Limitations of adaptive mesh refinement techniques for singularly perturbed problems with a moving interior layer

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Abstract

In a composed domain on an axis \( \mathbb{R} \) with the moving interface boundary between two subdomains, we consider an initial value problem for a singularly perturbed parabolic reaction–diffusion equation in the presence of a concentrated source on the interface boundary. Monotone classical difference schemes for problems from this class converge only when \( \varepsilon \gg N^{-1} + N_0^{-1} \), where \( \varepsilon \) is the perturbation parameter, \( N \) and \( N_0 \) define the number of mesh points with respect to \( x \) (on segments of unit length) and \( t \). Therefore, in the case of such problems with moving interior layers, it is necessary to develop special numerical methods whose errors depend rather weakly on the parameter \( \varepsilon \) and, in particular, are independent of \( \varepsilon \) (i.e., \( \varepsilon \)-uniformly convergent methods).

In this paper we study schemes on adaptive meshes which are locally condensing in a neighbourhood of the set \( \gamma^* \), that is, the trajectory of the moving source. It turns out, that in the class of difference schemes consisting of a standard finite difference operator on rectangular meshes which are (a priori or a posteriori) locally condensing in \( x \) and \( t \), there are no schemes that converge \( \varepsilon \)-uniformly, and in particular, even under the condition \( \varepsilon \approx N^{-2} + N_0^{-2} \), if the total number of the mesh points between the cross-sections \( x_0 \) and \( x_0 + 1 \) for any \( x_0 \in \mathbb{R} \) has order of \( NN_0 \). Thus, the adaptive mesh refinement techniques used directly do not allow us to widen essentially the convergence range of classical numerical methods. On the other hand, the use of

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condensing meshes but in a local coordinate system fitted to the set $\gamma^*$ makes it possible to construct schemes which converge $\varepsilon$-uniformly for $N, N_0 \to \infty$; such a scheme converges at the rate $O(N^{-1} \ln N + N_0^{-1})$.

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1. Introduction

Numerical analysis of heat and mass transfer with fixed concentrated sources in media characterized by small coefficients of heat conductivity/diffusion often bring us to diffraction boundary value problems for singularly perturbed partial differential equations. Here the singular perturbation parameter $\varepsilon$ is a coefficient multiplying the highest derivatives of the equations. The solutions of such problems for small values of the parameter $\varepsilon$ typically exhibit boundary and transition (interior) layers, moreover, for fixed finite values of $\varepsilon$ their derivatives are discontinuous at the points where the concentrated sources act. The singular behaviour of the solutions is complicated in the case of moving concentrated sources. So, the solutions of the reduced (for $\varepsilon = 0$) problems have discontinuities of the first kind on the trajectories of the moving sources.

In this paper we consider an initial value problem on an axis $\mathbb{R}$ for a singularly perturbed parabolic reaction–diffusion equation in a composed domain with a moving interface boundary between two subdomains; the concentrated source acts on the interface boundary. Note that the solution of such a problem, in contrast to boundary-value problems, has no boundary-layer singularities. However, singularities generated by the moving concentrated source still occur that give rise to difficulties in the numerical solution (see, e.g., Theorem 2). Namely, classical finite difference schemes for this problem converge only when $\varepsilon \gg N^{-1} + N_0^{-1}$, where $N$ and $N_0$ define the number of nodes in the grids with respect to $x$ (on segments of unit length) and $t$.

Therefore, in the case of problems with moving transition layers it is necessary to develop special numerical methods whose errors depend rather weakly on the parameter $\varepsilon$ and, in particular, are independent of $\varepsilon$ (i.e., $\varepsilon$-uniformly convergent methods). To this end, it seems expedient to apply the techniques based on locally condensing meshes that have been earlier proposed (see, for example, [3,6,7,9] and the bibliography therein) for several singularly perturbed problems with stationary boundary or transition layers; for the case of a regular boundary value problem with singularities in its solution also see [12]). So, in the case of regular problems whose solutions have singularities, the effect of improving the accuracy of a numerical solution can be achieved by a priori or a posteriori local grid refinement in those subregions where the errors in the approximate solution are large (see, e.g., [2,5,12]). However, the direct use of this approach in the case of singularly perturbed problems with moving concentrated sources is not sufficiently effective.

We study the class of difference schemes consisting of a standard finite difference operator on adaptive meshes which are locally refined in a neighbourhood of the set $\gamma^*$, that is, the trajectory of the moving source. At first sight, such adaptive grid refinement can resolve the layer phenomena
numerically in a completely satisfactory manner. Nevertheless, it turns out that in the case of rectangular meshes which are (a priori or a posteriori) locally condensing in \( x \) and \( t \) there are no schemes of the above class which converge \( \varepsilon \)-uniformly and, in particular, even for \( \varepsilon^{1/2} \approx N^{-1} + N_0^{-1} \) (see, e.g., the conclusion of Theorem 3) if the total number of the mesh points between the cross-sections \( x_0 \) and \( x_0 + 1 \) for any \( x_0 \in \mathbb{R} \) has order of \( N N_0 \). Thus, the adaptive mesh techniques used directly do not allow us to widen essentially the convergence range of classical numerical methods. The consideration of Kolmogorov widths \( \{d_P\} \), where \( P = N N_0 \) allowed us to find necessary conditions for the \( \varepsilon \)-uniform convergence (as \( P \to \infty \)) of optimal approximations to the solutions of initial value problems; these requirements are put in the basis for developing special numerical methods. As a practical conclusion, the use of condensing meshes in a local coordinate system fitted to the set \( \gamma^* \) (in the nearest vicinity of the singularity) make it possible to construct schemes that converge \( \varepsilon \)-uniformly for \( N, N_0 \to \infty \) (see, e.g., Remark 7 in Section 6). A similar \( \varepsilon \)-uniformly convergent scheme, but for a problem in a homogeneous domain, was considered in [10,11]; see also Remark 7 in Section 6.

2. Problem formulation. The objective of research

2.1.

In an infinite composed domain with the moving interface boundary between its subdomains, we consider an initial value problem for a singularly perturbed parabolic equation in the presence of a concentrated source acting on the interface boundary.

Let the domain \( \tilde{G} \) with boundary \( S = \tilde{G} \setminus G \), where \( G = \mathbb{R} \times (0, T] \), be decomposed into non-overlapping subdomains

\[
\tilde{G} = \tilde{G}^1 \cup \tilde{G}^2, \quad \tilde{G}^1 \cap \tilde{G}^2 = \emptyset,
\]

in each of which we consider the equation

\[
Lu(x,t) \equiv \left\{ \varepsilon a(x,t) \frac{\partial^2 u}{\partial x^2} - c(x,t) - p(x,t) \frac{\partial}{\partial t} \right\} u(x,t) = f(x,t), \quad (x,t) \in G^k, \quad k = 1, 2, \tag{2.2a}
\]

where \( a(x,t) = a^k(x,t), \ldots, f(x,t) = f^k(x,t), \ (x,t) \in \tilde{G}^k \),

\[
G^1 = \{(x,t) : x < \beta(t), \ t \in (0, T]\}, \quad G^2 = \{(x,t) : x > \beta(t), \ t \in (0,T]\}, \tag{2.3}
\]

the interface boundary between the subdomains \( \gamma^* = \{(x,t) : x = \beta(t), \ t \in (0,T]\} \) is sufficiently smooth. On the set \( S \) the function \( u(x,t) \) satisfies the initial condition

\[
u(x,t) = \varphi(x), \quad (x,t) \in S, \tag{2.2b}
\]

and on the interface boundary \( \gamma^* \) it obeys the conjugation condition

\[
[u(x,t)] = 0, \quad l u(x,t) \equiv \varepsilon \left[ a(x,t) \frac{\partial}{\partial x} u(x,t) \right] = -q(t), \quad (x,t) \in \gamma^*.
\]
Here $\varepsilon$ is a parameter taking arbitrary values from the half-interval $(0,1]$; $a^k(x,t), c^k(x,t), p^k(x,t), f^k(x,t), (x,t) \in \tilde{G}_k, k = 1, 2, \phi(x), x \in \mathbb{R}, \beta(t)$ and $q(t), t \in [0, T]$ are sufficiently smooth functions, the function $\beta(t)$ specifies the velocity of motion of the interface boundary, and $q(t)$ defines the power of the concentrated source. The symbol $[v(x,t)]$ in (2.2c) denotes the jump of the function $v(x,t)$ when passing through $\gamma^*$ from the set $G^1$ into the set $G^2$:

$$[u(x^*,t)] = \lim_{x \to x^*_+} u(x,t) - \lim_{x \to x^*_-} u(x,t), \quad a(x,t) \frac{\partial}{\partial x} u(x^*,t)$$

$$= \lim_{x \to x^*_+} a^2(x,t) \frac{\partial}{\partial x} u(x,t) - \lim_{x \to x^*_-} a^1(x,t) \frac{\partial}{\partial x} u(x,t), \quad (x^*,t) \in \gamma^*.$$  

We suppose that the problem data satisfy the following conditions:

\begin{align*}
0 < a_0 \leq a^k(x,t) \leq a^0, & \quad 0 < c^k(x,t) \leq c_0, \quad 0 < p_0 \leq p^k(x,t) \leq p^0, \quad (x,t) \in \tilde{G}_k, \\
0 < v_0 \leq (d/dt)\beta(t) \equiv v(t) \leq v^0, & \quad t \in [0, T], \quad |f^k(x,t)| \leq M, \quad (x,t) \in \tilde{G}_k, \\
|\phi(x)| \leq M, & \quad x \in \mathbb{R}, \quad |q(t)| \leq M, \quad t \in [0, T], \quad k = 1, 2. \quad (2.4)
\end{align*}

For simplicity, we assume that the compatibility conditions are fulfilled at the point $\gamma^0 = (\beta(0),0)$ to ensure sufficient smoothness of the solution of problem (2.2) on each of the subsets $\tilde{G}_k$ (for fixed values of the parameter $\varepsilon$); we suppose $S^k = \tilde{G}_k \setminus G^k, k = 1, 2$.

As $\varepsilon \to 0$, in a neighbourhood of the set $\gamma^*$ (on the right from it) there appears a transition layer decreasing exponentially when the point $(x,t)$ recedes away from $\gamma^*$ to the right. The solution of the reduced problem is a function being sufficiently smooth outside the set $\gamma^*$ and having a discontinuity of the first kind at $\gamma^*$.

2.2.

The errors in the solutions of finite difference schemes based on classical difference approximations to problem (2.2), (2.1) depend on the parameter $\varepsilon$ and become small only for those values of $\varepsilon$ that essentially exceed the “effective” mesh widths with respect to $x$ and $t$. So, by virtue of estimate (4.7), the classical difference scheme (4.4), (4.6) (see Section 4) converges under the condition

$$\varepsilon \gg N^{-1} + N_0^{-1}, \quad (2.5)$$

where the values $N + 1$ and $N_0 + 1$ is the number of mesh points with respect to $x$ (on a unit interval) and $t$, respectively. If this condition is violated, the solutions of the difference scheme do not converge to the solution of problem (2.2), (2.1).

By this argument, we are interested in constructing special difference schemes whose errors do not depend on the value of the parameter $\varepsilon$. In particular, it is of interest to develop such schemes that converge under a weaker condition than condition (2.5).

For the initial value problem (2.2), (2.1), using the condensing mesh method we are thus to construct $\varepsilon$-uniformly convergent schemes and also similar schemes, namely, schemes convergent for the values of $\varepsilon$ much less than in (2.5), which is our purpose in this paper.

\footnote{Here and below $M, M_i$ (or $m$) denote sufficiently large (small) positive constants which do not depend on $\varepsilon$ and on the discretization parameters. Throughout the paper, the notation $L_{(j,k)}(M_{(j,k)}, G_{(j,k)})$ means that these operators (constants, grids) are introduced in equation $(j,k)$.}
3. A priori estimates

In this section we give a priori estimates for the solution of problem (2.2), (2.1) used in our constructions (see also [3, 6, 7, 9]).

On the set $\tilde{G}^2$, the solution can be decomposed into its regular and singular components

$$u(x, t) = U(x, t) + W(x, t), \quad (x, t) \in \tilde{G}^2.$$  \hspace{1cm} (3.1a)

It is convenient to transform problem (2.2), (2.1) to the variables $\tilde{\xi} = x - \beta(t), \ t$ as follows

$$\tilde{L}\tilde{u}(\tilde{\xi}, t) = \tilde{f}(\tilde{\xi}, t), \quad (\tilde{\xi}, t) \in \tilde{G}^{(k)},$$

$$[\tilde{u}(\tilde{\xi}, t)] = 0, \quad \tilde{I}\tilde{u}(\tilde{\xi}, t) = -\tilde{q}(t), \quad (\tilde{\xi}, t) \in \tilde{\gamma}^*,$$

$$\tilde{u}(\tilde{\xi}, t) = \tilde{\phi}(\tilde{\xi}), \quad (\tilde{\xi}, t) \in \tilde{S}. \quad (3.2)$$

Here $\tilde{G}^{(k)}, \tilde{\gamma}^*, \tilde{S}$ are images of the sets $G^{(k)}, \gamma^*, S$ so that $\tilde{\gamma}^* = \{(\tilde{\xi}, t) : \tilde{\xi} = 0, \ t \in (0, T]\}, \ \tilde{\phi}(\tilde{\xi}, t) = v(\tilde{\xi} + \beta(t), t)$;

$$\tilde{L}_{(3.2)} \equiv \varepsilon\tilde{a}(\tilde{\xi}, t) \frac{\partial^2}{\partial \tilde{\xi}^2} + \beta'(t) \tilde{p}(\tilde{\xi}, t) \frac{\partial}{\partial \tilde{\xi}} - \tilde{c}(\tilde{\xi}, t) - \tilde{p}(\tilde{\xi}, t) \frac{\partial}{\partial t}, \quad (\tilde{\xi}, t) \in \tilde{G}^k,$$

$$\tilde{I}_{(3.2)} \tilde{u}(\tilde{\xi}, t) = \varepsilon \left[ \tilde{a}(\tilde{\xi}, t) \frac{\partial}{\partial \tilde{\xi}} \tilde{u}(\tilde{\xi}, t) \right], \quad (\tilde{\xi}, t) \in \tilde{\gamma}^*,$$

$$\tilde{a}(\tilde{\xi}, t) = \tilde{a}^k(\tilde{\xi}, t), \ldots, \tilde{f}(\tilde{\xi}, t) = \tilde{f}^k(\tilde{\xi}, t), \quad (\tilde{\xi}, t) \in \tilde{G}^k, \ k = 1, 2.$$  

The solution of problem (3.2) can be differentiated with respect to $t$ on $\tilde{G}$ and with respect to $\tilde{\xi}$ on $\tilde{G}^2$, and is $\varepsilon$-uniformly bounded together with its derivatives with respect to $t$ on $\tilde{G}^2$. We write the function $\tilde{u}(\tilde{\xi}, t)$ on $\tilde{G}^2$ in the form of a sum of the functions

$$\tilde{u}(\tilde{\xi}, t) = \tilde{U}(\tilde{\xi}, t) + \tilde{W}(\tilde{\xi}, t), \quad (\tilde{\xi}, t) \in \tilde{G}^2, \hspace{1cm} (3.1b)$$

where $\tilde{U}(\tilde{\xi}, t)$ and $\tilde{W}(\tilde{\xi}, t)$ are the regular and singular (interior layer) components of the solution. The function $\tilde{U}(\tilde{\xi}, t)$ is the restriction onto $\tilde{G}^2$ of the function $\tilde{U}^0(\tilde{\xi}, t), (\tilde{\xi}, t) \in \tilde{G}$, which is the solution of the problem

$$\tilde{L}^0 \tilde{U}^0(\tilde{\xi}, t) = \tilde{f}^0(\tilde{\xi}, t), \quad (\tilde{\xi}, t) \in \tilde{G},$$

$$\tilde{U}^0(\tilde{\xi}, t) = \phi^0(\tilde{\xi}), \quad (\tilde{\xi}, t) \in \tilde{S},$$

the operator $\tilde{L}^0$ and the functions $\tilde{f}^0(\tilde{\xi}, t), \phi^0(\tilde{\xi})$ are continuations of the operator $\tilde{L}_{(3.2)}$ and of the functions $\tilde{f}(\tilde{\xi}, t), \phi(\tilde{\xi})$ from the sets $\tilde{G}^2, \tilde{S}_0$ onto the sets $\tilde{G}$ and $\tilde{S}$, which preserve the smoothness.
and boundedness properties, i.e.

\[ 0 < a_0 \leq \tilde{a}^0(\xi, t) \leq a_0, \quad 0 \leq \tilde{c}^0(\xi, t) \leq c_0, \]
\[ 0 < p_0 \leq \tilde{p}^0(\xi, t) \leq p_0, \quad (\xi, t) \in \tilde{G}, \]
\[ |\tilde{f}^0(\xi, t)| \leq M, \quad (\xi, t) \in \tilde{G}, \quad |\tilde{\phi}^0(\xi)| \leq M, \quad \xi \in \tilde{S}_0, \]

where \( \tilde{a}^0(\xi, t) = \tilde{a}^2(\xi, t), \ldots, \tilde{f}^0(\xi, t) = \tilde{f}^2(\xi, t), (\xi, t) \in \tilde{G}^2, \tilde{\phi}^0(\xi) = \tilde{\phi}^2(\xi), \xi \in \tilde{S}_0. \)

The function \( \tilde{W}(\xi, t) \) is the solution of the problem

\[
\tilde{L} \tilde{W}(x, t) = 0, \quad (\xi, t) \in \tilde{G},
\]
\[
\tilde{W}(\xi, t) = \tilde{u}(\xi, t) - \tilde{U}(\xi, t), \quad (\xi, t) \in \tilde{S}^2.
\]

For the functions \( \tilde{U}(\xi, t), \tilde{W}(\xi, t) \) we obtain the estimates

\[
\left| \frac{\partial^{k_1 + k_0}}{\partial \xi^{k_1} \partial t^{k_0}} \tilde{U}(\xi, t) \right| \leq M, \]
\[
\left| \frac{\partial^{k_1 + k_0}}{\partial \xi^{k_1} \partial t^{k_0}} \tilde{W}(\xi, t) \right| \leq M e^{-k_1} \exp(-m_1 e^{-1} \xi), \quad (\xi, t) \in \tilde{G}^2, \quad k_1 + 2k_0 \leq 4, \tag{3.3}
\]

where \( m_1 \) is any number from the interval \((0, m_0)\),

\[ m_0 = \min\left[(a^2(x, t))^{-1} p^2(x, t)(d/dt)\beta(t)\right]. \]

Returning to the variables \( x, t, \) we find

\[
\left| \frac{\partial^{k_1 + k_0}}{\partial x^{k_1} \partial t^{k_0}} U(x, t) \right| \leq M, \]
\[
\left| \frac{\partial^{k_1 + k_0}}{\partial x^{k_1} \partial t^{k_0}} W(x, t) \right| \leq M e^{-k_1 - k_0} \exp(-m_1 e^{-1}(x - \beta(t))), \quad (x, t) \in \tilde{G}^2; \]
\[ k_1 + 2k_0 \leq 4, \quad m_1 = m_{1(3.3)}. \tag{3.4}
\]

On the set \( \tilde{G}^1 \) we have the estimate

\[
\left| \frac{\partial^{k_1 + k_0}}{\partial x^{k_1} \partial t^{k_0}} u(x, t) \right| \leq M, \quad (x, t) \in \tilde{G}^1, \quad k_1 + 2k_0 \leq 4. \tag{3.5}
\]

**Theorem 1.** Let \( a, c, p, f \in C^{4+\alpha}(\tilde{G}^k), \varphi \in C^{4+\alpha}(S_0^k), q \in C^{2+\alpha/2}([0, T]), \beta \in C^{3+\alpha/2}([0, T]), \) and also \( u \in C^{4+\alpha, 2+\alpha/2}(\tilde{G}^k), \alpha > 0, k = 1, 2, \) and let condition (2.4) hold. Then the solution of the initial value problem (2.2), (2.1) and its components from representation (3.1) satisfy estimates (3.3)–(3.5).
4. Classical difference schemes

Let us give a classical difference scheme for problem (2.2), (2.1) and show some difficulties arising in the numerical solution of the problem for small values of the parameter $\varepsilon$.

4.1. We consider a difference scheme based on “direct” approximation of the conjugation condition (2.2c). For this we need meshes which contain nodes on the set $\gamma^n$ at each time level $t = t^j$ of the difference scheme. Let us construct such meshes.

On the domain $\tilde{G}$, we introduce rectangular (base) meshes, on the basis of which we will construct the required grid sets. Let

$$\tilde{G}_h = \omega_1 \times \tilde{o}_0,$$  \hspace{1cm} (4.1)

where $\omega_1$ and $\tilde{o}_0$ are grids on the axis $x$ and the segment $[0,T]$ respectively; $\omega_1$ and $\tilde{o}_0$ are grids with any distribution of the nodes satisfying only the condition $h \leq MN^{-1}$, $h_t \leq MN_0^{-1}$, where $h = \max_j h^j$, $h^j = x_{i+1}^j - x_i^j$, $x_i^j, x_{i+1}^j \in \omega_1$, $h_t = t_{j+1}^j - t_j^j$, $t_j^j, t_{j+1}^j \in \tilde{o}_0$. Here $N + 1$ and $N_0 + 1$ are the maximal number of nodes on a segment of unit length on the axis $x$ and the number of nodes in the grid $\tilde{o}_0$, respectively. It is of great interest to consider also difference schemes on the simplest meshes, which are uniform with respect to both $x$ and $t$:

$$\tilde{G}_h = \tilde{G}_{h(4.1)},$$  \hspace{1cm} (4.2)

where $\omega_1$ and $\tilde{o}_0$ are uniform grids with step-sizes $h = N^{-1}$ and $h_t = TN_0^{-1}$.

On the set $\tilde{G}$ we construct the mesh $\tilde{G}^*_h = \tilde{G}^*_h(\tilde{G}_{h(4.1)})$ generated by the base mesh $\tilde{G}_{h(4.1)}$. On the time level $t = t^j \in \tilde{o}_0$ we introduce the grid set $\tilde{G}^*_{h^n} = G^*_{h^n} \cup S^*_{0h}$, $G^*_{h^n} = G_{h^n}^{(s)n} \cup G_{h^n}^{(s)n}$, $S^*_{0h} \equiv \{x = \alpha(t^n), t^n\}$; the set $G_{h}^{(s)n}$ is formed by those nodes $(x^i,t^n) \in \tilde{G}_{h(4.1)}$, $(x^i,t^n) \notin \gamma^n$, for which the segments $x^i \times [t^{n-1},t^n]$ entirely belongs to either $\tilde{G}^1$ or $\tilde{G}^2$; the set $S^*_{0h}$ consists of the nodes $(x^i,t^{n-1}) \in \tilde{G}_{h(4.1)}$ for which $(x^i,t^n) \in G^*_{h^n}$. We define the mesh $\tilde{G}^*_h$ by

$$\tilde{G}^*_h = \bigcup_{n=1}^{N_0} \tilde{G}^*_{h^n}.$$  \hspace{1cm} (4.3)

We approximate problem (2.2), (2.1) by the implicit difference scheme [8]

$$Az(x,t) \equiv \{a_0(x,t)\delta_{\tilde{x}} z(x,t) - c(x,t) - p(x,t)\delta_y z(x,t) = f(x,t), \quad (x,t) \in G_{h}^{(s)n},$$  \hspace{1cm} (4.4a)

$$t^n z(x,t) \equiv c \{a_0(x,t)\delta_{\tilde{x}} z(x,t) - a_1(x,t)\delta_{\tilde{y}} z(x,t)\} = -q(t), \quad (x,t) \in \gamma^n,$$  \hspace{1cm} (4.4b)

$$z(x,t) = \begin{cases} z_{n-1}^{(s)}(x,t), & t^{n-1} > 0, \\ \varphi(x), & t^n = 0, \end{cases} \quad (x,t) \in \tilde{G}^*_{h^n}, \quad n = 1,\ldots,N_0.$$  \hspace{1cm} (4.4c)

Here $z^n(x,t) = z(x,t)$ for $(x,t) \in G_{h^n}$, $\tilde{z}(x,t)$, $x \in \tilde{G}$, $t = t^n \in \tilde{o}_0$ is the linear, in $x$, interpolant constructed from the values of $z^n(x,t)$, $(x,t) \in G_{h^n}$; $\delta_{\tilde{x}} z(x,t)$, $\delta_{\tilde{y}} z(x,t)$ are the second and first difference derivatives; $\delta_{\tilde{x}}^2 z(x,t) = 2(h^2 + h^{-1})^{-1} \{\delta_x - \delta_{\tilde{x}}\} z(x,t)$, $x = x^i$, $h^{-1}$ and $h^l$ are the left and right “arms” of the three-point stencil on $G_{h^n}^*$ (for the operator $\delta_{\tilde{x}}$) with center at the node...
\((x^i, t^j) \in G_h^{(s)}\). The function
\[
z(x, t) = \begin{cases} z^n(x, t), & (x, t) \in G_h^{sn}, \\ z^{n-1}(x, t), & (x, t) \in \mathcal{S}_0^{sn}, \\ (x, t) \in \tilde{G}_h^{*n} \\ \end{cases}, \quad (x, t) \in \tilde{G}_h^*
\]
will be called the solution of scheme (4.4), (4.3).

For the difference scheme (4.4), (4.3) the maximum principle is valid [8].

4.2.

Let \(z(x, t), (x, t) \in \tilde{G}_h\) be a solution of some difference scheme. We say that the estimate
\[
|u(x, t) - z(x, t)| \leq M|\varepsilon^{-v_i} N^{-v_1} + \varepsilon^{-v_j} N_0^{-v_0}|, \quad (x, t) \in \tilde{G}_h,
\]
where \(v_i, v_j \geq 0\), is \textit{unimprovable} with respect to the entering values of \(N, N_0, \varepsilon\) when the estimate
\[
|u(x, t) - z(x, t)| \leq M|\varepsilon^{-z_i} N^{-z_1} + \varepsilon^{-z_j} N_0^{-z_0}|, \quad (x, t) \in \tilde{G}_h
\]
is, generally speaking, false if the following relations are fulfilled: \(z_i \geq v_i, z^j \leq v^j_1\), and also \(z_1 + z_0 - z_1 = z_0 > v_1 + v_0 - v^j_1 = v_0\).

By using the majorant function technique we find the estimate
\[
|z(x, t)| \leq M[1 + \varepsilon^{-1}], \quad (x, t) \in \tilde{G}_h^*.
\]

In the case of the mesh
\[
\tilde{G}_h^* = \tilde{G}_h^*(\tilde{G}_h^{(4,2)}),
\]
the solution of problem (4.4) is bounded under the (unimprovable) condition \(N^{-1}, N_0^{-1} = O(\varepsilon)\). Under this condition we obtain the (unimprovable) estimate
\[
|u(x, t) - z(x, t)| \leq M\varepsilon^{-1}[N^{-1} + N_0^{-1}], \quad (x, t) \in \tilde{G}_h^{(4,6)};
\]
thus, scheme (4.4), (4.6) converges under the (unimprovable) condition
\[
N^{-1}, N_0^{-1} = o(\varepsilon).
\]

\textbf{Theorem 2.} Let the solution of the initial value problem (2.2), (2.1) and its components from representation (3.1) satisfy the a priori estimates (3.3)–(3.5). Then condition (4.8) is necessary and sufficient for the convergence of difference scheme (4.4), (4.6) as \(N, N_0 \to \infty\). For the discrete solutions the estimates (4.5) and (4.7) are valid. The estimate (4.7) is unimprovable with respect to the values of \(N, N_0, \varepsilon\).

\textbf{Remark 1.} Based on the mesh \(\tilde{G}_h\), we construct a triangulation of the domain \(\tilde{G}\); triangular elements obtained by dividing elementary quadrangles in halves by a diagonal have vertices at the nodes from \(\tilde{G}_h\) (see, e.g., [4]). In the case of difference scheme (4.4), (4.6), the function \(\tilde{z}(x, t), (x, t) \in \tilde{G}\), i.e., the linear interpolant of \(z(x, t)\) on triangular elements, satisfies the error estimate similar to (4.7):
\[
|u(x, t) - \tilde{z}(x, t)| \leq M\varepsilon^{-1}[N^{-1} + N_0^{-1}], \quad (x, t) \in \tilde{G},
\]
which is unimprovable with respect to the values of \(N, N_0, \varepsilon\) for \(N^{-1}, N_0^{-1} = O(\varepsilon)\).
5. On the construction of \( \varepsilon \)-uniformly convergent schemes on locally condensing meshes

Note that the singularity inherent in the initial value problem (2.2), (2.1) does not extend to the set \( \tilde{G}^1 \) and exponentially decreases on \( \tilde{G}^2 \) when the point \((x,t)\) recedes away from the set \( \gamma^* \) (see the second estimate in (3.4)). The singular component \( W(x,t) \) for \( x \geq \beta(t) + \sigma \) does not exceed the value \( \tilde{M} \delta \), where \( \delta \) is a sufficiently small number, when \( \sigma = m_1^{-1} \varepsilon \ln \delta^{-1} \), with \( 0 < m_1 < m_0 \), \( m_0 = \min \{ \varepsilon^2 (a^2(x,t))^{-1} p^2(x,t)(d/dt)\beta(t) \} \). The residual of the difference scheme on the solution of the initial value problem is large but only in this neighbourhood, which is sufficiently narrow for small values of \( \varepsilon \).

5.1.

Bearing in mind the possible use of schemes on sufficiently arbitrary locally condensing meshes for solving the initial value problem, it would be convenient to introduce into consideration balanced meshes, that is, meshes with any distribution of their nodes (in \( x \) and \( t \)) but having the total number of the mesh points of order \( O(\sqrt{T}) \) in a unit vicinity of the set \( \gamma^* \), which is the same order as that in the case of uniform meshes with respect to \( x \) and \( t \). Thus, the amount of computational work (proportional to the number on the mesh points at which it is necessary to find the solution of the grid problem) for balanced meshes is of the same order just as for uniform meshes. Balanced meshes are not in general the tensor product of one-dimensional meshes with respect to \( x \) and \( t \).

5.2.

We consider a class of difference schemes composed of classical approximations to the initial value problem (2.2), (2.1) and "piecewise uniform" locally condensing meshes, i.e., meshes which are uniform both in the nearest neighbourhood of the curve \( \gamma^* \) and outside its somewhat greater neighbourhood.

5.2.1.

For simplicity, assume that \( \beta(t) = t \). Let the following mesh have been constructed in some way:
\[
\tilde{G}^*_h = \tilde{G}^*_h(\rho_1),
\]
where \( \rho_1 > 0 \) is a parameter chosen below, which defines the distribution of the mesh points. This mesh is uniform on each of the sets \( G^1_h = G^1_h(\rho_1) \) and \( G^2_h = G^2_h(M\rho_1) \), where \( G^1_h(\rho_1) = \{(x,t) : x \in (\beta(t),\beta(t) + \rho_1), t \in (0,T)\} \) is the right \( \rho_1 \)-neighbourhood of the set \( \gamma^* \). The meshes \( G^2_h = G^1_h \cap \tilde{G}^*_{h(5.1)} \) have step-sizes \( h_i \) and \( h_{1i} \) in \( x \) and \( t \) respectively, \( i = 1, 2 \). We consider for simplicity that the stencils of four-point implicit schemes having, as a center, the nodes from \( G^2_h \) are regular, i.e., their left, right and "lower" arms equal \( h_1 \) and \( h_{1i} \) respectively.

Let us consider fragments of the grid problem from the class of difference schemes on the meshes (5.1), namely, the fragments on the sets \( \tilde{G}^1_{ih} \) and \( \tilde{G}^2_{2h} \). Let \( z^i_1(x,t), (x,t) \in \tilde{G}^1_{ih} \) be the solution of the grid problem
\[
A_{(4.4)}z^i_1(x,t) = f(x,t), \quad (x,t) \in \tilde{G}^1_{ih},
\]
\[
z^i_2(x,t) = u(x,t), \quad (x,t) \in S^i_{ih}, \quad i = 1, 2,
\]
where \( u(x,t), (x,t) \in \tilde{G} \) is the solution of problem (2.2), (2.1).
For the functions $z_2^i(x,t), (x,t) \in \tilde{G}_{ih}$ we have the estimates

$$|u(x,t) - z_1^i(x,t)| \leq M[(\varepsilon + h_1)^{-2}h_1^2 + (\varepsilon + h_{1t})^{-1}h_{1t}], \quad (x,t) \in \tilde{G}_{ih};$$

$$|u(x,t) - z_2^i(x,t)| \leq M\left\{[(\varepsilon + h_2)^{-2}h_2^2 + (\varepsilon + h_{2t})^{-1}h_{2t}] \max_{S_{2h}}|W(x,t)| + h_2^2 + h_{2t}\right\}, \quad (x,t) \in \tilde{G}_{2h},$$

where $W(x,t)$ is the singular component of the solution $u(x,t)$; estimates (5.3a) and (5.3b) are unimprovable with respect to the entering values of $h_1, h_{1t}, \varepsilon$ and $h_2, h_{2t}, \varepsilon$ respectively.

In order that the function $z_2^i(x,t)$ converges $\varepsilon$-uniformly, it is necessary that the value $\rho_1$ satisfies the condition $\varepsilon = o(\rho_1)$. (5.4)

The following estimate for the function $z_1^2(x,t)$, which is the same with respect to the convergence order as the optimal estimate relatively to $h_1, h_{1t}$ for the fixed, equal to $MN_{N_0}$, number of nodes of the mesh $\tilde{G}_{2h}$, can be obtained under the condition $(\varepsilon + h_{1t})^{-1}h_{1t} = (\varepsilon + h_1)^{-2}h_1^2$:

$$|u(x,t) - z_1^2(x,t)| \leq M\varepsilon^{-4/3}\rho_1^{2/3}(NN_0)^{-2/3}[1 + \varepsilon^{-4/3}\rho_1^{2/3}(NN_0)^{-2/3}]^{-1}, \quad (x,t) \in \tilde{G}_{ih};$$

this estimate is unimprovable with respect to the entering values of $\rho_1, (NN_0)^{-1}, \varepsilon$. It follows from estimate (5.5) under condition (5.4) that the function $z_1^2(x,t)$ does not converge $\varepsilon$-uniformly for $N, N_0 \to \infty$.

Thus, the error analysis demonstrates that there are no piecewise uniform meshes $\tilde{G}_{(5.1)}$ on which the solutions of problems (5.2) for $i = 1, 2$ converge $\varepsilon$-uniformly to the solution of problem (2.2), (2.1). In the case of the auxiliary problem (5.2) on meshes (5.1), similarly to the above considerations, we make sure of the fact that there exist no meshes on which the solutions of these problems converge even under the condition

$$N^{-1} + N_0^{-1} \geq \varepsilon^{1/2}.$$

5.2.2.

From here it follows that this non-existence result remains valid also in the case of standard difference approximations to problem (2.2), (2.1) and the family of meshes (5.1) as well as the family of meshes

$$\tilde{G}_{(5.1)}^*,$$

which are uniform in the right $\rho$-neighbourhood of the set $\gamma^*$, where $m_\varepsilon \leq \rho \leq M_\varepsilon$.

**Theorem 3.** *For the initial value problem (2.2), (2.1), in the class of balanced difference schemes composed of standard finite difference operators on locally condensing grids (5.7) there are no schemes convergent under condition (5.6).*
Remark 2. If we use the grid equations (4.4b) in order to approximate the conjugation conditions (2.2c), in the case of meshes (5.1) and (5.7) there exist no balanced schemes convergent under the condition $N^{-1} + N_0^{-1} \geq \varepsilon^{2/3}$.

Remark 3. It follows from the given considerations that the use of locally condensing meshes for problem (2.2), (2.1) does not allow us to weaken essentially the convergence condition (4.8) for classical difference schemes; it is impossible to reduce the order of the parameter $\varepsilon$ in condition (2.5) more than twice on a class of sufficiently common locally condensing meshes, unless the stencil used is non-rectangular.

Thus, the direct use of adaptive mesh refinement techniques with no taking account of orientation of the transition layer is not sufficiently effective to solve numerically problems from this class of singularly perturbed problems with moving concentrated sources.

6. Necessary conditions for $\varepsilon$-uniform convergence

Let us consider approximations to the solution of the singularly perturbed problem (2.2), (2.1) using Kolmogorov widths (see e.g., [1] and references therein).

We are interested in approximating the set $\mathcal{U}$, i.e., the set of solutions to the class of problem (2.2), (2.1) in the space $X$ of continuous functions endowed with the maximum norm. The solutions are assumed to be sufficiently smooth on each of the subdomains $\tilde{G}_1$ and $\tilde{G}_2$; more precisely, the solutions and their components from representation (3.1) satisfy estimates (3.3)–(3.5).

Let $\tilde{G}_h$ be a set of points (“grid”) on $\tilde{G}$. Such grids $\tilde{G}_h$ can be either structured (generated by some regular family of lines) or unstructured. By $P$ we denote the minimal number of mesh points on a unit segment from $\tilde{G}$ between the cross-sections $x_0$ and $x_0 + 1$, $x_0 \in \mathbb{R}$. Let $T_p$ be a triangulation of $\tilde{G}$ generated by the grid $\tilde{G}_h$ [4]: the nodes of the grid $\tilde{G}_h$ are vertices of triangular elements, moreover, the triangular elements are formed by segments of straight lines passing through the nodes of $\tilde{G}_h$.

Let some grid function $u_h(x,t)$ be defined on set $\tilde{G}_h$; we denote its linear interpolant by $\bar{u}_h(x,t)$, $(x,t) \in \tilde{G}$. The set of such interpolants for the fixed triangulation $T_p$ is denoted by $U_h^P$; the collection of all admissible set of grids $\tilde{G}_h$ (but with the number of nodes on a unit segment from $\tilde{G}$ being equal to $P$) and triangulations $T_p$ based on these grids is denoted by $\mathcal{T}_P$. This collection $\mathcal{T}_P$ and the set of interpolants $U_h^P$ (for every triangulation from $\mathcal{T}_P$) define the space $X$.

We are now in a position to define the quantity $d_p(\mathcal{U},X)$ (so-called Kolmogorov width) by

$$d_p(\mathcal{U},X) = \inf_{T_p} \sup_{u \in \mathcal{U}} \inf_{\bar{u}_h \in U_h^P} \| u - \bar{u}_h \|,$$

(6.1)

where $\| \cdot \|$ is the norm in $C$.

Let $T_j^l$ be a triangular element from the partition $T_p$, $\rho_1(T_j^l)$ and $\rho_2(T_j^l)$ are the radii of respectively inscribed and circumscribed circles for the element $T_j^l$. The triangulation $T_p$ is called isotropic if the condition

$$\rho_1^{-1}(T_j^l) \rho_2(T_j^l) \leq M, \quad j = 1, \ldots, J$$

is satisfied, however, the values of $\rho_1^{-1}(T_j^l)$ and $\rho_2(T_j^l)$ can vary significantly from element to element, and anisotropic (with the coefficient of anisotropy $\eta \geq M_0$ where $M_0$ can be sufficiently
large) if we have the condition
\[
\rho_j^{-1}(T^j P) \rho_2(T^j P) \leq M \eta, \quad j = 1, \ldots, J,
\]
the constant \(M\) is independent of \(\varepsilon, P\). Here \(J = J(P)\) is the number of elements in the triangulation \(T_P\), \(J \approx P\). We assume that the collection \(\mathcal{T}_P\) is defined by the value \(\eta\); \(d_P(\mathcal{U}, X) = d_P(\mathcal{U}, X; \eta)\). In a similar way we introduce into consideration isotropic and anisotropic triangulations on subsets of \(\tilde{G}\), in particular, on the set \(\tilde{G}^2\). When the Kolmogorov width (denoted by \(d_P(\mathcal{U}, X; \tilde{G}^0)\) correspondingly) is considered on a subset \(\tilde{G}^0 \subset \tilde{G}\), the quantity \(\|u - \tilde{u}\|\) in (6.1) is calculated only on triangular elements entirely belonging to \(\tilde{G}^0\). Triangular elements for which the condition \(X \rightarrow \infty\) holds, for example, for \(P \rightarrow \infty\) and/or \(\varepsilon \rightarrow 0\), is called essentially anisotropic.

Let us first estimate the width in the case when the interface boundary \(\gamma^*\) is a segment of a straight line.

In the case of isotropic triangulations the width satisfies the lower bound
\[
d_P(\mathcal{U}, X) \geq m(1 + \varepsilon P)^{-1}. \tag{6.2}
\]
On the subset \(\tilde{G}^0\) defined as
\[
\tilde{G}^0 = \{(x, t) : x \in (\beta(t), \beta(t) + M \varepsilon), \ t \in (0, T]\}, \tag{6.3}
\]
which is the right \(M\varepsilon\)-neighbourhood of \(\gamma^*\), we obtain the estimate
\[
d_P(\mathcal{U}, X; \tilde{G}^0) \geq m(1 + \varepsilon P)^{-1} \tag{6.4}
\]
which is unimprovable with respect to the values of \(P, \varepsilon\).

The following estimate is valid on anisotropic elements:
\[
d_P(\mathcal{U}, X) \geq m(1 + \varepsilon P)^{-1}. \tag{6.5}
\]
On the set \(\tilde{G}^0_{(6.3)}\) we have
\[
d_P(\mathcal{U}, X; \tilde{G}^0) \geq m(1 + \varepsilon P)^{-1}, \tag{6.6}
\]
this estimate is unimprovable with respect to the entering values of \(P, \varepsilon, \eta\).

**Theorem 4.** In the case of the linear interface boundary \(\gamma^*\) the Kolmogorov widths \(d_P(\mathcal{U}, X)\) and \(d_P(\mathcal{U}, X; \tilde{G}^0)\) satisfy estimates (6.2), (6.4) on the isotropic triangulation \(T_P\) and estimates (6.5), (6.6) on the anisotropic triangulation \(T_P\).

**Remark 4.** The use of essentially anisotropic triangulations is necessary in order that the width \(d_P(\mathcal{U}, X)\) converges \(\varepsilon\)-uniformly.

Let the interface \(\gamma^*\) be a curve of bounded curvature. In the anisotropic case we define the width \(d^*_P(\mathcal{U}, X)\) by
\[
d^*_P(\mathcal{U}, X) = \inf_{\eta} d_P(\mathcal{U}, X; \eta).
\]

**Theorem 5.** In the case of the curvilinear interface boundary \(\gamma^*\) the width \(d^*_P(\mathcal{U}, X)\) considered on \(\tilde{G}^0\) satisfies the estimate
\[
d^*_P(\mathcal{U}, X; \tilde{G}^0) \geq m(1 + \varepsilon^{1/2} P)^{-1},
\]
which is unimprovable with respect to \(P, \varepsilon\).
**Remark 5.** To construct $\varepsilon$-uniform approximations to the width $d_P^*(\mathcal{U},X)$, it is necessary to apply triangular elements which are curvilinear and essentially anisotropic in the neighbourhood of the interior layer (with the local direction of the anisotropy conforming with the curve $\gamma^*$).

**Remark 6.** It is possible to introduce local coordinates on the set $\widetilde{G}^2$ (in the nearest neighbourhood of the interior layer), in which the curve $\gamma^*$ becomes rectilinear, and then to construct a triangulation (on triangular elements with linear sides) on which the width converges $\varepsilon$-uniformly in the interior layer region (see, e.g., Remark 4 to Theorem 4). In the original variables such a triangulation is generated by curvilinear triangular elements. Based on this local triangulation, the triangulation of the whole set $\widetilde{G}$ can be constructed so that the width considered on $\widetilde{G}$ converges $\varepsilon$-uniformly.

Similar statements remain valid in the case when one uses interpolants approximating the solution of problem (2.2), (2.1) with a higher order of accuracy in comparison with linear interpolants. Clearly, it is expedient to put the requirements implied by the last observations into the basis for constructing special numerical methods as follows.

**Remark 7.** (Schemes on a stencil fitted to the interior layer) The necessary convergence conditions for the widths $d_P(\mathcal{U},X), d_P^*(\mathcal{U},X)$ discussed in Remarks 4, 5 (these conditions are also necessary for $\varepsilon$-uniform convergence of finite difference or finite element methods), can be complied with meshes condensing (in the interior layer) along a normal to the interface boundary $\gamma^*$, for example, on Bakhvalov-type meshes or simpler piecewise uniform meshes. To construct special schemes with an improved convergence condition in comparison with (4.8), one can transform the initial value problem (2.2), (2.1) to the variables connected with the moving source, in which the source becomes already fixed. For the problem in these new variables one can construct a finite difference scheme on rectangular meshes (in particular, a scheme convergent $\varepsilon$-uniformly) and then return back to the old variables. The resulting meshes (i.e., meshes moving in agreement with the source) is no longer rectangular in the variables $x,t$. This, generally speaking, implies certain inconveniences for constructing the grid domains and finding the numerical solutions. However, it is possible to construct a similar scheme only in a sufficiently small neighbourhood of the source; outside this neighbourhood one can use standard (e.g., uniform) meshes and finite difference operators. Such an approach (in the case of piecewise uniform meshes) leads to a finite difference scheme that converges $\varepsilon$-uniformly at the rate $O(N^{−1}\ln N + N_0^{−1})$. Note that the meshes used in [10,11] satisfy the derived necessary conditions for $\varepsilon$-uniform convergence of numerical approximations as $P \to \infty$.

The idea to use Kolmogorov widths was proposed by N.S. Bakhvalov more than twenty five years ago when discussing special numerical methods for singularly perturbed problems.

**References**