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On the maximum size of a minimal k-edge connected augmentation

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ABSTRACT

We present a short proof of a generalization of a result of Cheriyan and Thurimella: a simple graph of minimum degree k can be augmented to a k-edge connected simple graph by adding $\leqslant \frac{kn}{k+1}$ edges, where n is the number of nodes. One application (from the previous paper) is an approximation algorithm with a guarantee of $1 + \frac{2}{k+1}$ for the following NP-hard problem: given a simple undirected graph, find a minimum-size k-edge connected spanning subgraph. For the special cases of $k = 4, 5, 6$, this is the best approximation guarantee known.

1. Introduction

Our goal is to study an extremal question in graph connectivity that has a well-known application in the area of approximation algorithms; also, we present a short proof for a generalization of a key result on this topic. Our first result is on the edge connectivity of simple, undirected graphs; we also have a result for undirected multigraphs. Let n and m denote the number of nodes and edges. For a graph $G = (V, E)$ and $S \subseteq V$, $\delta(S)$ denotes the *cut* with *shores* S and $V - S$, i.e., $\delta(S)$ is the set consisting of edges that have one end in S and the other end in $V - S$. By a *k-cut* we mean a cut that consists of exactly k edges, and by a $k^{\ominus}$*-cut* we mean a cut that has $\leqslant k$ edges. Recall that a graph G is called *k-edge connected* if every cut $\delta(S)$, where $\emptyset \neq S \subset V$, has $\geqslant k$ edges. We study the following question:

> *Given a simple graph* (V, M) *of minimum degree d, what is the maximum size of an edge set* F (*where* $M \cap F = \emptyset$) *such that the graph* $G = (V, M \dot\cup F)$ *stays simple and every edge in* F *belongs to some* $k^{\ominus}$*-cut of* G?

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A key special case of the question was answered by [2, Theorem 4.3] which proved an upper bound on $|F|$ of $\frac{k(n-1)}{k+1}$ when $d = k$ and the resulting simple graph G is required to be k-edge connected, that is, a graph of minimum degree k can be augmented to a simple k-edge connected graph by adding at most this number of edges.

We discuss two applications. (At the moment, these are the only applications known to us.) The first one is to the problem of finding an approximately minimum-size k-edge connected spanning subgraph of a given simple graph $G = (V, E)$. Let *opt* denote the minimum size. For $k \geqslant 2$, computing *opt* is NP-hard. A polynomial-time algorithm in [2] achieves an approximation guarantee of $1 + \frac{2}{k+1}$ by first finding a minimum-size subgraph (V, M) of minimum degree k (this can be done in polynomial-time, via matching algorithms), and then adding an inclusionwise-minimal set of edges $F \subseteq E - M$ such that the resulting graph is k-edge connected. The minimality of F implies that every edge in F belongs to a k-cut of the resulting graph. The approximation guarantee follows because $opt \geqslant kn/2$, $opt \geqslant |M|$, and $|F| \leqslant \frac{kn}{k+1} \leqslant \frac{2\,opt}{k+1}$. Another application is to an edge-connectivity analogue of Mader's "cycle theorem" for k-node connected graphs [10, Theorem 1]. An edge e of a k-edge connected graph G is called *critical* if e belongs to a k-cut of G, that is, if $G - e$ is *not* k-edge connected; analogously, an edge e of a k-node connected graph G is called *critical* (w.r.t. k-node connectivity) if $G - e$ is *not* k-node connected. Mader's theorem [10, Theorem 1] states that in a k-node connected graph, a cycle consisting of critical (w.r.t. k-node connectivity) edges must be incident to a node of degree k. An immediate consequence is that if $G = (V, E)$ is k-node connected and (V, M) is a subgraph of minimum degree k, then the number of critical (w.r.t. k-node connectivity) edges in $E - M$ is at most $n - 1$. Whereas, [2, Theorem 4.3] gives a bound of $\frac{k(n-1)}{k+1}$ for the analogous number for the k-edge connectivity of simple graphs.

We briefly discuss the research on approximation algorithms for minimum-size k-edge connected spanning subgraphs. This line of research was initiated by Khuller and Vishkin [9]. Subsequently, many papers have been published on this topic; see the survey by Khuller [8], and for more recent publications, see the references in [4]. Consider the problem restricted to simple graphs, i.e., assume that the input graph is simple. The algorithm in [2] (discussed above) achieves an approximation guarantee of $1 + \frac{2}{k+1}$. Recently, Gabow and Gallagher [4] presented an approximation algorithm with a guarantee of $1 + \frac{1}{2k} + O(\frac{1}{k^2})$; this improves on the guarantee of [2] for $k \geqslant 7$. The algorithm of [4] is based on Jain's iterative rounding method [6]. One drawback of this method is that a large linear programming problem has to be solved. In contrast, the methods in [2] and in this paper are based on simple combinatorial algorithms. For the special but important cases of $k = 2$ and $k = 3$, better approximation guarantees are known. Jothi, Raghavachari and Varadarajan [7] presented a 5/4-approximation algorithm for $k = 2$, and Gubbala and Raghavachari [5] presented a 4/3-approximation algorithm for $k = 3$. (The approximation algorithms and guarantees of [7] and [5] apply for both simple graphs and multigraphs.) To the best of our knowledge, for the special cases of $k = 4, 5$ and 6, there has been no improvement on the approximation guarantee of [2].

1.1. Our results

Our main contribution is a short and simple proof of the following generalization of [2, Theorem 4.3]:

Theorem 1. *Let d, k be positive integers where $n > d \geqslant k$, and let $G = (V, M \dot\cup F)$ (where $M \cap F = \emptyset$) be a simple graph such that* (i) *the graph (V, M) has minimum degree d, and* (ii) *each edge in F belongs to some $k^{\ominus}$-cut of G. Then $|F| \leqslant k\lfloor \frac{n}{d+1} \rfloor - k$, and this bound is tight.*

We extend this result to the case $d < k$ by noting that a graph of minimum degree $d < k$ can be made into a graph of minimum degree k by adding $\leqslant (k - d)(n - 1)$ edges.

Corollary 2. *For $d < k$ (and the other notation as in the above theorem), we have $|F| \leqslant (k - d)n + k\lfloor \frac{n}{k+1} \rfloor - 2k + d$.*

Possibly, the upper bound is not tight. For the case of $d = 0$, the complete bipartite graph $K_{n-k,k}$ shows that $|F| \geqslant k(n-k)$, whereas our upper bound is $kn + k\lfloor \frac{n}{k+1} \rfloor - 2k$. For the case of $d = k-1 \geqslant 1$, we have a lower bound (an example) with $|F| \geqslant 2k\lfloor \frac{n-(k+1)}{h} \rfloor$, where $h = (k+1) + \lceil \sqrt{k+1} \rceil$.

Our question arises also in the setting of multigraphs, and we settle this by a simple proof that gives tight bounds.

Theorem 3. *Let $d, k \geqslant 0$ be integers, and let $G = (V, M \dot\cup F)$ (where $M \cap F = \emptyset$) be a multigraph such that* (i) *the graph (V, M) has minimum degree d, and* (ii) *each edge in F belongs to some $k^{\ominus}$-cut of G. If $d \leqslant k$, then $|F| \leqslant (k - \frac{d}{2})n - k$, otherwise $|F| \leqslant \frac{kn}{2} - k$. Moreover, both these bounds are tight for even n.*

2. Proofs

Let the graph (or multigraph) $G = (V, M \dot\cup F)$ be as in the theorems, that is, (V, M) has minimum degree d, $M \cap F = \emptyset$, and each edge in F belongs to a $k^{\ominus}$-cut of G.

We call an edge in F (in M) an *F-edge* (an *M-edge*). Call a node set $S \subseteq V$ a *good set* if $\emptyset \neq S \neq V$ and $\delta(S)$ is a $k^{\ominus}$-cut. A good set S is said to *cover* an edge if the cut $\delta(S)$ contains the edge. It is well known that there exists a laminar family of good sets $\mathcal{L} = \{A_1, A_2, \dots, A_t\}$ that covers all the edges that belong to $k^{\ominus}$-cuts (i.e., each such edge is in $\delta(A_i)$ for some $A_i \in \mathcal{L}$); this follows from the construction of Gomory–Hu trees [3, Chap. 3.5.2]. (In more detail, there exists a laminar family of sets such that for every pair of nodes s, t, one of the sets in the laminar family is a shore of a minimum s, t cut.) For a laminar family $\mathcal{L}$, let $V(\mathcal{L})$ denote $\bigcup\{A_i \mid A_i \in \mathcal{L}\}$. For any set A_i in a laminar family $\mathcal{L}$, define the *core* ϕ_i to be $A_i - \bigcup\{A_j \mid A_j \in \mathcal{L},\ A_j \subsetneq A_i\}$ (ϕ_i is the set of nodes in A_i but not in any set of $\mathcal{L}$ that is a proper subset of A_i), and define the *level* ℓ_i to be zero if A_i is an inclusionwise-minimal set of $\mathcal{L}$, and $1 + \max\{\ell_j \mid A_j \in \mathcal{L},\ A_j \subsetneq A_i\}$ otherwise. Observe that $A_i = \phi_i$ iff $\ell_i = 0$. For any core ϕ_i, we call an edge e in $\delta(\phi_i)$ either an *up edge* if $e \in \delta(A_i)$, or a *down edge* if $e \notin \delta(A_i)$; thus an up edge has exactly one end in A_i, and a down edge has both ends in A_i.

2.1. A proof of Theorem 1

Proof of Theorem 1. Fix $n = |V|$, d, and k, where $n > d \geqslant k$. Let V, M, and F satisfy the conditions in the theorem, and let $\mathcal{L}$ be a laminar family of good sets covering the edges in F; moreover, assume that $|F|$ is maximum, $|M|$ is maximum, and $\mathcal{L}$ is inclusionwise minimal. Let this minimal laminar family be $\mathcal{L} = \{A_1, \dots, A_t\}$.

The minimality of $\mathcal{L}$ implies that for each $A_i \in \mathcal{L}$ the $k^{\ominus}$-cut $\delta(A_i)$ has an F-edge. Moreover, there exists an F-edge in $\delta(\phi_i) \cap \delta(A_i)$; otherwise, all the F-edges in $\delta(A_i)$ are covered by good sets in $\mathcal{L}$ that are proper subsets of A_i. A key observation is that each core ϕ_i $(i = 1, \dots, t)$ induces a clique in the graph (V, M). To justify this, note that $\mathcal{L}$ does not cover any edge with both ends in the same core, so none of these edges can belong to F. If there is a nonadjacent pair of nodes in some core ϕ_i, then we may add an M-edge between them; this preserves all the conditions; then we get a contradiction to the maximality of $|M|$.

The theorem follows from the next claim.

Claim. *Each core in $\mathcal{L}$ contains $\geqslant d+1$ nodes.*

Proof. By way of contradiction, suppose the claim fails. Let ϕ_i be a core with the smallest level ℓ_i and with $p := |\phi_i| < d+1$. Then we have

$$\left|\delta(\phi_i) \cap M\right| \geqslant p(d+1-p) \geqslant d \geqslant k,$$

because the graph is simple, each node in ϕ_i is incident with $\geqslant d$ edges of M, and only $p-1$ of these M-edges have both ends in ϕ_i; also, for each $p = 1, \dots, d$ we have $p(d+1-p) \geqslant d$. Suppose that the level ℓ_i is zero, that is, suppose $\phi_i = A_i$; then we get a contradiction to the minimality of $\mathcal{L}$, since $\delta(A_i)$ has $\geqslant k$ edges of M and so cannot have any edges of F. Hence, the level ℓ_i must be $\geqslant 1$, and there exist one or more down M-edges incident to nodes in ϕ_i.

Suppose that the core ϕ_i has a node v^* that is incident to both a down M-edge, call it e, and an up F-edge, call it f. Then we swap these two edges between M and F, i.e., we replace F by $(F - \{f\}) \cup \{e\}$ and M by $(M - \{e\}) \cup \{f\}$. It is easily seen that the new M and the new F satisfy the conditions of the theorem, and $\mathcal{L}$ covers the new F. (To see this, let $e = v^*x$, and note that there is a good set $A_j \in \mathcal{L}$ with $\ell_j < \ell_i$ and $x \in \phi_j$; then e is covered by A_j, and x is in the clique of (V, M) induced by ϕ_j, where $|\phi_j| \geqslant d + 1$ (by choice of ϕ_i, ℓ_i), hence, x is incident to $\geqslant d$ edges of the new M.)

If there is no such node $v^* \in \phi_i$, then note that there is a node $u \in \phi_i$ that is incident to an up F-edge, call it $f = uy$, and also there is a node $w \in \phi_i$ that is incident to at least one down M-edge and to no up edge. (To see the last part, suppose that each node of ϕ_i that is incident to a down M-edge is also incident to an up M-edge; then we get a contradiction since $\delta(\phi_i)$ contains $\geqslant k$ up M-edges, since $d - (p - 1)$ up M-edges are incident to u and $\geqslant 1$ up M-edge is incident to every other node in ϕ_i.) In this case, we "replace" the F-edge $f = uy$ by a new F-edge $f' = wy$, i.e., we remove f from F and add $f' = wy$ to F. It is easily seen that M and the new F satisfy the conditions of the theorem, $\mathcal{L}$ covers the new F, and the graph stays simple. Now, the node w satisfies the conditions on v^*, so we proceed as above, i.e., we swap two edges incident to v^* between M and F.

Clearly, these edge swaps between M and F can be repeated until $\delta(\phi_i)$ has no up F-edge. At that point, we get a contradiction to the minimality of $\mathcal{L}$ (since $A_i \in \mathcal{L}$ is redundant). This proves the claim. □

To obtain the theorem, assume that $|\mathcal{L}| \geqslant 1$, and focus on $|V(\mathcal{L})| = |A_1 \cup \cdots \cup A_t|$. We claim that $|V(\mathcal{L})| \leqslant n - (d + 1)$; to see this, pick a good set A_j with level $\ell_j = 0$ and pick a node v^* in $A_j = \phi_j$; then replace every good set $A_i \in \mathcal{L}$ that contains v^* by its complement $V - A_i$; the resulting family of good sets $\mathcal{L}'$ stays laminar, covers F, and stays minimal; moreover, $V(\mathcal{L}')$ contains none of the nodes in ϕ_j, hence, $|V(\mathcal{L}')| \leqslant n - (d + 1)$. Finally, examine the good sets $A_i \in \mathcal{L}'$ by increasing levels, and note that each good set contributes at most k new edges to F and contributes $|\phi_i|$ new nodes to $V(\mathcal{L}')$, hence,

$$\left|\mathcal{L}'\right| \leqslant \left\lfloor \frac{|V(\mathcal{L}')|}{d+1} \right\rfloor \leqslant \left\lfloor \frac{n-(d+1)}{d+1} \right\rfloor, \quad \text{and} \quad |F| \leqslant k\left|\mathcal{L}'\right| \leqslant k\left\lfloor \frac{n-(d+1)}{d+1} \right\rfloor.$$

To see that the bound on $|F|$ is tight for $n = (t + 1)(d + 1)$, consider the k-edge connected graph $G = (V, E)$ and edge set M obtained by taking $t + 1$ copies of the $(d + 1)$-clique, $C_0, C_1, \ldots, C_t$, and for each $i = 1, \ldots, t$, choosing an arbitrary node w_i in C_i and adding k (nonparallel) edges between w_i and C_0. Take $M = \bigcup_{i=0}^{t} E(C_i)$. Note that $F = E - M$, so $|F| = \frac{k}{d+1}(n - (d + 1))$. This construction extends to all $n > d \geqslant k$: fix $t + 1 = \lfloor \frac{n}{d+1} \rfloor$, and put the "extra" $n - (t + 1)(d + 1)$ nodes into C_0 (which becomes a bigger clique). □

2.2. A lower bound for Corollary 2

Here, assuming $k \geqslant 2$, we present an example of a k-edge connected simple graph $G = (V, E)$ such that there is subgraph (V, M) of minimum degree $(k - 1)$ such that each edge in $F := E - M$ is in a k-cut of G and $|F| \geqslant 2k\lfloor \frac{n-(k+1)}{h} \rfloor$, where $h = (k + 1) + \lceil \sqrt{k + 1} \rceil$.

Our construction uses the following k-edge connected simple graph H. Let r and q be integers such that

$r^2 \geqslant q \geqslant k, \quad k \geqslant r,$

q is an even number, and

r is the smallest positive integer satisfying these conditions.

Thus, we may fix $q = k$ or $q = k + 1$, and $r = \lceil \sqrt{q} \rceil$. Let Q and R be (disjoint) sets of nodes, with $|Q| = q$ and $|R| = r$, and let $R = \{a_1, a_2, \ldots, a_r\}$. Let $V(H) = Q \cup R$; thus, we have $|V(H)| = q + r \leqslant (k + 1) + \lceil \sqrt{k + 1} \rceil = h$. We define the edge set of H by defining $H[Q]$ and $H[R]$, and moreover, H has an edge between each node in Q and each node in R. (By $H[Q]$ and $H[R]$, we mean the subgraphs of H induced by Q and R, respectively.) We take $H[R]$ to be the complete graph K_r. We take $H[Q]$ to be a $(k-r)$-regular graph; thus, if $k-r=0$, then $H[Q]$ has no edges; moreover, if $k-r=1$, then $H[Q]$ consists of $q/2$ disjoint copies of the complete graph K_2 (thus the edge set of $H[Q]$ forms a perfect matching); finally, if $k-r \geqslant 2$, then we take $H[Q]$ to be a $(k-r)$-regular $(k-r)$-edge connected graph. (There exist ℓ-regular ℓ-edge connected simple graphs on q nodes, q an even number, for all integers $2 \leqslant \ell < q$; see the construction in [1, Chapter 3.3].) It can be seen that H is k-edge connected, and moreover, every node in Q has degree k. Next, we partition the edge set $E(H)$ into $F(H)$ and $M(H)$ as follows: first, partition Q into r sets $Q_1, Q_2, \ldots, Q_r$ such that each set has $\leqslant r$ nodes; for each node v in Q_j, $j=1,\ldots,r$, place the edge va_j in $F(H)$ (note that a_j is the jth node in R). Thus, we have $|F(H)|=q$. Observe that the remaining edges of H give a subgraph of minimum degree $(k-1)$, since each node $v \in Q$ is incident to exactly $(k-1)$ edges of $E(H)-F(H)$, and each node in R is incident to at least $(r-1)+(k-r)$ edges of $E(H)-F(H)$.

To construct G, assume that $n \geqslant h+k+1$. We take $t = \lfloor \frac{n-(k+1)}{h} \rfloor$ copies of H, and call them $H_1, H_2, \ldots, H_t$. We put the remaining nodes into a complete subgraph G_0; observe that G_0 has at least $k+1$ nodes and it is k-edge connected; we place all the edges of G_0 into M. For each H_i, $i=1,\ldots,t$, we add k edges between the nodes in $R(H_i)$ and $V(G_0)$, where $R(H_i)$ denotes the copy of R in H_i; the k edges in $\delta(V(H_i))$ are placed in F. Thus, we have $M = E(G_0) \cup \bigcup_{i=1}^{t} M(H_i)$, and $F = \bigcup_{i=1}^{t} (F(H_i) \cup \delta(V(H_i)))$; clearly, $|F| = t(q+k) \geqslant 2kt$. It can be seen that G is k-edge connected, and that (V, M) is a subgraph of minimum degree $(k-1)$.

If n is an integral multiple of $(r+q)$, then we can replace G_0 by another copy of H; then, we have $|F| \geqslant \frac{2kn}{(r+q)} - k$.

2.3. A proof of Theorem 3

Proof of Theorem 3. Recall that $\mathcal{L} = \{A_1, A_2, \ldots, A_t\}$ is a laminar family of good sets that covers F, and for each $A_i \in \mathcal{L}$ we denote the core of A_i by ϕ_i. Also, define $\phi_{t+1} := V - V(\mathcal{L}) = V - (A_1 \cup \cdots \cup A_t)$.

Consider the first part: if $d \leqslant k$, then $|F| \leqslant (k - \frac{d}{2})n - k$. We play the following dollar game. First, we give $k - \frac{d}{2}$ dollars to every core ϕ_i $(i=1,\ldots,t)$ for each of its nodes. In return, we demand that each ϕ_i $(i=1,\ldots,t)$ should pay one dollar for each of its up F-edges, and send 50 cents along each of its up M-edges.

Let us make sure that this demand can be met, provided the ϕ_i carry out the transactions in any order determined by increasing levels, starting from level zero. Observe that for a given ϕ_i there is no difficulty if $p := |\phi_i| \geqslant 2$. Indeed, in this case ϕ_i gets $p \cdot (k - \frac{d}{2}) \geqslant 2k - d \geqslant k$ dollars for its nodes, which is certainly enough to pay for and/or send money along ϕ_i's up edges, since the number of those edges is at most k. Now, assume that $p=1$. Let us denote by q the number of up M-edges. Then, when it is ϕ_i's turn to pay its dues, it has $k - \frac{d}{2}$ dollars for its node plus at least $\frac{(d-q)}{2}$ dollars it has received along its down M-edges. This makes a total of at least $k - \frac{q}{2}$ dollars. This is easily seen to be at most the amount ϕ_i is required to pay and/or send up.

To complete the proof, let us count the money we have invested in the ϕ_i $(i=1,\ldots,t)$, and the money we collect back from ϕ_{t+1}. The difference between these two sums is clearly an upperbound on $|F|$.

Let $b := |\phi_{t+1}| = |V - V(\mathcal{L})|$. Then we have invested $x := (n-b) \cdot (k - \frac{d}{2})$ dollars. If $b \geqslant 2$ then x is at most the claimed upper bound on $|F|$. If $b=1$ then $V - V(\mathcal{L})$ is a singleton set, say $\{v_n\}$, and we collect at least $\frac{d}{2}$ dollars along the M-edges incident to node v_n. Again, $x - \frac{d}{2}$ is at most the claimed upper bound. This proves the first part.

Consider the second part of the theorem: if $d > k$, then $|F| \leqslant \frac{kn}{2} - k$. We play a dollar game similar to that described in the proof of the first part, except for one difference: every core ϕ_i $(i=1,\ldots,t)$ receives $\frac{k}{2}$ dollars for each of its nodes, and is required to send $\frac{k}{2d}$ dollars along each of its up M-edges. As before, ϕ_i is required to pay a dollar for each of its up F-edges. For a singleton ϕ_i with q up M-edges (and hence $\geqslant d-q$ down M-edges), note that ϕ_i gets $\geqslant \frac{k}{2} + (d-q)\frac{k}{2d} = k - \frac{kq}{2d}$ dollars (it receives $\frac{k}{2}$ dollars and gets $\frac{k}{2d}$ dollars along each down M-edge), and this is sufficient for ϕ_i to pay out $\leqslant (k-q) + \frac{kq}{2d}$ dollars, since $k - q(1 - \frac{k}{2d}) \leqslant k - \frac{kq}{2d}$ for $k < d$. The rest of the proof is analogous to that of the first part, and is omitted.

To see that the bound on $|F|$ is tight for an even number $n = 2q$, consider the multigraph (V, M) formed by q disjoint copies of the multigraph consisting of two nodes and d edges; in other words, (V, M) is partitioned into q connected components, each with two nodes and d parallel edges. First, suppose that $d > k$. Then, we start with a set F_0 of $q - 1$ edges such that $(V, M \cup F_0)$ is connected (that is, F_0 corresponds to a spanning tree of the auxiliary graph where we have a node for each connected component of (V, M)), and then we obtain F by replacing each edge in F_0 by k parallel edges. Clearly, every edge in F is in a k-cut, and we have $|F| = k(q - 1) = \frac{kn}{2} - k$. Now, suppose that $d \leqslant k$ (the first part of the theorem). We use a similar construction, but for each connected component of (V, M), we add $k - d$ parallel F-edges in the component. Thus, we have $|F| = k(q - 1) + (k - d)q = kn - \frac{dn}{2} - k$. □

Remark. Notice that in the case of $d = k$ the statements and the proofs of both the parts in Theorem 3 become identical.

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