ON POWERS AND CENTERS OF CHORDAL GRAPHS

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A graph is chordal if every cycle of length strictly greater than three has a chord. A necessary and sufficient condition is given for all powers of a chordal graph to be chordal. In addition, it is shown that for connected chordal graphs the center (the set of all vertices with minimum eccentricity) always induces a connected subgraph. A relationship between the radius and diameter of chordal graphs is also established.

1. Introduction

Consider an undirected graph $G = (V, E)$ with vertex set $V$ and edge set $E$. It will be assumed that $G$ has no loops or multiple edges. A graph $G$ is chordal if every cycle in $G$ of length $\geq 3$ possesses a chord: namely, an edge joining two non-consecutive vertices on the cycle.

The class of chordal graphs includes trees, $k$-trees, complete graphs and interval graphs. Chordal graphs are known to be perfect [8] and they play an important role in elimination schemes for sparse systems of linear equations [13]. Certain problems that are known to be NP-hard for general graphs can be solved in polynomial time for chordal graphs [7].

This paper studies certain properties of the powers of chordal graphs as well as the centers of chordal graphs. Previous investigations into these and related topics can be found in [1, 2, 5, 6, 10, 11, 12].

2. Powers of chordal graphs

For any graph $G = (V, E)$ and integer $k \geq 1$, the graph $G^k$ has vertex set $V$ and edges joining vertices $x, y \in V$ whenever $d(x, y) \leq k$, where $d(x, y)$ is the shortest distance in $G$ between $x$ and $y$.

In an earlier paper, the authors [11] noted that the square of a chordal graph is not necessarily chordal. The graph of Fig. 1 furnishes the smallest such example, where the dotted lines indicate a chordless cycle in $G^2$. In the same paper the authors established that if $G$ is chordal, so are $G^3$ and $G^5$. It was also conjectured

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that any odd power of a chordal graph is again chordal. This conjecture has recently been shown to be true by Balakrishnan and Paulraja [2].

**Theorem 1** (Balakrishnan and Paulraja [2]). *If G is chordal, then so is \( G^{2k+1} \) for any \( k \geq 1 \).*

However, Duchet gives a stronger result.

**Theorem 2** (Duchet [5]). *If \( G^k \) is chordal, then so is \( G^{k+2} \) for any \( k \geq 1 \).*

Thus, from the above result, it follows that if \( G \) and \( G^2 \) are both chordal, then all powers of \( G \) are chordal. Now, the question arises when is \( G^2 \) chordal?

Balakrishnan and Paulraja give a sufficient condition for \( G^2 \) to be chordal for any graph \( G \).

**Theorem 3** ([1]). *If \( G \) is any graph having no induced subgraphs isomorphic to \( K_{1,3} \) or \( C_5 \), then \( G^2 \) is chordal.*

In this section, we give a necessary and sufficient condition for the square of a chordal graph to be chordal. This then provides a necessary and sufficient condition for all powers of a chordal graph to be chordal.

Before proving the main theorem, we list a well-known property of chordal graphs, introduce certain definitions, and state a lemma that derives from theorem 2.1 of [11].

**Property 1.** *Let \( C \) be a cycle of a chordal graph \( G \). Then for each edge \( uv \in C \), there

![Fig. 1. A chordal graph whose square is not chordal.](image-url)
exists a $w$ of $C$, such that $wuv$ is a triangle: that is, $uw$, $vw$ and $wu$ are distinct edges of $G$.

**Definition 1.** A cycle $C_n = (v_1, v_2, ..., v_n)$ is a **chordal cycle** if the induced subgraph $(C_n)$ is chordal.

**Definition 2.** A subgraph $S_n$ of $G$ is a **sunflower** if it consists of a chordal cycle $C_n = (v_1, v_2, ..., v_n)$ together with a set of $n$ independent vertices $\{u_1, u_2, ..., u_n\}$ such that for each $i$, $u_i$ is adjacent to only $v_i$ and $v_j$, where $j = i - 1 \mod n$.

**Definition 3.** A sunflower $S_n$ of $G$ is called a **suspended sunflower** in $G$ if there exists a vertex $w \in S_n$, such that $w$ is adjacent to at least one pair of vertices $u_j$ and $u_k$, where $j \neq k \pm 1 \mod n$. Fig. 2 provides an example of a suspended sunflower $S_7$.

![Fig. 2. A suspended sunflower $S_7$.](image)
Lemma 1 ([11]). Suppose $G$ is chordal but $G^2$ is not chordal, and let $C_n = (u_1, u_2, \ldots, u_n)$, $n \geq 4$, be a chordless cycle in $G^2$. Then no edge of $C_n$ is an edge of $G$.

Theorem 4. If $G$ is chordal and $G^2$ is not chordal, then $G$ has at least one sunflower $S_n$, $n \geq 4$, which is not suspended in $G$.

Proof. Let $C_n = (u_1, u_2, \ldots, u_n)$, $n \geq 4$, be a chordless cycle in $G^2$. By the above lemma no edge of $C_n$ is an edge of $G$, i.e., every edge $u_iu_{i+1}$ in $C_n$ is an edge of $G^2$ and thus can be extended to a two-step path $u_iu_1u_{i+1}$ in $G$. Extending in this way all the edges of $C_n$ in $G^2$, we get a cycle $C_n' = (u_1, u_2, u_3, \ldots, u_n)$. All the $v_i$'s are distinct, and disjoint from the $u_j$'s, otherwise $C_n$ will have a chord in $G^2$. Note that no $u_i$ is adjacent to $u_j$ in $G$. Also $v_i$ is adjacent in $G$ to $u_i$ and $u_{i+1}$ (mod $n$) and to no other $u$'s, otherwise we get a chord of $C_n$ in $G^2$. Consider the edge $u_iu_j$; by Property I, there must exist a vertex in $C_n'$ joined in $G$ to $u_i$ and $v_j$. The only such possible vertex then is $u_{i-1}$ (mod $n$). Hence, $v_1v_2v_3\ldots v_nv_1 \in E$ and so $Z = (u_1, u_2, \ldots, u_n)$ is a cycle in $G$. Since $G$ is chordal, $Z$ must be a chordal cycle. Thus $Z$ together with $\{u_1, u_2, \ldots, u_n\}$ forms a sunflower $S_n$, $n \geq 4$. Also, since $(u_1, u_2, \ldots, u_n)$ is a chordless cycle in $G^2$, no pair $u_j, u_k$ of vertices with $j \neq k \pm 1$ (mod $n$) can be adjacent in $G^2$. Hence, no such pair $u_j$ and $u_k$ is adjacent in $G$ to any vertex $w$ in $G$. Thus, the sunflower $S_n$ is not suspended. The following converse of Theorem 4 is readily verified also.

Theorem 5. If $G$ contains a sunflower $S_n$, $n \geq 4$, which is not suspended, then $G^2$ is not chordal.

Combining the above results, we state

Theorem 6. The square of a chordal graph $G$ is chordal iff every sunflower $S_n$, $n \geq 4$, of $G$ is suspended.

Certain corollaries follow immediately from Theorems 2 and 6.

Corollary 1. If $G$ is chordal with no sunflower, then $G^k$ is chordal for all $k \geq 1$.

Corollary 2. If $G$ is a tree, then $G^k$ is chordal for all $k \geq 1$.

$G$ is called a block graph if each block of $G$ induces a complete subgraph. Since a block graph $G$ is chordal and does not contain a sunflower, we have the following result, obtained independently by Jamison [10].

Corollary 3. If $G$ is a block graph, then $G^k$ is chordal for all $k \geq 1$. 

3. Centers of chordal graphs

In this section we establish properties of the center of a chordal graph \( G = (V, E) \) and demonstrate a relationship between the radius and diameter of chordal graphs. First, some appropriate terminology is established. The eccentricity \( e(u) \) of any vertex \( u \in V \) is given by

\[
e(u) = \max\{d(x, u) : x \in V\}.
\]

The radius \( r(G) \) of graph \( G \) is the minimum eccentricity of any vertex in \( G \), and the center \( C(G) \) is the set of all vertices \( u \) such that \( e(u) = r(G) \). The diameter \( d(G) \) is the maximum eccentricity of any vertex in \( G \).

A set \( S \subseteq V \) is an \( x-y \) separator for distinct nonadjacent vertices \( x, y \notin S \) if \( x \) and \( y \) are in different connected components of the subgraph induced by \( V - S \). In other words, every path between \( x \) and \( y \) contains a vertex of \( S \). It has been shown that in a chordal graph every minimal \( x-y \) separator induces a complete subgraph [4, 13].

**Theorem 7.** If \( G \) is a connected chordal graph, then the induced subgraph \( \langle C(G) \rangle \) is connected.

**Proof.** We may assume that \( C(G) \neq V \), since otherwise the result holds trivially. Suppose that \( \langle C(G) \rangle \) is not connected, so there exist distinct vertices \( c_1, c_2 \in C(G) \) that are not connected by a path in \( C(G) \). Thus \( Z = V - C(G) \neq \emptyset \) is a \( c_1-c_2 \) separator, and let \( Z_0 \subseteq Z \) be a minimal \( c_1-c_2 \) separator. By the result cited above \( \langle Z_0 \rangle \) is complete.

Since \( G \) is connected, there is a path in \( G \) joining \( c_1 \) and \( c_2 \), and it must contain some \( z_0 \in Z_0 \). Let \( v \in V \) be such that \( e(z_0) = d(v, z_0) \), and let \( P(v, z_0) \) denote a shortest path between \( v \) and \( z_0 \), of length \( e(z_0) \). Similarly, let \( P(v, c_1) \) and \( P(v, c_2) \) denote shortest paths from \( v \) to \( c_1 \), and \( v \) to \( c_2 \), respectively. Suppose \( a \) is the last common vertex on paths \( P(v, z_0) \) and \( P(v, c_1) \), and \( b \) is the last common vertex on paths \( P(v, z_0) \) and \( P(v, c_2) \).

Thus, \( P(c_1, a) \cup P(a, b) \cup P(b, c_2) \) is a path joining \( c_1 \) and \( c_2 \), and hence must contain some vertex \( z_1 \in Z_0 \) (possibly \( z_1 = z_0 \)). In any event, since \( \langle Z_0 \rangle \) is complete, \( d(z_1, z_0) \leq 1 \). Now either \( z_1 \in P(v, c_1) \) or \( z_1 \in P(v, c_2) \). In the former case then using the triangle inequality and the fact that \( z_1 \neq c_1 \) \( [Z_0 \cap C(G) = \emptyset] \) gives

\[
e(z_0) = d(v, z_0) \leq d(v, z_1) + d(z_1, z_0) \leq d(v, z_1) + 1
\]

Now since \( c_1 \in C(G) \) is a vertex of minimum eccentricity, then also \( z_0 \in C(G) \). This however is impossible because \( z_0 \in Z_0 \subseteq Z = V - C(G) \). In a similar way, the case \( z_1 \in P(v, c_2) \) leads to a contradiction, whence the theorem is proved.

It is well known that the center of any graph lies in a block [3]. In general, though,
the induced subgraph of the center need not necessarily be connected. Fig. 3 exhibits
a graph $G$ where $C(G) = \{v_3, v_4\}$ and $\langle C(G) \rangle$ is not connected. Theorem 7 asserts
that if $G$ is chordal and connected, this situation cannot occur. In the special case
of a tree, the center consists of either a single vertex or two adjacent vertices [9] and
so is connected, as predicted by the above theorem.

Define sets $C^0(G) = V$ and $C^k(G) = C(\langle C^{k-1}(G) \rangle)$ for $k \geq 1$. Then by Theorem 7,
$\langle C^k(G) \rangle$ is chordal and connected for all $k \geq 1$ if $G$ is chordal and connected. Because
$V \supseteq C(G) \supseteq C^2(G) \supseteq \cdots$, the following corollary to Theorem 7 is immediate.

**Corollary 4.** If $G$ is chordal and connected, then there exists a smallest $k \geq 1$ such
that $\langle C^k(G) \rangle = \langle C^{k-1}(G) \rangle$ is chordal and connected.

A (connected chordal) graph $G$ is said to be self-centered if $k = 1$ in the above cor-
ollary: that is, $G = \langle C(G) \rangle$. Clearly, $K_n$ is self-centered for all $n > 1$ since every
vertex has eccentricity 1. Fig. 4 illustrates a self-centered graph where every vertex
has eccentricity 2. Equivalently, $r(G) = d(G) = 2$ for this graph.

It is well known [3] that $r(G) \leq d(G) \leq 2r(G)$ holds for any connected graph $G$.
It will be shown (in Theorem 8) that for connected chordal graphs, $2r(G) - 3 \leq
d(G) \leq 2r(G)$. First, it will be useful to make the following observation.

**Lemma 2.** Suppose $\langle S \rangle$ is a complete induced subgraph of $G = (V, E)$. Given
$s_1, s_2 \in S$, then $|d(t, s_1) - d(t, s_2)| \leq 1$ for all $t \in V$, and $|e(s_1) - e(s_2)| \leq 1$.

**Proof.** Since $\langle S \rangle$ is complete, $d(s_1, s_2) \leq 1$, and the stated results follow immediate-
ly from the triangle inequality.

**Theorem 8.** If $G$ is chordal and connected, then
\[
\frac{1}{2} d(G) \leq r(G) \leq \left\{\frac{1}{2} d(G)\right\} + 1.
\]

Fig. 3. Graph $G$ with $\langle C(G) \rangle$ not connected.
Proof. Inasmuch as the first inequality holds for all graphs [3], we need only demonstrate the second inequality. Since this inequality holds when $G$ is complete $[r(G) = d(G) = 1]$, it will be supposed that $G$ is not complete.

Let $a$ and $b$ be a pair of diametrical vertices: i.e., $d(a, b) = d(G) = D$. Suppose $v$ is a ‘midpoint’ of some shortest path $P(a, b)$ between $a$ and $b$: namely, $v \in P(a, b)$ and

$$d(v, a) \geq \frac{1}{2} D, \quad d(v, b) \geq \frac{1}{2} D.$$  \hspace{1cm} (1)

Since $a$ and $b$ are not adjacent ($G$ is not complete) and the removal of $v$ separates $a$ and $b$ along $P(a, b)$, there exists a minimal $a$-$b$ separator set $S$ containing $v$. Let $A$ and $B$ be distinct connected components of $(V - S)$ with $a \in A$ and $b \in B$. Since $G$ is chordal then $\langle S \rangle$ is complete. We claim that

$$d(x, v) \leq \frac{1}{2} D + 1, \quad x \in V. \hspace{1cm} (2)$$

If $x \in S$, then since $\langle S \rangle$ is complete $d(x, v) \leq 1 \leq \frac{1}{2} D + 1$ and so (2) holds. Assume without loss of generality that $x \in B$. Because $x$ and $b$ are in different connected components of $\langle V - S \rangle$, the shortest path $P(x, b)$ must contain some vertex $w \in S$.

Case I: $d(v, b) \leq d(w, b)$.
Since $d(x, v) \leq d(x, w) + d(w, v) \leq d(x, w) + 1$, then using the triangle inequality, the fact that $\langle S \rangle$ is complete and relation (1) give

$$d(x, v) + d(v, b) \leq d(x, w) + d(w, b) + 1 = d(x, b) + 1 \leq D + 1,$$

$$d(x, v) \leq D + 1 - d(v, b) \leq D + 1 - \frac{1}{2} D = \left\lfloor \frac{1}{2} D \right\rfloor + 1.$$

Case II: $d(x, v) \leq d(x, w)$.
A similar argument gives $d(v, b) \leq d(w, b)$, and $d(x, v) + d(v, b) \leq d(x, b) + 1 \leq D + 1$. This again implies as in Case I that $d(x, v) \leq \left\lfloor \frac{1}{2} D \right\rfloor + 1$.

Case III: $d(v, b) = d(w, b) + 1$, $d(x, v) = d(x, w) + 1$. Notice that by Lemma 2 either $d(v, b) \leq d(w, b)$ or $d(v, b) = d(w, b) + 1$ holds since $v, w \in S$; likewise, either $d(x, v) \leq d(x, w)$ or $d(x, v) = d(x, w) + 1$. As a result, Case III is the only remaining case to check.

Fig. 4. A self-centered graph $G$ with $r(G) = d(G) = 2$. 
Now let $e$ denote the last common vertex on shortest paths $P(a, x)$ and $P(a, u)$, and let $f$ denote the last common vertex on shortest paths $P(x, a)$ and $P(x, w)$.

Subcase III(a). $v \in P(a, x)$.

Since $v \in P(a, x)$, $d(x, v) + d(v, a) = d(x, a) \leq D$. Thus by (1), $d(x, v) \leq D - d(v, a) \leq D - \lfloor \frac{1}{2}D \rfloor = \{ \frac{1}{2}D \}$, and so assertion (2) holds.

Subcase III(b). Vertices $e, f, u, w$ form a triangle of edges in $G$. In view of the assumption $d(x, v) = d(x, w) + 1$, the only possibility is that $f = w$. Hence

\[ d(x, w) + d(w, e) + d(e, a) = d(x, a) \leq D, \]
\[ d(x, w) + 1 \leq D - d(e, a) \]
\[ = D - (d(v, a) - 1) \leq D - \lfloor \frac{1}{2}D \rfloor + 1 = \lfloor \frac{1}{2}D \rfloor + 1. \]

Subcase III(c). Vertices $e, f, u, w$ induce a cycle of length $\geq 3$. More precisely, the union of shortest path segments $P(e, u) \cup P(u, w) \cup P(w, f) \cup P(f, e)$ defines a (simple) cycle $C$ of length $\geq 3$. Since edge $uw \in C$, Property I of chordal graphs implies that there must be some vertex $u \neq u, w$ such that $uw \in E, uw \in E$. Let $r, s$ denote vertices on $P(e, u)$ and $P(f, w)$ adjacent to vertices $u, w$, respectively.

Because $P(e, u)$ and $P(f, w)$ are shortest paths, the only possibilities for vertex $u$ are: $u = s, u = r$ or $u \notin P(e, f)$. The first possibility is precluded by the assumption $d(x, v) = d(x, w) + 1$. In the second possibility

\[ d(a, w) \leq d(a, r) + d(r, w) = d(a, r) + 1 = d(a, v), \]
\[ d(w, b) = d(v, b) - 1, \]

and using the fact that $v \in P(a, b)$ yields

\[ d(a, b) \leq d(a, w) + d(w, b) \leq d(a, v) + d(v, b) - 1 = d(a, b) - 1, \]
a contradiction. The remaining possibility $u \in P(e, f)$ is then the only feasible one. Now we must have either (i) or (ii):

(i) $d(x, u) \leq \lfloor \frac{1}{2}D \rfloor$.

Here, $d(x, u) \leq d(x, u) + d(u, v) = d(x, u) + 1 \leq \lfloor \frac{1}{2}D \rfloor + 1$, and assertion (2) is verified.

(ii) $d(x, u) \geq \lfloor \frac{1}{2}D \rfloor + 1$.

Since $d(a, x) \leq D$ and $u \in P(a, x)$, then $d(a, u) \leq \lfloor \frac{1}{2}D \rfloor - 1$. Thus

\[ d(a, w) \leq d(a, u) + d(u, w) = d(a, u) + 1 = \lfloor \frac{1}{2}D \rfloor \leq d(a, u), \]
\[ d(w, b) = d(v, b) - 1, \]

whence $d(a, b) \leq d(a, w) + d(w, b) \leq d(a, v) + d(v, b) - 1 = d(a, b) - 1$, a contradiction. So this case cannot occur.

In all possible cases, then, we have shown that relation (2) obtains: $d(x, v) \leq \lfloor \frac{1}{2}D \rfloor + 1$, for all $x \in V$. Therefore, $e(v) \leq \lfloor \frac{1}{2}D \rfloor + 1$ and also $r(G) \leq e(v) \leq \lfloor \frac{1}{2}D \rfloor + 1$ holds, whence the theorem is proved.

**Corollary 5.** If a chordal graph $G$ is self-centered, then $r(G) = d(G) \leq 3$. 
Proof. A self-centered graph $G$ has $r(G) = d(G)$. If $r(G)$ is even, then Theorem 8 gives $r(G) \leq \frac{1}{2}r(G) + 1$ so $r(G) = 2$. If $r(G)$ is odd, Theorem 8 gives $r(G) = 1$ or $r(G) = 3$.

References