Gap functions and existence of solutions to generalized vector variational inequalities

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Abstract

In this paper, the gap function for a new class of generalized vector variational inequalities with point-to-set mappings (for short, GVVI) is introduced and the necessary and sufficient conditions for the GVVI are established. In order to derive the existence of solutions for the GVVI, we also introduce the concept of $\eta$-$h$-$C(x)$-pseudomonotonicity. By considering the existence of solutions for vector variational inequalities (for short, VVI) with a single-valued function and a continuous selection theorem, we obtain the existence theorem for the GVVI under the assumption of $\eta$-$h$-$C(x)$-pseudomonotonicity. The results presented in this paper extend and unify corresponding results of other authors.

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1. Introduction

A vector variational inequality (for short, VVI) in a finite-dimensional Euclidean space was introduced first by Giannessi [1]. This is a generalization of a scalar variational inequality to the vector case by virtue of multi-criterion consideration. Throughout the last twenty years of development, existence theorems of solutions of VVI in various situations have been studied by many authors (see, for example [2–21],

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and the references therein). At the same time, VVI has found many of its applications in vector optimization [5,19], approximate vector optimization [6], vector equilibria [3,7,22] and vector traffic equilibria [20].

Gap functions play a crucial role in transforming a variational inequality problem into an optimization problem. Thus, powerful optimization solution methods and algorithms can be applied for finding solutions of variational inequalities. In [23], gap functions for several kinds of prevariational inequalities are investigated. Furthermore, a class of gap functions for inequality constrained prevariational inequalities is considered via a nonlinear Lagrangian. Yang and Yao [21] introduced the gap function and established necessary and sufficient conditions for the existence of a solution for the VVI. They investigated also the existence of solutions for the generalized VVI with a point-to-set mapping by virtue of the existence of a solution of the VVI with a single-valued function and a continuous selection theorem.

Inspired and motivated by above research works, in this paper, the gap function for a new class of generalized vector variational inequalities with point-to-set mappings (for short, GVVI) is introduced and the necessary and sufficient conditions for the GVVI are established. In order to derive the existence of solutions for the GVVI, we also introduce the concept of \( \eta \)-}\( h \)-\( C(x) \)-pseudomonotonicity. By considering the existence of solutions for vector variational inequalities (for short, VVI) with a single-valued function and a continuous selection theorem, we obtain the existence theorem for the GVVI under the assumption of \( \eta \)-}\( h \)-\( C(x) \)-pseudomonotonicity. The results presented in this paper extend and unify corresponding results of [14,21].

2. Gap functions to the GVVI

Let \( K \) be a nonempty compact subset of topological vector space \( X \) and \( Y \) another topological vector space. Let \( L(X, Y) \) be the space of all continuous linear mappings from \( X \) to \( Y \) and we will denote by \( \langle t, x \rangle \) the values of a linear operator \( t \in L(X, Y) \) at \( x \in X \). Furthermore, let \( C \subset Y \) be a pointed, closed and convex cone in \( Y \) with apex at the origin and \( \text{int } C \neq \emptyset \). Assume \( T : K \rightarrow 2^{L(X, Y)} \) is a point-to-set mapping with a compact set \( T(x) \) for each \( x \in K \); \( \eta : K \times K \rightarrow K \) and \( h : K \times K \rightarrow Y \) are two vector-valued functions satisfying \( \eta(x, x) = 0 \) and \( h(x, x) = 0 \) for each \( x \in K \), respectively. In this section, we consider the following three generalized vector variational inequality problems (for short, GVVI):

\[
\begin{align*}
\text{find} & \; x^* \in K \; \text{and} \; t^* \in T(x^*) \; \text{such that} \\
\langle t^*, \eta(y, x^*) \rangle + h(y, x^*) & \notin \text{int } C, \quad \forall y \in K; \quad (2.1) \\
\text{find} & \; x^* \in K \; \text{and} \; t^* \in T(x^*) \; \text{such that} \\
\langle t^*, \eta(y, x^*) \rangle + h(y, x^*) & \notin \text{int } C \setminus \{0\}, \quad \forall y \in K; \quad (2.2) \\
\text{find} & \; x^* \in K \; \text{such that,} \; \forall y \in K, \; \exists \; t^*(y) \in T(x^*) \; \text{such that} \\
\langle t^*(y), \eta(y, x^*) \rangle + h(y, x^*) & \notin \text{int } C, \quad \forall y \in K. \quad (2.3)
\end{align*}
\]

The following problems are special cases of GVVI.

(i) If \( \eta(x, y) = x - y \) and \( h(x, y) = 0 \) for any \( x, y \in K \), then (2.1), (2.2) and (2.3) reduce to (2.4), (2.5) and (2.6), respectively, which have been studied by Yang and Yao in [21].
Find \( x^* \in K \) and \( t^* \in T(x^*) \) such that
\[
\langle t^*, y - x^* \rangle \notin -\text{int } C, \quad \forall y \in K; \quad (2.4)
\]
find \( x^* \in K \) and \( t^* \in T(x^*) \) such that
\[
\langle t^*, y - x^* \rangle \notin -C \setminus \{0\}, \quad \forall y \in K; \quad (2.5)
\]
find \( x^* \in K \) such that, \( \forall y \in K, \exists t^*(y) \in T(x^*) \) such that
\[
\langle t^*(y), y - x^* \rangle \notin -\text{int } C, \quad \forall y \in K. \quad (2.6)
\]

(ii) If \( T : K \to L(X, Y) \) is a single-valued mapping, \( \eta(x, y) = x - y \) and \( h(x, y) = f(x) - f(y) \) for any \( x, y \in K \), where \( f : K \to Y \) is a vector-valued function, then (2.1) and (2.3) reduce to (2.7) and (2.2) reduces to (2.8).

Find \( x^* \in K \) such that
\[
\langle T(x^*), y - x^* \rangle + f(y) - f(x^*) \notin -\text{int } C, \quad \forall y \in K; \quad (2.7)
\]
find \( x^* \in K \) such that
\[
\langle T(x^*), y - x^* \rangle + f(y) - f(x^*) \notin -C \setminus \{0\}, \quad \forall y \in K. \quad (2.8)
\]

**Remark 2.1.** It is easy to see that any solution of the GVVI (2.2) is a solution of the GVVI (2.1), and any solution of the GVVI (2.1) is a solution of the GVVI (2.3). But the converse is not true in general.

In the rest of this section, let \( Y \) be an \( n \)-dimensional vector space \( R^n \), let
\[
R^n_+ = \{(x_1, x_2, \ldots, x_n) \in R^n | x_i \geq 0, i = 1, 2, \ldots, n\}
\]
be the non-negative orthant of \( R^n \) and let \( C = R^n_+ \). Next, we will introduce the concept of gap functions for the GVVI with point-to-set mappings.

**Definition 2.1.** Consider the GVVI. Let \( K_1 \) be the domain of the GVVI. A function \( \phi : K_1 \to R \) is said to be a gap function for the GVVI if it satisfies the following properties:

(i) \( \phi(x) \leq 0, \forall x \in K_1; \)

(ii) \( \phi(x^*) = 0 \), if and only if \( x^* \) solves the GVVI.

Let \( x, y \in K \) and \( t \in T(x) \). Denote
\[
\langle t, \eta(y, x) \rangle + h(y, x) = [(\langle t, \eta(y, x) \rangle + h(y, x))]_1, \ldots, [(\langle t, \eta(y, x) \rangle + h(y, x))]_n;
\]
i.e., \([\langle t, \eta(y, x) \rangle + h(y, x)]_i\) is the \( i \)th component of \( \langle t, \eta(y, x) \rangle + h(y, x) \), \( i = 1, 2, \ldots, n \). Now, we define two mappings \( \varphi_1 : K \times L(X, R^n) \to R \) and \( \varphi : K \to R \) as follows:
\[
\varphi_1(x, t) = \min_{y \in K} \max_{1 \leq i \leq n} \langle \langle t, \eta(y, x) \rangle + h(y, x) \rangle_i, \quad (2.9)
\]
\[
\varphi(x) = \max \{\varphi_1(x, t) | t \in T(x)\}. \quad (2.10)
\]
Since \( K \) and \( T(x) \) are compact, \( \varphi(x) \) is well defined. Again since \( \eta(x, x) = h(x, x) = 0 \) for any \( x \in K \), for \( x \in K \) and \( t \in T(x) \), it is easy to see that
\[
\varphi_1(x, t) = \min_{y \in K} \max_{1 \leq i \leq n} \langle \langle t, \eta(y, x) \rangle + h(y, x) \rangle_i \leq 0.
\]
**Theorem 2.1.** The function \( \varphi(x) \) defined by (2.10) is a gap function for the GVVI (2.1).

**Proof.** (i) Since
\[
\varphi_1(x, t) \leq 0, \quad \forall x \in K, t \in T(x),
\]
we have
\[
\varphi(x) = \max \{ \varphi_1(x, t) \mid t \in T(x) \} \leq 0, \quad \forall x \in K.
\]
(ii) If \( \varphi(x^*) = 0 \), then there exists \( t^* \in T(x^*) \) such that \( \varphi_1(x^*, t^*) = 0 \). Consequently, we obtain
\[
\min_y \max_{1 \leq i \leq n} \left( \langle t^*, \eta(y, x^*) \rangle + h(y, x^*) \right)_i = 0.
\]
It follows that, for any \( y \in K \),
\[
\max_{1 \leq i \leq n} \left( \langle t^*, \eta(y, x^*) \rangle + h(y, x^*) \right)_i \geq 0,
\]
which implies that, for any \( y \in K \),
\[
\langle t^*, \eta(y, x^*) \rangle + h(y, x^*) \notin -\text{int} \ R^n_+,
\]
i.e., \( x^* \) is a solution of the GVVI (2.1). Conversely, if \( x^* \) solves the GVVI (2.1), then there exists \( t^* \in T(x^*) \) such that
\[
\langle t^*, \eta(y, x^*) \rangle + h(y, x^*) \notin -\text{int} \ R^n_+, \quad \forall y \in K,
\]
from which it follows that for any \( y \in K \),
\[
\max_{1 \leq i \leq n} \left( \langle t^*, \eta(y, x^*) \rangle + h(y, x^*) \right)_i \geq 0.
\]
Thus, we have
\[
\varphi_1(x^*, t^*) = \min_y \max_{1 \leq i \leq n} \left( \langle t^*, \eta(y, x^*) \rangle + h(y, x^*) \right)_i \geq 0.
\]
(2.12)
Now, (2.11) and (2.12) imply that
\[
\varphi_1(x^*, t^*) = 0.
\]
Again from (2.11), we obtain
\[
\varphi_1(x^*, t) \leq 0, \quad \forall t \in T(x^*).
\]
Since
\[
\varphi(x^*) = \max \{ \varphi_1(x^*, t) \mid t \in T(x^*) \},
\]
it follows from \( \varphi_1(x^*, t^*) = 0 \) that \( \varphi(x^*) = 0 \). This completes the proof.

From Remark 2.1 and Theorem 2.1, it is easy to see the following result holds.

**Corollary 2.1.** If \( x^* \) is a solution of the GVVI (2.2), then \( \varphi(x^*) = 0 \).

Furthermore, Theorem 2.1 presented in [21] is a special case of Theorem 2.1.

**Corollary 2.2** ([21]). The function \( \varphi(x) \) defined by (2.10) is a gap function for the VVI (2.4).

Next, we will consider the gap function for the GVVI (2.3). First, for \( x \in K \), denote
\[
B_x = \{ t \mid t : K \to T(x) \},
\]
i.e., \( B_x \) is the set of all operators \( t \) from \( K \) to \( T(x) \). Let \( x \in K \) and \( t \in B_x \), then
\[
 t(y) \in T(x), \quad \forall y \in K.
\]
As functions \( \varphi_1 \) and \( \varphi \) defined by (2.9) and (2.10), respectively, we define two mappings \( \varphi_1^* : K \times L(X, R^n) \to R \) and \( \varphi^* : K \to R \) as follows:
\[
\varphi_1^*(x, t) = \min_{y \in K} \max_{1 \leq i \leq n} ((t(y), \eta(y, x)) + h(y, x))_i,
\]
\[
\varphi^*(x) = \max\{\varphi_1^*(x, t) \mid t \in B_x\}.
\]

(2.13)

**Theorem 2.2.** The function \( \varphi^*(x) \) defined by (2.13) is a gap function for the GVVI (2.3).

**Proof.** (i) It is clear that
\[
\varphi_1^*(x, t) = \min_{y \in K} \max_{1 \leq i \leq n} ((t(y), \eta(y, x)) + h(y, x))_i \leq 0, \quad \forall x \in K, t \in B_x,
\]
and hence
\[
\varphi^*(x) = \max\{\varphi_1^*(x, t) \mid t \in T(x)\} \leq 0, \quad \forall x \in K.
\]

(ii) If \( \varphi^*(x^*) = 0 \), then there exists \( t^* \in B_{x^*} \) such that \( \varphi_1^*(x^*, t^*) = 0 \). Consequently, we obtain
\[
\min_{y \in K} \max_{1 \leq i \leq n} ((t^*(y), \eta(y, x^*)) + h(y, x^*))_i = 0.
\]
It follows that, for any \( y \in K \),
\[
\max_{1 \leq i \leq n} ((t^*(y), \eta(y, x^*)) + h(y, x^*))_i \geq 0,
\]
which implies that, for any \( y \in K \),
\[
\langle t^*(y), \eta(y, x^*) \rangle + h(y, x^*) \notin -\text{int} R^n_+,
\]
i.e., \( x^* \) is a solution of the GVVI (2.3). Conversely, assume \( x^* \) solves the GVVI (2.3). Since \( x^* \) is a solution of the GVVI (2.3), for any \( y \in K \), there is a \( t^*(y) \in T(x^*) \) such that
\[
\langle t^*(y), \eta(y, x^*) \rangle + h(y, x^*) \notin -\text{int} R^n_+,
\]
from which it follows that
\[
\max_{1 \leq i \leq n} ((t^*(y), \eta(y, x^*)) + h(y, x^*))_i \geq 0.
\]
Thus, an operator \( t^* \) from \( K \) to \( T(x^*) \) has been defined. Then, \( t^* \in B_{x^*} \) and
\[
\max_{1 \leq i \leq n} ((t^*(y), \eta(y, x^*)) + h(y, x^*))_i \geq 0, \quad \forall y \in K.
\]

Hence,
\[
\varphi_1^*(x^*, t^*) = \min_{y \in K} \max_{1 \leq i \leq n} ((t^*(y), \eta(y, x^*)) + h(y, x^*))_i \geq 0.
\]
(2.15)

Thus, (2.14) and (2.15) imply that
\[
\varphi_1^*(x^*, t^*) = 0.
\]
Again by (2.14), we have
\[ \varphi^*_t(x^*, t) \leq 0, \quad \forall t \in B_{x^*}. \]

Since
\[ \varphi^*(x^*) = \max\{\varphi^*_t(x^*, t) \mid t \in B_{x^*}\}, \]
it follows from \( \varphi^*_t(x^*, t) = 0 \) that \( \varphi^*(x^*) = 0 \). This completes the proof.

From Remark 2.1 and Theorem 2.2, it is easy to see the following result holds.

**Corollary 2.3.** If \( x^* \) is a solution of the GVVI (2.1) or the GVVI (2.2), then \( \varphi^*(x^*) = 0 \).

Furthermore, Theorem 2.2 presented in [21] is a special case of Theorem 2.2.

**Corollary 2.4** ([21]). The function \( \varphi^*(x) \) defined by (2.13) is a gap function for the VVI (2.6).

### 3. Existence of solutions for the GVVI

Let \( X \) and \( Y \) be two real Hausdorff topological vector spaces, and let \( K \) be a nonempty weakly compact and convex subset of \( X \). Let \( L(X, Y) \) be as in Section 2. Let \( C : K \to 2^Y \) be a point-to-set mapping such that for each \( x \in K \), \( C(x) \) is a point, closed and convex cone in \( Y \) with apex at the origin and \( \text{int} \ C(x) \neq \emptyset \). Assume that \( F : K \to 2^{L(X, Y)} \) is a point-to-set mapping, \( f : K \to L(X, Y) \), \( \eta : K \times K \to K \) and \( h : K \times K \to Y \). In this section, we consider the GVVI with moving cone \( C(x) \): find \( x^* \in K \) and \( t^* \in F(x^*) \) such that
\[ (t^*, \eta(y, x^*)) + h(y, x^*) \notin \text{int} \ C(x^*), \quad \forall y \in K. \]

If \( F \) is a single-valued mapping, then GVVI (3.1) reduces to the following vector variational inequality problem (for short, VVI): find \( x^* \in K \) such that
\[ (f(x^*), \eta(y, x^*)) + h(y, x^*) \notin \text{int} \ C(x^*), \quad \forall y \in K. \]

If \( \eta(y, x) = y - x \) and \( h(y, x) = 0 \) for all \( x, y \in K \), then VVI (3.2) reduces to the following VVI: find \( x^* \in K \) such that
\[ (f(x^*), y - x^*) \notin \text{int} \ C(x^*), \quad \forall y \in K, \]
which has been studied by Chen [4].

If \( \eta(y, x) = y - g(x) \) and \( h(y, x) = 0 \) for all \( x, y \in K \), where \( g : K \to K \), then VVI (3.2) reduces to the following VVI: find \( x^* \in K \) such that
\[ (f(x^*), y - g(x^*)) \notin \text{int} \ C(x^*), \quad \forall y \in K, \]
which has been studied by Siddiqi, Ansari and Khaliq [16].

Recall that \( f \) is called a selection of \( F \) on \( K \) if
\[ f(x) \in F(x), \quad \forall x \in K. \]

Furthermore, the function \( f \) is called a continuous selection of \( F \) on \( K \) if \( f \) is a selection of \( F \) on \( K \) and it is also continuous on \( K \).

**Lemma 3.1.** If \( f \) is a selection of \( F \) on \( K \), then every solution of the VVI (3.2) is a solution of the GVVI (3.1).
Definition 3.2. Let \( C \) completes the proof.

Proof. Suppose that \( x^* \in K \) is solution of the VVI (3.2). Then,
\[
(f(x^*), \eta(y, x^*)) + h(y, x^*) \notin \text{int} C(x^*), \quad \forall y \in K.
\]
Let \( t^* = f(x^*) \). Since \( f \) is a selection of \( F \) on \( K \), we obtain \( t^* \in F(x^*) \) and
\[
(t^*, \eta(y, x^*)) + h(y, x^*) \notin \text{int} C(x^*), \quad \forall y \in K.
\]
That is to say, \( x^* \in K \) is a solution of the GVVI (3.1).

Now, we introduce the following \( \eta \)-\( h \)-\( C(x) \)-pseudomonotonicity.

Definition 3.1. (i) \( f \) is said to be \( \eta \)-\( h \)-\( C(x) \)-pseudomonotonic if, for every pair of points \( x, y \in K \), we have that
\[
(f(x), \eta(y, x)) + h(y, x) \notin \text{int} C(x) \quad \text{implies that} \quad (f(y), \eta(y, x)) + h(y, x) \notin \text{int} C(x);
\]
(ii) \( F \) is said to be \( \eta \)-\( h \)-\( C(x) \)-pseudomonotonic if, for every pair of points \( x, y \in K \) and for all \( t' \in F(x) \), \( t'' \in F(y) \), we have that
\[
(t', \eta(y, x)) + h(y, x) \notin \text{int} C(x) \quad \text{implies that} \quad (t'', \eta(y, x)) + h(y, x) \notin \text{int} C(x).
\]

Remark 3.1. If \( \eta(y, x) = y - x \) and \( h(y, x) = 0 \) for all \( x, y \in K \), then the \( \eta \)-\( h \)-\( C(x) \)-pseudomonotonicity reduces to the \( C(x) \)-pseudomonotonicity defined in [9].

Lemma 3.2. Let \( f \) be a selection of \( F \) on \( K \). If \( F \) is \( \eta \)-\( h \)-\( C(x) \)-pseudomonotonic, then \( f \) is also \( \eta \)-\( h \)-\( C(x) \)-pseudomonotonic.

Proof. Since \( f \) is a selection of \( F \) on \( K \), it follows from Definition 3.1 that the conclusion holds. This completes the proof.

Definition 3.2. Let \( K \) be a nonempty subset of topological vector space \( X \). A point-to-set mapping \( T : K \rightrightarrows 2^X \) is called KKM-mapping if, for every finite subset \( \{x_1, x_2, \ldots, x_n\} \) of \( K \), \( \text{co}\{x_1, x_2, \ldots, x_n\} \) is contained in \( \bigcup^n_{i=1} T(x_i) \), where \( \text{co} \) denotes the convex hull.

Lemma 3.3 ([24]). Let \( K \) be a nonempty subset of Hausdorff topological vector space \( X \). Let \( G : K \rightrightarrows 2^X \) be a KKM-mapping, such that for any \( y \in K \), \( G(y) \) is closed and \( G(y^*) \) is compact for some \( y^* \in K \). Then there exists \( x^* \in K \) such that \( x^* \in G(y) \) for all \( y \in K \), i.e., \( \cap_{y \in K} G(y) \neq \emptyset \).

Lemma 3.4. Let \( Y \) be a topological vector space with a pointed, closed and convex cone \( C \) such that \( \text{int} C \neq \emptyset \). Then \( \forall x, y, z \in Y \), we have \( x - y \in -C \) and \( x \notin \text{int} C \implies y \notin \text{int} C \).

Proof. If \( y \in \text{int} C \), then \( x = x - y + y \in -C - \text{int} C \subseteq \text{int} C \), a contradiction of our assumption.

Definition 3.3. Let \( f : K \rightrightarrows L(X, Y), \eta : K \times K \rightrightarrows K \) and \( h : K \times K \rightrightarrows Y \). We say that
(i) \( h(x, y) \) is \( C(x) \)-convex with respect to \( x \) if, for any given \( y \in K \),
\[
h(tx_1 + (1 - t)x_2, y) = th(x_1, y) + (1 - t)h(x_2, y) - C(y), \quad \forall x_1, x_2 \in K, t \in [0, 1];
\]
(ii) \( \eta(x, y) \) is affine with respect to \( x \) if, for any given \( y \in K \),
\[
\eta(tx_1 + (1 - t)x_2, y) = t\eta(x_1, y) + (1 - t)\eta(x_2, y), \quad \forall x_1, x_2 \in K, t \in R,
\]
with \( x = tx_1 + (1 - t)x_2 \in K \);
(iii) \( f \) is hemicontinuous with respect to \( \eta \) on \( K \) if, for all \( x, y \in K \), the function
\[
\nu(t) = \langle f(x + t(y - x)), \eta(y, x) \rangle
\]
is continuous on \([0, 1]\).

**Remark 3.2.** It is easy to prove that \( h(x, y) \) is \( C(x) \)-convex with respect to \( x \) if and only if for any given \( y \in K \),
\[
h\left( \sum_{i=1}^{n} t_i x_i, y \right) \in \sum_{i=1}^{n} t_i h(x_i, y) - C(x)
\]
for all \( x_i \in K \) and \( t_i \in [0, 1] \) \((i = 1, \ldots, n)\) with \( \sum_{i=1}^{n} t_i = 1 \). Also, \( \eta(x, y) \) is affine with respect to \( x \) if and only if for any given \( y \in K \),
\[
\eta\left( \sum_{i=1}^{n} t_i x_i, y \right) = \sum_{i=1}^{n} t_i \eta(x_i, y)
\]
for all \( x_i \in K \) and \( t_i \in R \) \((i = 1, \ldots, n)\) with \( \sum_{i=1}^{n} t_i x_i \in K \) and \( \sum_{i=1}^{n} t_i = 1 \).

**Definition 3.4.** Let \( W : X \rightarrow 2^Y \) be a point-to-set mapping. The graph of \( W \), denoted by \( \text{Graph}(W) \), is
\[
\text{Graph}(W) = \{(x, z) \in X \times Y \mid x \in X, z \in W(x)\}.
\]

**Theorem 3.1.** Let \( X \) and \( Y \) be two real Hausdorff topological vector spaces, and let \( K \) be a nonempty weakly compact and convex subset of \( X \). Let \( C \) be a point-to-set mapping such that for each \( x \in K \), \( C(x) \) is a point, closed and convex cone in \( Y \) with apex at the origin and \( \text{int} C(x) \neq \emptyset \). Let \( f : K \rightarrow L(X, Y) \), \( \eta : K \times K \rightarrow K \) and \( h : K \times K \rightarrow Y \). Assume that the following conditions hold:

(i) \( \eta(x, x) = h(x, x) = 0 \) for each \( x \in K \);
(ii) \( \eta(y, x) \) is affine with respect to \( y \), \( h(y, x) \) is \( C(y) \)-convex with respect to \( y \);
(iii) \( \eta(y, x) \) and \( h(y, x) \) are continuous with respect to \( x \);
(iv) \( f \) is \( \eta \)-\( h \)-\( C(x) \)-pseudomonotonic and hemicontinuous with respect to \( \eta \) on \( K \);
(v) \( W : K \rightarrow Y \) is defined by \( W(x) = Y \setminus (-\text{int} C(x)) \), such that the graph \( \text{Graph} W \) of \( W \) is weakly closed in \( X \times Y \). Then, the VVI (3.2) has a solution.

First, we give the following Lemma.

**Lemma 3.5.** If all conditions in Theorem 3.1 hold, then the VVI (3.2) is equivalent to the following VVI: find \( x^* \in K \) such that
\[
(f(y), \eta(y, x^*)) + h(y, x^*) \notin -\text{int} C(x^*), \quad \forall y \in K.
\]

**Proof.** Since \( f \) is \( \eta \)-\( h \)-\( C(x) \)-pseudomonotonic, it is easy to see that every solution of the VVI (3.2) is also a solution of the VVI (3.3). Conversely, let \( x^* \in K \) be a solution of the VVI (3.3), then
\[
(f(y), \eta(y, x^*)) + h(y, x^*) \notin -\text{int} C(x^*), \quad \forall y \in K.
\]
For any given \( y \in K \) and \( t \in (0, 1) \), set \( y_t = (1 - t)x^* + ty \). It follows that
\[
(f(y_t), \eta(y_t, x^*)) + h(y_t, x^*) \notin -\text{int} C(x^*).
\]

(3.4)
Since $\eta(y, x)$ is affine with respect to $y$, $h(y, x)$ is $C(y)$-convex with respect to $y$, and $\eta(x, x) = h(x, x) = 0$ for each $x \in K$, we get
\[
\{(f(y_t), \eta(y_t, x^*)) + h(y_t, x^*)\} = \{(f(y_t), \eta(y_t, x^*)) + h(y, x^*)\}
\]
\[
= \{(f(y_t), \eta(y_t, x^*)) + h(y_t, x^*)\} - t\{(f(y_t), \eta(y, x^*)) + h(y, x^*)\}
\]
\[
= -(1 - t)\{(f(y_t), \eta(x^*, x^*)) + h(x^*, x^*)\}
\]
\[
\in -C(x^*).
\] 
By Lemma 3.4, it follows from (3.4) and (3.5) that
\[
t\{(f(y_t), \eta(y, x^*)) + h(y, x^*)\} \notin \text{int } C(x^*).
\]
Since $C(x^*)$ is a convex cone, we have
\[
\langle f(y_t), \eta(y, x^*) \rangle + h(y, x^*) \notin -\text{int } C(x^*),
\]
i.e.,
\[
\langle f(y_t), \eta(y, x^*) \rangle + h(y, x^*) \in W(x^*).
\] 
Again since $f$ is hemicontinuous with respect to $\eta$ on $K$ and $W(x^*)$ is weakly closed, from (3.6), we obtain
\[
\langle f(x^*), \eta(y, x^*) \rangle + h(y, x^*) \in W(x^*), \quad \text{as } t \to 0,
\]
that is to say, $x^*$ is a solution of the VVI (3.2). This completes the proof.

**The Proof of Theorem 3.1.** Define the point-to-set mappings $F_1, F_2 : K \to 2^K$ by
\[
F_1(y) = \{x \in K \mid \langle f(x), \eta(y, x) \rangle + h(y, x) \notin -\text{int } C(x)\}
\]
and
\[
F_2(y) = \{x \in K \mid \langle f(y), \eta(y, x) \rangle + h(y, x) \notin -\text{int } C(x)\}
\]
for each $y \in K$, respectively. The proof of which consists of four steps.

Step 1. We show that $F_1$ is a KKM-mapping. Note that $F_1(y) \neq \emptyset$ for each $y \in K$, since $y \in F_1(y)$. Let $z$ be in the convex hull of any finite subset $\{y_1, \ldots, y_n\}$ of $K$. Then, $z = \sum_{i=1}^{n} \lambda_i y_i \in K$ for some non-negative $\lambda_i$, $1 \leq i \leq n$, with $\sum_{i=1}^{n} \lambda_i = 1$. Suppose that
\[
z \notin \bigcup_{i=1}^{n} F_1(y_i).
\]
Then
\[
z \notin F_1(y_i), \quad \forall i = 1, \ldots, n,
\]
and thus,
\[
\langle f(z), \eta(y_i, z) \rangle + h(y_i, z) \in -\text{int } C(z), \quad \forall i = 1, \ldots, n.
\]
Since $C(z)$ is a convex cone, we have,
\[
\sum_{i=1}^{n} \lambda_i \{(f(z), \eta(y_i, z)) + h(y_i, z)\} \in -\text{int } C(z).
\]
Again since \( \eta(y, x) \) is affine with respect to \( y \), \( h(y, x) \) is \( C(y) \)-convex with respect to \( y \), and \( \eta(x, x) = h(x, x) = 0 \) for each \( x \in K \), one has

\[
0 = \langle f(z), \eta(z, z) \rangle + h(z, z) = \langle f(z), \sum_{i=1}^{n} \lambda_i \eta(y_i, z) \rangle + h \left( \sum_{i=1}^{n} \lambda_i y_i, z \right) \in \sum_{i=1}^{n} \lambda_i \langle f(z), \eta(y_i, z) \rangle + \sum_{i=1}^{n} \lambda_i h(y_i, z) - C(z) \subseteq -\text{int} \ C(z) - C(z) \subseteq -\text{int} \ C(z).
\]

Thus, \( 0 \in -\text{int} \ C(z) \), which is a contradiction. Therefore, \( F_1 \) is a KKM-mapping.

Step 2. Since \( f \) is \( \eta \)-\( h \)-\( C(x) \)-pseudomonotonic, it follows that \( F_1(y) \subseteq F_2(y) \) for all \( y \in K \), and hence, \( F_2 \) is also a KKM-mapping.

Step 3. We show that \( F_2(y) \) is weakly closed and weakly compact for all \( y \in K \) and \( \cap_{y \in K} F_2(y) \neq \emptyset \). In fact, let \( \{x_\alpha\} \) be a net of \( F_2(y) \) such that \( x_\alpha \) converges weakly to \( x_0 \in K \). For each \( \alpha \), since \( x_\alpha \in F_2(y) \), we obtain

\[
\langle f(y), \eta(y, x_\alpha) \rangle + h(y, x_\alpha) \notin -\text{int} \ C(x_\alpha).
\]

We note that \( f(y) \in L(X, Y) \) is continuous. Since \( \eta \) is continuous with respect to the second argument, we have

\[
\langle f(y), \eta(y, x_\alpha) \rangle \to \langle f(y), \eta(y, x_0) \rangle.
\]

It follows from the continuity of \( h(\cdot, x) \) that the net

\[
(x_\alpha, \langle f(y), \eta(y, x_\alpha) \rangle + h(y, x_\alpha)) \to (x_0, \langle f(y), \eta(y, x_0) \rangle + h(y, x_0)).
\]

Therefore, \( (x_0, \langle f(y), \eta(y, x_0) \rangle + h(y, x_0)) \in \text{Graph} \ W \), which is weakly closed in \( X \times Y \). Hence \( \langle f(y), \eta(y, x_0) \rangle + h(y, x_0) \in \text{Graph} \ W \), i.e., \( \langle f(y), \eta(y, x_0) \rangle + h(y, x_0) \notin -\text{int} \ C(x_0) \) and \( F_2(y) \) is weakly closed. Since \( K \) is weakly compact, \( F_2(y) \) is also weakly compact for all \( y \in K \). By Step 2, we know that \( F_2 \) is a KKM-mapping. Therefore, from Lemma 3.3, we have \( \cap_{y \in K} F_2(y) \neq \emptyset \).

Step 4. We prove that the VVI (3.2) has a solution. From Lemma 3.5, we have \( \cap_{y \in K} F_1(y) = \cap_{y \in K} F_2(y) \), and by Step 3, we obtain \( \cap_{y \in K} F_2(y) \neq \emptyset \). Then \( \cap_{y \in K} F_1(y) \neq \emptyset \), that is to say, the VVI (3.2) has a solution. This completes the proof.

Remark 3.3. From Theorem 3.1, it is easy to see that the set of solutions for the VVI (3.1) is \( \cap_{y \in K} F_1(y) = \cap_{y \in K} F_2(y) \), which is nonempty weakly closed and weakly compact.

Theorem 3.2. Let \( X, Y, C \) and \( W \) be as in Theorem 3.1 and let \( K \) be a nonempty, closed and convex subset of \( X \). Let \( f : K \to L(X, Y) \), \( \eta : K \times K \to K \) and \( h : K \times K \to Y \). Assume that conditions (i)–(v) in Theorem 3.1 hold and the following coercive condition on \( K \) is satisfied: there exists a weakly compact subset \( D \) of \( X \) and \( y_0 \in D \cap K \) such that

\[
\langle f(y_0), \eta(y_0, x) \rangle + h(y_0, x) \in -\text{int} \ C(x), \quad \text{for all} \quad x \in K \setminus D.
\]

Then, the VVI (3.2) has a solution.
Proof. As the proof in Theorem 3.1, we only need to prove that \( F_2(y_0) \) is weakly compact. From the coercive condition, it is clear that \( F_2(y_0) \subseteq D \). Consider Step 3 in the proof of Theorem 3.1, \( F_2(y_0) \) is also weakly compact. This completes the proof.

**Theorem 3.3.** Let \( X, Y, K, C \) and \( W \) be as in Theorem 3.1. Let \( F : K \rightarrow 2^{L(X,Y)} \), \( \eta : K \times K \rightarrow K \) and \( h : K \times K \rightarrow Y \). Assume that conditions (i)–(iii) and (v) in Theorem 3.1 hold and the following assumptions are satisfied:

- (iv) \( F \) is \( \eta \)-\( h \)-pseudo-monotone;
- (vi) there is a continuous selection \( f \) of \( F \) on \( K \).

Then, the GVVI (3.1) has a solution.

**Proof.** By the assumption, there is a continuous function \( f : K \rightarrow L(X,Y) \) such that

\[
f(x) \in F(x), \quad \forall x \in K.
\]

It follows from Lemma 3.2 that \( f \) is also \( \eta \)-\( h \)-pseudo-monotone. Then, all conditions in Theorem 3.1 are satisfied. Thus, there exists a solution \( x^* \) of VVI (3.2). By Lemma 3.1, \( x^* \) solves the GVVI (3.1). This completes the proof.

**Remark 3.4.** From Theorem 3.3, we use the method of a continuous selection to obtain the existence Theorem of the GVVI (3.1), which is totally different from the method used in [9] or [14].

**Theorem 3.4.** Let \( X, Y, C \) and \( W \) be as in Theorem 3.3 and let \( K \) be a nonempty, closed and convex subset of \( X \). Let \( F : K \rightarrow 2^{L(X,Y)} \), \( \eta : K \times K \rightarrow K \) and \( h : K \times K \rightarrow Y \). Assume that all conditions in Theorem 3.3 hold and the continuous selection \( f \) of \( F \) satisfies the coercive condition on \( K \) defined in Theorem 3.2. Then, the GVVI (3.1) has a solution.

**Proof.** It follows from Theorems 3.2 and 3.3 that the conclusion holds. This completes the proof.

**Corollary 3.1.** Assume that all conditions in Theorem 3.3 hold, except condition (vi), which is replaced by

- (vi)' \( F : K \rightarrow 2^{L(X,Y)} \) is continuous on \( K \).

Then, the GVVI (3.1) has a solution.

**Proof.** It follows from the selection theorem of [25] that there exists a continuous selection \( f : K \rightarrow L(X,Y) \) such that

\[
f(x) \in F(x), \quad \forall x \in K.
\]

By Theorem 3.3, the conclusion holds. This completes the proof.

**Remark 3.5.** Theorem 3.3 and Corollary 3.1 generalize the corresponding results in [21].

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References