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Practical Foundations of Mathematics

Paul Taylor, Cambridge Studies in Advanced Mathematics, Vol. 59, Cambridge University Press, Cambridge, 1999. xi+572 pages, price £50 paperback, ISBN 0-521-63107-6

The book presents a very well-chosen selection of mathematical notions and techniques as they are relevant for the foundations of Mathematics and Computer Science. The somewhat mysterious word "practical" in the title simply insinuates that the author doesn't want "foundations" to be understood in any "foundational" sense. The aim rather is to exhibit and study the mathematical principles behind logic and induction as needed and used for the formalisation of (the main parts of) Mathematics and Computer Science. However, instead of classical logic and set theory, the book is based on and provides an introduction to constructive logic and type theory, the latter being much closer to mathematical practice than set theory. Constructive logic is also discussed from the point of view of propositions-as-types where proofs appear as sort of functional programs thus bridging the gap between logic and computation. The mathematical structures employed for providing a mathematical underpinning are mainly categorical in nature thus allowing for general and concise formulations. Whereas till the last chapter the underlying logic is intuitionistic higher-order logic (à la Church, i.e. based on functions and a type of propositions) in the final chapter one finds a thorough discussion of the set-theoretic axiom of Replacement and strong arguments in favour of type-theoretic universes as capturing the essence of the axiom of Replacement in a type theoretic framework. As opposed to most texts on constructive type theory which are very syntactically inclined in the current book the use of type-theoretic language is much more informal which is in accordance with the informal way set theory is used in mathematics.

Next, we survey the contents of the book in more detail.

The first chapter introduces the basics of *First-Order Reasoning* with an emphasis on a box calculus providing a graphical visualisation of the handling of contexts in natural deduction.

The second chapter on *Types and Induction* discusses sum, product and function types, the paradigm of propositions-as-types and the technique of structural induction exemplified by natural numbers and lists.

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Chapter 3 on *Posets and Lattices* explains the basic notions of lattices and domains as well as closure operators and Galois connections. The aim is to present the order-theoretic version of categorical concepts introduced later on in greater generality.

Chapter 4 on *Cartesian Closed Categories* provides a detailed study of products and exponentials, i.e. the categorical semantics of simply typed λ-calculus on which functional programming languages like SML and HASKELL are based.

Chapter 5 on *Limits and Colimits* introduces these basic categorical structures and discusses also their relation to programming. Moreover, one can find there a detailed account of extensivity of sums, factorisation systems and regular categories in one place whereas previously one had to consult several sources to get these informations.

Chapter 6 on *Structural Recursion* discusses a general recursion theorem in categorical terms where the emphasis is on "glueing attempts". As sort of digressions one can find a detailed discussion of tail recursion and a brief account of ordinals and how they may look like from a constructive point of view.

Chapter 7 on *Adjunctions* introduces free constructions and pairs of adjoint functors (including Freyd's Adjoint Functor Theorem) and monads (including Beck's Monadicity Theorem). In the final section one finds a detailed account of the gluing technique from categorical logic which is used for proving the completeness of the equational theory of simply typed λ-calculus and the disjunction and existence properties of constructive logic.

Chapter 8 discusses *Algebra with Dependent Types*. After introducing the syntax of dependent types one finds a detailed discussion of its term model which serves as a motivation for the categorical semantics of dependent type theories as given by categories together with a class of display maps.

Finally, in Chapter 9, titled *The Quantifiers*, the author explains how the main ingredients of logic find their place within type theory. After introducing the important concept of indexed or fibred category existential and universal quantifiers are introduced as left and right adjoints to weakening, respectively, an idea going back to work of F.W. Lawvere from the end of 1960s. Furthermore, the concept of comprehension (also going back to Lawvere) is explained by the requirement that for the fibration P representing the logic it holds that $P \dashv T \dashv C$, where T selects the terminal object in each fibre and C turns a predicate A on X into the subtype $\{x \in X | A(x)\}$ of X. Higher-order logic is introduced via a generic predicate Tr on the type of propositions for which it is required that any predicate on X is equivalent to $Tr(\chi(x))$ for some map χ from X to the type of propositions. Finally, there is a section about *universes* containing a proof of Gödel's Incompleteness Theorem and a discussion of the set-theoretic axiom of Replacement. The conclusion of the latter is that type-theoretic universes (à la Martin-Löf, albeit possibly impredicative) allow one to define families of types by recursion over some inductive index type and that's precisely what the axiom of Replacement is used for in set theory. Moreover, such a universe can be used for constructing a model of the fragment without this universe, i.e. for consistency proofs, just as in ZF where Replacement allows one to construct a model for Zermelo's set theory, i.e. ZF without the axiom of Replacement. Thus, via universes one may consider hierarchies of stronger and stronger system each of them proving the consistency of the previous ones but not of itself as this were in contradiction to Gödel's Incompleteness Theorem.

From this survey of contents it should be clear that the book provides a coherent picture of "modern" categorical foundations as opposed to the somewhat "old-fashioned" set-theoretic ones. The former are as strong as the latter as long as classical logic is assumed. However, the advantage of categorical foundations, besides being closer to mathematical practice, is that they aren't biased towards classical logic like traditional set theory.

Therefore, the book can be strongly recommended to everybody interested in foundations, typically rather a mathematically inclined computer scientist than a "working mathematician" who in general has less need for and, accordingly, less interest in foundations. However, it would be nice if the presentation were on a more elementary level. This surely would have been possible because the things presented are actually fairly elementary in character. Alas, the author couldn't refrain from making numerous allusions which cannot be understood on the basis of what has been presented so far. For example, the author discusses locally presentable categories before giving a systematic introduction to limits and colimits. The author's profound knowledge of sheaves and toposes lurks behind the surface at many places and often breaks through. Alas, it isn't made clear what the naive reader is expected to understand and what are the comments to be appreciated by the specialist. This results in a mixture of very elementary and very advanced passages which, I fear, may repell a student who just starts to learn about foundations. Moreover, sometimes proofs aren't explicit enough to my taste (e.g. of Beck's Monadicity Theorem) or a bit too complicated (as e.g. the presentation of interpretations of dependent type theories in Chapter 7). A definitive advantage of the book are the numerous exercises on various levels of difficulty, including advanced ones of interest also to the specialist. These numerous exercises are very helpful to anybody who wants to give a course on "modern foundations". To such a person this book definitely can be recommended. To the "naive" reader I can recommend this book as well albeit under the proviso that he is patient enough to accept that many parts can be understood only after consulting the literature or at least subsequent chapters of the book.

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