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Journal of Mathematical Analysis and Applications

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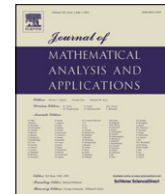
Connectedness of planar self-affine sets associated with non-consecutive collinear digit sets

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ARTICLE INFO

ABSTRACT

In the paper, we focus on the connectedness of planar self-affine sets $T(A, \mathcal{D})$ generated by an integer expanding matrix A with $|\det(A)| = 3$ and a collinear digit set $\mathcal{D} = \{0, 1, b\}v$, where $b > 1$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. We discuss the domain of the digit b to determine the connectedness of $T(A, \mathcal{D})$. Especially, a complete characterization is obtained when we restrict b to be an integer. Some results on the general case of $|\det(A)| > 3$ are obtained as well.

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1. Introduction

Let $M_n(\mathbb{Z})$ denote the set of $n \times n$ matrices with integer entries, let $A \in M_n(\mathbb{Z})$ be expanding, i.e., all eigenvalues of A have moduli strictly larger than 1. Assume $|\det(A)| = q$, and a finite set $\mathcal{D} = \{d_1, \ldots, d_q\} \subset \mathbb{R}^n$ with cardinality q, we call it a q-digit set. It is well known that there exists a unique *self-affine set* $T := T(A, \mathcal{D})$ [1] satisfying

$$T = A^{-1}(T + \mathcal{D}) = \left\{ \sum_{i=1}^{\infty} A^{-i} d_{j_i} : d_{j_i} \in \mathcal{D} \right\}.$$

T is called a *self-affine tile* if such T has a nonvoid interior.

The geometric and topological properties of $T(A, \mathcal{D})$ have been studied extensively. One of the interesting aspects is the connectedness, in particular the disklikeness (i.e., homeomorphic to a closed unit disk). It was asked by Gröchenig and Haas [2] that given an expanding integer matrix $A \in M_n(\mathbb{Z})$, whether there exists a digit set $\mathcal{D}$ such that $T(A, \mathcal{D})$ is a connected tile and they partially solved the question in $\mathbb{R}^2$. Hacon et al. [3] proved that any self-affine tile $T(A, \mathcal{D})$ with 2-digit set is always pathwise connected. A systematical study about this question was done by [4–6] which mainly concerned consecutive collinear digit sets of the form $\{0, 1, \ldots, q - 1\}v, v \in \mathbb{Z}^n \setminus \{0\}$ via the algebraic property of the characteristic polynomial of the matrix A. For some other case, Laarakker and Curry [7] considered the connectedness of $T(A, \mathcal{D})$ generated by an expanding matrix with rational eigenvalues and a so-called centered canonical digit set. In $\mathbb{R}^2$, the disklikeness is an interesting topic, Bandt and Gelbrich [8], Bandt and Wang [9], and Leung and Lau [10] investigated the disklikeness of self-affine tiles in terms of the neighbors of T. A translation of the tile $T + l$ is called a neighbor of T if $T \cap (T + l) \neq \emptyset$, where l is a lattice point (see Section 2 for details). Deng and Lau [11], and Kirat [12] discussed the connectedness and disklikeness of some other planar self-affine tiles with non-collinear digit sets.

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Here we need to point out that, in [4] Kirat and Lau obtained an interesting result in one dimensional space:

Proposition 1.1. *For $A = [q]$, and $\mathcal{D} = \{d_1, \ldots, d_{|q|}\} \subset \mathbb{R}$ with $q \in \mathbb{Z}$, $|q| \geq 2$, then $T(A, \mathcal{D})$ is an interval (a connected tile) if and only if, up to a translation, $\mathcal{D} = \{0, a, \ldots, (|q| - 1)a\}$ for some $a > 0$.*

Unfortunately, this result cannot be extended to a higher dimensional space in general. By using matrix expansion and eigenvalue arguments, Tan [13] constructed a counterexample:

Proposition 1.2. *Suppose $A \in M_2(\mathbb{Z})$ with $|\det(A)| = 3$ is expanding, whose characteristic polynomial is $f(x) = x^2 - x - 3$, and $\mathcal{D} = \{0, 1, b\}v$, where $b > 1$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. Then $T(A, \mathcal{D})$ is connected if $8/5 \leq b \leq 8/3$, and $T(A, \mathcal{D})$ is disconnected if $b < \frac{\sqrt{13}-1}{2}$ or $b > \frac{\sqrt{13}+5}{2}$.*

As a generalization of Proposition 1.1, we can only have the following.

Theorem 1.3. *Let $A \in M_n(\mathbb{Z})$ be expanding with $|\det(A)| = q \geq 2$, whose characteristic polynomial $f(x) = x^n \pm q$, the digit set is $\mathcal{D} = \{d_1, d_2, \ldots, d_q\}v$ where $d_i \in \mathbb{R}$ and $v \in \mathbb{R}^n \setminus \{0\}$. Then the self-affine set $T(A, \mathcal{D})$ is connected if and only if, up to a translation, $\mathcal{D}$ is of the form $\{0, 1, 2, \ldots, (q - 1)\}av$ for some $a > 0$.*

However, more results on this kind of non-consecutive collinear digit sets have not been obtained yet. Motivated by that, we try to investigate this property in detail.

For an expanding integer matrix $A \in M_2(\mathbb{Z})$, it is known by [8] that the characteristic polynomial of A is given by

$$f(x) = x^2 + px + q, \quad \text{with } |p| \leq q, \text{ if } q \geq 2; \qquad |p| \leq |q + 2|, \quad \text{if } q \leq -2.$$

When $|\det(A)| = 3$, there are 10 eligible characteristic polynomials of A as follows:

$$x^2 \pm 3; \qquad x^2 \pm x + 3; \qquad x^2 \pm 2x + 3; \qquad x^2 \pm 3x + 3; \qquad x^2 \pm x - 3.$$

Our main purpose in this paper is to study the self-affine sets generated by A with $|\det(A)| = 3$ and $\mathcal{D} = \{0, 1, b\}v$ with $b > 1$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. We get some criteria for b to determine the connectedness of $T(A, \mathcal{D})$. First if b is an integer, we have the following.

Theorem 1.4. *Let $|\det(A)| = 3$, the characteristic polynomial of A be $f(x) = x^2 + px \pm 3$ and $\mathcal{D} = \{0, 1, m\}v$ where $2 \leq m \in \mathbb{Z}$ such that $\{v, Av\}$ is linearly independent. Then we have*

 (i) *when $m = 2$, $T(A, \mathcal{D})$ is always a connected tile;*
 (ii) *when $m \geq 4$, $T(A, \mathcal{D})$ is always a disconnected set;*
(iii) *when $m = 3$, $T(A, \mathcal{D})$ is a connected set if $f(x) = x^2 \pm 2x + 3$ or $x^2 \pm 3x + 3$ or $x^2 \pm x - 3$; $T(A, \mathcal{D})$ is a disconnected set if $f(x) = x^2 \pm 3$ or $x^2 \pm x + 3$.*

Although there are many calculations in the proof, the method is elementary. To get our desired results, we apply the Hamilton–Cayley theorem to find the potential neighbors of T and to rule out the ineligible ones by a neighbor-generating algorithm. It seems that this method can be applied to the more general situation. By modifying the proof, we obtain

Theorem 1.5. *Assume that $A \in M_2(\mathbb{Z})$ with $|\det(A)| = 3$ is expanding, a digit set $\mathcal{D} = \{0, 1, b\}v$, where $b > 1$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. Then we have*

Case 1: $f(x) = x^2 \pm x + 3$ T is disconnected if $b \geq 67/25$ or $b \leq 67/42$;

Case 2: $f(x) = x^2 \pm 2x + 3$ T is disconnected if $b \geq 37/10$ or $b \leq 37/27$;

Case 3: $f(x) = x^2 \pm 3x + 3$ T is disconnected if $b \geq 33/10$ or $b \leq 33/23$;

Case 4: $f(x) = x^2 \pm x - 3$ T is disconnected if $b > 19/5$ or $b < 19/14$.

It should be remarked that we mainly discuss each case under the assumption of $b \geq 2$. If $1 < b \leq 2$ then $b/(b - 1) \geq 2$, with the same argument we can consider the digit set $\mathcal{D}' = \{0, 1, b/(b - 1)\}(b - 1)v$ which replaces the original one, since $T(A, \mathcal{D})$ is connected if and only if $T(A, \mathcal{D}')$ is connected (just by a translation). So the above related numbers come out.

Conjecture 1.6. *We conjecture that there exists a critical value $c \geq 2$ (dependent on the characteristic polynomial of A) such that $T(A, \mathcal{D})$ is connected if and only if $c/(c - 1) < b \leq c$.*

Since our estimates about b are rough, we can only solve the conjecture partially. While if we suppose that b is an integer, then a positive answer is given as in Theorem 1.4.

The rest of this paper is organized as follows. In Section 2, we introduce some basic and useful results which play an important role in our proofs. In Section 3, we first discuss the integer digit set case where b is an integer and then prove Theorems 1.3 and 1.4. Theorem 1.5 will be shown in Section 4. In the last section we consider the general planar self-affine sets with $|\det(A)| > 3$ and non-consecutive collinear digit sets, some sufficient conditions for $T(A, \mathcal{D})$ to be connected are obtained.

2. Preliminaries

In this section, we prepare some elementary knowledge about the geometric properties of self-affine sets which will be used frequently in the paper.

We define

$$\mathcal{E} = \{(d_i, d_j) : (T + d_i) \cap (T + d_j) \neq \emptyset, d_i, d_j \in \mathcal{D}\}$$

to be the set of edges for the digit set $\mathcal{D}$. We say that d_i and d_j are $\mathcal{E}$-connected if there exists a finite sequence (or path) $\{d_{j_1}, \ldots, d_{j_k}\} \subset \mathcal{D}$ such that $d_i = d_{j_1}, d_j = d_{j_k}$ and $(d_{j_l}, d_{j_{l+1}}) \in \mathcal{E}, 1 \leq l \leq k - 1$.

It is easy to check that $(d_i, d_j) \in \mathcal{E}$ if and only if $d_i - d_j = \sum_{k=1}^{\infty} A^{-k} v_k$ where $v_k \in \Delta\mathcal{D} := \mathcal{D} - \mathcal{D}$. Then we get a criterion of connectedness of a self-affine set by using a graph argument on $\mathcal{D}$:

Proposition 2.1 ([4]). *A self-affine set T with a digit set $\mathcal{D}$ is connected if and only if any two $d_i, d_j \in \mathcal{D}$ are $\mathcal{E}$-connected.*

When the digit set $\mathcal{D}$ can be written as $\{d_1 v, \ldots, d_q v\}$ for some non-zero vector $v \in \mathbb{R}^n$ and $d_1 < d_2 < \cdots < d_q$, $d_i \in \mathbb{R}$, $\mathcal{D}$ is said to be collinear. If $d_{i+1} - d_i = constant (= 1)$, $\mathcal{D}$ is called *consecutive collinear digit set*. Let $D = \{d_1, \ldots, d_q\}$, $\Delta D = D - D = \{d = d_i - d_j : d_i, d_j \in D\}$. Then $\mathcal{D} = Dv$ and $\Delta\mathcal{D} = \Delta Dv$. It is easy to see that the connectedness of T is invariant under a translation of the digit set, hence we always assume that $d_1 = 0$. The radix expansion of a point $x = \sum_{i=1}^{\infty} a_i A^{-i} v \in \mathbb{R}^2$ is given by $0.a_1 a_2 a_3 \ldots$ An overbar denotes repeating digits as in $0.12\overline{301} = 0.12301301301 \ldots$ Likewise, $a_{-2} a_{-1} a_0 . a_1 a_2 a_3 \ldots$ represents a point

$$a_{-2} A^2 v + a_{-1} A v + a_0 v + \sum_{i=1}^{\infty} a_i A^{-i} v.$$

Let $\mathbb{Z}[x]$ denote the class of polynomials with integer coefficients. We call a polynomial $f(x) \in \mathbb{Z}[x]$ an *expanding polynomial* if all its roots have moduli strictly bigger than 1. Note that a matrix $A \in M_n(\mathbb{Z})$ is expanding if and only if its characteristic polynomial is expanding.

We say that a monic polynomial $f(x) \in \mathbb{Z}[x]$ with $|f(0)| = q$ has the *Height Reducing Property* (HRP) if there exists $g(x) \in \mathbb{Z}[x]$ such that

$$g(x)f(x) = x^k + a_{k-1} x^{k-1} + \cdots + a_1 x \pm q,$$

where $|a_i| \leq q - 1, i = 1, \ldots, k - 1$.

This property was introduced by Kirat and Lau [4] to study the connectedness of self-affine tiles associated with consecutive collinear digit sets. It was proved that:

Proposition 2.2. *Let $A \in M_n(\mathbb{Z})$ with $|\det(A)| = q$ be expanding and let $\mathcal{D} = \{0, 1, 2, \ldots, (q-1)\}v$ be a consecutive collinear digit set in $\mathbb{R}^n$. Suppose the characteristic polynomial $f(x)$ of A has HRP, then T is connected.*

In [5], Kirat et al. conjectured that all expanding integer monic polynomials have HRP. Akiyama and Gjini [14] solved it up to degree 4. However it is still open for arbitrary degree. Recently, He et al. [6] have developed an algorithm to check HRP for any monic polynomial, and Akiyama et al. [15] have studied HRP of algebraic integers on canonical number systems from another perspective.

In the sequel, we always consider the planar self-affine set $T(A, \mathcal{D})$ associated with a collinear digit set of the form $\{d_1 v, \ldots, d_q v\}$, where v is a vector in $\mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. It is known that $\{v, Av\}$ is always linearly independent provided $|\det(A)| = 3$ (or any prime number) and $v \in \mathbb{Z}^2 \setminus \{0\}$. Denote the characteristic polynomial of A by $f(x) = x^2 + px + q$, where $p, q \in \mathbb{Z}$. Then we can regard A as the companion matrix of $f(x)$ for simplicity, i.e.,

$$A = \begin{bmatrix} 0 & -q \\ 1 & -p \end{bmatrix}.$$

Let $\Delta = p^2 - 4q$ be the discriminant. Define α_i, β_i by

$$A^{-i} v = \alpha_i v + \beta_i A v, \quad i = 1, 2, \ldots$$

Applying the Hamilton–Cayley theorem $f(A) = A^2 + pA + qI = 0$, where I is a 2×2 identity matrix, and the definitions of α_i, β_i above, we have a lemma:

Lemma 2.3 ([16]). *Let α_i, β_i be defined as the above. Then $q\alpha_{i+2} + p\alpha_{i+1} + \alpha_i = 0$ and $q\beta_{i+2} + p\beta_{i+1} + \beta_i = 0$, i.e.,*

$$\begin{bmatrix} \alpha_{i+1} \\ \alpha_{i+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/q & -p/q \end{bmatrix}^i \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}; \qquad \begin{bmatrix} \beta_{i+1} \\ \beta_{i+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/q & -p/q \end{bmatrix}^i \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

and $\alpha_1 = -p/q, \alpha_2 = (p^2 - q)/q^2; \beta_1 = -1/q, \beta_2 = p/q^2$. Moreover for $\Delta \neq 0$, we have

$$\alpha_i = \frac{q(y_1^{i+1} - y_2^{i+1})}{\Delta^{1/2}} \quad \text{and} \quad \beta_i = \frac{-(y_1^i - y_2^i)}{\Delta^{1/2}},$$

where $y_1 = \frac{-p+\Delta^{1/2}}{2q}$ and $y_2 = \frac{-p-\Delta^{1/2}}{2q}$ are the two roots of $qx^2 + px + 1 = 0$.

Since $f(x) = x^2 + px + q$ is expanding, the two roots y_1, y_2 of $x^2 f(x^{-1}) = qx^2 + px + 1$ have moduli less than 1. It follows that the following two series converge:

$$\tilde{\alpha} := \sum_{i=1}^{\infty} |\alpha_i|, \qquad \tilde{\beta} := \sum_{i=1}^{\infty} |\beta_i|.$$

From the previous lemma, it is easy to say, when $\Delta < 0$ (hence $q \geq 2$), then

$$|\alpha_i| \leq \frac{2q|y_1^{i+1}|}{|\Delta^{1/2}|} = \frac{2q^{-(i-1)/2}}{(4q - p^2)^{1/2}} \quad \text{and} \quad |\beta_i| \leq \frac{2|y_1^i|}{|\Delta^{1/2}|} = \frac{2q^{-i/2}}{(4q - p^2)^{1/2}}.$$

Hence, the upper bounds of $\tilde{\alpha}, \tilde{\beta}$ are estimated by

$$\tilde{\alpha} \leq \sum_{i=1}^{n-1} |\alpha_i| + \frac{2q^{-(n-1)/2}}{(1 - q^{-1/2})(4q - p^2)^{1/2}}, \tag{2.1}$$

$$\tilde{\beta} \leq \sum_{i=1}^{n-1} |\beta_i| + \frac{2q^{-n/2}}{(1 - q^{-1/2})(4q - p^2)^{1/2}}. \tag{2.2}$$

Since $q \geq 2$, we can find very accurate upper estimates of $\tilde{\alpha}$ and $\tilde{\beta}$ by taking $n = 13$ or any larger integer. This is the most important tool in our proofs.

Let $L := \{\gamma v + \delta Av : \gamma, \delta \in \mathbb{Z}\}$ be the *lattice* generated by $\{v, Av\}$. For $l \in L \setminus \{0\}$, $T + l$ is called a *neighbor* of T if $T \cap (T + l) \neq \emptyset$. It is easy to show that $T + l$ is a neighbor of T if and only if l can be expressed in the form

$$l = \sum_{i=1}^{\infty} b_i A^{-i} v \in T - T, \quad \text{where } b_i \in \Delta D.$$

After a few calculations, we have the following.

Lemma 2.4 ([16]). *If $T + l$ is a neighbor of T, where $l = \gamma v + \delta Av = \sum_{i=1}^{\infty} b_i A^{-i} v$, then $|\gamma| \leq \max_i |b_i| \tilde{\alpha}$ and $|\delta| \leq \max_i |b_i| \tilde{\beta}$. Moreover, this implies that $T + l'$ is another neighbor of T satisfying $l' = Al - b_1 v = \gamma' v + \delta' Av$ with $\gamma' = -(q\delta + b_1)$ and $\delta' = \gamma - p\delta$.*

By repeatedly using Lemma 2.4, we can construct a sequence of neighbors of T: $\{T + l_n\}_{n=0}^{\infty}$, where $l_0 = l$ and $l_n = \gamma_n v + \delta_n Av$, $n \geq 1$ by the following inductive formula:

$$\begin{bmatrix} \gamma_n \\ \delta_n \end{bmatrix} = A^n \begin{bmatrix} \gamma \\ \delta \end{bmatrix} - \sum_{i=1}^{n} A^{i-1} \begin{bmatrix} b_{n+1-i} \\ 0 \end{bmatrix}. \tag{2.3}$$

Moreover, $|\gamma_n| \leq \max_i |b_i| \tilde{\alpha}$ and $|\delta_n| \leq \max_i |b_i| \tilde{\beta}$ hold for any $n \geq 0$.

Lemma 2.5. *Let $T_1 = T(A, \mathcal{D})$ and $T_2 = T(-A, \mathcal{D})$. Then $T_1 + l$ is a neighbor of T_1 if and only if $T_2 + l$ is a neighbor of T_2.*

Proof. If $l \in T_1 - T_1$, then

$$l = \sum_{i=1}^{\infty} b_i A^{-i} v = \sum_{i=1}^{\infty} b_{2i}(-A)^{-2i} v + \sum_{i=1}^{\infty} (-b_{2i-1})(-A)^{-2i+1} v,$$

i.e., $l \in T_2 - T_2$ and vice versa. $\square$

An immediate corollary of the lemma follows.

Corollary 2.6. *If the characteristic polynomial of the expanding matrix A is $x^2 + px + q$ and that of B is $x^2 - px + q$. Then the self-affine set $T(A, \mathcal{D})$ is connected if and only if $T(B, \mathcal{D})$ is connected.*

3. Integer collinear digit set: $\mathcal{D} = \{0, 1, m\}v, 2 \leq m \in \mathbb{Z}$

In the section, we mainly consider the connectedness of the self-affine set $T(A, \mathcal{D})$ associated with a special digit set $\mathcal{D} = \{0, 1, m\}v$, where $2 \leq m \in \mathbb{Z}$.

Recall that, for $|\det(A)| = 3$, there are 10 eligible characteristic polynomials of A of the form: $f(x) = x^2 + px \pm 3$. We are going to discuss them separately.

3.1. When $p = 0$

We first provide a general result.

Theorem 3.1. *Let $A \in M_n(\mathbb{Z})$ be expanding with $|\det(A)| = q \geq 2$, whose characteristic polynomial $f(x) = x^n \pm q$, the digit set is $\mathcal{D} = \{d_1, d_2, \ldots, d_q\}v$ where $d_i \in \mathbb{R}$ and $v \in \mathbb{R}^n \setminus \{0\}$. Then the self-affine set $T(A, \mathcal{D})$ is connected if and only if, up to a translation, $\mathcal{D}$ is of the form $\{0, 1, 2, \ldots, (q-1)\}av$ for some $a > 0$.*

Proof. Consider $f(x) = x^n - q$ only. Since $f(A) = 0$, we have $A^{-n} = q^{-1}I$. Let $y = \sum_{i=1}^{\infty} a_i A^{-i} v \in T$, where $a_i \in D = \{d_1, d_2, \ldots, d_q\}$. Then

$$y = \sum_{k=0}^{n-1} \left(\sum_{j=1}^{\infty} a_{jn-k} A^{-jn} \right) A^k v = \sum_{k=0}^{n-1} \left(\sum_{j=1}^{\infty} a_{jn-k} q^{-j} \right) A^k v.$$

If T is connected, then for every $k \in \{1, 2, \ldots, n-1\}$, along the direction $A^k v$, the coordinate set

$$\left\{ \sum_{j=1}^{\infty} a_{jn-k} q^{-j} : a_{jn-k} \in D \right\}$$

is an interval, which is equivalent to say D is a translation of $\{0, 1, 2, \ldots, (q-1)\}a$ for some $a > 0$ by Proposition 1.1. Consequently, $\mathcal{D}$ is of the form $\{0, 1, 2, \ldots, (q-1)\}av$.

The converse is also true by Proposition 2.2, since $f(x)$ above has HRP. □

Corollary 3.2. *Assume that $A \in M_2(\mathbb{Z})$ with $|\det(A)| = 3$ is expanding, its characteristic polynomial is $f(x) = x^2 \pm 3$, $\mathcal{D} = \{0, 1, b\}v$ is a collinear digit set, where $1 < b \in \mathbb{R}$ and $v \in \mathbb{R}^2 \setminus \{0\}$. Then $T(A, \mathcal{D})$ is connected if and only if $b = 2$.*

3.2. When $p \neq 0$ and $m \geq 4$

We will show that, in all cases, $(T + v) \cap (T + mv) = \emptyset$ and $T \cap (T + mv) = \emptyset$. Hence by Proposition 2.1, T is disconnected $\left(\text{see Fig. 1 where we take the vector } v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$.

Theorem 3.3. *Let the characteristic polynomial of A be $f(x) = x^2 + px \pm 3$ ($p \neq 0$) and $\mathcal{D} = \{0, 1, m\}v$, $m \geq 4$ be the digit set. Then $T(A, \mathcal{D})$ is disconnected.*

Proof. It suffices to prove the cases for $p > 0$ by Corollary 2.6.

Case 1: $f(x) = x^2 + x + 3$. Let $T + l$ be a neighbor of T, then $l = \gamma v + \delta A v = \sum_{i=1}^{\infty} b_i A^{-i} v$, $b_i \in \Delta D = \{0, \pm 1, \pm(m-1), \pm m\}$. By Lemma 2.3 and (2.1), (2.2), we have

$$\tilde{\alpha} < 0.88; \qquad \tilde{\beta} < 0.63. \tag{3.1}$$

Then $|\gamma| \leq m\tilde{\alpha} < 0.88m$, $|\delta| \leq m\tilde{\beta} < 0.63m$.

Suppose $(T + v) \cap (T + mv) \neq \emptyset$, then $(m-1)v = \sum_{i=1}^{\infty} b_i A^{-i} v$, $b_i \in \Delta D$. By (2.3), we obtain $l_1 = -(3(m-1) + b_2)v - (m - 1 + b_1)Av$, where $|3(m-1) + b_2| < 0.88m$. Since $|3(m-1) + b_2| \geq 3(m-1) - m = 2m - 3$, it follows $m < 3/1.12$, which contradicts $m \geq 4$.

Suppose $T \cap (T + mv) \neq \emptyset$, similarly, we get $l_1 = -(3m + b_2)v - (m + b_1)Av$, where $|3m + b_2| < 0.88m$. Since $|3m + b_2| \geq 3m - m = 2m$, a contradiction follows.

Case 2: $f(x) = x^2 + 2x + 3$. Let $T + l$ be a neighbor of T, then $l = \gamma v + \delta A v = \sum_{i=1}^{\infty} b_i A^{-i} v$, where $b_i \in \Delta D$. By Lemma 2.3 and (2.1), (2.2), we have

$$\tilde{\alpha} < 1.17, \qquad \tilde{\beta} < 0.73. \tag{3.2}$$

Then $|\gamma| \leq m\tilde{\alpha} < 1.17m$, $|\delta| \leq m\tilde{\beta} < 0.73m$.
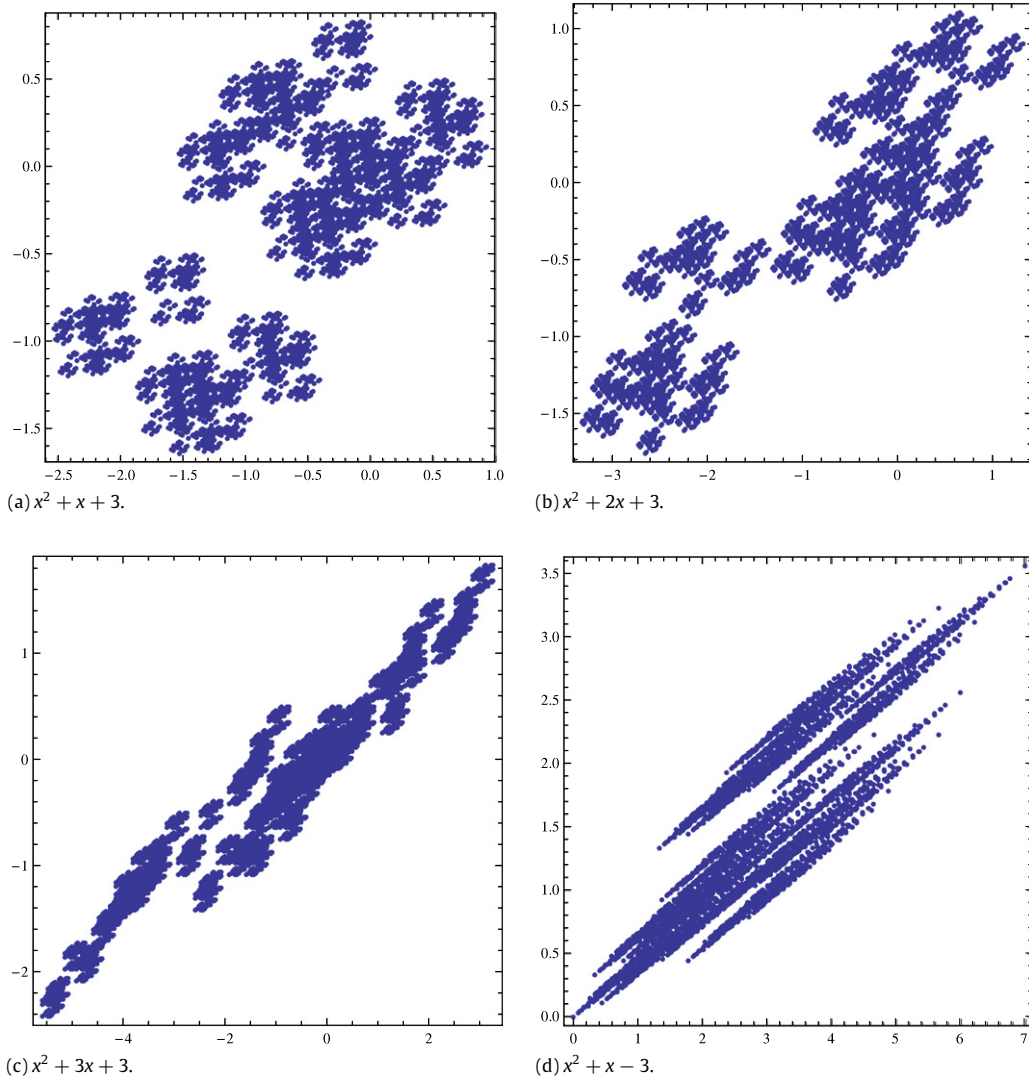
Suppose $(T + v) \cap (T + mv) \neq \emptyset$, then $(m-1)v = \sum_{i=1}^{\infty} b_i A^{-i} v$, $b_i \in \Delta D$. By (2.3), we obtain $l_1 = -(3(m-1) + b_2)v - (2(m-1) + b_1)Av$. By (3.2), then $2m - 3 = 3m - 3 - m \leq |3(m-1) + b_2| < 1.17m$, it follows that $m < 4$ which contradicts the assumption $m \geq 4$.

Suppose $T \cap (T + mv) \neq \emptyset$, similarly, by (2.3), we find a neighbor $T - (3m + b_2)v - (2m + b_1)Av$. So $2m = 3m - m \leq 3m + b_2 \leq |3m + b_2| < 1.17m$, then $m < 0$, a contradiction follows.

Case 3: $f(x) = x^2 + 3x + 3$. Let $T + l$ be a neighbor of T, then $l = \gamma v + \delta A v = \sum_{i=1}^{\infty} b_i A^{-i} v$, where $b_i \in \Delta D$. By Lemma 2.3 and (2.1), (2.2), we have

$$\tilde{\alpha} < 2.24, \qquad \tilde{\beta} < 1.08. \tag{3.3}$$

Then $|\gamma| \leq m\tilde{\alpha} < 2.24m$, $|\delta| \leq m\tilde{\beta} < 1.08m$.

(a) $x^2 + x + 3$.

(b) $x^2 + 2x + 3$.

(c) $x^2 + 3x + 3$.

(d) $x^2 + x - 3$.

Fig. 1. When $p \neq 0$, $m = 4$.

Suppose $T \cap (T + mv) \neq \emptyset$, then $mv = \sum_{i=1}^{\infty} b_i A^{-i} v$, $b_i \in \Delta D$. By (2.3), we obtain $l_1 = -(3m + b_2)v - (3m + b_1)Av$, where $|3m + b_1| < 1.08m$. Then $b_1 < -1.92m$, which is not in ΔD. This is ridiculous.

Suppose $(T + v) \cap (T + mv) \neq \emptyset$, similarly, by (2.3), $l_2 = (9m - 9 + 3b_1 - b_3)v + (6m - 6 + 3b_1 - b_2)Av$, where $|9m - 9 + 3b_1 - b_3| < 2.24m$. Since $|9m - 9 + 3b_1 - b_3| \geq 5m - 9$, it follows $m < 9/2.76 < 4$, which contradicts the assumption $m \geq 4$.

Case 4: $f(x) = x^2 + x - 3$. In this case, the discriminant $\Delta > 0$. By Lemma 2.3, we have $\alpha_1 = 1/3$, $\alpha_2 = 4/9$, $\beta_1 = 1/3$, $\beta_2 = 1/9$, and

$$\begin{bmatrix} \alpha_{i+1} \\ \alpha_{i+2} \end{bmatrix} = B^i \begin{bmatrix} 1/3 \\ 4/9 \end{bmatrix}; \qquad \begin{bmatrix} \beta_{i+1} \\ \beta_{i+2} \end{bmatrix} = B^i \begin{bmatrix} 1/3 \\ 1/9 \end{bmatrix},$$

where $B = \begin{bmatrix} 0 & 1 \\ 1/3 & 1/3 \end{bmatrix}$. Hence $\alpha_i \geq 0$ and $\beta_i \geq 0$ for all i. It yields that

$$\tilde{\alpha} = \begin{bmatrix} 1 & 0 \end{bmatrix} \sum_{i=0}^{\infty} B^i \begin{bmatrix} 1/3 \\ 4/9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} (I - B)^{-1} \begin{bmatrix} 1/3 \\ 4/9 \end{bmatrix} = 2.$$

Similarly, we also have $\tilde{\beta} = 1$.

Let $T + l$ be a neighbor of T, then $l = \gamma v + \delta Av = \sum_{i=1}^{\infty} b_i A^{-i} v$, where $b_i \in \Delta D$. By Lemma 2.4,

$$|\gamma| \leq m\tilde{\alpha} = 2m, \qquad |\delta| \leq m\tilde{\beta} = m. \tag{3.4}$$

(a) $x^2 + x + 3$.

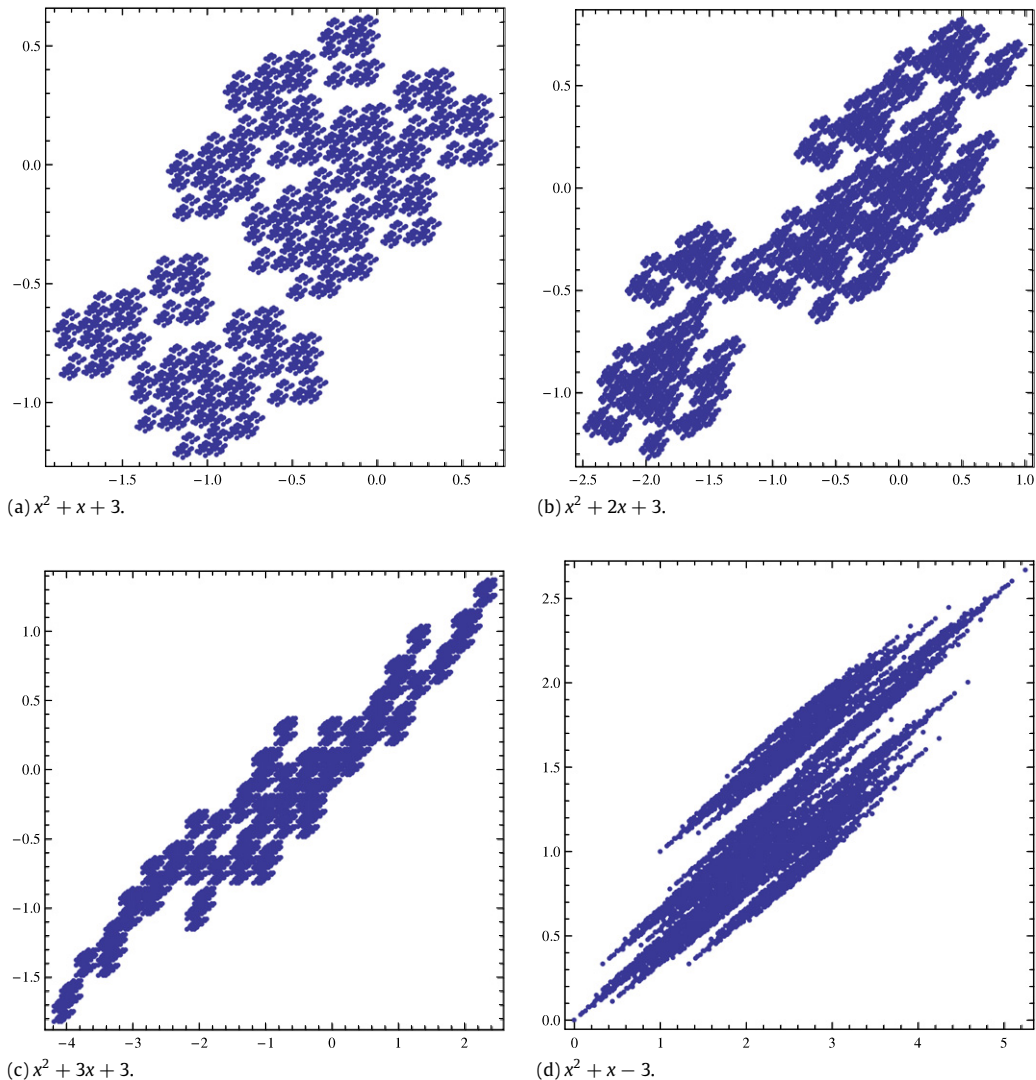(b) $x^2 + 2x + 3$.

(c) $x^2 + 3x + 3$.

(d) $x^2 + x - 3$.

Fig. 2. When $p \neq 0$, $m = 3$.

Suppose $(T + v) \cap (T + mv) \neq \emptyset$, then $(m - 1)v = \sum_{i=1}^{\infty} b_i A^{-i} v$, $b_i \in \Delta D$. By (2.3), $l_1 = -(3m - 3 + 3b_1 + b_3)v + (4m - 4 + b_1 - b_2)Av$, where $|4m - 4 + b_1 - b_2| \leq m$. Since $|4m - 4 + b_1 - b_2| \geq 2m - 4$, it follows that $m = 4$, $b_1 = -4$, $b_2 = 4$. Hence $l_1 = -(b_3 - 3)v + 4Av$. Using (2.3) again, we get $l_2 = (12 - b_4)v - (1 + b_3)Av$. Then $b_4 = 4$ from $|12 - b_4| \leq 8$, and $l_2 = 8v - (1 + b_3)Av$. Repeatedly using (2.3), then $l_3 = -(3 + 3b_3 + b_5)v + (9 + b_3)Av$, where $|9 + b_3| \leq 4$. Since $|9 + b_3| \geq 5$, a contradiction follows.

Suppose $T \cap (T + mv) \neq \emptyset$, similarly, $T - (3m + 3b_1 + b_3)v + (4m + b_1 - b_2)Av$ is a neighbor of T by (2.3), where $|4m + b_1 - b_2| \leq m$. Since $|4m + b_1 - b_2| \geq 2m$, it is impossible.

Therefore, in all cases, $T(A, \mathcal{D})$ is disconnected and the theorem follows. □

3.3. When $p \neq 0$ and $m \leq 3$

$\left(\text{See Fig. 2 where we take the vector } v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$

Theorem 3.4. *Let $\mathcal{D} = \{0, 1, m\}v$ be a digit set with $2 \leq m \in \mathbb{Z}$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. If the characteristic polynomial of the matrix A is $f(x) = x^2 \pm x - 3$ or $x^2 \pm 2x + 3$ or $x^2 \pm 3x + 3$. Then $T(A, \mathcal{D})$ is a connected set if $m = 2$ or 3. Moreover, $T(A, \mathcal{D})$ is a connected tile if and only if $m = 2$.*

If $f(x) = x^2 \pm x + 3$. Then $T(A, \mathcal{D})$ is a connected set if and only if $m = 2$.

Proof. Since $|\det(A)| = 3$ is a prime, $\{v, Av\}$ is independent. Applying Theorem 3.1 in [4], we know $T(A, \mathcal{D})$ is a tile if and only if $\{0, 1, m\}$ is a complete set of coset representatives of $\mathbb{Z}_3$, that is, $m = 2 + 3k$ for $k \in \mathbb{Z}$. Moreover, the connectedness of T implies $m \le 4$ by Theorem 3.3. Therefore $m = 2$.

To show the other parts, by Theorem 3.3, it suffices to verify the cases for $m = 3$, where the digit set becomes $\mathcal{D} = \{0, 1, 3\}v$, and $\Delta\mathcal{D} = \Delta Dv = \{0, \pm 1, \pm 2, \pm 3\}v$.

For $f(x) = x^2 - x - 3$, we have $f(A) = A^2 - A - 3I = 0$ which implies $v = A^{-1}v + 3A^{-2}v \in T - T$, that is, $T \cap (T + v) \ne \emptyset$. On the other hand, $0 = f(A)(2A - I) = 2A^3 - 3A^2 - 5A + 3I = (2A - 3I)(A^2 - I) - 3A$, it follows that $2A - 3I = 3A^{-1}(I - A^{-2})^{-1} = 3\sum_{i=0}^{\infty} A^{-2i-1}$. Consequently,

$$2v = 3A^{-1}v + 3\sum_{i=1}^{\infty} A^{-2i}v \in T - T.$$

That is, in radix expansion, $2 = 0.3\overline{30}$. Hence $(T + v) \cap (T + 3v) \ne \emptyset$. Therefore T is connected by Proposition 2.1.

For $f(x) = x^2 + 2x + 3$, we have $f(A) = A^2 + 2A + 3I = 0$ which implies $v = -2A^{-1}v - 3A^{-2}v \in T - T$, that is $T \cap (T + v) \ne \emptyset$.

On the other hand, from $0 = (A - I)(2A^2 + I)(A^2 + 2A + 3I)$, we obtain $2A^2 + 2A + 3I = (A^3 - I)^{-1}(3A^2 - 3A)$. It follows that $223 = 0 \cdot \overline{3(-3)0}$ and the radix expansion of $2v$ is

$$2 = 0 \cdot (-2)(-3)\overline{3(-3)0}.$$

Hence $(T + v) \cap (T + 3v) \ne \emptyset$. Therefore T is connected.

For $f(x) = x^2 + 3x + 3$, we have $f(A) = A^2 + 3A + 3I = 0$. Then $I = -3A^{-1} - 3A^{-2}$, so $v = -3A^{-1}v - 3A^{-2}v \in T - T$, that is, $T \cap (T + v) \ne \emptyset$.

On the other hand, from $0 = (2A - I)(A^2 + 3A + 3I) = 2A^3 + 5A^2 + 3A - 3I$, we obtain $2A^2 + 3A = 3(A + I)^{-1} = 3\sum_{i=1}^{\infty}(-1)^{i-1}A^{-i}$. It follows that $230 = 0 \cdot \overline{3(-3)}$ and the radix expansion of $2v$ is

$$2 = 0 \cdot (-3)0\overline{3(-3)}.$$

Hence $(T + v) \cap (T + 3v) \ne \emptyset$. Therefore T is connected.

For $f(x) = x^2 + x + 3$, we can use the same method in the proof of Theorem 3.3 to show that $T(A, \mathcal{D})$ is disconnected. Indeed, let $l = \gamma v + \delta Av = \sum_{i=1}^{\infty} b_i A^{-i}v$, where $b_i \in \Delta D = \{0, \pm 1, \pm 2, \pm 3\}$, then $T + l$ is a neighbor of T. By (3.1), we have estimates

$$|\gamma| < 2.64, \qquad |\delta| < 1.89.$$

Suppose $(T + v) \cap (T + 3v) \ne \emptyset$, $2v = \sum_{i=1}^{\infty} b_i A^{-i}v$. By (2.3), it follows that $l_1 = \gamma_1 v + \delta_1 Av = -(6 + b_2)v - (2 + b_1)Av$, where $|6 + b_2| < 2.64$, which is impossible since $|b_2| \le 3$.

Suppose $T \cap (T + 3v) \ne \emptyset$, then $3v = \sum_{i=1}^{\infty} b_i A^{-i}v$. By (2.3), it follows that $l_1 = -(9 + b_2)v - (3 + b_1)Av$, where $|9 + b_2| < 2.64$, which is also impossible as $|b_2| \le 3$. Therefore T is disconnected. $\square$

4. General collinear digit set: $\mathcal{D} = \{0, 1, b\}v$, $1 < b \in \mathbb{R}$

In this section, we discuss the case when $|\det(A)| = 3$ and $\mathcal{D} = \{0, 1, b\}v$, where $1 < b \in \mathbb{R}$. By modifying the proof of Theorem 3.3, we prove Theorem 1.5 which is restated in the following.

Theorem 4.1. *Assume that $A \in M_2(\mathbb{Z})$ with $|\det(A)| = 3$ is expanding, a digit set $\mathcal{D} = \{0, 1, b\}v$, where $b > 1$ and $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent. Then we have*

Case 1: $f(x) = x^2 \pm x + 3$ T is disconnected if $b \ge 67/25$ or $b \le 67/42$;

Case 2: $f(x) = x^2 \pm 2x + 3$ T is disconnected if $b \ge 37/10$ or $b \le 37/27$;

Case 3: $f(x) = x^2 \pm 3x + 3$ T is disconnected if $b \ge 33/10$ or $b \le 33/23$;

Case 4: $f(x) = x^2 \pm x - 3$ T is disconnected if $b > 19/5$ or $b < 19/14$.

Proof. In each case, we mainly consider the situation of $b \ge 2$ and get one desired inequality. For otherwise, if $1 < b \le 2$, then $b/(b - 1) \ge 2$, we replace the original digit set with $\mathcal{D}' = \{0, 1, b/(b - 1)\}v$. Hence the other inequality follows. If T is connected, then

$$(b - y)v = \sum_{k=1}^{\infty} b_k A^{-k}v \quad \text{holds for } y = 0 \text{ or } 1$$

where $b_k \in \Delta D = \{0, \pm 1, \pm(b - 1), \pm b\}$. That is, $T + (b - y)v$ is a neighbor of T. By making use of the same estimates of the neighbors as in the previous section, we prove the theorem case by case.

Case 1: $f(x) = x^2 + x + 3$. By (2.3) and (3.1), we obtain $l_1 = -(3(b-y) + b_2)v - (b-y+b_1)Av$ and $|3(b-y)+b_2| < 0.88b$. Since $|3(b-1)+b_2| \geq 3(b-1) - b = 2b - 3$, it follows that $b < 67/25$. On the other hand, it yields $b > 67/42$ from $b/(b-1) < 67/25$.

Case 2: $f(x) = x^2 + 2x + 3$. By (2.3) and (3.2), we have $l_1 = -(3(b-y)+b_2)v - (2(b-y)+b_1)Av$, and $|3(b-y)+b_2| < 1.17b$. Since $|3(b-y)+b_2| \geq 2b - 3$, it follows $b < 3/0.83 < 37/10$. We obtain $b > 37/27$ from $b/(b-1) < 37/10$.

Case 3: $f(x) = x^2 + 3x + 3$. By using (2.3) and (3.3) repeatedly, we have $l_1 = -(3(b-y)+b_2)v - (3(b-y)+b_1)Av$ and $l_2 = (9(b-y)+3b_1-b_3)v + (6(b-y)+3b_1-b_2)Av$, where $|9(b-y)+3b_1-b_3| < 2.24b$. Since $|9(b-y)+3b_1-b_3| \geq 5b-9$, it follows $b < 33/10$. From $b/(b-1) < 33/10$ we get $b > 33/23$.

Case 4: $f(x) = x^2 + x - 3$. By using (2.3) and (3.4) repeatedly, we can get another neighbor $T + (-21(b-y) - 12b_1 + 3b_2 - 3b_3 - b_5)v + (19(b-y) + 7b_1 - 4b_2 + b_3 - b_4)Av$ and $|19(b-y) + 7b_1 - 4b_2 + b_3 - b_4| \leq b$. Since $|19(b-y)+7b_1-4b_2+b_3-b_4| \geq 6b - 19$, it follows that $b \leq 19/5$. It follows from $b/(b-1) \leq 19/5$ that $b \geq 19/14$. $\quad\square$

5. Other results on $|\det(A)| > 3$

In previous sections, we investigated the connectedness of certain planar self-affine sets $T(A, \mathcal{D})$ generated by an expanding integer matrix A with $|\det(A)| = 3$ and a non-consecutive collinear digit set $\mathcal{D}$. Generally, it is very difficult to determine the connectedness of $T(A, \mathcal{D})$ for $|\det(A)| > 3$ since the structure of $\triangle\mathcal{D}$ can become more complicated. In the section, we try to give some partial answers.

Let $f(x) = x^2 + px \pm q$ be the characteristic polynomial of an expanding integer matrix A where $q \geq 2$. Let $v \in \mathbb{R}^2$ such that $\{v, Av\}$ is linearly independent and $\mathcal{D} = Dv$ be a q-digit set such that $D = \{0 = d_1, d_2, \ldots, d_q\} \subset \mathbb{Z}$ in the increasing order with $d_{i+1} - d_i = 1$ or 2 for all i, $d_{j+1} - d_j = 1$ for at least one j and $d_{k+1} - d_k = 2$ for at least one k. We have the following sufficient conditions for this $T(A, \mathcal{D})$ to be connected. Since we have solved the case when $p = 0$ (Theorem 3.1) and it is known that $T(A, \mathcal{D})$ is connected if and only if $T(-A, \mathcal{D})$ is connected (Corollary 2.6), we may assume $p > 0$ in this section.

Theorem 5.1. *Let* $f(x) = x^2 + px + q$ *with* $q \geq 2$ *and* $2p > q + 2$. *Assume* $\{0, \pm1, \pm2, \ldots, \pm q\} \subset \triangle D$. *Then* T *is connected if* $2p - 2 \in \triangle D$ *and* $2q - p \in \triangle D$.

Proof. By using $0 = f(A)(A - I) = A^3 + (p-1)A^2 + (q-p)A - qI$, we have

$$1 = 0 \cdot \overline{(1-p)(p-q)(q-1)}. \tag{5.1}$$

Hence

$$T \cap (T + v) \neq \emptyset.$$

Similarly, from $A + (p-1)I = -(q - p + 1)(A + I)^{-1}$ we obtain

$$1 = 0 \cdot (1-p)\overline{[-(q-p+1)](q-p+1)}. \tag{5.2}$$

Adding (5.1) and (5.2), we get

$$2 = 0 \cdot (2 - 2p)\overline{(2p - 2q - 1)(2q - p)(-q)1(p-2)(q - 2p + 2)}.$$

We can see easily that $|2p - 2q - 1|, q, 1, |p - 2|, |q - 2p + 2| \leq q$. So if, in addition, $2p - 2 \in \triangle D$ and $2q - p \in \triangle D$, then it follows that

$$T \cap (T + 2v) \neq \emptyset.$$

Therefore T is connected. $\quad\square$

Example 5.2. Let $f(x) = x^2 + 5x + 6$ and $D = \{0, 1, 2, 4, 6, 8\}$. Assume $\mathcal{D} = Dv$ is a digit set such that $\{v, Av\}$ is linearly independent. We can deduce from Theorem 5.1 that $T(A, \mathcal{D})$ is connected. $\left(\text{see Fig. 3(a) where we take the vector } v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

Theorem 5.3. *Let* $f(x) = x^2 + px - q$ *with* $q \geq 2$ *and* $2p > q - 2$. *Assume* $\{0, \pm1, \pm2, \ldots, \pm(q-1)\} \subset \triangle D$. *Then* T *is connected if* $2p + 1 \in \triangle D$ *and* $2q - p - 2 \in \triangle D$.

Proof. By using $0 = f(A)$, we have

$$1 = 0 \cdot \overline{(-p)(q-1)}. \tag{5.3}$$
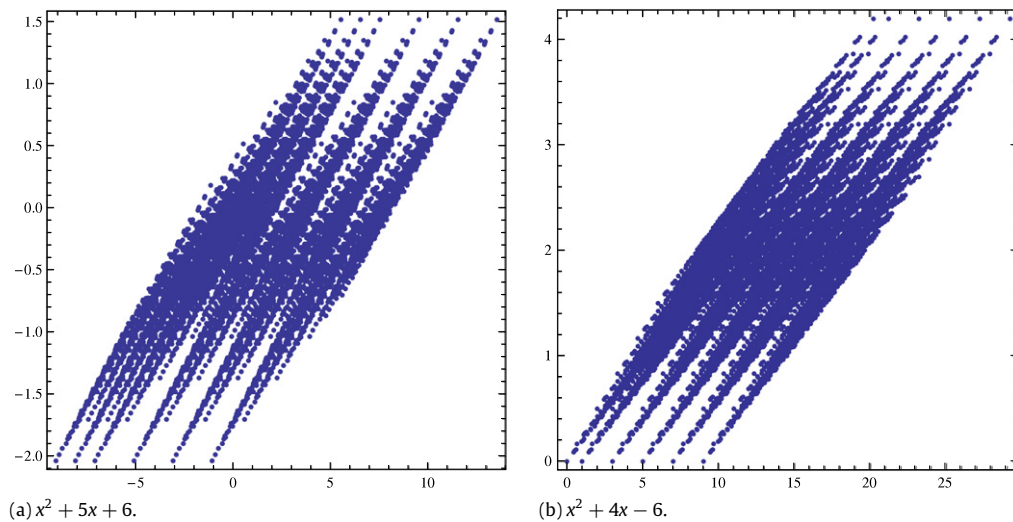
Hence

$$T \cap (T + v) \neq \emptyset.$$

Fig. 3. Two connected self-affine sets.

Similarly, from $A + (p-1)I = -(q-p-1)(A-I)^{-1}$ we obtain

$$1 = 0 \cdot [-(p+1)]\overline{(q-p-1)}. \tag{5.4}$$

Adding (5.3) and (5.4), we get

$$2 = 0 \cdot (-2p-1)\overline{(2q-p-2)(q-2p-1)}.$$

We can see easily that $|q - 2p - 1| \le q - 1$. So if, in addition, $2p + 1 \in \Delta D$ and $2q - p - 2 \in \Delta D$, then it follows that

$$T \cap (T + 2v) \neq \emptyset.$$

Therefore T is connected. $\quad\square$

Example 5.4. Let $f(x) = x^2 + 4x - 6$ and $D = \{0, 1, 3, 5, 7, 9\}$. Assume $\mathcal{D} = Dv$ is a digit set such that $\{v, Av\}$ is linearly independent. We can deduce from Theorem 5.3 that $T(A, \mathcal{D})$ is connected. $\left(\text{see Fig. 3(b) where we take the vector } v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

Acknowledgments

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