JOURNAL OF DIFFERENTIAL EQUATIONS 123, 480–503 (1995)

# Construction and Qualitative Behavior of the Solution of the Perturbated Riemann Problem for the System of One-Dimensional Isentropic Flow with Damping

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Received February 16, 1993; revised May 5, 1994

## 1. INTRODUCTION

We investigate the following hyperbolic conservation laws with damping

$$\begin{aligned} &v_t - u_x = 0 \\ &u_t + p(v)_x = -\alpha u, \alpha > 0, p'(v) < 0 \end{aligned} \tag{1.1}$$

which can be used to describe the compressible flow through porous media where $u$ is the velocity, $v > 0$ is the specific volume and $p(v)$ is the pressure, with the discontinuous initial value

$$(u_0(x), v_0(x)) = \begin{cases} (u_-(x), v_-(x)), & x < 0 \\ (u_+(x), v_+(x)), & x > 0, \end{cases} \tag{1.2}$$

where $u_\mp(x)$, $v_\mp(x)$ are smooth function such that $\lim_{x \to 0}(u_\mp(x), v_\mp(x)) = (u^\mp, v^\mp)$ and $v^\mp > 0$.

For the system without the damping term, i.e.,

$$\begin{aligned} &v_t - u_x = 0 \\ &u_t + p(v)_x = 0, p'(v) < 0 \end{aligned} \tag{1.3}$$

the same kind of discontinuous initial value problem (1.3), (1.2) has been studied in [LZ] where $u_\mp(x)$, $v_\mp(x)$ are considered as perturbated Riemann data. It has been proved under certain restriction on the initial data (1.2) that the problem (1.3), (1.2) admits a unique global solution in a class of piecewise continuous and piecewise smooth functions and the solution has a global structure similar to that of the corresponding Riemann problem with Riemann data $(u^-, v^-)$ and $(u^+, v^+)$. The

* Project supported by the National Science Foundation of China.



0022-0396/95 $12.00

Riemann problem for (1.3) has been studied very well. (see [CH], [LI1], [DD], [SM], etc.])

However, the Riemann problem for the system (1.1) with damping term has not been well studied in the literature since there is no self-similar solution anymore and it becomes much more complicated. It is well-known that this problem is of significance. It plays a special role in the study of the existence, uniqueness and asymptotic behavior of the solution for the general initial-value problem.

To study the discontinuous initial value problem of (1.1), (1.2) is also a part of the program to show that the system (1.1) can be modeled by

$$v_t = \frac{-1}{\alpha} p(v)_{xx}$$

$$u = \frac{-1}{\alpha} p(v)_x$$

time asymptotically even in the case when shock waves may develop in the solutions of (1.1) (see [HL] and [HT] for detail).

We investigate the perturbated Riemann problem (1.1), (1.2) and construct the globally defined piece-wise continuous and piece-wise smooth solutions with showing the qualitative behavior in a sequence of papers. This paper handles the case in which the two states $(u^-, v^-)$ and $(u^+, v^+)$ are connected by a backword shock curve and a forward shock curve subsequently in the phase plane. For other kind of relation between $(u^-, v^-)$ and $(u^+, v^+)$, the result can be found in [TX]. For simplicity, we take a typical form of the state function $p(v)$ to discuss, namely, $p(v) = av^{-\gamma}$, $a > 0$, $1 < \gamma < 3$, which is the state function for polytropic gas. The case $\gamma = 1$ has been discussed in [HT].

Under certain restriction on the strength of the initial discontinuity and the perturbation in the initial data, expressed as condition A and conditon B respectively, explained in Section 2 and Section 3, we prove that the problem (1.1), (1.2) admits a unique global discontinuous solution on $t \geqslant 0$ in a class of piecewise continuous and piecewise smooth functions which only contains two entropy shocks—a backward shock and a forward shock. The shocks do not disappear for any finite time and do disappear with the strength decay exponentially fast when time $t$ tends to infinity. We prove the result for unperturbated Riemann data in Section 2 first. The result for perturbated Riemann data is obtained in Section 3.

For general Cauchy problem of inhomogeneous hyperbolic system, the discussion on the existence of weak solutions can be also found in [DH], $[LI]_2$, [YW], [DCL], [HM], [LT], etc.

## 2. Construction and Behavior of Solution for Riemann Problem

The unperturbated initial data discussed in this section is called as Riemann data, namely

$$(u(0,x), v(0,x)) = \begin{cases} (u^-, v^-), & x<0 \\ (u^+, v^+), & x>0. \end{cases} \tag{2.1}$$

We take $a=1$ and $\alpha=1$ in (1.1) for convenience. Then the characteristic speed takes the form $\lambda = -\sqrt{\gamma}\cdot v^{-(\gamma+1)/2}$ and $\mu = \sqrt{\gamma}\cdot v^{-(\gamma+1)/2}$ for the backward and forward family respectively.

Introduce Riemann invariants

$$\begin{aligned} r &= u-\phi \\ s &= u+\phi, \end{aligned} \tag{2.2}$$

where

$$\phi = \frac{2\sqrt{\gamma}}{\gamma-1} v^{(1-\gamma)/2}. \tag{2.3}$$

Thus

$$\begin{aligned} u &= \frac{s+r}{2} \\ v &= \left[\frac{\gamma-1}{4\sqrt{\gamma}}(s-r)\right]^{2/(1-\gamma)} \end{aligned} \tag{2.4}$$

and the system (1.1) can be written as

$$\begin{aligned} r_t + \lambda \gamma_x &= -\tfrac{1}{2}(r+s) \\ s_t + \mu s_x &= -\tfrac{1}{2}(r+s) \end{aligned} \tag{2.5}$$

which is equivalent to (1.1) wherever the solution is smooth.

By the notation of

$$\begin{aligned} {}' &= \frac{\partial}{\partial t} + \lambda\frac{\partial}{\partial x} \\ {}^{\backprime} &= \frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x} \end{aligned} \tag{2.6}$$

(2.5) becomes

$$
\begin{aligned}
r' &= -\tfrac{1}{2}(r+s) \\
s' &= -\tfrac{1}{2}(r+s).
\end{aligned}
\tag{2.7}
$$

A discontinuity in a solution of (1.1) is characterized by the left-hand, respectively right-hand, values at the discontinuity, and by the discontinuity's speed. These quantities are related among them by the well-known Rankine–Hugoniot condition, which, in the present case, implies that

$$
\begin{aligned}
\frac{dx_1(t)}{dt} &= -\sqrt{\frac{p(v_R)-p(v_L)}{v_L-v_R}} \\
u_L-u_R &= \sqrt{(p(v_R)-p(v_L))/(v_L-v_R)}\cdot(v_L-v_R)
\end{aligned}
\tag{2.8}
$$

or

$$
\begin{aligned}
\frac{dx_2(t)}{dt} &= \sqrt{\frac{p(v_R)-p(v_L)}{v_L-v_R}} \\
u_L-u_R &= -\sqrt{(p(v_R)-p(v_L))/(v_L-v_R)}\cdot(v_L-v_R),
\end{aligned}
\tag{2.9}
$$

where $(u_R, v_R)=(u, v)(t, x_i(t)+0)$, $(u_L, v_L)=(u, v)(t, x_i(t)-0)$, $i=1$ in (2.8) and $i=2$ in (2.9) respectively.

A discontinuity is called a backward shock (or forward shock) if (2.8) (or (2.9)) holds and the entropy condition is satisfied as well, namely, $v_R<v_L$ (or $v_R>v_L$) (2.10) (or (2.11))

By using (2.2) and (2.3), it is easy to rewrite the above formulae in $(r, s)$ variables in the following form where the exact expression will be given later.

$$
\begin{aligned}
\frac{dx_1(t)}{dt} &= f_1(s_R, r_R; s_L, r_L) \\
h_1(s_R, r_R; s_L, r_L) &= 0
\end{aligned}
\tag{2.12}
$$

and

$$
s_R-r_R>s_L-r_L, \tag{2.13}
$$

or

$$
\begin{aligned}
\frac{dx_2(t)}{dt} &= f_2(s_R, r_R; s_L, r_L) \\
h_2(s_R, r_R; s_L, r_L) &= 0
\end{aligned}
\tag{2.14}
$$

and

$$s_R - r_R < s_L - r_L. \tag{2.15}$$

In the coming discussion, we may choose either $(u, v)$ or $(r, s)$ according to the convenience.

For the initial data (2.1) in the present paper, there exists a state $(r_0, s_0)$ (or $(u_0, v_0)$ in $(u, v)$ variables) such that

$$\begin{aligned} &h_1(s_0, r_0; s^-, r^-) = 0, \\ &h_2(s^+, r^+; s_0, r_0) = 0, \qquad \text{and} \\ &\qquad s_0 - r_0 > s^- - r^-, \\ &\qquad s_0 - r_0 > s^+ - r^+, \end{aligned} \tag{2.16}$$

where $(r^+, s^+)$ and $(r^-, s^-)$ are determined by $(u^+, v^+)$ and $(u^-, v^-)$ through (2.2) and (2.3). This means that the two states $(u^-, v^-)$ and $(u^+, v^+)$ are connected by a backward shock curve and a forward one subsequently in the phase plane.

Now we construct the global discontinuous solution for (1.1) (2.1) and (2.16).

Just like the case of $\gamma = 1$ in [HT], we first use the initial data to obtain the unique global $C^1$ solution

$$(u_-(t, x), v_-(t, x)) = (u^- e^{-t}, v^-) \tag{2.17}$$

on the domain

$$\hat{R}_- = \left\{ (t, x) \mid t \geqslant 0, x \leqslant \xi_-(t), \frac{d\xi_-(t)}{dt} = \lambda(v_-(t, \xi_-(t))) \right\}$$

and

$$(u_+(t, x), v_+(t, x)) = (u^+ e^{-t}, v^+) \tag{2.18}$$

on the domain

$$\hat{R}_+ = \left\{ (t, x) \mid t \geqslant 0, x \geqslant \xi_+(t), \frac{d\xi_+(t)}{dt} = \mu(v_+(t, \xi_+(t))) \right\}$$

for the corresponding initial value problem.

By the local existence theorem in [LY], this discontinuous initial value problem (1.1), (2.1) admits a unique discontinuous solution at least on a local domain $R(\delta) = \{(t, x) \mid 0 \leqslant t \leqslant \delta, -\infty < x < \infty\}$ in a class of piecewise continuous and piecewise smooth functions and this solution contains only a backward shock $x=x_1(t)$ and a forward shock $x=x_2(t)$ passing through the origin.

In view of the entropy condition on the initial discontinuities, it is known that $x=x_1(t)$ must be located on the left side of $x=\xi_-(t)$ while $x=x_2(t)$ on the right side of $x=\xi_+(t)$. Therefore, the solution on the left side of $x=x_1(t)$ and the right side of $x=x_2(t)$ will be furnished by $(u_-(t,x), v_-(t,x))$ and $(u_+(t,x), v_+(t,x))$ respectively. Thus, in order to construct the globally defined discontinuous solution for (1.1) (2.1), we only need to solve the following free boundary problem on the angular domain $R=\lim_{T\to\infty} R(T)$, where $R(T)=\{(t,x) \mid 0\leqslant t\leqslant T, x_1(t)\leqslant x\leqslant x_2(t)\}$.

(FBP): On the free boundary $x=x_1(t)(x_1(0)=0, t\geqslant 0)$

$$\begin{aligned} &\frac{dx_1(t)}{dt}=-\left\{\frac{v^{-\gamma}-v_-^{-\gamma}}{v_--v}\right\}^{1/2} \\ &(r+s)-(r_-+s_-)=-2\{(v_--v)/(v^{-\gamma}-v_-^{-\gamma})\}^{1/2}\cdot(v_--v), \end{aligned} \tag{2.19}$$

where all of the values are taken at $(t, x_1(t))$.

On the free boundary $x=x_2(t)(x_2(0)=0, t\geqslant 0)$

$$\begin{aligned} &\frac{dx_2(t)}{dt}=\left\{\frac{v^{-\gamma}-v_+^{-\gamma}}{v_+-v}\right\}^{1/2} \\ &(r+s)-(r_++s_+)=-2\{(v_+-v_-)/(v^{-\gamma}-v_+^{-\gamma})\}^{1/2}\cdot(v-v_+) \end{aligned} \tag{2.20}$$

where all of the values are taken at $(t, x_2(t))$. Moreover, the entropy condition has to be satisfied on the free boundaries, namely,

$$(s-r)(t, x_1(t))>(s_--r_-)(t, x_1(t))(t\geqslant 0) \tag{2.21}$$

$$(s-r)(t, x_2(t))>(s_+-r_+)(t, x_2(t))(t\geqslant 0). \tag{2.22}$$

Without loss of generality, we assume that

$$u^\pm>0 \qquad \text{and} \qquad 0<v^-\leqslant v^+<1 \tag{2.23}$$

The other cases can be discussed similarly.

In order to guarantee that the problem (1.1) (2.1) admits a unique global discontinuous solution on $t\geqslant 0$ which only contains two entropy shocks, we introduce condition A which makes restriction on the strength of initial shocks and guarantees that the gradient of $u$ and $v$ on the inner side of the two shocks-free boundaries never becomes large.

THEOREM 2.1. *Under condition A stated below, the free boundary problem* (1.1), (2.19), (2.20) *admits a global classical solution* $(r, s)\in C^1$ *in*

*R with* $x_i(t)\in C^2$ $(i=1,2)$ *on which* (2.21) *and* (2.22) *hold respectively. Furthermore, this solution possesses the following properties. For any* $(t,x)\in R$,

$$s_*\leqslant s\leqslant s^-,\qquad r_*\leqslant r\leqslant r_0,\qquad v_0\leqslant v\leqslant \max\{v(t,x_1(t)),v(t,x_2(t))\} \tag{2.24}$$

$$\min\left\{\min_{\substack{x_2(\tau;\,t,\,x)\\ \alpha\leqslant\tau\leqslant t}}\frac{2v}{3-\gamma},\left(s_x+\frac{2v}{3-\gamma}\right)(\alpha,x_1(\alpha))\right\}$$
$$\leqslant s_x+\frac{2v}{3-\gamma}\leqslant\max\left\{\max_{\substack{x_2(\tau;\,t,\,x)\\ \alpha\leqslant\tau\leqslant t}}\frac{2v}{3-\gamma},\left(s_x+\frac{2v}{3-\gamma}\right)(\alpha,x_1(\alpha))\right\} \tag{2.25}$$

$$\min\left\{\min_{\substack{x_1(\tau;\,t,\,x)\\ \beta\leqslant\tau\leqslant t}}\frac{2v}{3-\gamma},\left(r_x+\frac{2v}{3-\gamma}\right)(\beta,x_2(\beta))\right\}$$
$$\leqslant r_x+\frac{2v}{3-\gamma}\leqslant\max\left\{\max_{\substack{x_1(\tau;\,t,\,x)\\ \beta\leqslant\tau\leqslant t}}\frac{2v}{3-\gamma},\left(r_x+\frac{2v}{3-\gamma}\right)(\beta,x_2(\beta))\right\}, \tag{2.26}$$

*where*

$$s_*=\frac{2\sqrt{\gamma}}{\gamma-1}(\max\{v^+,v^-\})^{(1-\gamma)/2}=\frac{1}{2}\max\{s^+-r^+,s^--r^-\}, \tag{2.27}$$

$$r_*=-\frac{2\sqrt{\gamma}}{\gamma-1}v_0^{(1-\gamma)/2}=-\frac{s_0-\gamma_0}{2} \tag{2.28}$$

*and* $x_1(\tau;t,x)(x_2(\tau;t,x))$ *denotes the backward (forward) characteristic passing through* $(t,x)$ *such that*

$$\begin{aligned}x_1(\alpha)&=x_2(\alpha;t,x)\\ x_2(\beta)&=x_1(\beta;t,x)\end{aligned} \tag{2.29}$$

*Moreover, for any point* $(t,x_1(t))$, $0\leqslant t<\infty$, *it holds that*

$$(v^--v_0)\,e^{-A_3t}\leqslant v_--v\leqslant(v^--v_0)\,e^{-A_4t} \tag{2.30}$$

*and for any point* $(t,x_2(t))$, $0\leqslant t<\infty$, *it holds that*

$$(v^+-v_0)\,e^{-A_1t}\leqslant v_+-v\leqslant(v^+-v_0)\,e^{-A_2t}, \tag{2.31}$$

*where* $v_-=v_-(t,x_1(t))$, $v_+=v_+(t,x_2(t))$, $A_i$ *is a positive constant* $(i=1,2,3,4)$.

The condition A can be stated as follows. Denote

$$K_1(\zeta)=\left(\sqrt{\gamma}+\sqrt{\frac{1-\zeta^{-\gamma}}{\zeta-1}}\right)\left[1+\frac{\gamma(\zeta-1)+(1-\zeta^{-\gamma})}{2\sqrt{\gamma(\zeta-1)(1-\zeta^{-\gamma})}}\right]$$

$$K_2(\zeta)=\left(\sqrt{\gamma}-\sqrt{\frac{1-\zeta^{-\gamma}}{\zeta-1}}\right)\left[-1+\frac{\gamma(\zeta-1)+(1-\zeta^{-\gamma})}{2\sqrt{\gamma(\zeta-1)(1-\zeta^{-\gamma})}}\right] \tag{2.32}$$

$$K(\zeta,\eta)=[K_1(\zeta)]^{-1}\left[(3-\gamma)\sqrt{(\zeta-1)(1-\zeta^{-\gamma})}-K_2(\zeta)(\zeta\eta-1)\right]+1$$

$$\sigma=\frac{\gamma-1}{\gamma+1}.$$

*Condition* A. Let $\zeta_0^+=v^+/v_0$, $\zeta_0^-=v^-/v_0$, where $\zeta_0^+\geqslant\zeta_0^->1$.

1. For any $\zeta\in(1,\zeta_0^-]$,

$$K(\zeta,\zeta_0^+)-\max\left\{\sqrt{2\sigma}\left(\frac{\zeta\zeta_0^+}{\zeta_0^-}\right)^{(3-\gamma)/8},\frac{\zeta}{\zeta_0^-}\right\}>0$$

$$K\left(\zeta,\frac{1}{\zeta_0^-}\right)-\frac{\zeta\zeta_0^+}{\zeta_0^-}<0. \tag{2.33}$$

For any $\zeta\in(1,\zeta_0^+]$,

$$K(\zeta,1)-\max\left\{\sqrt{2\sigma}\,[\zeta]^{(3-\gamma)/8},\frac{\zeta}{\zeta_0^+}\right\}>0$$

$$K\left(\zeta,\frac{1}{\zeta_0^+}\right)-\zeta<0. \tag{2.34}$$

2. For any $\zeta\in(1,\zeta_0^+]$, it holds that

$$1-\frac{2}{(3-\gamma)}\left[\sqrt{\frac{\gamma}{(\zeta-1)(1-\zeta^{-\gamma})}}-\frac{1}{\zeta-1}\right]\left(1-\frac{\zeta}{\zeta_0^+}\right)>0. \tag{2.35}$$

3.

$$\zeta_0^+<(2\sigma)^{-2/(3-\gamma)}. \tag{2.36}$$

4.

$$\left.\frac{K_1(\zeta)}{\sqrt{(\zeta-1)(1-\zeta^{-\gamma})}}\right|_{\zeta=\zeta_0^\pm}>\left.\frac{K_2(\zeta)}{\sqrt{(\zeta-1)(1-\zeta^{-\gamma})}}\right|_{\zeta=\zeta_0^\pm} \tag{2.37}$$

*Remark* 2.2. It is easy to see that all of the inequalities in condition A, except $(2.33)_1$ and $(2.34)_1$, hold automaticly as $\gamma \to 1$ since $K_1(\zeta) \to (\sqrt{\zeta}+1)^3/(2\zeta)$, $K_2(\zeta) \to (\sqrt{\zeta}-1)^3/(2\zeta)$ and $K(\zeta) \to \zeta/(\sqrt{\zeta}+1)^3 \cdot [(6\sqrt{\zeta} + 2/\sqrt{\xi}) - (\sqrt{\zeta}-1)^3 \zeta]$, as $\gamma \to 1$. Thus the condition A reduces to

$$\begin{aligned} &6\sqrt{\zeta}+\frac{2}{\sqrt{\zeta}} > (\sqrt{\zeta}-1)^3 \frac{\zeta_0^+}{\zeta_0^-} + (\sqrt{\zeta}+1)^3 \cdot \frac{1}{\zeta_0^-}, \qquad \text{for} \quad \zeta \in (1, \zeta_0^-] \\ &6\sqrt{\zeta}+\frac{2}{\sqrt{\zeta}} > (\sqrt{\zeta}-1)^3 + (\sqrt{\zeta}+1)^3 \cdot \frac{1}{\zeta_0^+}, \qquad \text{for} \quad \zeta \in (1, \zeta_0^+]. \end{aligned} \tag{*}$$

It is obvious that there exists a constant $b > 1$ such that (*) holds if $\zeta_0^+ \leqslant b$. Furthermore, it can be claimed, by a lengthy but straightforward calculation, that for any fixed $\gamma \in (1, 3)$, there exists a constant $b(\gamma)(b(\gamma) > 1)$ such that the condition A is satisfied if $\zeta_0^+ \leqslant b(\gamma)$.

To prove Theorem 2.1, we need the following lemmas.

LEMMA 2.3. *Suppose that the classical solution of* (1.1), (2.19), (2.20) *exists in* $R(T)$, *then it holds along* $x = x_1(t)$ *that*

$$\begin{aligned} &\left(\sqrt{\gamma}\, v^{-(1+\gamma)/2} + \sqrt{\frac{v^{-\gamma}-v_-^{-\gamma}}{v_- - v}}\right) \\ &\quad \cdot \left\{1 + \frac{[\gamma v^{-1-\gamma}(v_- - v) + (v^{-\gamma}-v_-^{-\gamma})]}{2\sqrt{\gamma(v_- - v)(v^{-\gamma}-v_-^{-\gamma})}}\, v^{(1+\gamma)/2}\right\} s_x \\ &= 2\sqrt{(v_- - v)(v^{-\gamma}-v_-^{-\gamma})} - \left[\sqrt{\gamma}\cdot v^{-(1+\gamma)/2} - \sqrt{\frac{v^{-\gamma}-v_-^{-\gamma}}{v_- - v}}\right] \\ &\quad \cdot \left\{-1 + \frac{[\gamma v^{-1-\gamma}(v_- - v) + (v^{-\gamma}-v_-^{-\gamma})]}{2\sqrt{\gamma(v_- - v)(v^{-\gamma}-v_-^{-\gamma})}} \cdot v^{(1+\gamma)/2}\right\} r_x \end{aligned} \tag{2.38}$$

*and along* $x = x_2(t)$ *that*

$$\begin{aligned} &\left(\sqrt{\gamma}\, v^{-(1+\gamma)/2} + \sqrt{\frac{v^{-\gamma}-v_+^{-\gamma}}{v_+ - v}}\right) \\ &\quad \cdot \left\{1 + \frac{[\gamma v^{-1-\gamma}(v_+ - v) + (v^{-\gamma}-v_+^{-\gamma})]}{2\sqrt{\gamma(v_+ - v)(v^{-\gamma}-v_+^{-\gamma})}}\, v^{(1+\gamma)/2}\right\} r_x \\ &= 2\sqrt{(v_+ - v)(v^{-\gamma}-v_+^{-\gamma})} - \left[\sqrt{\gamma}\cdot v^{-(1+\gamma)/2} - \sqrt{\frac{v^{-\gamma}-v_+^{-\gamma}}{v_+ - v}}\right] \\ &\quad \cdot \left\{-1 + \frac{[\gamma v^{-1-\gamma}(v_+ - v) + (v^{-\gamma}-v_+^{-\gamma})]}{2\sqrt{\gamma(v_+ - v)(v^{-\gamma}-v_+^{-\gamma})}} \cdot v^{(1+\gamma)/2}\right\} s_x \end{aligned} \tag{2.39}$$

*Proof.* Differentiate $(2.19)_2$ with respect to $t$, we obtain (2.38) with the help of (2.19), (2.5), and the knowledge about $(u_-(t, x_1(t)), v_-(t, x_1(t)))$. Similarly, we obtain (2.39).

Let

$$f = \gamma^{1/4} \frac{(\gamma+1)}{4} v^{(\gamma-3)/4}$$

$$g = \frac{4\gamma^{1/4}}{(3-\gamma)} \cdot v^{(3-\gamma)/4} = cf^{-1} \left(c = \frac{\gamma+1}{3-\gamma}\right)$$

$$w = \mu^{1/2} r_x + \frac{1}{2} g$$

$$y = \mu^{1/2} s_x + \frac{1}{2} g.$$

We denote the following statement by $(H_1)$ and $(H_2)$ respectively.

$(H_1)$ $(r, s) \in C^1$ on $x = x_i(t)$, $x_i(t) \in C^2$ $(i = 1, 2)$, $(t \geqslant 0)$.

$(H_2)$ Along $x = x_1(t)(t \geqslant 0)$, it holds that

$$y_0 = y(\alpha, x_1(\alpha)) > \sqrt{\frac{\sigma}{2} g(v(\alpha, x_1(\alpha)))\, g(\max\{v^-, v^+\})}\ (\alpha \geqslant 0)$$

$$\frac{2}{3-\gamma} v_0 \leqslant s_x + \frac{2v}{3-\gamma} \leqslant \frac{2}{3-\gamma} \max\{v^-, v^+\}$$

$[(-1/(v_- - v))] \cdot (d(v_- - v)/dt) \in [A_4, A_3]$, for positive constants $A_3 > A_4 > 0$.

Along $x = x_2(t)(t \geqslant 0)$, it holds that

$$w_0 = w(\beta, x_2(\beta)) > \sqrt{\frac{\sigma}{2} g(v(\beta, x_2(\beta))) \cdot g(\max\{v^-, v^+\})}\ (\beta \geqslant 0)$$

$$\frac{2}{3-\gamma} v_0 \leqslant r_x + \frac{2v}{3-\gamma} \leqslant \frac{2}{3-\gamma} \max\{v^-, v^+\}$$

$[(-1/(v_+ - v))] \cdot (d(v_+ - v)/dt) \in [A_2, A_1]$, for positive constants $A_1 > A_2 > 0$.

In $R$, it holds that $v_0 \leqslant v \leqslant \max\{v^-, v^+\}$

*Remark* 2.4. $(H_2)$ implies the entropy condition

$$(s-r)(\alpha, x_1(\alpha)) > (s_- - r_-)(\alpha, x_1(\alpha))$$

$$(s-r)(\beta, x_2(\beta)) > (s_+ - r_+)(\beta, x_2(\beta)).$$

LEMMA 2.5. *Under the condition $A_3$, the classical solution defined in $R(T)$ satisfies the estimates cited in* (2.24), (2.25), (2.26), (2.30), *and* (2.31) *if* $(H_1)$ *and* $(H_2)$ *hold for* $t \in [0, T]$.

*Proof.* The approach to obtain $(r, s)$-estimation, used in [HT] for the case of $\gamma = 1$, does not depend on $\gamma$, so it is still valid here. It is similar to that case one obtains the $v$-estimation as well, by following $s_x$- (and/or $r_x$-) estimate. The detail is omitted.

Now we give the $r_x$-estimation by upper-lower solution method. For any given $(t, x) \in R(T)$, it is not difficult to know that

$$\begin{aligned} w' &= -f\left(w - \frac{1}{2}g\right)\left(w - \frac{\gamma-1}{\gamma+1}g\right), \\ &\quad \text{along} \quad x = x_1(\tau; t, x), \beta \leqslant \tau \leqslant t; \\ y' &= -f\left(y - \frac{1}{2}g\right)\left(y - \frac{\gamma-1}{\gamma+1}g\right), \\ &\quad \text{along} \quad x = x_2(\tau; t, x), \alpha \leqslant \tau \leqslant t, \end{aligned} \tag{2.40}$$

where $x_1(\beta; t, x) = x_2(\beta)$, $x_2(\alpha; t, x) = x_1(\alpha)$.

With the help of $v$-bounds in $(\mathrm{H}_2)$, condition $A_3$ means that

$$\inf_{R(T)} g > \sqrt{2\sigma} \sup_{R(T)} g. \tag{2.41}$$

Denote

$$F^* = \max_{\substack{x_1(\tau; t, x) \\ \beta \leqslant \tau \leqslant t}} f, \qquad F_* = \min_{\substack{x_1(\tau; t, x) \\ \beta \leqslant \tau \leqslant t}} f;$$

let $z$ be the solution of

$$z' = -F^*\left(z - \frac{c}{2F^*}\right)\left(z - \frac{\sigma c}{F^*}\right), \qquad \beta \leqslant \tau \leqslant t,$$

with initial data $z = w_0$ at $\tau = \beta$. Due to $(\mathrm{H}_2)$,

$$w_0 > \sqrt{\frac{\sigma}{2} g(v(\beta, x_2(\beta))) \cdot g(\max\{v^-, v^+\})} \geqslant \sqrt{\frac{\sigma c^2}{2F^* F_*}}. \tag{2.42}$$

This, with (2.41) together, implies that

$$\min\left\{w_0, \frac{c}{2F^*}\right\} \leqslant z \leqslant \max\left\{w_0, \frac{c}{2F^*}\right\}. \tag{2.43}$$

Since $(2.40)_1$ can be written as

$$w' = -f\left(w - \frac{c}{2f}\right)\left(w - \frac{\sigma c}{f}\right),$$

it turns out that

$$w \leqslant \max\{w_0, \tfrac{1}{2} \max_{\substack{x_1(\tau;\, t,\, x) \\ \beta \leqslant \tau \leqslant t}} g\} \tag{2.44}$$

in view of (2.41) and (2.42).

Denote $u = z - w$. It holds then that

$$\begin{aligned} \tilde{u}' &= -f\tilde{u}(z+w) + \left(\frac{1}{2}+\sigma\right) c\tilde{u} + (f - F^*)\left(z^2 - \frac{\sigma c^2}{2F^* f}\right) \\ \tilde{u}_0 &= \tilde{u}(\alpha, x_1(\alpha)) = 0. \end{aligned} \tag{2.45}$$

With the fact $\sigma \in (0, \frac{1}{2})$ and (2.43), it is easy to show that

$$z^2 - \frac{\sigma c^2}{2F^* f} > 0$$

and hence $\tilde{u} \leqslant 0$ as the solution of (2.45), namely

$$z \leqslant w. \tag{2.46}$$

The $r_x$-estimate (2.26) follows then from (2.44) and (2.46). It is similar to get (2.25).

By using Lemma 2.3, it is not difficult to claim the next Lemma.

LEMMA 2.6. *Suppose that the classical solution of* (1.1), (2.19), (2.20) *exists in* $R(T)$. *Then it holds that*

$$\begin{aligned} \frac{-1}{(v_- - v)} \cdot \frac{d(v_- - v)}{dt} &= \left\{\sqrt{\gamma}\,(\zeta - 1)\left[1 + \frac{\gamma(\zeta-1) + (1-\zeta^{-\gamma})}{2\sqrt{\gamma(\zeta-1)(1-\zeta^{-\gamma})}}\right]\right\}^{-1} \\ &\quad \cdot \left[\left(\sqrt{\gamma} - \sqrt{\frac{1-\zeta^{-\gamma}}{\zeta-1}}\right)\frac{r_x}{v} + \sqrt{(\zeta-1)(1-\zeta^{-\gamma})}\right] \end{aligned}$$

*along* $x = x_1(t)$, *where* $\zeta = v^-/v$; *and it holds that*

$$\begin{aligned} \frac{-1}{(v_+ - v)} \cdot \frac{d(v_+ - v)}{dt} &= \left\{\sqrt{\gamma}\,(\xi - 1)\left[1 + \frac{\gamma(\xi-1) + (1-\xi^{-\gamma})}{2\sqrt{\gamma(\xi-1)(1-\xi^{-\gamma})}}\right]\right\}^{-1} \\ &\quad \cdot \left[\left(\sqrt{\gamma} - \sqrt{\frac{1-\xi^{-\gamma}}{\xi-1}}\right)\frac{s_x}{v} + \sqrt{(\xi-1)(1-\xi^{-\gamma})}\right] \end{aligned}$$

*along* $x = x_2(t)$, *where* $\xi = v^+/v$.

LEMMA 2.7. *There exists a small constant* $\delta_0$ *such that the solution of* (1.1), (2.19), (2.20) *exists in* $R(\delta_0)$ *for which* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *hold under condition* $A_1$, $A_3$ *and* $A_4$.

*Proof.* The existence follows from the local existence as mentioned before ([LY]) and it is obvious for $(\mathrm{H}_1)$ to hold. Moreover, it can be shown, with the help of the continuity of the local solution and Lemma 2.3, that

$$\lim_{t\to 0}\left(s_x+\frac{2v}{3-\gamma}\right)(t, x_1(t))$$
$$=\frac{2v_0}{3-\gamma}+2v_0$$
$$\cdot\frac{K_1(\zeta_0^+)\sqrt{(\zeta_0^- -1)[1-(\zeta_0^-)^{-\gamma}]}-K_2(\zeta_0^-)\sqrt{(\zeta_0^+ -1)[1-(\zeta_0^+)^{-\gamma}]}}{K_1(\zeta_0^-)\,K_1(\zeta_0^+)-K_2(\zeta_0^-)\,K_2(\zeta_0^+)}. \tag{2.47}$$

Thus, condition $A_4$ guarantees that

$$\lim_{t\to 0}\left(s_x+\frac{2v}{3-\gamma}\right)(t, x_1(t))>\frac{2v_0}{3-\gamma}.$$

On the other hand, due to $(2.33)_2$ in condition $A_1$ (as $\xi\to\xi_0^-$), (2.47) leads to that

$$\lim_{t\to 0}\left(s_x+\frac{2v}{3-\gamma}\right)(t, x_1(t))<\frac{2}{3-\gamma}\max\{v^-, v^+\}$$

It is similar to obtain the formular on $r_x$, namely

$$\lim_{t\to 0}\left(r_x+\frac{2v}{3-\gamma}\right)(t, x_2(t))$$
$$=\frac{2v_0}{3-\gamma}+2v_0$$
$$\cdot\frac{K_1(\xi_0^-)\sqrt{(\zeta_0^+ -1)[1-(\zeta_0^+)^{-\gamma}]}-K_2(\zeta_0^+)\sqrt{(\zeta_0^- -1)[1-(\zeta_0^-)^{-\gamma}]}}{K_1(\zeta_0^+)\,K_1(\zeta_0^-)-K_2(\zeta_0^+)\,K_2(\zeta_0^-)} \tag{2.48}$$

Thus, it follows by using condition $A_4$ and $A_1((2.34)_2)$, that

$$\frac{2v_0}{3-\gamma}<\lim_{t\to 0}\left(r_x+\frac{2v}{3-\gamma}\right)(t, x_2(t))<\frac{2}{3-\gamma}\max\{v^-, v^+\}.$$

Since $v$ is quite close to $v_0$ for small $t$, the estimates on $s_x$ and $r_x$, obtained in the previous discussion in Lemma 2.5, certainly still hold for small $t$. Then, it follows that there exists a small constant $\delta_0$ such that $v_0\leqslant v\leqslant\max\{v^-, v^+\}$ in $R(\delta_0)$ by the same argument as mentioned in Lemma 2.5.

Due to (2.47), (2.48), and (2.37), it is not difficult to show that $r_x\geqslant 0$, $s_x\geqslant 0$ in $R(\delta_0)$ as $\delta_0$ small enough. In view of this fact, with the boundedness of $v$ and the boundedness of $r_x$ on $x=x_1(t)$ and $s_x$ on $x=x_2(t)$ together, the diminution of the discontinuities can be easily obtained from Lemma 2.6. At last, it can be shown, by using the formulae in Lemma 2.3 and the condition $A_3$, that $y(\alpha, x_1(\alpha))>\sqrt{(\sigma/2)\,g(v(\alpha, x_1(\alpha)))\cdot g(\max\{v^-, v^+\})}$ for $0\leqslant\alpha\leqslant\delta_0$ and $w(\beta, x_2(\beta))>\sqrt{(\sigma/2)\,g(v(\beta, x_2(\beta)))\cdot g(\max\{v^-, v^+\})}$ for $0\leqslant\beta\leqslant\delta_0$, if $\delta_0$ is small enough. The Lemma 2.7 follows now from the above discussion.

LEMMA 2.8. *The solution constructed by extension preserves* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *in each step under condition* $A_1-A_3$. *Namely, if* $(\mathrm{H}_1)$ *and* $(H_2)$ *hold in* $R(T_0)$ *where the classical solution of* (1.1), (2.19), (2.20) *is defined, then the solution defined in* $R(T_0+\delta)$ *still satisfies* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$, *provided* $\delta$ *small enough.*

*Proof.* Denote $T^*=T_0+\delta$. It is not difficult to see that $(\mathrm{H}_1)$ holds and there is a constant $0<\varepsilon_0<1$ such that $v\in[v_0(1-\varepsilon_0),\ (1+\varepsilon_0)\cdot\max\{v^-, v^+\}]$ in $R(T^*)$, since $\delta$ is small enough and $(\mathrm{H}_2)$ holds in $R(T_0)$. Because of the same reason, the strength of discontinuity along $x=x_i(t)$ $(i=1, 2)$ still diminishes exponentially for $t\in(T_0, T^*]$, though might be in a slightly changed rate. This implies that $v\in[v_0, v^-)$ on $x=x_1(t)$ and $v\in[v_0, v^+)$ on $x=x_2(t)$ for $t\in(T_0, T^*]$. The step of $t$ can be chosen so small that for any $(t_1, x_2(t_1))$ with $t_1\in(T_0, T^*]$, the corresponding $(t_2, x_1(t_2))$, determined by $x_2(t_1; t_2, x_1(t_2))=x_2(t_1)$, satisfies that

$$t_2\leqslant T_0,$$

$$y_0(t_2, x_1(t_2))\geqslant\sqrt{\frac{\sigma}{2}g(v(t_2, x_1(t_2)))\cdot g(\max_{R(T^*)} v)}$$

$$\min_{R(T^*)} g\geqslant\sqrt{2\sigma}\max_{R(T^*)} g.$$

It follows then, by the similar argument as used in Lemma 2.5, that

$$\min\left\{\left(s_x+\frac{2v}{3-\gamma}\right)(t_2, x_1(t_2)), \frac{2}{3-\gamma}\min_{\substack{x_2(\tau;\, t_1,\, x_2(t_1))\\ t_2\leqslant\tau\leqslant t_1}} v\right\}$$

$$\leqslant\left(s_x+\frac{2}{3-\gamma}v\right)(t_1, x_2(t_1))$$

$$\leqslant\max\left\{s_x+\frac{2}{3-\gamma}(t_2, x_1(t_2)), \frac{2}{3-\gamma}\max_{\substack{x_2(\tau;\, t_1,\, x_2(t_1))\\ t_2\leqslant\tau\leqslant t_1}} v\right\}. \tag{2.48}$$

It can be shown by (2.4) and (2.7) that

$$v'=\left(s_x+\frac{2v}{3-\gamma}\right)-\frac{2v}{3-\gamma}, \tag{2.49}$$

where $(s_x+2v/(3-\gamma))$ can be estimated similarly to (2.48), namely

$$\min\left\{\left(s_x+\frac{2}{3-\gamma}v\right)(\alpha, x_1(\alpha)), \frac{2}{3-\gamma}\min_{\substack{x_2(\tau;\, t,\, x)\\ \alpha\leqslant\tau\leqslant t}} v\right\}$$

$$\leqslant\left(s_x+\frac{2}{3-\gamma}v\right)(t, x)$$

$$\leqslant\max\left\{\left(s_x+\frac{2}{3-\gamma}v\right)(\alpha, x_1(\alpha)), \frac{2}{3-\gamma}\max_{\substack{x_2(\tau;\, t,\, x)\\ \alpha\leqslant\tau\leqslant t}} v\right\}. \tag{2.50}$$

here $(t, x)\in R(T^*)$, $t\leqslant T^*$, $\alpha\leqslant T_0$.

Then, it turns out that $\max_{R(T^*)} v\leqslant\max\{v^-, v^+\}$ and $\min_{R(T^*)} v\geqslant v_0$, due to (2.49), (2.50), and the fact that

$$\frac{2}{3-\gamma}v_0\leqslant\left(s_x+\frac{2}{3-\gamma}v\right)(t_2, x_1(t_2))\leqslant\frac{2}{3-\gamma}\max\{v^-, v^+\}.$$

Therefore, one obtains

$$\frac{2}{3-\gamma}v_0\leqslant\left(s_x+\frac{2}{3-\gamma}v\right)(t_1, x_2(t_1))\leqslant\frac{2}{3-\gamma}\max\{v^-, v^+\}. \tag{2.51}$$

Lemma 2.3 reads that

$$r_x+\frac{2}{3-\gamma}v=\frac{2v}{3-\gamma}K\left(\zeta, \frac{s_x+2v/(3-\gamma)}{2v/(3-\gamma)}\right)$$

at $(t_1, x_2(t_1))$, where $\zeta = v_+/v$. Thus, condition $A_1$ with (2.51) show that

$$\frac{2}{3-\gamma} v_0 \leqslant \left(r_x + \frac{2}{3-\gamma} v\right)(t_1, x_2(t_1)) \leqslant \frac{2}{3-\gamma} \max\{v^-, v^+\} \tag{2.52}$$

and

$$w_0 = w(t_1, x_2(t_1)) > \sqrt{\sigma \cdot g(v(t_1, x_2(t_1))) \cdot g(\max\{v^-, v^+\})/2} \tag{2.53}$$

Similarly, we get the $s_x$-estimation along $x = x_1(t)$ for $t \in (T_0, T^*]$. Therefore, all of the estimates in Lemma 2.5 hold in $R(T^*)$ under the condition $A_3$. We will conclude this lemma with a uniform estimation about the discontinuity diminishness next.

Let $\zeta = v_+/v$ along $x = x_2(t)$. Lemma 2.6, condition $A_2$ and the bounds of $v$ lead to that

$$\begin{aligned}
&\left(\sqrt{\gamma} - \sqrt{\frac{1-\zeta^{-\gamma}}{\zeta - 1}}\right)\frac{s_x}{v} + \sqrt{(\zeta-1)(1-\zeta^{-\gamma})} \\
&\quad > \sqrt{(\zeta-1)(1-\zeta^{-\gamma})} \cdot \left\{1 - \frac{\sqrt{\gamma} - \sqrt{(1-\zeta^{-\gamma})/(\zeta-1)}}{\sqrt{(\zeta-1)(1-\zeta^{-\gamma})}}\right. \\
&\quad \left. \cdot \frac{2}{(3-\gamma)}\left[1 - \frac{\zeta}{\max\{\zeta_0^-, \zeta_0^+\}}\right]\right\}
\end{aligned}$$

and henceforth there exists a positive constant $\bar{A}_2$ such that

$$\frac{-1}{(v_+ - v)} \frac{d(v_+ - v)}{dt} > \bar{A}_2.$$

The upper bound for the diminish is obvious and the discussion for $x = x_1(t)$ is the same as above.

The main theorem 2.1 follows then from all of the lemmas.

## 3. Perturbated Riemann Problem

We consider certain perturbated initial data (1.2) in this section, namely

$$(u(0, x), v(0, x)) = \begin{cases} (u_-(x), v_-(x)), & x < 0 \\ (u_+(x), v_+(x)), & x > 0 \end{cases}$$

or in $(r, s)$-variables (3.1)

$$(r(0, x), s(0, x)) = \begin{cases} (r_-(x), s_-(x)), & x < 0 \\ (r_+(x), s_+(x)), & x > 0 \end{cases}$$

The functions $u_{\mp}(x)$, $v_{\mp}(x)$, $r_{\mp}(x)$ and $s_{\mp}(x)$ are smooth with

$$\lim_{x \to 0^{\mp}} (u_{\mp}(x), v_{\mp}(x)) = (u^{\mp}, v^{\mp})$$

or

$$\lim_{x \to 0^{\mp}} (r_{\mp}(x), s_{\mp}(x)) = (r^{\mp}, s^{\mp})$$

where $(r^-, s^-)$ and $(r^+, s^+)$ satisfy the condition (2.16). Without loss of generality, we assume that $0 < v_{\mp}(x) < 1$, $u_{\mp}(x) > 0$.

We will construct the global discontinuous solution for (1.1) (3.1) under the following restriction on the perturbation in (3.1), expressed as condition B which plays the same role as condition A for Riemann initial data in Section 2.

Denote

$$\underline{Q^-} = \min\left\{ \inf_{x \leqslant 0}\left(r'_-(x) + \frac{2v_-(x)}{3-\gamma}\right), \inf_{x \leqslant 0}\left(s'_-(x) + \frac{2v_-(x)}{3-\gamma}\right), \inf_{x \leqslant 0} \frac{2}{3-\gamma} v_-(x)\right\}$$

$$\overline{Q^-} = \max\left\{ \sup_{x \leqslant 0}\left(r'_-(x) + \frac{2v_-(x)}{3-\gamma}\right), \sup_{x \leqslant 0}\left(s'_-(x) + \frac{2v_-(x)}{3-\gamma}\right), \sup_{x \leqslant 0} \frac{2}{3-\gamma} v_-(x)\right\}$$

$$\underline{Q^+} = \min\left\{ \inf_{x \geqslant 0}\left(r'_+(x) + \frac{2v_+(x)}{3-\gamma}\right), \inf_{x \geqslant 0}\left(s'_+(x) + \frac{2v_+(x)}{3-\gamma}\right), \inf_{x \geqslant 0} \frac{2}{3-\gamma} v_+(x)\right\}$$

$$\overline{Q^+} = \max\left\{ \sup_{x \geqslant 0}\left(r'_+(x) + \frac{2v_+(x)}{3-\gamma}\right), \sup_{x \geqslant 0}\left(s'_+(x) + \frac{2v_+(x)}{3-\gamma}\right), \sup_{x \geqslant 0} \frac{2}{3-\gamma} v_+(x)\right\}.$$

*Condition* B. For some positive constant $\varepsilon$, it holds that

1.

$$\frac{2v_0}{3-\gamma} < (1-\varepsilon)\frac{2}{3-\gamma} v^{\pm} \leqslant \underline{Q^{\pm}} \leqslant \overline{Q^{\pm}} \leqslant (1+\varepsilon)\frac{2}{3-\gamma} v^{+}$$

Denote $\zeta_0^-$ and $\zeta_0^+$ by $\zeta_0^- = v^-/v_0$ and $\zeta_0^+ = v^+/v_0$ respectively, where $\zeta^+ \geqslant \zeta^- > 1$. Denote $\omega = (1+\varepsilon)/(1-\varepsilon)$

2. For any $\zeta \in (1, \zeta_0^- \omega]$,

$$\bar{K}\left(\zeta, \frac{\zeta_0^+}{\zeta_0^-}\omega, \frac{1}{\omega}\right) - \max\left\{\sqrt{2\sigma}\left(\frac{\zeta\zeta_0^+\omega}{\zeta_0^-}\right)^{(3-\gamma)/8}, \frac{\zeta}{\zeta_0^-}\cdot\frac{1}{1-\varepsilon}\right\} > 0$$

$$\bar{K}\left(\zeta, \frac{1}{\zeta_0^-(1+\varepsilon)}, \omega\right) - \frac{\zeta\zeta_0^+\omega}{\zeta_0^-} < 0.$$

3. For any $\zeta \in (1, \zeta_0^+ \omega]$,

$$\bar{K}\left(\zeta, \omega, \frac{1}{\omega}\right) - \max\left\{\sqrt{2\sigma}(\zeta \cdot \omega)^{(3-\gamma)/8}, \frac{\zeta}{\zeta_0^+}\frac{1}{1-\varepsilon}\right\}$$

$$\bar{K}\left(\zeta, \frac{1}{\zeta_0^+(1+\varepsilon)}, \omega\right) - \zeta\omega < 0.$$

4. For any $\zeta \in (1, \zeta_0^+ \omega]$,

$$\frac{(3-\gamma)}{2\zeta}\sqrt{(\zeta-1)(1-\zeta^{-\gamma})} + \left(\sqrt{\gamma} - \sqrt{\frac{1-\zeta^{-\gamma}}{\zeta-1}}\right)\left(\frac{1-\varepsilon}{\zeta_0^+} - \frac{1}{\zeta}\right)$$

$$- \frac{\varepsilon}{1-\varepsilon}\left\{\left(\sqrt{\gamma}\zeta^{-(1+\gamma)/2} + \sqrt{\frac{1-\zeta^{-\gamma}}{\zeta-1}}\right)\right.$$

$$\left.\cdot\left[(\zeta^{-(1+\gamma)/2} - 1) + \frac{\sqrt{\gamma(\zeta-1)(1-\zeta^{-\gamma})}}{2}\right]\right\}$$

$$- \frac{\varepsilon}{1+\varepsilon}\left(-\sqrt{\gamma}\zeta^{-(1+\gamma)/2} + \sqrt{\frac{1-\zeta^{-\gamma}}{\zeta-1}}\right)$$

$$\cdot\left[(\zeta^{-(1+\gamma)/2} - 1) - \frac{\sqrt{\gamma(\zeta-1)(1-\zeta^{-\gamma})}}{2}\right] > 0.$$

5. $\zeta_0^+(1+\varepsilon) < (2\sigma)^{2/(3-\gamma)}$

6.

$$K_5(\zeta_0^+, \zeta_0^-) + K_6(\zeta_0^+, \zeta_0^-) \leqslant \zeta_0^+ [K_1(\zeta_0^+) K_1(\zeta_0^-) - K_2(\zeta_0^+) K_2(\zeta_0^-)]$$

$$K_5(\zeta_0^+, \zeta_0^-) - K_6(\zeta_0^+, \zeta_0^-) \geqslant K_1(\zeta_0^+) K_1(\zeta_0^-) - K_2(\zeta_0^+) K_2(\zeta_0^-)$$

$$K_5(\zeta_0^-, \zeta_0^+) + \varepsilon K_6(\zeta_0^-, \zeta_0^+) \leqslant \zeta_0^+ \cdot [K_1(\zeta_0^+) K_1(\zeta_0^-) - K_2(\zeta_0^+) K_2(\zeta_0^-)]$$

$$K_5(\zeta_0^-, \zeta_0^+) - \varepsilon K_6(\zeta_0^-, \zeta_0^+) \geqslant K_1(\zeta_0^+) K_1(\zeta_0^-) - K_2(\zeta_0^+) K_2(\zeta_0^-),$$

where

$$K_3(\zeta) = \left(\sqrt{\gamma}\zeta^{-(\gamma+1)/2} + \sqrt{\frac{1-\zeta^{-\gamma}}{\zeta-1}}\right)$$

$$\cdot\left[1 + \frac{\gamma(\zeta-1)\zeta^{-1-\gamma} + (1-\zeta^{-\gamma})}{2\sqrt{\gamma(\zeta-1)(1-\zeta^{-\gamma})}} \cdot \zeta^{(1+\gamma)/2}\right]$$

$$K_4(\zeta) = \left(-\sqrt{\gamma}\zeta^{-(\gamma+1)/2} + \sqrt{\frac{1-\zeta^{-\gamma}}{\zeta-1}}\right)$$

$$\cdot\left[-1+\frac{\gamma\zeta^{-1-\gamma}(\zeta-1)+(1-\zeta^{-\gamma})}{2\sqrt{\gamma(\zeta-1)(1-\zeta^{-\gamma})}}\cdot\zeta^{(1+\gamma)/2}\right]$$

$$\bar{K}(\zeta,\mu,\omega)=[K_1(\zeta)]^{-1}[-K_2(\zeta)\mu+K_3(\zeta)\omega-K_4(\zeta)/\omega]\cdot\zeta$$

$$K_5(\zeta_1,\zeta_2)=\zeta_1K_1(\zeta_2)[K_3(\zeta_1)-K_4(\zeta_1)]$$
$$-\zeta_2K_2(\zeta_1)[K_3(\zeta_2)-K_4(\zeta_2)]$$

$$K_6(\zeta_1,\zeta_2)=\zeta_1K_1(\zeta_2)[K_3(\zeta_1)+K_4(\zeta_1)]-K_2(\zeta_1)[K_3(\zeta_2)-K_4(\zeta_2)].$$

*Remark* 3.1. When $\varepsilon$ tends to zero, the condition $B_1$ holds automatically if $r'_{\mp}(x)\equiv 0$ and $s'_{\mp}(x)\equiv 0$ and the others become the corresponding items in condition A. This means that the condition B is a direct generalization of condition A. Moreover, it can be claimed, by a careful calculation, that for any fixed number $\hat{b}$,

$$0<\hat{b}<b(\gamma)$$

where $b(\gamma)$ is defined in Remark 2.2, there exists a positive constant $\varepsilon(\hat{b})$ such that all of inequalities in condition B hold for this $\varepsilon$ if

$$\zeta_0^+\leqslant\hat{b}$$

and

$$\operatorname{osc} v_{\mp}(x)+|r'_{\mp}(x)|+|s'_{\mp}(x)|<\varepsilon_0 \qquad (\text{for } x\leqslant 0 \text{ or } x\geqslant 0 \text{ respectively}),$$

where $\varepsilon_0$ depends on $\varepsilon(\hat{b})$.

Similar to Section 2, by using the initial data on $x\geqslant 0$ and $x\leqslant 0$ we solve the corresponding initial value problem for (1.1). Taking the similar discussion as for the case $\gamma=1$ (see [HT]), one obtains the estimates cited in the following theorem except the $r_{x^-}$ (and/or $s_{x^-}$) estimates which are given by condition $B_5$ through the same procedure as in proving (2.25) and (2.26) in Lemma 2.5.

THEOREM 3.2. *Under condition $B_5$, there exists a unique classical solution $(r_-(t,x), s_-(t,x))\in C^1$ and a unique classical solution $(r_+(t,x), s_+(t,x))\in C^1$ in the region $\hat{R}^-$ and $\hat{R}^+$ respectively such that in $\hat{R}^-$: $s_*^-\leqslant s\leqslant\sup_{x\leqslant 0}s_-(x)$, $r_*^-\leqslant r$,*

$$\frac{2v}{3-\gamma}\in[\underline{Q^-},\overline{Q^-}],\qquad r_x+\frac{2v}{3-\gamma}\in[\underline{Q^-},\overline{Q^-}],\qquad s_x+\frac{2v}{3-\gamma}\in[\underline{Q^-},\overline{Q^-}].$$

*In $\hat{R}_+$: $s_*^+\leqslant s\leqslant\sup_{x\geqslant 0}s_+(x)$, $r_*^+\leqslant r$,*

$$\frac{2v}{3-\gamma}\in[\underline{Q^+},\overline{Q^+}],\qquad r_x+\frac{2v}{3-\gamma}\in[\underline{Q^+},\overline{Q^+}],\qquad s_x+\frac{2v}{3-\gamma}\in[\underline{Q^+},\overline{Q^+}].$$

*Where the subscript with the solution* $(r, s)$ *is omitted,* $(s_*^-, r_*^-)$ *and* $(s_*^+, r_*^+)$ *are defined by*

$$s_*^- = \frac{2\sqrt{\gamma}}{\gamma-1}(\overline{Q^-})^{(1-\gamma)/2}, \qquad r_*^- = -\frac{2\sqrt{\gamma}}{\gamma-1}(\underline{Q^-})^{(1-\gamma)/2},$$

$$s_*^+ = \frac{2\sqrt{\gamma}}{\gamma-1}(\overline{Q^+})^{(1-\gamma)/2}, \qquad r_*^+ = -\frac{2\sqrt{\gamma}}{\gamma-1}(\underline{Q^+})^{(1-\gamma)/2},$$

It is the same as in Section 2 that the discontinuous initial value problem (1.1), (3.1) admits a unique discontinuous solution at least on a local domain $R(\delta)$ in a class of piecewise smooth functions and this solution contains only a backward shock $x = x_1(t)$ and a forward shock $x = x_2(t)$ passing through the origin. Moreover, the solution on the left side of $x = x_1(t)$ and on the right side of $x = x_2(t)$ is furnished by $(r_-(t, x), s_-(t, x))$ and $(r_+(t, x), s_+(t, x))$ respectively, and one is required to solve the same free boundary problem (FBP) on the angular domain $R$ as in Section 2 in order to construct the globally defined discontinuous solution for (1.1), (3.1) which contains only two shocks.

LEMMA 3.3. *Suppose that the classical solution of* (1.1), (2.19), (2.20) *exists in* $R(T)$, *then along* $x = x_1(t)$ *it holds that*

$$K_1(\zeta)\left(s_x + \frac{2v}{3-\gamma}\right)$$
$$= -K_2(\zeta)\cdot\left(r_x + \frac{2v}{3-\gamma}\right) + K_3(\zeta)\left[(s_-)_x + \frac{2v_-}{3-\gamma}\right]$$
$$-K_4(\zeta)\cdot\left[(r_-)_x + \frac{2v_-}{3-\gamma}\right],$$

$$\frac{-2\sqrt{\gamma}[1 + (\gamma v^{-1-\gamma}\cdot\bar{A} + \bar{B})/(2\sqrt{\gamma\bar{A}\bar{B}})\cdot v^{-(1+\gamma)/2}]}{(v_- - v)}\,\frac{d(v_- - v)}{dt}$$
$$= 2v^{(1+\gamma)/2}\cdot\sqrt{\frac{\bar{B}}{\bar{A}}} + 2\cdot\frac{(\sqrt{\gamma}\cdot v^{-(1+\gamma)/2} - \sqrt{\bar{B}/\bar{A}})}{\bar{A}}\,r_x$$
$$-\frac{[\sqrt{\gamma}(v_-)^{-(1+\gamma)/2} + \sqrt{\bar{B}/\bar{A}}][(v_-)^{(1+\gamma)/2} - v^{(1+\gamma)/2} + \sqrt{\gamma\bar{A}\bar{B}}/2]}{\bar{A}}\cdot(s_-)_x$$
$$+\frac{[-\sqrt{\gamma}(v_-)^{-(1+\gamma)/2} + \sqrt{\bar{B}/\bar{A}}][(v_-)^{(1+\gamma)/2} + v^{(1+\gamma)/2} - \sqrt{\gamma\bar{A}\bar{B}}/2]}{\bar{A}}(r_-)_x,$$

*where* $\zeta = v_-/v$, $\bar{A} = v_- - v$, $\bar{B} = v^{-\gamma} - (v_-)^{-\gamma}$, $v_- = v_-(t, x_1(t))$, $r_- = r_-(t, x_1(t))$, $s_- = s_-(t, x_1(t))$, *etc.; and along* $x = x_2(t)$ *it holds that*

$$K_1(\xi)\left(r_x + \frac{2v}{3-\gamma}\right)$$

$$= -K_2(\xi)\cdot\left(s_x + \frac{2v}{3-\gamma}\right) + K_3(\xi)\left[(r_+)_x + \frac{2v_+}{3-\gamma}\right] - K_4(\xi)\cdot\left[(s_+)_x + \frac{2v_+}{3-\gamma}\right],$$

$$\frac{-2\sqrt{\gamma}[1 + (\gamma v^{-1-\gamma}\cdot A + B)/(2\sqrt{\gamma AB})\cdot v^{-(1+\gamma)/2}}{(v_+ - v)}\,\frac{d(v_+ - v)}{dt}$$

$$= 2v^{(1+\gamma)/2}\cdot\sqrt{\frac{B}{A}} + 2\cdot\frac{(\sqrt{\gamma}\cdot v^{-(1+\gamma)/2} - \sqrt{B/A})}{A}\,s_x$$

$$-\frac{[\sqrt{\gamma}(v_+)^{-(1+\gamma)/2} + \sqrt{B/A}][(v_+)^{(1+\gamma)/2} - v^{(1+\gamma)/2} + \sqrt{\gamma AB}/2]}{A}\cdot(r_+)_x$$

$$+\frac{[-\sqrt{\gamma}(v_+)^{-(1+\gamma)/2} + \sqrt{B/A}][(v_+)^{(1+\gamma)/2} + v^{(1+\gamma)/2} - \sqrt{\gamma AB}/2]}{A}(s_+)_x,$$

*where* $\xi = v_+/v$, $A = v_+ - v$, $B = v^{-\gamma} - (v_+)^{-\gamma}$, $v_+ = v_+(t, x_2(t))$, $r_+ = r_+(t, x_2(t))$, $s_+ = s_+(t, x_2(t))$, *etc.*

THEOREM 3.4. *Under condition B, the free boundary problem* (1.1), (2.19), (2.20) *admits a global classical solution* $(r, s) \in C^1$ *in* $R$ *with* $x_i(t) \in C^2$ $(i = 1, 2)$ *on which* (2.21), (2.22) *hold respectively. Furthermore, this solution possesses the following properties. For any* $(t, x) \in R$,

$$\min\{s_*^+, s_*^-\} \leqslant s \leqslant \sup_{x \leqslant 0} s_-(x), \quad r_* \leqslant r \leqslant r^*, \quad v_0 \leqslant v \leqslant \frac{3-\gamma}{2}\max\{\overline{Q^+}, \overline{Q^-}\}, \tag{3.2}$$

$$\min\left\{\min_{\substack{x_2(\tau; t, x)\\ \alpha \leqslant \tau \leqslant t}} \frac{2v}{3-\gamma}, \left(s_x + \frac{2v}{3-\gamma}\right)(\alpha, x_1(\alpha))\right\}$$

$$\leqslant s_x + \frac{2v}{3-\gamma} \tag{3.3}$$

$$\leqslant \max\left\{\max_{\substack{x_2(\tau; t, x)\\ \alpha \leqslant \tau \leqslant t}} \frac{2v}{3-\gamma}, \left(s_x + \frac{2v}{3-\gamma}\right)(\alpha, x_1(\alpha))\right\}$$

$$\min\left\{\min_{\substack{x_1(\tau;t,x)\\ \beta\le\tau\le t}}\frac{2v}{3-\gamma},\left(r_x+\frac{2v}{3-\gamma}\right)(\beta,x_2(\beta))\right\}$$

$$\le r_x+\frac{2v}{3-\gamma} \tag{3.4}$$

$$\max\left\{\max_{\substack{x_1(\tau;t,x)\\ \beta\le\tau\le t}}\frac{2v}{3-\gamma},\left(r_x+\frac{2v}{3-\gamma}\right)(\beta,x_1(\beta))\right\},$$

*where the notation of* $x_i(\tau;t,x)$, $(\beta,x_2(\beta))$ *and* $(\alpha,x_1(\alpha))$ *is defined in the same way as in Theorem* 2.1, $r_*=-(2\sqrt{\gamma}/(\gamma-1))\,v_0^{(1-\gamma)/2}=-(s_0-r_0)/2$, $r^*$ *is determined by* $s^*-r^*=(4\sqrt{\gamma}/(\gamma-1))\,v_0^{(1-\gamma)/2}$ *and* $h_2(s_R,r_R;s^*,r^*)=0$ *with* $s_R=\sup_{x\ge0}s_+(x)$, $r_R=\sup_{x\ge0}s_+(x)-(4\sqrt{\gamma}/(\gamma-1))((3-\gamma)/2)\,\overline{Q^+})^{(1-\gamma)/2}$.

*Moreover, for any point* $(t,x_1(t))(t\ge0)$, *it holds that*

$$(v^--v_0)\,e^{-A_3t}\le v_--v\le(v^--v_0)\,e^{-A_4t} \tag{3.5}$$

*and for any point* $(t,x_2(t))(t\ge0)$, *it holds that*

$$(v^+-v_0)\,e^{-A_1t}\le v_+-v\le(v^+-v_0)\,e^{-A_2t} \tag{3.6}$$

*where* $v_-=v_-(t,x_1(t))$, $v_+=v_+(t,x_2(t))$, $A_i(i=1,2,3,4)$ *is a positive constant with* $A_3>A_4>0$, $A_1>A_2>0$.

Let us denote the following statement by $H_2'$

$(H_2')$ Along $x=x_1(t)$, it holds that

$$\frac{-1}{(v_--v)}\frac{d(v_--v)}{dt}\in[A_4,A_3]$$

$$y_0=y(\alpha,x_1(\alpha))>\sqrt{\frac{\sigma}{2}g(v(\alpha,x_1(\alpha)))\cdot g\left(\frac{3-\gamma}{2}\max\{\overline{Q^-},\overline{Q^+}\}\right)}$$

$$\frac{2v_0}{3-\gamma}\le s_x+\frac{2v}{3-\gamma}\le\max\{\overline{Q^+},\overline{Q^-}\}$$

and along $x=x_2(t)$ it holds that

$$\frac{-1}{(v_+-v)}\frac{d(v_+-v)}{dt}\in[A_2,A_1]$$

$$w_0=w(\beta,x_2(\beta))>\sqrt{\frac{\sigma}{2}g(v(\beta,x_2(\beta)))\cdot g\left(\frac{3-\gamma}{2}\max\{\overline{Q^-},\overline{Q^+}\}\right)}$$

$$\frac{2v_0}{3-\gamma}\le r_x+\frac{2v}{3-\gamma}\le\max\{\overline{Q^+},\overline{Q^-}\}.$$

Moreover, it holds in $R$ that $v_0 \leqslant v \leqslant (3-\gamma)/2 \cdot \max\{\overline{Q^+}, \overline{Q^-}\}$. We prove the Theorem 3.4 next by the same framework as used in Section 2.

First, it can be shown by the same argument as in Lemma 2.5 (with the only change of $(3-\gamma)/2 \cdot \overline{Q^-}$ and $(3-\gamma)/2 \cdot \overline{Q^+}$ in place of $v^-$ and $v^+$ respectively) that under condition $B_5$, the classical solution defined in $R(T)$ satisfies all of the estimates cited in (3.2)–(3.6) if $(H_1)$ and $(H_2')$ hold in $R(T)$.

It is also similar to Section 2 to prove that $(H_1)$ and $(H_2')$ hold locally in $t$. In fact, it can be found out that

$$\lim_{t\to 0}\left(r_x + \frac{2v}{3-\gamma}\right)(t, x_2(t))$$
$$= \frac{K_1(\zeta_0^-)[K_3(\zeta_0^+)\, R^+ - K_4(\zeta_0^+)\, S^+] - K_2(\zeta_0^+)[K_3(\zeta_0^-)\, S^- - K_4(\zeta_0^-)\, R^-]}{K_1(\zeta_0^+)\, K_1(\zeta_0^-) - K_2(\zeta_0^+)\, K_2(\zeta_0^-)}$$

where

$$R^\pm = r'_\pm(x) + \frac{2v^\pm}{3-\gamma}, \qquad S^\pm = s'_\pm(x) + \frac{2v^\pm}{3-\gamma}.$$

Due to condition $B_1$ and $B_6$, it follows from the above expression that

$$\frac{2v_0}{3-\gamma} \leqslant \lim_{t\to 0}\left(r_x + \frac{2v}{3-\gamma}\right)(t, x_2(t)) \leqslant \frac{2}{3-\gamma}\max\{v^+, v^-\}.$$

Similarly, it can be obtained that

$$\frac{2v_0}{3-\gamma} \leqslant \lim_{t\to 0}\left(s_x + \frac{2v}{3-\gamma}\right)(t, x_1(t)) \leqslant \frac{2}{3-\gamma}\max\{v^+, v^-\}.$$

The other in the discussion for $(H_1)$, $(H_2')$ being hold locally in $t$ is the same as in Section 2. At last, to show that the solution constructed by extension reserves $(H_1)$ and $(H_2')$ in each step under condition $B_2$, $B_3$, and $B_4$, we take the same frame as in Lemma 2.8.

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