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Jordan Homomorphisms of Semiprime Rings

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1. INTRODUCTION

A Jordan homomorphism ϕ of a ring T into a ring R is an additive mapping of T into R such that $\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x)$ for all $x, y \in T$. Homomorphisms and antihomomorphisms afford obvious examples, and in case ϕ is onto and R is prime it was shown by Herstein (2) that these are in fact the only examples. Since Herstein's result is central to this paper we state it here as

THEOREM 1.1 (Herstein). *Let R be a 2-torsion free prime ring and let ϕ be a Jordan homomorphism of a ring T onto R. Then ϕ is either a homomorphism or an antihomomorphism.*

(We remark that Herstein originally proved his theorem in the case that the characteristic of R was unequal to 2 or 3, and that the assumption of characteristic of R unequal to 3 was subsequently removed by Smiley [5]).

It is our aim in this paper to study Jordan homomorphism onto semiprime rings R. We shall assume throughout this paper that R is 2-torsion free, and, except for emphasis, we shall not usually make explicit reference to this assumption.

The following simple example points the way to what the conjecture should be in the case that R is semiprime. Take S to be a prime ring with nontrivial involution $*$, set $R = S \oplus S$, and define $\phi: R \to R$ according to $\phi(s, t) = (s, t^*)$. Then ϕ is clearly the "direct" sum of a homomorphism $\sigma_1: R \to R$ (given by $(s, t) \to (s, o)$) and an antihomomorphism $\sigma_2: R \to R$ (given by $(s, t) \to (o, t^*)$).

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With this example in mind we make precise what we mean by the notion of direct sum of two additive mappings.

DEFINITION. A mapping ϕ: $T \to R$ is called a *direct sum* of mappings ϕ_1: $T \to R$ and σ_2: $T \to R$ if there exist ideals V_1 and V_2 of R such that $V_1 \cap V_2 = 0$, σ_1 maps T into V_1, and σ_2 maps T into V_2. We write ϕ as $\sigma_1 \oplus \phi_2$.

The above example suggests the following conjecture: every Jordan homomorphism ϕ of T onto a semiprime ring R is the direct sum of a homomorphism σ_1 of T into R and an antihomomorphism σ_2 of T into R. A simple example (suggested to us by Kaplansky) shows, however, this conjecture to be false. Let A be the elements of constant term 0 in the free noncommutative algebra in two generators x and y over a field F, and let $*$ be the involution on A determined by $x \to x$ and $y \to y$. Let $R = A \oplus A \oplus F$, the ring obtained by adjoining an identity to the ring $A \oplus A$ by the usual process. Then R is semiprime and $\phi(a, b, \lambda) = (a, b^*, \lambda)$ defines a Jordan automorphism of R. We show that ϕ cannot be written as a sum σ_1 and σ_2, direct or not, where σ_1: $R \to R$ is a homomorphism and σ_2: $R \to R$ is an antihomomorphism. Suppose to the contrary that $\phi = \sigma_1 + \sigma_2$. The only nonzero idempotent in R is the identity $1 = (0, 0, 1)$. Noting that $\phi(1)$, $\sigma_1(1)$, and $\sigma_2(1)$ must be idempotents we may conclude, without loss of generality, that $\phi(1) = 1$, $\sigma_1(1) = 1$, $\sigma_2(1) = 0$. It follows that $\sigma_2(R) = 0$ and that $\phi = \sigma_1$, an obvious contradiction.

Our conjecture turns out to be true, however, if we modify it in either of two ways. First, we might no longer insist that σ_1 and σ_2 be defined on all of T but rather just on a "large piece" of T. More precisely we draw the reader's attention to the notion of essential ideal: an ideal U of a ring T is *essential* if $U \cap I \neq 0$ for every nonzero ideal I of T. We then conjecture: if ϕ is a Jordan homomorphism of T onto a semiprime ring R then there exists an essential ideal U of T such that the restriction of ϕ to T is the direct sum of a homomorphism σ_1: $U \to R$ and an antihomomorphism σ_2: $U \to R$. We verify this conjecture (Theorem 2.7) in section two.

Secondly we might no longer insist that the images of σ_1 and σ_2 be contained in R but rather contained in some "slightly larger" ring A. In section three we see that the so-called central closure of a semiprime ring is the appropriate ring to use. Among other results we prove (Theorem 3.9) that if ϕ is a Jordan isomorphism of a semiprime ring T onto a semiprime ring R then ϕ can be extended to a Jordan isomorphism Φ of the central closure T' of T onto the central closure R' of R, and Φ in turn (Theorem 3.8) is the direct sum of a homomorphism σ_1: $T' \to R'$ and an antihomomorphism σ_2: $T' \to R'$.

We close this section by listing a few more or less well-known results which we shall need in the sequel. The proofs are either in the literature or can be easily done using standard arguments.

THEOREM 1.2. *Let R be a semiprime ring such that* $[[R, R], R] = 0$. *Then R is commutative.*

THEOREM 1.3. *If $a \neq 0$ and b are elements of a prime ring R such that $axb = bxa$ for all $x \in R$, then $b = \lambda a$ for some $\lambda \in C$, the extended centroid of R (see, e.g.,* [4], *Theorem* 1).

THEOREM 1.4. *Every nonzero Jordan ideal of a semiprime ring R contains a nonzero ideal of R (see* [3], *Theorem* 1.1).

Remark 1.5. Let U be an essential ideal of a semiprime ring R. If $b \in R$ such that $bu + ub = 0$ for all $u \in U$, then $b = 0$.

2. RESTRICTION TO AN ESSENTIAL IDEAL

Let ϕ be a Jordan homomorphism of a ring T onto a 2-torsion free semiprime ring R. Our aim in this section is to show that there is an essential ideal E of T such that the restriction of ϕ to E is the direct sum of a homomorphism $\sigma_1: E \to R$ and an antihomomorphism $\sigma_2: E \to R$. We shall first obtain this result when T is also semiprime and ϕ is a Jordan isomorphism and then show that the general result follows easily from this special case. Therefore, until otherwise indicated, we assume that ϕ is a Jordan isomorphism of a semiprime ring T onto a 2-torsion free semiprime ring R. Necessarily T is 2-torsion free.

Since T is semiprime there exist prime ideals $\{Q_\alpha \mid \alpha \in \mathscr{A}\}$ of T such that $\bigcap_\alpha Q_\alpha = 0$. Without loss of generality we may assume that the prime rings $T_\alpha = T/Q_\alpha$ are each 2-torsion free. Indeed, let $\mathscr{B} = \{\beta \in \mathscr{A} \mid T_\beta \text{ is 2-torsion free}\}$ and $\mathscr{C} = \{\gamma \in \mathscr{A} \mid T_\gamma \text{ is of characteristic } 2\}$. Thus $2T \subseteq \bigcap_\gamma Q_\gamma$ and, if $x \in \bigcap_\beta Q_\beta$, $2x \in \bigcap_\alpha Q_\alpha = 0$, forcing $x = 0$.

Our first lemma already describes the essential ideal E we are seeking and is perhaps of some independent interest.

LEMMA 2.1. *Let I be the ideal of T generated by $[[T, T], T]$. Then there exists a central ideal J of T such that $I \cap J = 0$ and $E = I \oplus J$ is an essential ideal of T.*

Proof. By Zorn's Lemma we may pick an ideal J of T such that $I \cap J = 0$ and $E = I \oplus J$ is essential. As indicated before we have $\bigcap_\alpha Q_\alpha = 0$, Q_α a prime ideal of T. We set $T_\alpha = T/Q_\alpha$ and let I_α and J_α denote respectively the images of I and J in T_α. For each α either $I_\alpha = 0$ or $J_\alpha = 0$, since $I_\alpha J_\alpha = 0$ and T_α is a prime ring. If $I_\alpha = 0$ then in particular $[[T_\alpha, T_\alpha], T_\alpha] = 0$ and so by Theorem 1.2 T_α is commutative and $[J, T] \subseteq Q_\alpha$. If $J_\alpha = 0$ then $J \subseteq Q_\alpha$ and so again $[J, T] \subseteq Q_\alpha$. Thus $[J, T] \subseteq \bigcap_\alpha Q_\alpha = 0$ and J is central.

We fix the essential ideal $E = I \oplus J$ given by Lemma 2.1. If $\bigcap Q_\alpha = 0$, Q_α a prime ideal of T, we claim that without loss of generality for each α, E is not contained in Q_α. Indeed, let $\mathscr{B} = \{\beta \mid E \nsubseteq Q_\beta\}$ and $\mathscr{C} = \{\gamma \mid E \subseteq Q_\gamma\}$. If $\bigcap Q_\beta \neq 0$ then $0 \neq \bigcap Q_\beta \cap E \subseteq \bigcap Q_\alpha = 0$, a contradiction. We thereby fix a set of prime ideals $\{Q_\alpha \mid \alpha \in \mathscr{A}\}$ of T such that $\bigcap_\alpha Q_\alpha = 0$ and for each α, $E \nsubseteq Q_\alpha$.

We next remark that for each $\alpha \in \mathscr{A}$ we have a Jordan isomorphism of R onto the prime ring T/Q_α given by $\tau_\alpha\phi^{-1}$, where τ_α is the natural homomorphism of T onto T/Q_α. By Theorem 1.1, since we have characteristic $T/Q_\alpha \neq 2$, we may conclude that $\tau_\alpha\phi^{-1}$ is either a homomorphism or an antihomomorphism. In either case the kernel of $\tau_\alpha\phi^{-1}$ is a prime ideal P_α of R and in fact it is clear that $P_\alpha = \phi(Q_\alpha)$. Therefore, as $E \nsubseteq Q_\alpha$, we have for each α, $\phi(E) \nsubseteq P_\alpha$.

We set $R_\alpha = R/P_\alpha$ and form the direct product $S = \prod_\alpha R_\alpha$. We let η_α denote the natural homomorphism of R onto R_α and let $\eta = \prod_\alpha \eta_\alpha$ be the isomorphism of R into S. For each $\alpha \in \mathscr{A}$, $\phi_\alpha = \eta_\alpha\phi$ is a Jordan homomorphism of T onto R_α. Therefore by Theorem 1.1 each ϕ_α is either a homomorphism or an antihomomorphism. On J, therefore, each ϕ_α is simultaneously a homomorphism and an antihomomorphism. Now partition $\mathscr{A}$ as follows.

$$\mathscr{B} = \{\beta \in \mathscr{A} \mid \phi_\beta \text{ is a nonzero homomorphism on } I\}$$

$$\mathscr{C} = \{\gamma \in \mathscr{A} \mid \phi_\gamma \text{ is a nonzero antihomomorphism on } I \text{ but not a homomorphism on } I\}$$

$$\mathscr{D} = \{\delta \in \mathscr{A} \mid \phi_\delta \text{ is the zero homomorphism on } I\}.$$

We let ϵ_1 be the projection of S into itself given by

$$\{r_\alpha\} \to \{s_\alpha\}, \qquad \text{where} \quad s_\alpha = \begin{cases} r_\alpha & \text{if } \alpha \in \mathscr{B} \\ 0 & \text{if } \alpha \notin \mathscr{B}. \end{cases}$$

Similarly we define projections ϵ_2 and ϵ_3 of S determined by $\mathscr{C}$ and $\mathscr{D}$ respectively. It is clear that ϵ_1, ϵ_2, ϵ_3 are orthogonal idempotents whose sum is the identity mapping on S.

At this point we pause to state a result which, though easily achieved, we will need to refer to in the next section. Setting $\tau_1 = \epsilon_1 + \epsilon_3$ and $\tau_2 = \epsilon_2$, we note that

THEOREM 2.2. *$\tau_1\eta\phi$ is a homomorphism of T into S, $\tau_2\eta\phi$ is an antihomomorphism of T into S, and $\eta\phi = \tau_1\eta\phi + \tau_2\eta\phi$.*

However, for the purposes of this section we set $\phi_i = \epsilon_i\eta\phi$, $i = 1, 2, 3$, and note ϕ_1 and ϕ_3 are homomorphisms of T into S whereas ϕ_2 is an antihomomorphism of T into S. Our immediate problem is to prove that each ϕ_i maps E into $\eta(R)$.

LEMMA 2.3. *For $i = 1, 2, 3$, $\phi_i[T, T] \subseteq \eta(R)$.*

Proof. Set $\psi = \eta\phi$ and let $x, y \in T$. Then

$$\psi(x)\,\psi(y) = \phi_1(x)\,\phi_1(y) + \phi_2(x)\,\phi_2(y) + \phi_3(x)\,\phi_3(y). \tag{1}$$

On the other hand

$$\begin{aligned}\psi(xy) &= \phi_1(xy) + \phi_2(xy) + \phi_3(xy)\\ &= \phi_1(x)\,\phi_1(y) + \phi_2(y)\,\phi_2(x) + \phi_3(x)\,\phi_3(y).\end{aligned} \tag{2}$$

From (1) and (2) we obtain

$$\psi(xy) - \psi(x)\,\psi(y) = \phi_2(xy - yx)$$

which shows that $\phi_2[T, T] \subseteq \eta(R)$. An analogous argument then shows that $\phi_1[T, T] \subseteq \eta(R)$, and $\phi_3[T, T] = 0$ since $\phi_\delta(I) = 0$ for $\delta \in \mathscr{D}$.

LEMMA 2.4. *For* $i = 1, 2, 3$ $\phi_i(I) \subseteq \eta(R)$.

Proof. For $a \in [T, T]$ and $b \in T$ we see by Lemma 2.3 that

$$\begin{aligned}\phi_1(a)\,\phi_1[a, b] &= \phi_1(a)(\phi_1(a)\,\phi_1(b) - \phi_1(b)\,\phi_1(a))\\ &= \phi_1(a)\,\phi_1(ab) - \phi_1(ab)\,\phi_1(a) = \phi_1[a, ab] \in \eta(R).\end{aligned} \tag{3}$$

Linearization of (3) yields

$$\phi_1(a)\,\phi_1[x, b] + \phi_1(x)\,\phi_1[a, b] \in \eta(R). \tag{4}$$

Replacement of a by $[c, d]$ in (4) gives

$$\phi_1[c, d]\,\phi_1[x, b] + \phi_1(x)\,\phi_1[[c, d], b] \in \eta(R). \tag{5}$$

Since the first summand of (5) lies in $\eta(R)$ by Lemma 2.3, we see that

$$\phi_1(x)\,\phi_1[[c, d], b] \in \eta(R)$$

or

$$\phi_1\{x[[c, d], b]\} \in \eta(R). \tag{6}$$

Likewise

$$\phi_1\{[[c, d, b]x\} \in \eta(R). \tag{7}$$

From (6) and (7) we finally see that

$$\phi_1\{y[[c, d], b]x\} = \phi_1[y[[c, d], b], x] + \phi_1\{yx[[c, d], b]\}$$

is an element of $\eta(R)$, and so we have shown that $\phi_1(I) \subseteq \eta(R)$. In a similar fashion one proves that $\phi_2(I) \subseteq \eta(R)$, and, as $\phi_3(I) = 0$, the proof is complete.

LEMMA 2.5. $\phi_1(J) = \phi_2(J) = 0$, *and* $\phi_3(J) \subseteq \eta(R)$.

Proof. Suppose $\phi_1(J) \neq 0$. This means that there exists $\beta \in \mathscr{B}$ such that $\phi_\beta(J) \neq 0$. On the other hand the definition of $\mathscr{B}$ implies that $\phi_\beta(I) \neq 0$. But

$\phi_\beta(I)\phi_\beta(J) \subseteq \phi_\beta(IJ) = 0$, a contradiction to the primeness of R_β. A similar argument shows that $\phi_2(J) = 0$. Consequently $\phi_3(J) = (\eta\phi - \phi_1 - \phi_2)(J) = \eta\phi(J) \subseteq \eta(R)$.

We can now state

THEOREM 2.6. *Let ϕ be a Jordan isomorphism of a semiprime ring T onto a 2-torsion free semiprime ring R. Then there exists an essential ideal E of T such that the restriction of ϕ to E is a direct sum $\sigma_1 \oplus \sigma_2$, where σ_1 is a homomorphism of E into R and σ_2 is an antihomomorphism of E into R.*

Proof. Recalling our terminology $\tau_1\eta\phi = \phi_1 + \phi_3$ and $\tau_2\eta\phi = \phi_2$, we see by Lemma 2.4 and Lemma 2.5 that $\tau_i\eta\phi(E) \subseteq \eta(R)$, $i = 1, 2$. We claim that $V_1 = \tau_1\eta\phi(E)$ and $V_2 = \tau_2\eta\phi(E)$ are in fact ideals of $\eta(R)$ such that $V_1 \cap V_2 = 0$. Indeed, for $\tau_1\eta\phi(u) \in V_1$ and $\eta\phi(t) \in \eta(R)$ we have $\tau_1\eta\phi(u)\,\eta\phi(t) = \tau_1\eta\phi(u) \times [\tau_1\eta\phi(t) + \tau_2\eta\phi(t)] = \tau_1\eta\phi(ut) \in V_1$. Thus V_1, and similarly V_2, are ideals of $\eta(R)$. Clearly $V_1 \cap V_2 = 0$ and $\eta\phi\,|_E$ is the direct sum of $\tau_1\eta\phi\,|_E$ and $\tau_2\eta\phi\,|_E$. Finally, by applying the isomorphism η^{-1}: $\eta(R) \to R$ to $\tau_1\eta\phi\,|_E$ and $\tau_2\eta\phi\,|_E$, we obtain the desired conclusion.

We are now in a position to consider the general problem which was outlined at the beginning of this section.

THEOREM 2.7. *Let ϕ be a Jordan homomorphism of a ring T onto a 2-torsion free semiprime ring R. Then there exists an essential ideal E of T such that the restriction of ϕ to E is a direct sum $\sigma_1 \oplus \sigma_2$ where σ_1 is a homomorphism of E into R and σ_2 is an antihomomorphism of E into R.*

Proof. We first show that the kernel K of ϕ is an ideal of T. Indeed, there exist prime ideals $\{P_\alpha\}$ of R such that $\bigcap P_\alpha = 0$. We let η_α be the natural homomorphism of R onto $R_\alpha = R/P_\alpha$ and recall that $\phi_\alpha = \eta_\alpha\phi$ is either a homomorphism or an antihomomorphism of T onto R_α because of Theorem 1.1. Now let $a \in K$ and $x \in T$. Then either $\phi_\alpha(ax) = \phi_\alpha(a)\,\phi_\alpha(x)$ or $\phi_\alpha(ax) = \phi_\alpha(x)\,\phi_\alpha(a)$. In either case, since $\phi_\alpha(a) = \eta_\alpha\phi(a) = 0$, we have $\phi_\alpha(ax) = 0$. It follows that $\phi(ax) = 0$, i.e., $ax \in K$. Similarly $xa \in K$ and so K is an ideal.

Since K is an ideal, letting $\bar{T} = T/K$, we may define $\bar{\phi}$: $\bar{T} \to R$ by $\bar{\phi}(\bar{x}) = \phi(x)$, $x \in T$. $\bar{\phi}$ is well-defined and is clearly a Jordan isomorphism of $\bar{T}$ onto R. Suppose $\bar{a}\bar{T}\bar{a} = 0$ for some $a \in T$. Then $0 = \bar{\phi}(\bar{a})\,\bar{\phi}(\bar{x})\,\bar{\phi}(\bar{a}) = \phi(a)\,\phi(x)\,\phi(a)$ for all $x \in T$. Recalling that ϕ is an onto mapping we conclude that $\phi(a) = 0$, That is, $a \in K$. This shows that $\bar{T}$ is semiprime. The conditions of Theorem 2.6 thus prevail so as to insure the existence of an essential ideal $\bar{E}$ of $\bar{T}$ such that $\bar{\phi} = \bar{\sigma}_1 \oplus \bar{\sigma}_2$, where $\bar{\sigma}_1$ is a homomorphism of $\bar{E}$ into R and $\bar{\sigma}_2$ is an antihomomorphism of $\bar{E}$ into R.

We take $E = \rho^{-1}(\bar{E})$, where ρ is the natural homomorphism of T onto T/K, and remark that E is an essential ideal of T. Indeed, let V be a nonzero ideal of T. If $V \subseteq K$ we have immediately $V \cap E \supseteq V \cap K = V \neq 0$. If $V \nsubseteq K$ we may choose $v \in V$, $v \notin K$ such that $\bar{v} \in \bar{E}$. This means that $v = u + k$, $u \in E$, $k \in K \subseteq E$, and so $0 \neq v \in V \cap E$, and E is essential.

Finally, for $i = 1, 2$, we define $\sigma_i: E \to R$ according to $\sigma_i(u) = \bar{\sigma}_i(u)$, $u \in E$. It is then clear that $\phi|_E = \sigma_1|_E \oplus \sigma_2|_E$ and the proof of the theorem is complete.

We leave as an open question the following: in Theorem 2.7 must $\phi(E)$ be an associative subring of R? If not, perhaps there is a way to choose some essential ideal E of T so that $\phi(E)$ is an associative subring of R.

3. Extension to the Central Closure

We begin this section by defining the *extended centroid* and *central closure* for a semiprime ring. These notions and their basic properties were first established for prime rings by Martindale [4] and subsequently generalized to semiprime rings by Amitsur [1]. For completeness we provide the details of these constructions.

Let R be a semiprime ring and let $\mathscr{U}$ be the family of all essential ideals of R. We may consider $U \in \mathscr{U}$ and R to be right R-modules U_R and R_R and shall refer to a mapping $f: U_R \to R_R$ as a *right permissible* map. Such a map will be denoted by (f, U), and we let $\mathscr{F}$ be the totality of all possible right permissible maps. We define $(f, U) \sim (g, V)$ to mean that $f = g$ on some $W \subseteq U \cap V$. This defines an equivalence relation on $\mathscr{F}$ since $\mathscr{U}$ is closed under finite intersections. We remark that if (f, U) is right permissible and $W \in \mathscr{U}$ such that $W \subseteq U$ and $f(W) = 0$, then $f(U) = 0$. Indeed, for $u \in U$ and $w \in W$ we have $f(u)w = f(uw) = 0$, whence $f(u)W = 0$ and $f(u) = 0$. It follows from this remark that if $(f, U) \sim (g, V)$ then $f = g$ on $U \cap V$. We shall tacitly use this observation in the sequel.

We let $\overline{(f, U)}$ denote the equivalence class determined by (f, U) and let F be the set of all equivalence classes. Addition and multiplication in F are defined respectively by

$$\overline{(f, U)} + \overline{(g, V)} = \overline{(f + g, U \cap V)}$$

and

$$\overline{(f, U)}\,\overline{(g, V)} = \overline{(f(g), VU)}$$

and are easily seen to be well-defined. Under these operations F is readily seen to be a ring with 1.

We may also regard $U \in \mathscr{U}$ and R as R-bimodules ${}_RU_R$ and ${}_RR_R$. We shall refer to a mapping $f: {}_RU_R \to {}_RR_R$ as a *permissible* map and we let $\mathscr{C} \subseteq \mathscr{F}$ denote the set of all permissible maps. We then set $C = \{\lambda \in F \mid \lambda = \overline{(f, U)}$ for some $(f, U) \in \mathscr{C}\}$ and call C the *extended centroid* of R.

LEMMA 3.1. *The extended centroid C of a semiprime ring R is the center of F and is von Neumann regular. It contains (an isomorphic copy of) the centroid Γ of R.*

Proof. Let $(f, U) \in \mathscr{F}$, $(g, V) \in \mathscr{C}$, and set $W = U \cap V$. For $x, y \in W$ we have

$$fg(xy) = f[xg(y)] = f(x)\,g(y) = g[f(x)y] = gf(xy)$$

which shows that $fg = gf$ on the essential ideal W^2, i.e., C lies in the center of F. Conversely, let $\overline{(f, U)}$ be an element of the center of F, let $a \in R$, and let a_l be the left multiplication of R determined by a. Then $(a_l, R) \in \mathscr{F}$ and so $fa_l = a_l f$ on U. For $u, v \in U$ we have

$$\begin{aligned}[f(au) - af(u)]v &= f(auv) - af(uv) \\ &= fa_l(uv) - a_l f(uv) = 0,\end{aligned}$$

which shows that for all $u \in U$, $f(au) = af(u)$, i.e. $(f, U) \in \mathscr{C}$.

Next let $(f, U) \in \mathscr{C}$, note that $\ker f$ is an ideal of R, and choose an ideal J such that $\ker f \oplus J$ is essential in R and also contained in U. $f(J)$ is also an ideal of R and we may pick an ideal I such that $V = f(J) \oplus I$ is essential. We define $g\colon V \to R$ by $g(f(b) + a) = b$ for $b \in J$ and $a \in I$. g is well-defined since f is one-one on J. For $x \in \ker f$ and $b \in J$ we have

$$fgf(x + b) = fgf(b) = f(b) = f(x + b)$$

which shows that C is von Neumann regular. Finally, Γ is isomorphically embedded in C via the mapping $\gamma \to \overline{(f, R)}$, where $\gamma \in \Gamma$ and $f(r) = \gamma r$ for all $r \in R$.

The mapping $a \to (a_l, R)$ furnishes an isomorphic embedding of R into F. Indeed, if $\overline{(a_l, R)} = 0$ then $aR = 0$, forcing $a = 0$ by the semiprimeness of R. We identify R with its isomorphic image in F and we call the ring $A = RC$ the *central closure* of R. We define a semiprime ring to be *closed* if its centroid coincides with its extended centroid, or, equivalently, it is equal to its own central closure.

For future reference we remind the reader of two useful properties enjoyed by the elements of A. One is that if $q \in A$ is such that $qU = 0$ for some $U \in \mathscr{U}$ then $q = 0$. The other is that given $q \in A$ there exists $U \in \mathscr{U}$ such that $qU \subseteq R$. To see the latter write $q = \sum r_i\lambda_i$, $r_i \in R$, $\lambda_i = \overline{(f_i, U_i)} \in C$, and set $U = \bigcap U_i$. We make use of these observations in

THEOREM 3.2. *The central closure RC of a semiprime ring R is a closed semiprime ring.*

Proof. Suppose $qRCq = 0$. Pick $U \in \mathscr{U}$ such that $qU \subseteq R$. Then $(qU)R(qU) = 0$, whence $qU = 0$ by the semiprimeness of R. But this forces $q = 0$, and so RC is semiprime.

To show RC is closed it suffices to show that the embedding of C (which is certainly already contained in the centroid of RC) into the extended centroid of

RC is onto. To this end let $g: V \to RC$, V an essential ideal of RC, be a permissible map. We first claim that $U = V \cap R$ is an essential ideal of R. Indeed, if I is a nonzero ideal of T then IC is a nonzero ideal of RC and so $V \cap IC \neq 0$. We then may choose $0 \neq \sum a_i\lambda_i \in V$, where $a_i \in I$ and $\lambda_i \in C$. Now pick an essential ideal W such that for all i, $\lambda_i W \subseteq R$, and then pick $w \in W$ such that $0 \neq \sum a_i(\lambda_i w) \in V \cap R \cap I$.

Next we prove that $G = \{x \in U \mid g(x) \in R\}$ is an essential ideal of R. Let $a \neq 0 \in R$ and let $\langle a \rangle$ be the ideal of R generated by a. First, there exists $0 \neq b \in \langle a \rangle$ such that $b \in U$, since U is essential in R. If $g(b) = 0 \in R$ we see directly that $b \in G \cap \langle a \rangle$. If $0 \neq g(b) \in RC$ we may find an essential ideal W of R such that $0 \neq g(b)W \subseteq R$. Pick $w \in W$ such that $0 \neq g(b)w = g(bw) \in R$, whence $0 \neq bw \in G \cap \langle a \rangle$. Therefore the restriction f of g to the essential ideal G of R is a permissible map for R and so $\lambda = \overline{(f, G)}$ is an element of C. Clearly λ maps to the element $\overline{(g, V)}$ and our proof is complete.

Later in this section, when we try to lift a Jordan isomorphism ϕ of semiprime rings to a Jordan isomorphism of their central closures, we will have occasion to apply ϕ to an essential ideal V. At first glance all we know about $\phi(V)$ is that it is an essential Jordan ideal, and we will want to know that it at least contains an essential ideal. Our immediate goal, then, is to establish this fact.

A Jordan ideal U of a ring R is an additive subgroup of R such that $au + ua \in U$ for all $u \in U$, $a \in R$. A Jordan ideal U is called essential if for every nonzero Jordan ideal V or R, $U \cap V \neq 0$. We now suppose that U is an essential Jordan ideal of a semiprime ring R.

LEMMA 3.3. *Let $V = \{b \in R \mid aba = 0$ for all $a \in U\}$. Then V is a Jordan ideal of R and $U \cap V = 0$.*

Proof. By linearization we have $abc + cba = 0$ for $a, c \in U$ and $b \in V$. Therefore for $a \in U$, $b \in V$, $r \in R$ we see that

$$a(br + rb)a = ab(ra + ar) + (ar + ra)ba - [abar + raba] = 0,$$

which shows that V is a Jordan ideal of R. If $U \cap V \neq 0$ by Theorem 1.4 it contains a nonzero ideal I with the property that $a^3 = 0$ for all $a \in I \subseteq U \cap V$. This is a contradiction in a semiprime ring, and so $U \cap V = 0$.

LEMMA 3.4. *For $0 \neq a \in U$ there exists $b \in U$ such that $ab + ba \neq 0$.*

Proof. Suppose $ab + ba = 0$ for all $b \in U$. Then $0 = ab^2 + b^2a = ab^2 - bab = 2ab^2$, and so $ab^2 = 0$ for all $b \in U$. Therefore $bab = (ba + ab)b = 0$. In particular $a \in V$ (as defined in Lemma 3.3), and so $a \in U \cap V$, a contradiction to Lemma 3.3.

We define $U^+ = \sum_\alpha 2Rc_\alpha R$, where $c_\alpha = a_\alpha b_\alpha + b_\alpha a_\alpha$, $a_\alpha, b_\alpha \in U$.

THEOREM 3.5. *If U is an essential Jordan ideal of R then U^+ is an essential ideal of R contained in U.*

Proof. We follow Herstein ([3], p. 3) in first showing that $U^+ \subseteq U$. For $a, b \in U$ and $x \in R$ we have

$$a(xb - bx) + (xb - bx)a = (ax - xa)b + b(ax - xa) + x(ab + ba) - (ab + ba)x$$

which shows that $x(ab + ba) - (ab + ba)x \in U$. Setting $c = ab + ba$ and noting that $xc + xc \in U$ we obtain $2cx \in U$, whence $(2xc)y + y(2xc) \in U$. Since $2yxc \in U$ we have $2RcR \subseteq U$.

We now prove that U^+ is an essential ideal of R. For, given a nonzero ideal B of R, B is a Jordan ideal of R and hence $B \cap U \neq 0$. Thus there exists $a \neq 0 \in U \cap B$ and hence, by Lemma 3.4, there exists $b \in U$ such that $ab + ba \neq 0$. Therefore $2R(ab + ba)R \neq 0$ and is contained in $U^+ \cap B$.

We shall also later be confronted with additive mappings $f: U \to R$, where $U \in \mathscr{U}$, which are *J-permissible* in the sense that $f(xu + ux) = xf(u) + f(u)x$ for all $u \in U$ and $x \in R$. Our next immediate goal is to show that J-permissible maps are in fact permissible ones.

LEMMA 3.6. *Let R be a prime ring, U a nonzero ideal of R, and $f: U \to R$ a J-permissible map. Then (f, U) is permissible.*

Proof. For $a, b \in U$ we have

$$af(b) + f(b)a = f(a)b + bf(a). \tag{8}$$

Replacement in (8) of b by $ax + xa$, $x \in R$, yields

$$\begin{aligned} &a(f(a)x + xf(a)) + (f(a)x + xf(a))a \\ &\qquad = f(a)(ax + xa) + (ax + xa)f(a), \end{aligned} \tag{9}$$

and (9) reduces to

$$(af(a) - f(a)a)x = x(af(a) - f(a)a). \tag{10}$$

From (10) we have $af(a) - f(a)a \in Z \cap U$, where Z is the center of R.

Suppose $Z \cap U = 0$. Then $[a, f(a)] = 0$ for all $a \in U$, whence

$$f(a^2) = af(a), \qquad a \in U. \tag{11}$$

Linearization of (11) produces

$$f(ab + ba) = af(b) + bf(a), \qquad a, b \in U. \tag{12}$$

But $f(ab + ba) = af(b) + f(b)a$, which together with (12) gives

$$bf(a) = f(b)a, \qquad a, b \in U. \tag{13}$$

We set $b = ax + xa$, $x \in R$, in (13) and see that

$$(ax + xa)f(a) = (f(a)x + xf(a))a,$$

which leads to

$$axf(a) = f(a)\, xa \tag{14}$$

for all $x \in R$, since $[a, f(a)] = 0$. By Theorem 1.3 (14) enables us to say that for each $a \in U$, $f(a) = \lambda a$, where $\lambda = \lambda_a$ is an element of the extended centroid C of R.

Suppose $Z \cap U \neq 0$. For $a \in U$ and $\alpha \in U \cap Z$ we have $[a + \alpha, f(a + \alpha)] = [a, f(a)] + [a, f(\alpha)] \in Z$, whence $[a, f(\alpha)] \in Z$. But then

$$\begin{aligned}[a, f(\alpha)]^2 &= af(\alpha)[a, f(\alpha)] - f(\alpha)\, a[a, f(\alpha)] \\ &= a[f(\alpha)a, f(\alpha)] - f(\alpha)\, a[a, f(\alpha)] \\ &= [f(\alpha)a, f(\alpha)]a - f(\alpha)[a, f(\alpha)]a = 0\end{aligned} \tag{15}$$

Since Z is a field (15) shows that $[a, f(\alpha)] = 0$ for all $a \in U$, and consequently $f(\alpha) \in Z$ for all $\alpha \in U \cap Z$. Now choose $0 \neq \gamma \in Z \cap U$ and $a \in U$. Since $f(\gamma) \in Z$ we have

$$2f(\gamma a) = f(\gamma)a + af(\gamma) = 2f(\gamma)a. \tag{16}$$

Also we have

$$2f(\gamma a) = 2\gamma f(a). \tag{17}$$

From (16) and (17) we see that $\gamma f(a) = f(\gamma)a$, that is, $f(a) = \gamma^{-1}f(\gamma)a = \lambda a$, where $\lambda = \lambda_a = \gamma^{-1}f(\gamma) \in C$.

Summarizing the proof thus far, we have shown in any case that if $a \in U$ then $f(a) = \lambda a$ for suitable $\lambda = \lambda_a \in C$. We fix $a \neq 0 \in U$ and remark next that $W = \{x \in U \mid \lambda x \in R\}$ is a nonzero ideal of R containing a, since $\lambda a = f(a) \in R$. Now $g = f - \lambda$ is a J-permissible map of W into R. Therefore $\ker g$ and $g(W)$ are Jordan ideals of R such that $xy + yx = 0$, $x \in \ker g$, $y \in g(W)$. If $\ker g$ and $g(W)$ are both nonzero then by Theorem 1.4 they contain nonzero ideals K and L respectively. Then $K \cap L \neq 0$ and has the property that $2x^2 = 0$ and hence $x^2 = 0$ for all $x \in K \cap L$, a contradiction since R is prime. Since $0 \neq a \in \ker g$ we must conclude that $g(W) = 0$, that is, $f(w) = \lambda w$ for all $w \in W$. Therefore, for $x \in R$, we have $f(ax) = \lambda ax = f(a)x$ and $f(xa) = \lambda xa = xf(a)$. This completes the proof that (f, U) is permissible.

THEOREM 3.7. *Let R be a semiprime ring, U an essential ideal of R, and f: $U \to R$ a J-permissible map. Then (f, U) is permissible.*

Proof. As we saw in the remarks following Lemma 2.1 we may assume that there are prime ideals $\{P_\alpha\}$ such that $\bigcap P_\alpha = 0$ and for each α, $U \not\subseteq P_\alpha$. We set

$\bar{R} = R/P_\alpha$ and note that $\bar{U}$ is a nonzero ideal of $\bar{R}$. We then define $\tilde{f}: \bar{U} \to \bar{R}$ as follows:

$$\tilde{f}(\bar{a}) = \overline{f(a)}, \qquad \text{where} \quad a \in U.$$

In order to show that $\tilde{f}$ is well-defined, suppose $\bar{a} = 0$, where $a \in U$. Then $a \in P_\alpha \cap U$ and, using the fact that f is J-permissible, we have

$$f(au + ua) = f(a)u + uf(a) = af(u) + f(u)a \in P_\alpha$$

for all $u \in U$. Therefore in the prime ring $\bar{R}$ we see that $\overline{f(a)}\,\bar{u} + \bar{u}\,\overline{f(a)} = 0$ for all $\bar{u} \in \bar{U}$, in which case Remark 1.5 tells us that $\overline{f(a)} = 0$. From the definition of $\tilde{f}$ it is clear that $\tilde{f}$ is a J-permissible map of $\bar{U}$ into $\bar{R}$ and so, by Lemma 3.6, $(\tilde{f}, \bar{U})$ is permissible. It follows easily that $f(ax) - f(a)x$ and $f(xa) - xf(a)$ lie in P_α for $a \in U$, $x \in R$. Since $\bigcap_\alpha P_\alpha = 0$ we conclude finally that $f(ax) = f(a)x$ and $f(xa) = xf(a)$, that is, (f, U) is permissible.

In case R is a closed semiprime ring we are able to sharpen Theorem 2.7 as follows.

THEOREM 3.8. *Let ϕ be a Jordan homomorphism of a ring T onto a closed 2-torsion free semiprime ring R. Then ϕ is the direct sum of a homomorphism σ_1 of T into R and antihomomorphism σ_2 of T into R.*

Proof. As noted in the proof of Theorem 2.7 the kernel K of ϕ is an ideal of T. In the semiprime ring $\bar{T} = T/K$ we pick the essential ideal $\bar{E}$ given by Lemma 2.1. As shown in the proof of Theorem 2.7 the inverse image of $\bar{E}$ is an essential ideal E of T. Then $\phi(E) = \bar{\phi}(E)$ is an essential Jordan ideal of R, which by Theorem 3.5 contains an essential ideal V of R. As in section 2 we may choose prime ideals $\{P_\alpha\}$ of R such that $\bigcap P_\alpha = 0$ but for each α, $V \not\subseteq P_\alpha$. Also defined in section two was the isomorphism η of R into $S = \Pi R_\alpha$, and the projections τ_1 and τ_2 of S into itself. Theorem 2.2 says that $\tau_1\eta\phi$ is a homomorphism of T into S and $\tau_2\eta\phi$ is an antihomomorphism of T into S. By Lemma 2.3 and Lemma 2.4 $\tau_1\eta\phi$ and $\tau_2\eta\phi$ map E into $\eta(R)$, that is, $\tau_1\eta$ and $\tau_2\eta$ map $V \subseteq \phi(E)$ into $\eta(R)$. Therefore $\eta^{-1}\tau_i\eta$, $i = 1, 2$, maps V into R and, in fact, each $(\eta^{-1}\tau_i\eta, V)$ is permissible. Then $\lambda_i = \overline{(\eta^{-1}\tau_i\eta, V)}$ belongs to the extended centroid C of R and, since R is closed, λ_i actually lies in the centroid of R. So λ_i is such that $\lambda_i R \subseteq R$ and $\lambda_i(v) = \eta^{-1}\tau_i\eta(v)$ for all $v \in V$, that is, $\eta\lambda_i(v) = \tau_i\eta(v)$ for all $v \in V$. We wish to show that $\eta\lambda_i(a) = \tau_i\eta(a)$ for all $a \in R$. For $v \in V$ we know that

$$[\eta\lambda_i(a) - \tau_i\eta(a)]\,\eta(v) = \eta\lambda_i(av) - \tau_i\eta(av) = 0 \tag{18}$$

Now $\eta_\alpha(V)$ is a nonzero ideal of R_α since for each α, $V \not\subseteq P_\alpha$. Let ϵ_α be the projection of S onto R_α (thus $\eta_\alpha = \epsilon_\alpha\eta$) and apply ϵ_α to (18), obtaining $\epsilon_\alpha[\eta\lambda_i(a) - \tau_i\eta(a)]\,\eta_\alpha(v) = 0$ for all $v \in V$. Therefore for each α, $\epsilon_\alpha(\eta\lambda_i a - \tau_i\eta(a)) = 0$ since it annihilates the nonzero ideal $\eta_\alpha(V)$ of R_α, and consequently

$\tau_i\eta(a) = \eta\lambda_i a \in R$, $i = 1, 2$. It follows easily from this that $\sigma_1 = \eta^{-1}\tau_1\eta\phi$ is a homomorphism of T into R, $\sigma_2 = \eta^{-1}\tau_2\eta\phi$ is an antihomomorphism of T into R, and $\phi = \sigma_1 \oplus \sigma_2$.

THEOREM 3.9. *Let T and R be semiprime rings, with respective extended centroids D and C. Then any Jordan isomorphism ϕ of T into R can be extended to a Jordan isomorphism Φ of the TD onto RC. Then Φ is a direct sum of a homomorphism σ_1: $TD \to RC$ and an antihomomorphism σ_2: $TD \to RC$.*

Proof. We first show that ϕ induces an isomorphism of D onto C. For $\lambda \in D$ we write $\lambda = \overline{(f, U)}$. Since U is an essential Jordan ideal of T we see that $\phi(U)$ is an essential Jordan ideal of R. By Theorem 3.5, $V = \phi(U)^+$ is an essential ideal of R contained in $\phi(U)$. A mapping g: $V \to R$ is defined according to

$$g\colon \phi(u) \to \phi(f(u)), \quad \phi(u) \in V.$$

We show that g is J-permissible. Let $\phi(u) \in V$, for $u \in U$, and let $x \in R$. Writing $x = \phi(t)$, $t \in T$, we have

$$\begin{aligned} g(\phi(u)x + x\phi(u)) &= g(\phi(u)\,\phi(t) + \phi(t)\,\phi(u)) \\ &= g(\phi(ut + tu)) = \phi(f(ut + tu)) = \phi(f(u)t + tf(u)) \\ &= \phi(f(u))\,\phi(t) + \phi(t)\,\phi(f(u)) = g(\phi(u))x + xg(\phi(u)), \end{aligned}$$

and hence g: $V \to R$ is J-permissible. By Theorem 3.7 (g, V) is permissible, and so $\tilde{\lambda} = \overline{(g, V)}$ is an element of C. It is straightforward to verify that the mapping

$$\lambda \to \tilde{\lambda}, \qquad \lambda \in D$$

which we have just created is well-defined, that is, independent of the choice of essential ideal V contained in $\phi(U)^+$.

We claim that $\lambda \to \tilde{\lambda}$ is an isomorphism of D onto C. It is clear that $\lambda \to \tilde{\lambda}$ is additive and we proceed to verify that it is multiplicative. For $\lambda_1 = \overline{(f_1, U_1)}$ and $\lambda_2 = \overline{(f_2, U_2)}$ in D we pick $U \subseteq U_2$ such that $f_2(U) \subseteq U_1$ (for example, $U = U_1U_2$). Then $\tilde{\lambda}_1 = \overline{(g_1, V_1)}$ and $\tilde{\lambda}_2 = \overline{(g_2, V_2)}$ and we pick V to be essential and contained in $V_1 \cap V_2 \cap \phi(U)$. For $v = \phi(u) \in V$ we have

$$g_1g_2\phi(u) = g_1\phi(f_2(u)) = \phi(f_1f_2(u))$$

which shows that $\widetilde{\lambda_1\lambda_2} = \tilde{\lambda}_1\tilde{\lambda}_2$. Next suppose that $\lambda = \overline{(f, U)}$ is such that $\tilde{\lambda} = \overline{(g, V)} = 0$. By Theorem 3.5 we may pick an essential ideal W of T which is contained in $U \cap \phi^{-1}(V)$. For $w = \phi^{-1}(v) \in W$ we see that

$$\phi(f(w)) = g\phi(w) = g\phi\phi^{-1}(v) = g(v) = 0$$

and so $f(W) = 0$, that is, $\lambda = 0$ and $\lambda \to \tilde{\lambda}$ is one-one. Finally, let $\mu = (g, V) \in C$, choose U essential in T such that $U \subseteq \phi^{-1}(V)$, and define $f\colon U \to T$ according to $f\phi^{-1}(v) = g(v)$ for $\phi^{-1}(v) \in U$. Then $\lambda = \overline{(f, U)} \in D$; $\tilde{\lambda} = \mu$, and so $\lambda \to \tilde{\lambda}$ is onto.

We are now in a position to lift ϕ to a Jordan automorphism of TD onto RC. Indeed, we define

$$\Phi\colon \sum t_i\lambda_i \to \sum \phi(t_i)\,\tilde{\lambda}_i$$

for $t_i \in T$, $\lambda_i \in D$. We first prove that Φ is well-defined. Suppose $\sum t_i\lambda_i = 0$. Then there exists an essential ideal U of T such that $\lambda_i U \subseteq T$ and an essential ideal V of R contained in $\phi(U)$ such that $\tilde{\lambda}_i(v) = \tilde{\lambda}_i\phi(u) = \phi(\lambda_i - v)$ for all $v \in V$. Now we note that

$$\begin{aligned}
&\sum \phi(t_i)\,\tilde{\lambda}_i v + v \sum \phi(t_i)\,\tilde{\lambda}_i \\
&\quad \sum \phi(t_i)\,\tilde{\lambda}_i\phi(u) + \phi(u) \sum \phi(t_i)\,\tilde{\lambda}_i \\
&\quad = \sum \phi(t_i)\,\tilde{\lambda}_i\phi(u) + \sum \tilde{\lambda}_i\phi(u)\,\phi(t_i) \\
&\quad = \sum \phi(t_i)\,\phi(\lambda_i u) + \sum \phi(\lambda_i u)\,\phi(t_i) \\
&\quad = \sum \phi(t_i\lambda_i u + \lambda_i u t_i) \\
&\quad = \phi\left[\left(\sum t_i\lambda_i\right) u + u \left(\sum t_i\lambda_i\right)\right] = 0
\end{aligned}$$

for all $v \in V$. By Remark 1.5 $\sum \phi(t_i)\,\tilde{\lambda}_i$ must then be 0, and Φ is well-defined. Clearly Φ is additive, and for $t, s \in T$, $\lambda, \mu \in D$, the equations

$$\begin{aligned}
\Phi[(t\lambda)(s\mu) + (s\mu)(t\lambda)] &= \Phi[(ts + st)(\lambda\mu)] \\
&= \phi(ts + st)(\lambda\mu) = [\phi(t)\,\phi(s) + \phi(s)\,\phi(t)]\,\tilde{\lambda}\tilde{\mu} \\
&= (\phi(t)\tilde{\lambda})(\phi(s)\tilde{\mu}) + (\phi(s)\tilde{\mu})(\phi(t)\tilde{\lambda})
\end{aligned}$$

shows that Φ is a Jordan homomorphism. Φ is onto since ϕ is into $\lambda \to \tilde{\lambda}$ is onto. That Φ is one-one follows from the symmetric argument that ϕ^{-1} can be lifted to a well-defined map from RC into TD. It is obvious that Φ is an extension of ϕ. The last sentence of the theorem follows immediately from Theorem 3.8.

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