Gaussian Sum PHD Filtering Algorithm for Nonlinear Non-Gaussian Models

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Abstract

A new multi-target filtering algorithm, termed as the Gaussian sum probability hypothesis density (GSPHD) filter, is proposed for nonlinear non-Gaussian tracking models. Provided that the initial prior intensity of the states is Gaussian or can be identified as a Gaussian sum, the analytical results of the algorithm show that the posterior intensity at any subsequent time step remains a Gaussian sum under the assumption that the state noise, the measurement noise, target spawn intensity, new target birth intensity, target survival probability, and detection probability are all Gaussian sums. The analysis also shows that the existing Gaussian mixture probability hypothesis density (GMPHD) filter, which is unsuitable for handling the non-Gaussian noise cases, is no more than a special case of the proposed algorithm, which fills the shortage of incapability of treating non-Gaussian noise. The multi-target tracking simulation results verify the effectiveness of the proposed GSPHD.

Keywords: signal processing; Gaussian sum probability hypothesis density; simulation; nonlinear non-Gaussian; tracking

1 Introduction

The main objective of multi-target tracking is to jointly estimate the unknown and time-varying number of targets as well as their individual states from the history of noisy and cluttered observation sets. Most approaches to this problem involve data association techniques such as nearest neighbor (NN), joint probabilistic data association (JPDA), and multiple hypothesis tracking (MHT)[1-5], which constitute the bulk of the computational work in multi-target tracking algorithms.

Finite set statistics (FISST) provides a general systematic foundation for multi-target filtering based on the theory of random finite set (RFS), which performs filtering on set-valued observations and states without explicit connections among measurements and targets[6-16]. RFS considers sets as elements, which can be seen as the extension of the random variable and random vector. In simple single-target tracking, where there are no appearing or disappearing targets or spurious measurements (clutter), the states and measurements are both vectors, whose dimensions will not submit to changes. However, in multi-target tracking, the number of targets and the measured tracks will be time-varying with changing dimensions of the states and measurements, because of targets disappearing, spawning, spontaneous births, and clutter. By modeling the collection of individual targets as an RFS and the collection of individual observations as another RFS, the problem of dynamically estimating multiple targets in the presence of clutter and associated uncertainty can be cast in a Bayesian filtering framework [7,9,16]. Such a theoretically optimal approach

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to multi-target tracking is an elegant generalization of the single-target Bayesian filter, and circumvents the data association procedure.

Mahler may have been the first person to introduce the RFS approach of multi-target tracking to the tracking community more than a decade ago\cite{6-7}. Despite being theoretically solid, the RFS approach has been rejected by several tracking researchers and engineers owing to the involved intractable computations\cite{10}. Recently, the RFS formulation has drawn considerably more attention thanks to the enhancement of the computational capability and the developed computational approximation strategies: particle filter (PF), also known as sequential Monte Carlo (SMC) sampling algorithms\cite{17-20}, and the probability hypothesis density (PHD) approximation\cite{12-13,16}. SMC implementations of the RFS multi-target filtering recursion can be found in Ref.[9]. However, this method is still computationally intensive owing to the combinatorial nature of the densities, especially when the number of targets is large\cite{16-17}. The idea of Rao-Blackwellisation is applied in Ref.[10] to reduce the computational loads, where only some states are sampled, while the others are handled analytically.

As a novel RFS based filter, the PHD filter is a suboptimal but computationally tractable alternative to the RFS Bayesian multi-target filter\cite{12-13,16}. It is a recursion that propagates the first-order statistical moment, or intensity, of the states RFS in time. One such algorithm, known as the SMCPHD (or particle PHD) filter, was proposed in Ref.[9] and Ref.[13], which used the SMC technique to propagate the posterior intensity in time. The main drawbacks of this approach are the large number of particles and the unreliability of clustering techniques for extracting the state estimates\cite{16}. Another PHD algorithm, known as the Gaussian mixture PHD (GMPHD) filter, provides an analytic solution to the PHD recursion for linear Gaussian target dynamic model by approximating the intensity function with a weighted mixture of Gaussians\cite{16}. The GMPHD algorithm is also extended to nonlinear target dynamic models using approximation strategies from the extended and unscented Kalman filters. Since these techniques are all based on the Gaussian process and measurement noises, these may not be adequate to handle non-Gaussian models, which are more universal in practice.

This article proposes a solution to the PHD recursion for nonlinear non-Gaussian dynamic model, termed as the Gaussian sum probability hypothesis density (GSPHD) filter, a generalization of the GMPHD algorithm. It is perceived that under the conditions where the state noise, the measurement noise, target spawn intensity, new target birth intensity, target survival probability, and detection probability are expressed as Gaussian sums, the predictive and posterior intensities at any subsequent time step remain Gaussian sums if the initial prior intensities are Gaussian or a Gaussian sums. Simulation results are presented to demonstrate the validity of the proposed filter.

2 Model Description

2.1 Models for random vector filtering

Consider the general system model\cite{21} as follows

\[
x_k = f_k(x_{k-1}) + w_k
\]

\[
z_k = h_k(x_k) + v_k
\]

where \(x_k\) and \(z_k\) denote the state and measurement, respectively, \(w_k\) and \(v_k\) denote the process noise and the measurement noise, respectively, and both \(f_k\) and \(h_k\) denote the known nonlinear functions. Let \(x_{1:k} = \{x_1, \cdots, x_k\}\) and \(z_{1:k} = \{z_1, \cdots, z_k\}\). Here, the purpose is to estimate the posterior probability density function, \(p(x_k | z_{1:k})\), using the following equations

\[
p(x_k | z_{1:k-1}) = \int p(x_k | z_{1:k-1})p(x_{k-1} | z_{1:k-1})dx_{k-1}
\]

\[
p(x_k | z_{1:k}) = \frac{p(z_k | x_k)p(x_k | z_{1:k-1})}{\int p(z_k | x_k)p(x_k | z_{1:k-1})dx_k}
\]

2.2 Models for RFS filtering

Suppose that the number of targets and measurements are \(M(k)\) and \(N(k)\), respectively, at the time step \(k\). Then, RFSs \(X_k\) and \(Z_k\) can be
used to denote the multi-target states and measurements, respectively, as follows

\[ X_k = \{ x_{k,1}, \ldots, x_{k,M(k)} \} \]  
\[ Z_k = \{ z_{k,1}, \ldots, z_{k,N(k)} \} \]

Let \( p_{k,i}(x_{k,i}) \) denote the probability when a target still exists at time \( k \) and its previous state \( x_{k-1} \), \( f_{k|k-1}(x_k | x_{k-1}) \) the state transition density. Then, for a given state \( x_{k-1} \in X_{k-1} \) at time \( k-1 \), its behavior at the next time step is modeled as the RFS, i.e.,

\[ S_{k|k-1}(x_{k-1}) = \{ x_k \}, \text{Survival} \]
\[ \emptyset, \text{Disappear} \]  
(7)

In this way, the target states at time \( k \) can be described as the union of survival, spawned, and spontaneous birth targets by

\[ X_k = \bigcup_{\mathcal{C} \in X_{k-1}} S_{k|k-1}(\mathcal{C}) \bigcup_{\mathcal{C} \in X_{k-1}} G_{k|k-1}(\mathcal{C}) \bigcup B_k \]  
(8)

where \( G_{k|k-1}(\mathcal{C}) \) denotes the RFS of targets spawned from a target with previous state \( \mathcal{C} \), and \( B_k \) denotes the RFS of the spontaneous birth targets.

Assume that \( p_{D,k}(x_k) \) denotes the target detection probability, and \( h_k(z_k | x_k) \) the probability density obtained by an observation \( z_k \) of the state \( x_k \). Therefore, at time \( k \), each state \( x_k \in X_k \) generates an RFS as

\[ \Theta_k(x_k) = \{ x_k \}, \text{Detected} \]
\[ \emptyset, \text{Missed} \]  
(9)

The measurement \( Z_k \) is formed by the union of targets generated by measurements and clutter, i.e.,

\[ Z_k = \bigcup_{x_k \in X_k} \Theta_k(x_k) \bigcup M_k \]  
(10)

where \( M_k \) denotes the false measurements or clutter.

Let \( p_k(\cdot | Z_{1:k}) \) represent a multi-target posterior density, then \( \mu_{16,12} \),

\[ p_{k|k-1}(X_k | Z_{1:k-1}) = \int f_{k|k-1}(X_k | X_{k-1}) p_{k-1}(X_{k-1} | Z_{1:k-1}) \mu_{16,12} \]  
(11)

\[ p_k(X_k | Z_{1:k}) = \frac{h_k(Z_k | X_k) p_{k|k-1}(X_k | Z_{1:k-1})}{\int h_k(Z_k | X_k) p_{k|k-1}(X_k | Z_{1:k-1}) \mu_{16,12}} \]  
(12)

where \( \mu \) is an appropriate reference measure\(^{[16]} \). The details on the formulation of the multi-target filtering in RFS may be found in Ref.\([7]\), Ref.\([9]\) and Ref.\([16]\).

3 GSPHD Filter for Nonlinear Non-Gaussian Models

3.1 PHD filter

For an RFS \( X \) on \( Z \) with a probability distribution \( \mu \), its first order moment is a non-negative function \( \mu \) on \( X \), called the intensity or the PHD function, with the property for any closed subset \( \mathcal{G} \subset Z \)\(^{[12,16]} \),

\[ \int X \cap \mathcal{G} \mu dX = \int \mu (x) d\mathcal{G} \]

where \( |X| \) denotes the cardinality of \( X \). In other words, given the intensity \( \mu \), its integral over any region \( \mathcal{G} \) gives the expected number of elements in \( X \) that are in \( \mathcal{G} \). The local maxima of the intensity \( \mu \) are points in \( X \) with the highest local concentration of the expected number of elements, and hence can be used to generate estimates for the elements of \( X \).

Given the posterior intensity \( \mu_{k-1} \) at time \( k-1 \), the predicted intensity function \( \mu_{k|k-1} \) and the posterior intensity \( \mu_k \) at time \( k \) can be given, respectively, by\(^{[12]} \)

\[ \mu_{k|k-1}(x_k) = \int p_{k|k-1}(x_k | \mathcal{C}) \mu_{k|k-1}(\mathcal{C}) d\mathcal{C} + \int p_{k|k-1}(x_k | \mathcal{C}) \beta_{k|k-1}(\mathcal{C}) d\mathcal{C} + h_k(x) \]

\[ c_k(x) = \int p_{k|k-1}(x_k | \mathcal{C}) \kappa_k(z) + p_{k|k-1}(x_k | \mathcal{C}) \beta_{k|k-1}(\mathcal{C}) d\mathcal{C} \]

(13)

(14)

(15)

where \( \kappa_k(\cdot) \) is the intensity of the clutter RFS, \( \beta_k(\cdot) \) the intensity of the target RFS spawned by a target of previous state \( \mathcal{C} \) at time \( k-1 \); and \( p_{S,k}(\mathcal{C}) \) and \( p_{D,k}(x) \) are the survival and detection probabilities, respectively.

3.2 Gaussian sum approximation

Let \( N(x;\mu,\Sigma) \) denote the Gaussian distribu-
tion with mean $\mathbf{a}_i$ and covariance $\mathbf{B}_i$, then
\[ N(\mathbf{x}; \mathbf{a}_i, \mathbf{B}_i) = (2\pi)^{-\frac{m}{2}} |\mathbf{B}_i|^{-1/2} \exp \left\{ \frac{1}{2} (\mathbf{x} - \mathbf{a}_i)^T \mathbf{B}_i^{-1} (\mathbf{x} - \mathbf{a}_i) \right\} \]  
(16)
where $n$ denotes the dimension of $\mathbf{x}$. Any density $p(\mathbf{x})$ can be approximated as close as required by a linear combination of Gaussian densities\cite{21-22}.

Given any $\varepsilon > 0$, a positive integer $N$ can be found as follows\cite{22-23}
\[ \int p(\mathbf{x}) \sum_{i=1}^m \alpha_i N(\mathbf{x}; \mathbf{a}_i, \mathbf{B}_i) \, d\mathbf{x} \leq \varepsilon \quad (m > N) \]  
(17)
where $\alpha_i$ is the weight of each Gaussian with $\sum_{i=1}^m \alpha_i = 1$.

### 3.3 GSPHD algorithm

The distribution of process noise and measurement noise in Eq.(1) and Eq.(2) can be rewritten in terms of the following Gaussian sums\cite{21-22}:
\[ p(\mathbf{w}_k) = \sum_{j=1}^{N} \alpha_{w_k}^{(j)} N(\mathbf{w}_k; \mathbf{w}_k^{(j)}, \mathbf{Q}_k^{(j)}) \]  
(18)
\[ p(\mathbf{v}_k) = \sum_{j=1}^{N} \alpha_{v_k}^{(j)} N(\mathbf{v}_k; \mathbf{v}_k^{(j)}, \mathbf{R}_k^{(j)}) \]  
(19)
where
\[ \sum_{j=1}^{N} \alpha_{w_k}^{(j)} = \sum_{j=1}^{N} \alpha_{v_k}^{(j)} = 1 \]  
(20)
Then
\[ f_{k|k-1}(\mathbf{x}_k | \mathbf{x}_{k-1}) = \sum_{j=1}^{N} \alpha_{w_{k-1}}^{(j)} N(\mathbf{x}_k; f_k(\mathbf{x}_{k-1}) + \mathbf{w}_k^{(j)}, \mathbf{Q}_k^{(j)}) \]  
(21)
\[ h_k(\mathbf{z}_k | \mathbf{x}_k) = \sum_{j=1}^{N} \alpha_{v_k}^{(j)} N(\mathbf{z}_k; h_k(\mathbf{x}_k) + \mathbf{v}_k^{(j)}, \mathbf{R}_k^{(j)}) \]  
(22)
In the same way, the survival, detection probabilities and the intensities of the spawned and birth RFSs can be rewritten in terms of the following Gaussian sums:
\[ p_{S_k}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_{S_{k-1}}^{(j)} N(\mathbf{x}; \mathbf{m}_{S_{k-1}}^{(j)}, \mathbf{P}_{S_{k-1}}^{(j)}) \]  
(23)
\[ p_{D_k}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_{D_{k-1}}^{(j)} N(\mathbf{x}; \mathbf{m}_{D_{k-1}}^{(j)}, \mathbf{P}_{D_{k-1}}^{(j)}) \]  
(24)
\[ \beta_{k|k-1}(\mathbf{x} | \mathbf{z}_k) = \sum_{j=1}^{N} \alpha_{\beta_{k-1}}^{(j)} N(\mathbf{x}; \mathbf{m}_{\beta_{k-1}}^{(j)}, \mathbf{P}_{\beta_{k-1}}^{(j)}) \]  
(25)
\[ b_k(\mathbf{x}) = \sum_{j=1}^{N} \alpha_{b_k}^{(j)} N(\mathbf{x}; \mathbf{m}_{b_k}^{(j)}, \mathbf{P}_{b_k}^{(j)}) \]  
(26)

Before introducing the following propositions, it will be better to review two lemmas for Gaussian function.

**Lemma 1** Given $f$, $d$, $\mathbf{Q}$, $\mathbf{m}$, and $\mathbf{P}$ having appropriate dimensions, and $\mathbf{Q}$ and $\mathbf{P}$ being positive definite and $f$ differentiable, then
\[ \int N(\mathbf{x}; f(\mathbf{z}) + d, \mathbf{Q}) N(\mathbf{z}; \mathbf{m}, \mathbf{P}) \, d\mathbf{z} \approx N(\mathbf{x}; f(\mathbf{m}) + d, \mathbf{Q} + \mathbf{F} \mathbf{P} \mathbf{F}^T) \]  
(27)
where $\mathbf{F} = \frac{\partial f(\mathbf{z})}{\partial \mathbf{z}} |_{\mathbf{z} = \mathbf{m}}$.

**Lemma 2** Given $h$, $e$, $\mathbf{R}$, $\mathbf{m}$, and $\mathbf{P}$ having appropriate dimensions, and $\mathbf{R}$ and $\mathbf{P}$ being positive definite and $h$ differentiable, then
\[ N(\mathbf{z}; h(\mathbf{x}) + e, \mathbf{R}) N(\mathbf{x}; \mathbf{m}, \mathbf{P}) \approx q(z) N(\mathbf{x}; \hat{\mathbf{m}}, \hat{\mathbf{P}}) \]  
(28)
where
\[ q(z) = N(z; h(\mathbf{m}) + e, \mathbf{R} + \mathbf{H} \mathbf{P} \mathbf{H}^T) \]
\[ \hat{\mathbf{m}} = \mathbf{m} + \mathbf{K} (z - h(\mathbf{m}) - e), \hat{\mathbf{P}} = (\mathbf{I} - \mathbf{K} \mathbf{H}) \mathbf{P} \]
\[ \mathbf{K} = \mathbf{P} \mathbf{H}^T (\mathbf{H} \mathbf{P} \mathbf{H}^T + \mathbf{R}), \mathbf{H} = \frac{\partial h(\mathbf{x})}{\partial \mathbf{x}} |_{\mathbf{x} = \mathbf{m}} \]  
(29)

One understanding of Lemma 1 and Lemma 2 is by means of the extended Kalman filtering (EKF) equations\cite{22}. Eq.(27) can be treated as the time update step, where, $f(\mathbf{m}) + d$ on its right is the state prediction and $\mathbf{Q} + \mathbf{F} \mathbf{P} \mathbf{F}^T$ is the state prediction covariance, while Eq.(28) can be treated as the measurement update step, where, $\hat{\mathbf{m}}$ and $\hat{\mathbf{P}}$ on its right are state estimation and state estimation covariance, respectively. Moreover, $h(\mathbf{m}) + e$ and $\mathbf{R} + \mathbf{H} \mathbf{P} \mathbf{H}^T$ in $q(z)$ are the mean and covariance of the measurements when the state prediction is given. If $f$ and $h$ are linear, e.g. $f(\mathbf{m}) = \mathbf{F} \mathbf{m}$, $h(\mathbf{m}) = \mathbf{H} \mathbf{m}$, Eq.(27) and Eq.(28) can be reduced to the Kalman filtering (KF) equations, which are exactly the Lemma 1 and Lemma 2 in Ref.[16].

**Proposition 1** (Predict step) Suppose that the posterior intensity at time $k - 1$ is a Gaussian sum of the form
\[ c_{k-1}(\mathbf{x}) = \sum_{j=1}^{N} \alpha_{c_{k-1}}^{(j)} N(\mathbf{x}; \mathbf{m}_{c_{k-1}}^{(j)}, \mathbf{P}_{c_{k-1}}^{(j)}) \]  
(30)
then, after the prediction step of Eq.(14), the inten-
The proof of Proposition 1 can be found in appendix A.

**Proposition 2** (Update step) Suppose the predicted intensity at time $k$ is a Gaussian sum of the form

$$c_{k|k-1}(x) = \sum_{i=1}^{N_k} \omega_{k,i}(x) N(x; m_{i,k-1}^{(e)}, P_{i,k-1}^{(e)})$$

then, after the update step of Eq.(15), the posterior intensity at time $k$ remains a Gaussian sum given by

$$\sum_{i=1}^{N_k} \omega_{k,i}^{(e)} N(x; m_{i,k}^{(e)}, P_{i,k}^{(e)})$$

and

$$c_{k|k-1}(x) = \sum_{i=1}^{N_k} \omega_{k,i}^{(e)} N(x; m_{i,k}^{(e)}, P_{i,k}^{(e)})$$

The proof of Proposition 1 can be found in appendix A.
The proof of Proposition 2 can be found in appendix B.

By substituting Eqs.(32)-(35) into Eqs.(39)-(42), the following can be obtained
\[
N_k = (N_w N_{S,x} + N_{P,x} + N_{D,x}) + N_{V,x} N_{D,x}^* \quad \mid Z_k \rangle |N_{D,x} + N_{V,x} N_{D,x} | Z_k | N_{D,x} \quad \text{for completeness, the key steps of the GSPHD}
\]

For completeness, the key steps of the GSPHD filter can be summarized in Schema 1(Fig.1), where the first three sub-schemata are for prediction while the last three are for update and the middle at right is for the construction of update components. Note that Schema 1 shows only one cycle of the GSPHD filter from \( \{ \omega_k^{(1)}, m_k^{(1)}, p_k^{(1)}, \} \) to \( \{ \omega_k^{(i)}, m_k^{(i)}, p_k^{(i)} \} \).

\[
\begin{align*}
\text{for } i = 0 & : N_0, \\
\text{for } j = 1 : N_{k-1} & : 1 \text{ for } i = 1 : N_{k-1} \text{ for } j = 1 : N_{k-1}
\end{align*}
\]

When \( N_{w,k} = N_{v,k} = 1, f_k, h_k, f_{\beta,k-1} \) are linear and \( p_{S,x} (x) \) and \( p_{D,x} (x) \) are state-independent, i.e., \( p_{S,x} (x) = p_{S,x} \), \( p_{D,x} (x) = p_{D,x} \), and schema 1 can be reduced to Table 1 in Ref.[16] and to Table 2 in Ref.[16] when \( N_{w,k} = N_{v,k} = 1 \). From the above discussion, it can be concluded that the GMPHD in Ref.[16] is a special case of the proposed GSPHD. Similar to the GMPHD filter, the GSPHD filter also suffers from computational troubles caused by the increasing number of Gaussian components with the progress of time in Eq.(44). Therefore, the method from Ref.[16] is used to reduce the number of Gaussian components by truncating components having weak weights and merging the closest components into one.

4 Simulation Results

4.1 Linear non-Gaussian tracking model

Consider the following multi-target tracking model in a two dimensional situation[16]:
\[
x_k = \begin{bmatrix}
T & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} x_{k-1} + \begin{bmatrix}
\frac{T^2}{2} & T & 0 \\
0 & 0 & \frac{T^2}{2}
\end{bmatrix} w_k \]
\[
y_k = \begin{bmatrix}
1 & 0 & 0 & 0 & 1
\end{bmatrix} x_k + \begin{bmatrix}
v_{1,k} \\
v_{2,k}
\end{bmatrix} \]

where \( x_k \) and \( y_k \) denote the state and measurement at time \( k \), respectively, \( x_k = [x_{k,1}, x_{k,2}, x_{k,3}, x_{k,4}]^T \), here, \( x_{k,1} \) and \( x_{k,3} \) denote the \( x \) and \( y \)
position, respectively, $x_{k,2}$ and $y_{k,2}$ denote the $x$ and $y$ velocity, and $T$ denotes the sampling interval. Suppose the measurement noises and the intensity of birth targets are both Gaussian sums with two Gaussian components and the intensity of spawned targets and the process noises are Gaussian i.e.,

$$w_k = \begin{bmatrix} w_{1,k} \\ w_{2,k} \end{bmatrix} = N(w_k; w_k^{(1)}, Q_k^{(1)})$$

$$v_k = \begin{bmatrix} v_{1,k} \\ v_{2,k} \end{bmatrix} = \omega_{v,k}^{(1)} N(v_k; v_k^{(1)}, R_k^{(1)}) + \omega_{v,k}^{(2)} N(v_k; v_k^{(2)}, R_k^{(2)})$$

$$b_k(x) = \omega_{b,k}^{(1)} N(x; m_{b,k}^{(1)}, P_{b,k}^{(1)}) + \omega_{b,k}^{(2)} N(x; m_{b,k}^{(2)}, P_{b,k}^{(2)})$$

$$\beta_{k-1}(x; \zeta) = \omega_{\beta,k} N(x; \zeta, Q_{\beta,k-1})$$

The following data are used in simulations:

$$\omega_{v,k}^{(1)} = 0.8, \omega_{v,k}^{(2)} = 0.2, \omega_{b,k} = 0.05, \omega_{b,k}^{(1)} = 0.1, \omega_{b,k}^{(2)} = 0.1$$

$$m_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, v_k^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, v_k^{(2)} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, Q_k = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

$$R_k^{(1)} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}, R_k^{(2)} = \begin{bmatrix} 225 & 0 \\ 0 & 225 \end{bmatrix}$$

$$P_{b,k}^{(1)} = P_{b,k}^{(2)} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

$$Q_{\beta,k-1} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 400 & 0 \\ 0 & 0 & 100 \end{bmatrix}$$

$$m_{b,k}^{(1)} = [250 0 250 0]^T$$

$$m_{b,k}^{(2)} = [-250 0 -250 0]^T$$

The probability of the survival and detection are $p_{s,k} = 0.99$ and $p_{d,k} = 0.98$, respectively. Clutter is modeled on a Poisson RFS over the surveillance region with an average of 50 clutter points per scan.

Fig.2 shows the target trajectories in $x$-$y$ plane. Targets 1, 2, and 3 are born (denoted by circles) at time 1, 5, and 17 and die (by squares) at time 40, 32 and 50, and targets 1 and 2 are spontaneous birth targets; target 1 spawns target 3. The true and estimated tracks in $x$ and $y$ coordinates are given in Fig.3 separately. It can be seen from the position estimates shown in Figs.2-3 that the GSPHD filter is capable of providing accurate tracking performances.

### 4.2 Nonlinear non-Gaussian tracking model

Consider a bearing and range tracking model as follows \cite{16}

$$\bm{y}_k = \bm{F}(\omega_{k-1})\bm{y}_{k-1} + \bm{Gw}_{k-1}$$

$$\omega_k = \omega_{k-1} + \Delta\mu_{k-1}$$

$$\bm{z}_k = \begin{bmatrix} \arctan\left(\frac{\hat{p}_{x,k}}{\hat{p}_{y,k}}\right) \\ \sqrt{(\hat{p}_{x,k})^2 + (\hat{p}_{y,k})^2} \end{bmatrix} + \bm{e}_k$$

where the target state takes the form $\bm{x}_k = [\bm{y}_k^T \omega_k]^T$ and $\bm{y}_k = [\hat{p}_{x,k} \hat{p}_{y,k} \hat{p}_{x,k} \hat{p}_{y,k}]^T$ consisting of position $(\hat{p}_{x,k}, \hat{p}_{y,k})$ and velocity $(\hat{p}_{x,k}, \hat{p}_{y,k})$. Also,
where $\Delta = 1$, $w_k \sim N(\cdot, \sigma_w^2 I_2)$, $\sigma_w = 15$, and $u_k \sim N(\cdot, \sigma_u^2 I_2)$, $\sigma_u = \pi/180$, $e_k \sim \alpha_1 N(\cdot; m_k^1, R_k^1) + \alpha_2 N(\cdot; m_k^2, R_k^2)$ with $\alpha_1 = 0.9$, $\alpha_2 = 0.1$, $m_k^1 = [0 \ 0]^T$, $m_k^2 = [0.05 \ 30]^T$, $R_k^1 = \text{diag}((2\pi/180)^2, 20^2)$, $R_k^2 = \text{diag}((3\pi/180)^2, 30^2)$. Assume that no spawning happens as is the case in Ref.[16]. The probabilities of survival and detection are $p_s = 0.99$ and $p_d = 0.98$, respectively. Clutter is modeled on a Poisson RFS over the surveillance region with an average of 20 clutter points per scan.

Fig.4(a) and Fig.4(b) separately denote the tracks of $x$ and $y$ coordinates.

**5 Conclusions**

This article proposes a new PHD filter, namely, the GSPHD filter, which can be viewed as the generalized form of the existing GMPHD filter. It obviates the limits of the GMPHD filter to be applied to tracking models with non-Gaussian noises. It also derives the recursions for the weights, means, and covariances of the constituent Gaussian components of the posterior intensity in the GSPHD. Simulation results have verified the effectiveness of the proposed GSPHD. Future study is expected to focus on other sub-filtering methods besides the EKF, e.g. the unscented Kalman filter[24-25], the central difference filter[26], the polynomial predictive filter[27] and the convergence properties of the various PHD filters.

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Appendix A: Proof of Proposition 1

By substituting Eq.(21), Eq.(23), and Eqs.(25)-(26) into Eq.(14), the following can be obtained:

$$c_{b,k-1}(x) = c_{S,k-1}(x) + c_{b,k-1}(x) + b_{k}(x)$$

where

$$c_{S,k-1}(x) = \sum_{r=1}^{N_{S}} \alpha_{S,k}^{(r)} N(\zeta; m_{S,k}^{(r)}, P_{S,k}^{(r)})$$

$$c_{b,k-1}(x) = \sum_{l=1}^{N_{b}} \alpha_{b,k}^{(l)} N(x; f_{b,k}^{(l)}(\zeta) + w_{k}^{(l)}, Q_{b,k}^{(l)})$$

$$c_{b,k-1}(x) = \sum_{i=1}^{N_{b}} \alpha_{b,k}^{(i)} N(x; f_{b,k-1}^{(i)}(\zeta) + g_{b,k-1}^{(i)}(\zeta) N(\zeta; m_{b,k-1}^{(i)} + P_{b,k-1}^{(i)})$$

From Lemma 1 and Lemma 2,
Appendix B: Proof of Proposition 2

By substituting Eq.(22), Eq.(24), Eq.(31) into Eq.(15), it can be obtained
\[
c_{z, k}(x) = \sum_{z \in Z_k} \kappa_k(z) + \sum_{i=1}^{N_k} \sum_{j=1}^{N_ki+1} \sum_{l=1}^{N_k} \omega_k^{(i, j)} N(z; h_k(x) + v_k^{(j)}, R_k^{(j)}) \sum_{r=1}^{N_ki+1} \omega_r^{(i, k)} N(x; m_r^{(i, k)}, P_r^{(i, k)})
\]

where

\[
P_k^{(j)} = P_k^{(i, k)} - K_k^{(j)} P_k^{(i, k)}
\]

\[
K_k^{(j)} = P_k^{(i, k)} P_k^{(i, k)}^{-1} - K_k^{(j)} P_k^{(i, k)}
\]

\[
m_k^{(j, j)} = m_k^{(i, k)} + K_k^{(j)} [z - h_k(m_k^{(i, k)}) - v_k^{(j)}]
\]

\[
P_k^{(j, j)} = P_k^{(i, k)} - K_k^{(j)} P_k^{(i, k)}
\]

\[
K_k^{(j, j)} = P_k^{(i, k)} (H_k^{(j)})^T (H_k^{(j)} P_k^{(i, k)} (H_k^{(j)})^T + R_k^{(j)})^{-1}
\]

\[
H_k^{(j)} = \frac{\partial h_k(x)}{\partial x} |_{x=m_k^{(j, j)}}
\]

and

\[
\omega_k^{(j, j, j)} = \left[ \omega_k^{(i, j)} \omega_k^{(i, k)} \omega_k^{(i, k)} \right] N(z; h_k(m_k^{(i, k)})) + v_k^{(j)}, R_k^{(j)} + H_k^{(j)} P_k^{(i, k)} (H_k^{(j)})^T]
\]

Proposition 2 is thus proved.