, for a real number $\alpha > -1$, the operator $\mathscr{I}_\alpha$ was used recently by Owa and Srivastava (cf., e.g., [9, 10, 12]; see also [8; 13, pp. 181–182]).

The following (known or new) results will be required in our investigation.

LEMMA 1 (cf., e.g., Duren [3, p. 34, Theorem 3.2]). *If $f(z)$ is analytic in $\mathscr{U}$, and if*

$$\operatorname{Re}\{f(z)\} > 0 \qquad (z \in \mathscr{U}), \tag{1.11}$$

then $f(z) \in \mathscr{H}^p$ for all $p < 1$.

LEMMA 2 (Hardy and Littlewood [4, p. 415, Theorem 33]; see also Duren [3, p. 88, Theorem 5.12]). *If* $f'(z) \in \mathscr{H}^p$ $(0<p<1)$, *then*

$$f(z) \in \mathscr{H}^q \qquad (q = p/(1-p)), \tag{1.12}$$

the index q *being the best possible for each admissible value of* p.

Making use of Lemma 1 and Lemma 2, it is not difficult to prove

LEMMA 3. *If* $f(z) \in \mathscr{R}$, *then*

$$f(z) \in \mathscr{H}^p \qquad (0<p<\infty). \tag{1.13}$$

Remark 1. It is easily seen by considering the function [7, p. 532]

$$f(z) = -z - 2\log(z-1) \in \mathscr{R} \tag{1.14}$$

that the class $\mathscr{R}$ need not be contained in the class $\mathscr{H}^\infty$ of bounded analytic functions.

Upon substituting for $f(t)$ from (1.1) into Eqs. (1.7), (1.8), and (1.9), if we evaluate the resulting integrals in terms of Beta and Gamma functions, we shall arrive at

LEMMA 4. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathscr{A}$. *Then*

$$\mathscr{P}^\alpha f = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\alpha a_n z^n \qquad (\alpha > 0), \tag{1.15}$$

$$\mathscr{Q}_\beta^\alpha f = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n \qquad (\alpha>0; \beta>-1), \tag{1.16}$$

and

$$\mathscr{J}_\alpha f = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+1}{\alpha+n}\right) a_n z^n \qquad (\alpha > -1). \tag{1.17}$$

Remark 2. Comparing the definitions (1.8) and (1.9), or (more easily) the explicit representations (1.16) and (1.17), we observe that

$$\mathscr{J}_\alpha f = \mathscr{Q}_\alpha^1 f \qquad (\alpha > -1). \tag{1.18}$$

2. The Inclusion Theorems

Our first inclusion theorem, involving the classes $\mathscr{R}$ and $\mathscr{H}^{\infty}$, is stated as

THEOREM 1. *If $f(z)$ is in the class $\mathscr{R}$, then*

$$\mathscr{P}^{\alpha} f \in \mathscr{H}^{\infty} \tag{2.1}$$

at least for $\alpha > 1$.

Proof. Making use of the assertion (1.15) of Lemma 4, we obtain

$$\frac{d}{dz}\mathscr{P}^{\alpha} f = \frac{1}{z}\{-\mathscr{P}^{\alpha} f + 2\mathscr{P}^{\alpha-1} f\} \qquad (\alpha > 1), \tag{2.2}$$

which, in view of the elementary inequality:

$$\max\{A^{p}, B^{p}\} \leqslant (A+B)^{p} \leqslant 2^{p}(A^{p}+B^{p}) \qquad (0 < p < \infty;\, A \geqslant 0;\, B \geqslant 0), \tag{2.3}$$

readily yields

$$\left|\frac{d}{dz}\mathscr{P}^{\alpha} f\right|^{p} \leqslant \left(\frac{2}{r}\right)^{p} (|\mathscr{P}^{\alpha} f|^{p} + 2^{p}\, |\mathscr{P}^{\alpha-1} f|^{p})$$
$$(r = |z|;\, 0 < p < \infty;\, \alpha > 1). \tag{2.4}$$

For $p = 1$, a substantially improved inequality would follow directly from (2.2), and we have

$$\left|\frac{d}{dz}\mathscr{P}^{\alpha} f\right| \leqslant \frac{1}{r}(|\mathscr{P}^{\alpha} f| + 2\, |\mathscr{P}^{\alpha-1} f|) \qquad (r = |z|;\, \alpha > 1), \tag{2.5}$$

which may be compared with a special case of (2.4) when $p = 1$.

Now rewrite the definition (1.7) as

$$\mathscr{P}^{\alpha} f = \mathscr{P}^{\alpha} f(z) = \frac{2^{\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\log\frac{1}{t}\right)^{\alpha-1} f(zt)\, dt \qquad (\alpha > 0;\, f \in \mathscr{A}), \tag{2.6}$$

so that

$$\operatorname{Re}\left\{\frac{d}{dz}\mathscr{P}^{\alpha} f\right\} = \frac{2^{\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\log\frac{1}{t}\right)^{\alpha-1} t\, \operatorname{Re}\{f'(zt)\}\, dt \qquad (\alpha > 0). \tag{2.7}$$

Since $f \in \mathscr{R}$ (by hypothesis), it follows from (2.7) that

$$\mathscr{P}^{\alpha} f \in \mathscr{R} \qquad (\alpha > 0). \tag{2.8}$$

Thus, by Lemma 3, we have

$$\mathscr{P}^{\alpha} f \in \mathscr{H}^{p} \qquad (0<p<\infty;\ \alpha>0). \tag{2.9}$$

In view of the definitions (1.3) and (1.5), the inequality (2.5) yields

$$M_1\left(r, \frac{d}{dz} \mathscr{P}^{\alpha} f\right) \leqslant \frac{1}{r}\left\{M_1(r, \mathscr{P}^{\alpha} f)+2 M_1(r, \mathscr{P}^{\alpha-1} f)\right\} \qquad (r=|z|;\ \alpha>1) \tag{2.10}$$

and

$$\left\|\frac{d}{dz} \mathscr{P}^{\alpha} f\right\|_1 \leqslant\|\mathscr{P}^{\alpha} f\|_1+2\|\mathscr{P}^{\alpha-1} f\|_1 \qquad (\alpha>1). \tag{2.11}$$

Since $\alpha>1$, it follows readily from the inclusion relation (2.9) that

$$\mathscr{P}^{\alpha} f \in \mathscr{H}^1 \quad \text{and} \quad \mathscr{P}^{\alpha-1} f \in \mathscr{H}^1, \tag{2.12}$$

and hence (2.11) gives

$$\frac{d}{dz} \mathscr{P}^{\alpha} f \in \mathscr{H}^1 \qquad (\alpha>1). \tag{2.13}$$

Therefore, by appealing to a known result [3, p. 42, Theorem 3.11], we conclude from (2.12) that $\mathscr{P}^{\alpha} f\ (\alpha>1)$ is continuous in

$$\bar{\mathscr{U}}=\mathscr{U} \cup \partial \mathscr{U}=\{z: z \in \mathbb{C} \text{ and } |z| \leqslant 1\}. \tag{2.14}$$

Finally, since $\bar{\mathscr{U}}$ is compact, $\mathscr{P}^{\alpha} f$ is bounded in $\bar{\mathscr{U}}$. Thus $\mathscr{P}^{\alpha} f$ is a bounded analytic functon in $\mathscr{U}$, which leads us to the assertion (2.1) of Theorem 1. This evidently completes the proof of Theorem 1.

Remark 3. We have actually shown above that $\mathscr{P}^{\alpha} f\ (\alpha>1)$ is continuous on the closure $\bar{\mathscr{U}}$ of the open unit disk $\mathscr{U}$. This is not implied by the assertion of Theorem 1.

Incidentally, a result much stronger than Theorem 1 can be proven by making use of the inequality (2.4) itself. Indeed it follows from (2.4) and the definition (1.3) that

$$\left\{M_p\left(r, \frac{d}{dz} \mathscr{P}^{\alpha} f\right)\right\}^p \leqslant\left(\frac{2}{r}\right)^p\left[\{M_p(r, \mathscr{P}^{\alpha} f)\}^p+\{2 M_p(r, \mathscr{P}^{\alpha-1} f)\}^p\right]$$
$$(r=|z|;\ 0<p<\infty;\ \alpha>1), \tag{2.15}$$

which, on proceeding to the limit as $r \to 1-$, yields the assertion:

$$\frac{d}{dz}\mathscr{P}^{\alpha}f \in \mathscr{H}^{p} \qquad (f \in \mathscr{R}; 0 < p < \infty; \alpha > 1), \tag{2.16}$$

in view of the inclusion relation (2.9) involving the Hardy space $\mathscr{H}^p$ $(0 < p < \infty)$. By virtue of Lemma 2, the assertion (2.16) is much stronger than that provided by Theorem 1.

We next state

THEOREM 2. *If $f(z)$ is in the class $\mathscr{R}$, then*

$$\mathscr{Q}_{\beta}^{\alpha} f \in \mathscr{H}^{\infty} \tag{2.17}$$

at least for $\alpha > 1$ and $\beta > -1$.

Proof. The proof of Theorem 2 is much akin to that of Theorem 1 which we have detailed above. Indeed, in place of the relationships (2.2) and (2.6), we make use of the following consequences of (1.16) and (1.8), respectively:

$$\frac{d}{dz}\mathscr{Q}_{\beta}^{\alpha} f = \frac{1}{z}\{-(\alpha+\beta-1)\,\mathscr{Q}_{\beta}^{\alpha} f + (\alpha+\beta)\,\mathscr{Q}_{\beta}^{\alpha-1} f\} \quad (\alpha > 1; \beta > -1) \tag{2.18}$$

and

$$\operatorname{Re}\left\{\frac{d}{dz}\mathscr{Q}_{\beta}^{\alpha} f\right\} = \alpha\binom{\alpha+\beta}{\beta}\int_0^1 (1-t)^{\alpha-1}\, t^{\beta} f'(zt)\, dt \qquad (\alpha > 0; \beta > -1). \tag{2.19}$$

Applying the method of proof of Theorem 1 (and Theorem 2) mutatis mutandis, and making use of the following consequence of (1.17):

$$\frac{d}{dz}\mathscr{I}_{\alpha} f = -\frac{1}{z}\{\alpha \mathscr{I}_{\alpha} f - (\alpha+1) f(z)\} \qquad (\alpha > -1), \tag{2.20}$$

instead of (2.2) and (2.18), we obtain

THEOREM 3. *If $f(z)$ is in the class $\mathscr{R}$, then*

$$\mathscr{I}_{\alpha} f \in \mathscr{H}^{\infty} \tag{2.21}$$

for all values of $\alpha > -1$.

Remark 4. Although the operator $\mathcal{J}_\alpha$ is a special case of the operator $\mathcal{Q}_\beta^\alpha$ involved in Theorem 2 (see Remark 2 above), Theorem 3 does not seem to be derivable by specializing Theorem 2.

Remark 5. For a function $f(z)$ defined by (1.1) and belonging to the class $\mathcal{R}$, it is known that [7, p. 533, Theorem 1]

$$|a_n| \leqslant \frac{2}{n} \qquad (n = 2, 3, 4, \ldots). \tag{2.22}$$

Applying (2.22) and (1.15), we obtain

$$\begin{aligned} |\mathcal{P}^\alpha f(z)| &\leqslant |z| + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\alpha \frac{2}{n} |z|^n \\ &\leqslant 1 + \sum_{n=2}^{\infty} \frac{2^{\alpha+1}}{n(n+1)^\alpha} \\ &< \infty \qquad (|z| \leqslant 1; \alpha > 1), \end{aligned} \tag{2.23}$$

which leads us once again to the assertion of Theorem 1 (cf. Remark 3). Theorems 2 and 3 can also be proven similarly; the proof of Theorem 2, for example, would make use of (2.22), (1.16), and the order estimate:

$$\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} = O\left(\frac{1}{n^\alpha}\right) \qquad (n \to \infty) \tag{2.24}$$

for fixed α and β.

Remark 6. The above arguments show also that the power series for $\mathcal{P}^\alpha f(z)$, $\mathcal{Q}_\beta^\alpha f(z)$, and $\mathcal{J}_\alpha f(z)$ converge absolutely for $z \in \partial\mathcal{U}$ (cf. Remark 3), that is, for all points on the boundary of $\mathcal{U}$ defined by (2.14).

3. Application to Generalized Hypergeometric Functions

Let λ_j $(j = 1, \ldots, l)$ and μ_j $(j = 1, \ldots, m)$ be complex numbers such that

$$\mu_j \neq 0, -1, -2, \ldots \qquad (j = 1, \ldots, m).$$

Then the generalized hypergeometric function ${}_lF_m(z)$ is defined by (cf., e.g., [13, p. 333])

$$\begin{aligned} {}_lF_m(z) &\equiv {}_lF_m(\lambda_1, \ldots, \lambda_l; \mu_1, \ldots, \mu_m; z) \\ &= \sum_{n=0}^{\infty} \frac{(\lambda_1)_n \cdots (\lambda_l)_n}{(\mu_1)_n \cdots (\mu_m)_n} \frac{z^n}{n!} \qquad (l \leqslant m+1), \end{aligned} \tag{3.1}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n=0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n\in\mathbb{N}. \end{cases} \tag{3.2}$$

We note that the ${}_lF_m(z)$ series in (3.1) converges absolutely for $|z|<\infty$ if $l<m+1$, and for $z\in\mathscr{U}$ if $l=m+1$.

It is not difficult to apply the definitions (1.9), (3.1), and (3.2) in order to show that

$$\mathscr{J}_\alpha\{z\ {}_lF_m(\lambda_1, ..., \lambda_l;\ \mu_1, ..., \mu_m; z)\}$$
$$= z\ {}_{l+1}F_{m+1}(\lambda_1, ..., \lambda_l, \alpha+1; \mu_1, ..., \mu_m, \alpha+2; z). \tag{3.3}$$

Thus, by a repeated use of Theorem 3, we arrive at the following application involving generalized hypergeometric functions:

COROLLARY. *Let the function*

$$z\ {}_lF_m(\lambda_1, ..., \lambda_l; \mu_1, ..., \mu_m; z) \qquad (l\leqslant m+1)$$

be in the class $\mathscr{R}$.

Then the function

$$z\ {}_{l+s}F_{m+s}(\lambda_1, ..., \lambda_l, \alpha_1+1, ..., \alpha_s+1; \mu_1, ..., \mu_m, \alpha_1+2, ..., \alpha_s+2; z)$$

is in $\mathscr{H}^\infty$ *at least for* $\alpha_j>0$ $(j=1, ..., s)$.

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