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An Extension of Carlson's Theorem for Entire Functions of Exponential Type, II*

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1. INTRODUCTION AND STATEMENT OF RESULTS

1.1. Let $f(z)$ be regular on an unbounded connected subset E of the complex plane. If

$$M_f^*(r) = \sup_{|z|=r,\, z\in E} |f(z)| \tag{1}$$

then

$$\rho = \limsup_{r\to\infty} \frac{\log\log M_f^*(r)}{\log r} \tag{2}$$

is said to be the order of $f(z)$. The order of a bounded function is 0, by convention. It is clear that $\rho \geqslant 0$ if $M_f^*(r)$ is unbounded. If $0 < \rho < \infty$ we define the type of $f(z)$ to be

$$\tau = \limsup_{r\to\infty} r^{-\rho} \log M_f^*(r). \tag{3}$$

A function of order not exceeding ρ and of type at most τ if of order ρ is said to be of growth (ρ, τ). A function of growth $(1, \tau)$, $\tau < \infty$, is called a function of exponential type τ.

If $f(z)$ is of exponential type in a sector

$$S: |z| \geqslant 0, \qquad \alpha \leqslant \arg z \leqslant \beta, \tag{4}$$

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its indicator function is defined as

$$h_f(\theta) = \limsup_{r\to\infty} r^{-1} \log |f(re^{i\theta})|, \qquad \alpha \leqslant \theta \leqslant \beta. \tag{5}$$

For properties of the indicator function we refer the reader to Chapter 5 of [1]. One additional property which is not mentioned in [1] but which we shall need is that for a function of order 1 and type $\tau < \infty$ in the sector (4), we have

$$\tau = \max_{\alpha\leqslant\theta\leqslant\beta} h_f(\theta). \tag{6}$$

1.2. It was proved by Carlson ([2], also see [5, pp. 185–186]) that if $f(z)$ is of exponential type $\tau < \pi$ in $\operatorname{Re} z \geqslant 0$, and $f(z) = 0$ for $z = 0, 1, 2, 3,\ldots$, then $f(z) \equiv 0$. This result has been extended and generalized in various ways. For example

THEOREM A. *If $f(z)$ is of exponential type $<m\pi$ in* $\operatorname{Re} z \geqslant 0$, *and*

$$f(n) = 0, \qquad f'(n) = 0,\ldots, f^{(m-1)}(n) = 0, \qquad (n = 0, 1, 2,\ldots) \tag{7}$$

then $f(z) \equiv 0$.

THEOREM B. *Let $\{\lambda_n\}_{n=0}^{\infty}$ be an increasing sequence of positive numbers and let $\Lambda(t)$ denote the number of λ_n not exceeding t. If $f(z)$ is of exponential type in* $\operatorname{Re} z \geqslant 0$, $f(\lambda_n) = 0$, $n = 0, 1, 2,\ldots$,

$$\Lambda(x) \geqslant x + \delta(x), \tag{8}$$

and

$$\log |f(iy)f(-iy)| \leqslant 2\pi\{|y| + \sigma(|y|)\}, \qquad \text{and} \qquad -\int_1^R \sigma(t)\,dt \leqslant O(R^2) \tag{9}$$

then $f(z) \equiv 0$ if

$$\limsup_{R\to\infty} \int_1^R y^{-2}\{\delta(y) - \sigma(y)\}\,dy = +\infty. \tag{10}$$

Theorem A is implicitly contained in [4]. Later in the paper we will give a short proof of this result. For a proof of Theorem B we refer the reader to [1, p. 155]. Theorem B generalizes Carlson's theorem in two different ways. On the one hand, functions of order 1 type π are allowed if a supplementary hypothesis is satisfied. On the other hand, $f(z)$ need not be assumed to vanish at all the positive integers—indeed the zeros of $f(z)$ do not have to be at the integers.

In connection with our study of $(0, m)$-interpolation by entire functions of exponential type in points on the real axis we wanted to know to what extent an entire function of exponential type is determined by its values and those of its m-th derivative at the set of integers. We obtained [3] the following results.

THEOREM C. *If $f(z)$ is an entire function of exponential type $\tau < 2\pi$ such that*

$$f(n) = f''(n) = 0, \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{11}$$

then $f(z) = c \sin(\pi z)$ where c is a constant. Here $\tau = 2\pi$ is inadmissible.

THEOREM D. *Let m be an even integer $\geqslant 4$. If $f(z)$ is an entire function of exponential type $\tau < \pi \sec(\pi/m)$ such that*

$$f(n) = f^{(m)}(n) = 0, \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{12}$$

then $f(z) = c \sin(\pi z)$ where c is a constant. Here τ cannot be allowed to be $\pi \sec(\pi/m)$.

THEOREM E. *Let m be an odd integer $\geqslant 3$. If $f(z)$ is an entire function of exponential type $\tau < \pi \sec(\pi/2m)$ such that*

$$f(n) = f^{(m)}(n) = 0, \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{12'}$$

then $f(z) \equiv 0$. Here $\tau = \pi \sec(\pi/2m)$ is inadmissible.

THEOREM F. *Let m be an integer $\geqslant 2$, and λ an arbitrary number in $[0, 1)$. If $f(z)$ is an entire function of exponential type 2π such that*

(i) $|f(x)| \leqslant A + B \mid x \mid^{\lambda}$ *for all real x and certain constants A, B,*

(ii) $f(n) = f^{(m)}(n) = 0$, $(n = 0, \pm 1, \pm 2, \ldots)$

then

$$f(z) = \begin{cases} C_1 \sin(\pi z) + C_2 \sin(2\pi z) & \text{if} \quad m \text{ is even} \\ C \sin^2(\pi z) & \text{if} \quad m \text{ is odd,} \end{cases} \tag{13}$$

where C_1, C_2 and C are constants. Here λ cannot be allowed to be equal to 1.

Now a number of questions come to mind.

QUESTION No. 1. *Is it necessary to assume in Theorems* C, D, E *and* F *that $f(z)$, $f^{(m)}(z)$ vanish at all the positive and negative integers?*

We shall see that in the case of Theorems C, D, and E the hypothesis about the zeros of $f(z)$ and $f^{(m)}(z)$ can be considerably relaxed. For example, it is enough to assume that $f(z)$, $f^{(m)}(z)$ vanish at all the positive integers. Thus under the growth restrictions of these three theorems, $f(z)$, $f^{(m)}(z)$ vanish at the positive integers only if they vanish at all the positive and negative integers. The situation is different as far as Theorem F is concerned. Here we cannot say that $f(z)$ is of the form (13), unless we really know that $f(n)$, $f^{(m)}(n)$ vanish at *all* the integers $n = 0, \pm 1, \pm 2, \ldots$. This is one of the consequences of the following

THEOREM 1. *For a fixed integer $m \geqslant 2$, let*

$$A_{\mu,m}(z)$$
$$= (-1)^{\mu} \frac{\sin(\pi z)}{\pi(z-\mu)} + (-1)^{\mu} \sin(\pi z) \left\{ \sum_{\nu=1}^{[m/2]} - N_\nu \exp\left(\pi(z-\mu)\cot\frac{\nu\pi}{m}\right) \right.$$
$$\times \int_{z-\mu}^{\infty} \exp\left(-\pi\zeta\cot\frac{\nu\pi}{m}\right) H(\zeta)\, d\zeta$$
$$\left. + \sum_{\nu=[m/2]+1}^{m-1} N_\nu \exp\left(\pi(z-\mu)\cot\frac{\nu\pi}{m}\right) \int_{-\infty}^{z-\mu} \exp\left(-\pi\zeta\cot\frac{\nu\pi}{m}\right) H(\zeta)\, d\zeta \right\}$$
$$(\mu = 0, \pm 1, \pm 2, \ldots) \qquad (14)$$

where

$$N_\nu = \frac{1}{\omega'\left(\pi\cot\frac{\nu\pi}{m}\right)}, \quad (\nu = 1, 2, \ldots, m-1), \qquad \omega(z) = \prod_{\nu=1}^{m-1}\left(z - \pi\cot\frac{\nu\pi}{m}\right), \qquad (15)$$

and

$$H(z) = (-1)^{m+1} \frac{m!}{\pi z^{m+1}} \left\{ \sin(\pi z) - \sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1} \frac{(\pi z)^{2\nu-1}}{(2\nu-1)!} \right\}. \qquad (16)$$

An entire function $F_{\mu,m}(z)$ of exponential type 2π satisfies the conditions

(i) $|F_{\mu,m}(x)|$ *is bounded on the real axis*

(ii) $F_{\mu,m}(n) = \delta_{\mu,n} \qquad (n = 0, \pm 1, \pm 2, \ldots)$ (17)

(iii) $F_{\mu,m}^{(m)}(n) = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots)$

if and only if it has the form $A_{\mu,m}(z) + C_1 \sin(\pi z) + C_2 \sin(2\pi z)$ or $A_{\mu,m}(z) + C \sin^2(\pi z)$ according as m is even or odd. Here as usual

$$\delta_{\mu,n} = \begin{cases} 0 & \text{if} \quad n \neq \mu \\ 1 & \text{if} \quad n = \mu \end{cases}$$

and C_1, C_2, C are arbitrary constants.

Further, let

$$B_{\mu,m}(z)$$
$$= (-1)^{\mu} \sin(\pi z) \left\{ \sum_{\nu=1}^{[m/2]} - N_\nu \exp\left(\pi(z-\mu)\cot\frac{\nu\pi}{m}\right) \right.$$

$$\times \int_{z-\mu}^{\infty} \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) \frac{\sin(\pi\zeta)}{\pi\zeta}\, d\zeta$$

$$+ \sum_{\nu=[m/2]+1}^{m-1} N_\nu \exp\left(\pi(z-\mu) \cot \frac{\nu\pi}{m}\right) \int_{-\infty}^{z-\mu} \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) \frac{\sin(\pi\zeta)}{\pi\zeta}\, d\zeta\Bigg\}$$

$$(\mu = 0, \pm 1, \pm 2, \ldots). \qquad (18)$$

An entire function $G_{\mu,m}(z)$ *of exponential type* 2π *satisfies the conditions*

$$\begin{aligned} &\text{(a)} \quad |G_{\mu,m}(x)| \text{ is bounded on the real axis} \\ &\text{(b)} \quad G_{\mu,m}(n) = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots) \\ &\text{(c)} \quad G_{\mu,m}^{(m)}(n) = \delta_{\mu,n} \qquad (n = 0, \pm 1, \pm 2, \ldots) \end{aligned} \qquad (19)$$

if and only if it is of the form $B_{\mu,m}(z) + C_1 \sin(\pi z) + C_2 \sin(2\pi z)$ *or* $B_{\mu,m}(z) + C \sin^2(\pi z)$ *according as m is even or odd.*

QUESTION No. 2. *Is it necessary to assume in Theorems* C, D, *and* E *that* $f(z)$ *is entire, or is it perhaps enough to assume* $f(z)$ *to be regular in the right half-plane and of the same growth as before?*

It turns out (see Corollary 1) that the conclusion of Theorem C remains unchanged under this more general hypothesis. As for Theorems D and E we will see (see Corollaries 3, 4) that $f(z)$ must be a linear combination of certain functions $f_\nu(z)$ of the form

$$\sin(\pi z) \exp\left(\pi z \cot \frac{\nu\pi}{m}\right).$$

QUESTION No. 3. *Does it make any difference if we assume* $f(z)$ *to be regular and of exponential type in the half-plane* $\operatorname{Im}(ze^{-i\alpha}) \leqslant 0$, $0 < \alpha < \pi/2$, *which is not symmetrical with respect to the positive real axis on which the zeros of* $f(z)$ *and those of* $f^{(m)}(z)$ *are supposed to lie?*

It is interesting that the conclusions of Theorems D and E remain unchanged (see Corollaries 3′, 4′) if $f(z)$ instead of being entire is regular and of the same growth as before in the half-plane $\operatorname{Im}(ze^{-i\alpha}) \leqslant 0$ where $\pi/4 \leqslant \alpha < \pi/2$, provided m is sufficiently large. This, along with various other results answering the preceding questions will be deduced from Theorem 2 below. In order to state the theorem we need to introduce a

DEFINITION. Given a sequence of complex numbers $\{w_k\}_{k=1}^{\infty}$, such that

$$|w_1| \leqslant |w_2| \leqslant \cdots \leqslant |w_k| \leqslant |w_{k+1}| \leqslant \cdots, \qquad \lim_{k\to\infty} |w_k| = \infty,$$

let $n_w(r)$ denote the number of w_k's in $|z| \leqslant r$. We shall say that "$f(z)$ and $f^{(m)}(z)$

vanish at almost all the positive integers" if there exist two sequences of positive integers $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$ satisfying

$$\lim_{r\to\infty} \frac{n_\lambda(r)}{r} = 0, \qquad \lim_{r\to\infty} \frac{n_\mu(r)}{r} = 0, \tag{20}$$

such that $f(z)$ vanishes at all the positive integers except possibly at the λ_k's and $f^{(m)}(z)$ vanishes at all the positive integers except possibly at the μ_k's.

By considering the union of the sets $\{\lambda_1, \lambda_2, ..., \lambda_k, ...\}$ and $\{\mu_1, \mu_2, ..., \mu_k, ...\}$ we may suppose that the two sequences $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$ in the above definition are the same. Thus the statement "$f(z)$ and $f^{(m)}(z)$ vanish at almost all the positive integers" means that $f(z) = f^{(m)}(z) = 0$ at all the positive integers except possibly for a sequence $(\lambda_k)_{k=1}^{\infty}$ of positive integers such that

$$\lim_{r\to\infty} \frac{n_\lambda(r)}{r} = 0.$$

Hereafter α will be a real number such that $0 < \alpha \leqslant \pi/2$.

THEOREM 2. *Let $f(z)$ be regular and of exponential type in the half-plane* $\operatorname{Im}(ze^{-i\alpha}) \leqslant 0$ *such that*

$$\max\{h_f(\alpha), h_f(\alpha - \pi)\} = T_\alpha < 2\pi \sin \alpha. \tag{21}$$

If, for an integer $m \geqslant 2$, $f(z)$ and $f^{(m)}(z)$ vanish at "almost all" the positive integers, then

$$f(z) = \sum_{(m/2)(1-(2\alpha/\pi))<\nu<(m/2)(1+(2\alpha/\pi))} c_\nu f_\nu(z) \tag{22}$$

where

$$f_\nu(z) = \{\sin(\pi z)\} \exp\left(\pi z \cot \frac{\nu\pi}{m}\right), \qquad \frac{m}{2}\left(1 - \frac{2\alpha}{\pi}\right) < \nu < \frac{m}{2}\left(1 + \frac{2\alpha}{\pi}\right) \tag{23}$$

and c_ν's are constants.

In the case $m = 2$, the conclusion of Theorem 2 takes a particularly simple form.

COROLLARY 1. *Let $f(z)$ be regular and of exponential type in* $\operatorname{Im}(ze^{-i\alpha}) \leqslant 0$ *such that* (21) *holds. If $f(z)$ and $f''(z)$ vanish at "almost all" the positive integers, then $f(z) = c \sin(\pi z)$ where c is a constant. The function $f(z) = \sin(2\pi z)$ shows that here $T_\alpha = 2\pi \sin \alpha$ is inadmissible.*

It is clear that the functions $f_\nu(z)$ are entire. In fact, $f_\nu(z)$ is of order 1 and of type $\tau_\nu = \pi \operatorname{cosec}(\nu\pi/m)$. Hence taking $\alpha = \pi/2$ in Theorem 2 we can conclude

COROLLARY 2. *If $f(z)$ is an entire function of exponential type $<2\pi$ such that, for an integer $m \geqslant 2$, $f(z)$ and $f^{(m)}(z)$ vanish at "almost all" the positive integers, then*

$$f(z) = \sum_{m/6<\nu<5m/6} c_\nu f_\nu(z)$$

where

$$f_\nu(z) = \{\sin(\pi z)\} \exp\left(\pi z \cot \frac{\nu\pi}{m}\right), \qquad \frac{m}{6} < \nu < \frac{5m}{6}$$

and c_ν's are constants.

Note that Theorems D and E can be readily deduced from the preceding corollary.

If τ_ν^* denotes the type of the function

$$f_\nu(z) = \{\sin(\pi z)\} \exp\left(\pi z \cot \frac{\nu\pi}{m}\right)$$

in $\operatorname{Im}(ze^{-i\alpha}) \leqslant 0$, then

$$\begin{aligned}\tau_\nu^* &= \max_{\alpha-\pi\leqslant\theta\leqslant\alpha} h_f(\theta) \\ &= \begin{cases} \pi \operatorname{cosec} \dfrac{\nu\pi}{m} & \text{if} \quad 0 < \nu \leqslant m\left(1 - \dfrac{\alpha}{\pi}\right) \\ \pi \sin \alpha - \pi\left(\cot \dfrac{\nu\pi}{m}\right) \cos \alpha & \text{if} \quad m\left(1 - \dfrac{\alpha}{\pi}\right) < \nu < \dfrac{m}{2}\left(1 + \dfrac{2\alpha}{\pi}\right). \end{cases}\end{aligned}$$

Now we have to distinguish two cases.

Case I. If m is an even integer $\geqslant 4$, then

$$\tau_\nu^* \geqslant \pi \sec \frac{\pi}{m}$$

for all ν in $(0, m(1 - \alpha/\pi)]$ except for $\nu = m/2$ as well as for all $\nu \geqslant [\theta_*]$ where θ_* is the smallest root of the equation

$$\sin \alpha - \left(\cot \frac{\theta\pi}{m}\right) \cos \alpha - \sec \frac{\pi}{m} = 0$$

in $(m(1 - \alpha/\pi), \infty)$. Thus we have

COROLLARY 3. *Let m be an even integer $\geqslant 4$. If $f(z)$ is regular and of exponential type $<\pi \sec \pi/m$ in the half-plane* $\mathrm{Im}(ze^{-i\alpha}) \leqslant 0$ *such that* (21) *holds, and $f(z)$, $f^{(m)}(z)$ vanish at "almost all" the positive integers, then*

$$f(z) = c_{m/2} \sin(\pi z) + \sin(\pi z) \sum_{m(1-(\alpha/\pi))<\nu<\theta_*} c_\nu \exp\left(\pi z \cot \frac{\nu\pi}{m}\right)$$

where $c_{m/2}$, $c_\nu(m(1-\alpha/\pi) < \nu < \theta_*)$ are constants.

Case II. If m is an odd integer $\geqslant 3$, then

$$\tau_\nu^* \geqslant \pi \sec \frac{\pi}{2m}$$

for all ν in $(0, m(1-\alpha/\pi)]$ as well as for all $\nu \geqslant [\theta^*]$ where θ^* is the smallest root of the equation

$$\sin \alpha - \left(\cot \frac{\theta\pi}{m}\right) \cos \alpha - \sec \frac{\pi}{2m} = 0$$

in $(m(1-\alpha/\pi), \infty)$. Hence we have

COROLLARY 4. *Let m be an odd integer $\geqslant 3$. If $f(z)$ is regular and of exponential type $<\pi \sec \pi/2m$ in the half-plane* $\mathrm{Im}(ze^{-i\alpha}) \leqslant 0$ *such that* (21) *holds, and $f(z)$, $f^{(m)}(z)$ vanish at "almost all" the positive integers, then*

$$f(z) = \sin(\pi z) \sum_{m(1-(\alpha/\pi))<\nu<\theta^*} c_\nu \exp\left(\pi z \cot \frac{\nu\pi}{m}\right)$$

where c_ν's are constants.

Note that if $m \geqslant 2\pi/(\pi - 2\alpha)$ in Corollary 3, then $m(1-\alpha/\pi) \geqslant \theta_*$, whereas if $m \geqslant \pi/(\pi - 2\alpha)$ in Corollary 4, then $m(1-\alpha/\pi) \geqslant \theta^*$. Consequently we have

COROLLARY 3′. *Let m be an even integer $\geqslant 2\pi/(\pi - 2\alpha)$. If $f(z)$ is regular and of exponential type $<\pi \sec \pi/m$ in the half-plane* $\mathrm{Im}(ze^{-i\alpha}) \leqslant 0$ *such that* (21) (*which is automatically satisfied if $\pi/4 \leqslant \alpha < \pi/2$*) *holds, and $f(z)$, $f^{(m)}(z)$ vanish at "almost all" the positive integers, then $f(z) = c \sin(\pi z)$ where c is a constant.*

COROLLARY 4′. *Let m be an odd integer $\geqslant \pi/(\pi - 2\alpha)$. If $f(z)$ is regular and of exponential type $<\pi \sec \pi/2m$ in the half-plane* $\mathrm{Im}(ze^{-i\alpha}) \leqslant 0$ *such that* (21) (*which is automatically satisfied if $\pi/4 \leqslant \alpha < \pi/2$*) *holds, and $f(z)$, $f^{(m)}(z)$ vanish at "almost all" the positive integers, then $f(z) \equiv 0$.*

2. Some Lemmas

Notation. Let n be a positive integer. The plane region (a rhombus)

$$\left\{z = x + iy: \frac{1}{2}\sin\alpha \leqslant x\sin\alpha - y\cos\alpha \leqslant \left(n + \frac{1}{2}\right)\sin\alpha,\ |y| \leqslant \frac{n}{2}\sin\alpha\right\}$$

will be denoted by $\mathscr{R}_n$.

Lemma 1. *On the boundary $\partial\mathscr{R}_n$ of $\mathscr{R}_n$ we have*

$$|\sin(\pi z)| \geqslant \sin\left(\frac{\pi}{2}\sin\alpha\right). \tag{24}$$

Proof. Using the infinite product representation of $\sin(\pi z)$ it can be shown that (24) holds on circles γ_j centred at the points $j = 0, \pm 1, \pm 2,...$ and radius $\frac{1}{2}\sin\alpha$. Besides, it is easily seen that if N is an integer, then on all the four sides of the square C_N with corners at the points $\pm(N + \frac{1}{2}) \pm i(N + \frac{1}{2})$ we have

$$|\sin(\pi z)| \geqslant 1.$$

Hence by the minimum modulus principle (24) holds in the region D_N bounded by C_N and γ_j, $j = 0, \pm 1, \pm 2,..., \pm N$. If $N \geqslant n(1 + \frac{1}{2}\cos\alpha)$ then $\partial\mathscr{R}_n$ lies in D_N and so (24) holds on $\partial\mathscr{R}_n$ as well.

Lemma 2. *Let $f(z)$ be regular and of exponential type in the half-plane* $\text{Im}(ze^{-i\alpha}) \leqslant 0$. *If*

$$f(n) = 0, \qquad n = 1, 2,...$$

then

$$f(z) = \phi(z)\sin(\pi z)$$

where $\phi(z)$ is regular and of exponential type in $\text{Im}\{(z-1)\,e^{-i\alpha}\} \leqslant 0$.

Proof. Let $f(z)$ be of exponential type τ, i.e. for every $\epsilon > 0$ there exists a constant $K(\epsilon)$ such that

$$|f(z)| < K(\epsilon)\exp\{(\tau + \epsilon)\,|z|\}$$

if $\text{Im}(ze^{-i\alpha}) \leqslant 0$. In view of the preceding lemma, we have on $\partial\mathscr{R}_n$

$$|\phi(z)| \leqslant |f(z)|\,\text{cosec}\left(\frac{\pi}{2}\sin\alpha\right)$$

$$< K(\epsilon)\,\text{cosec}\left(\frac{\pi}{2}\sin\alpha\right)\exp\left\{(\tau + \epsilon)\left|\frac{1}{2} + n\left(1 + \frac{1}{2}e^{i\alpha}\right)\right|\right\}.$$

By the maximum modulus principle the same estimate holds inside $\mathscr{R}_n$. Hence, if $z \in \mathscr{R}_{n+1} \backslash \mathscr{R}_n$, then

$$\begin{aligned}
|\phi(z)| &\leqslant \max_{z\in\partial\mathscr{R}_{n+1}} |\phi(z)| \\
&< K(\epsilon)\operatorname{cosec}\left(\frac{\pi}{2}\sin\alpha\right)\exp\left\{(\tau+\epsilon)\left|\frac{1}{2}+(n+1)\left(1+\frac{1}{2}e^{i\alpha}\right)\right|\right\} \\
&\leqslant K(\epsilon)\operatorname{cosec}\left(\frac{\pi}{2}\sin\alpha\right) \\
&\quad\times\exp\left\{(\tau+\epsilon)\left|\frac{1}{2}+(n+1)\left(1+\frac{1}{2}e^{i\alpha}\right)\right|\left(\frac{|z|}{(n/2)\sin\alpha}\right)\right\} \\
&< K(\epsilon)\operatorname{cosec}\left(\frac{\pi}{2}\sin\alpha\right)\exp\{7(\operatorname{cosec}\alpha)(\tau+\epsilon)|z|\}. \qquad (25)
\end{aligned}$$

Besides, on $\mathscr{R}_1$

$$\begin{aligned}
|\phi(z)| &\leqslant \max_{z\in\partial\mathscr{R}_1} |\phi(z)| \\
&< K(\epsilon)\operatorname{cosec}\left(\frac{\pi}{2}\sin\alpha\right)\exp\left\{(\tau+\epsilon)\left(\frac{5}{2}+\frac{3}{2}\cos\alpha\right)^{1/2}\right\} \\
&\leqslant K(\epsilon)\operatorname{cosec}\left(\frac{\pi}{2}\sin\alpha\right)\exp\left\{(\tau+\epsilon)\left(\frac{5}{2}+\frac{3}{2}\cos\alpha\right)^{1/2}\frac{|z|}{\frac{1}{2}\sin\alpha}\right\} \\
&< K(\epsilon)\operatorname{cosec}\left(\frac{\pi}{2}\sin\alpha\right)\exp\{4(\operatorname{cosec}\alpha)(\tau+\epsilon)|z|\}. \qquad (26)
\end{aligned}$$

Inequalities (25) and (26) show that $\phi(z)$ is of exponential type in $\operatorname{Im}\{(z-\frac{1}{2})e^{-i\alpha}\} \leqslant 0$ and *a fortiori* in $\operatorname{Im}\{(z-1)e^{-i\alpha}\} \leqslant 0$.

Remark 1. We have shown above that if $|f(z)| < K(\epsilon)\exp\{(\tau+\epsilon)|z|\}$ in $\operatorname{Im}(ze^{-i\alpha}) \leqslant 0$ then $|\phi(z)| < K(\epsilon)\operatorname{cosec}((\pi/2)\sin\alpha)\exp\{7(\operatorname{cosec}\alpha)(\tau+\epsilon)|z|\}$ in $\operatorname{Im}\{(z-1)e^{-i\alpha}\} \leqslant 0$. From this it can be easily deduced that if $f(z)$ is regular in the half-plane $\operatorname{Re} z \geqslant -1$ where

$$|f(z)| < K(\epsilon)\exp\{(\tau+\epsilon)|z|\},$$

and $f(n) = 0$ for $n = 0, 1, 2,\ldots$, then

$$\left|\frac{f(z)}{\sin(\pi z)}\right| < e^{8(\tau+\epsilon)}K(\epsilon)\exp\{7(\tau+\epsilon)|z|\}$$

in the right half-plane. Thus, if $f(z)$ is an entire function of exponential type vanishing at all the positive and negative integers, then applying the above conclusion to the function $f(-z)$ as well we conclude that

$$f(z) = \phi(z) \sin(\pi z)$$

where $\phi(z)$ is an entire function of exponential type.

LEMMA 3 ([1], see 6.2.3). *If $f(z)$ is regular and of exponential type in the first quadrant, $|f(iy)| \leqslant M$ $(0 \leqslant y < \infty)$, and $|f(x)| \leqslant Ae^{cx}$ $(0 \leqslant x < \infty)$, then*

$$|f(x+iy)| \leqslant \max(M, A)\, e^{cx}, \qquad 0 \leqslant y < \infty, \quad 0 \leqslant x < \infty. \tag{27}$$

LEMMA 4. *Let $\Phi(z)$ be regular and of exponential type in the first quadrant. If*

$$|\Phi(iy)| = O(e^{ty}), \qquad 0 \leqslant y < \infty$$

then for a given positive a

$$|\Phi^{(k)}(a+iy)| \leqslant C_k e^{ty}, \qquad 0 \leqslant y < \infty, \quad k = 0, 1, 2, \ldots,$$

where C_k depends on a but not on y.

Proof. Consider the function

$$f(z) = e^{itz}\Phi(z)$$

which is of exponential type in the first quadrant. Besides, there exist constants A, c and M such that $|f(iy)| \leqslant M$ $(0 \leqslant y < \infty)$, and $|f(x)| \leqslant Ae^{cx}$ $(0 \leqslant x < \infty)$. Hence by the preceding lemma

$$|f(x+iy)| \leqslant \max(M, A)\, e^{cx}, \qquad 0 \leqslant x < \infty, \quad 0 \leqslant y < \infty.$$

Consequently,

$$|\Phi(x+iy)| \leqslant \max(M, A)\, e^{2ac} e^{ty}, \qquad 0 \leqslant x \leqslant 2a, \quad 0 \leqslant y < \infty. \tag{28}$$

For $y > a$, we have by Cauchy's integral formula

$$\begin{aligned} |\Phi^{(k)}(a+iy)| &= \left| \frac{k!}{2\pi i} \int_{|\zeta-(a+iy)|=a} \frac{\Phi(\zeta)}{\{\zeta-(a+iy)\}^{k+1}}\, d\zeta \right| \\ &\leqslant \frac{k!}{a^k} \max(M, A)\, e^{2ac} e^{t(y+a)} \\ &= C' e^{ty} \end{aligned}$$

where C' depends on a. Since

$$\max_{0\leqslant y\leqslant a} |\,\Phi^{(k)}(a+iy)| \leqslant C''$$

for some constant C'', the desired result follows.

From Lemma 4 we can easily deduce

LEMMA 4'. *Let $\Phi(z)$ be of exponential type in* $\operatorname{Im}\{(z-1)\,e^{-i\alpha}\} \leqslant 0$. *If*

$$|\,\Phi(1+re^{i\theta})| = O(e^{tr}), \qquad (\theta = \alpha, \alpha - \pi)$$

then

$$|\,\Phi^{(k)}(2+re^{i\theta})| = O(e^{tr}), \qquad (\theta = \alpha, \alpha - \pi).$$

LEMMA 5. *If $f(z)$ is an even entire function of exponential type given by*

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n{}^2}\right),$$

and if

$$\lim_{r\to\infty}\frac{n_\lambda(r)}{r} = 2B,$$

then

$$\lim_{r\to\infty}\frac{\log|f(re^{i\theta})|}{r} = \pi B\,|\sin\theta\,|, \qquad \theta \neq 0, \pi.$$

For a proof of this lemma we refer the reader to [1, p. 137].

3. PROOFS OF THEOREMS 1 AND 2

Proof of Theorem 1. Let us construct a function $F_{0,m}(z)$ of exponential type 2π such that

$$\begin{aligned}
&\text{(i)}\quad |\,F_{0,m}(x)| \text{ is bounded on the real axis}\\
&\text{(ii)}\quad F_{0,m}(0) = 1, \qquad F_{0,m}(n) = 0 \qquad (n = \pm 1, \pm 2, \pm 3,\ldots) \qquad\qquad (29)\\
&\text{(iii)}\quad F_{0,m}^{(m)}(n) = 0 \qquad\qquad\qquad\qquad (n = 0, \pm 1, \pm 2, \pm 3,\ldots).
\end{aligned}$$

Since $F_{0,m}(z)$ is supposed to vanish at all the positive and negative integers except at the origin where it assumes the value 1 it should be (see Remark 1) of the form $\sin(\pi z)/\pi z + \sin(\pi z)\,\phi(z)$ where $\phi(z)$ is entire and of exponential type. Using Leibnitz's theorem we obtain

$$z^{m+1}A_{0,m}^{(m)}(z)$$
$$= \left\{\sum_{\nu=0}^{[m/2]} (-1)^{\nu}\,\pi^{2\nu}\binom{m}{2\nu}\left(\frac{(-1)^{m-2\nu}\,(m-2\nu)!}{\pi}\,z^{2\nu} + z^{m+1}\phi^{(m-2\nu)}(z)\right)\right\}\sin(\pi z)$$
$$+ \left\{\sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1}\,\pi^{2\nu-1}\binom{m}{2\nu-1}\right.$$
$$\left.\times\left(\frac{(-1)^{m-2\nu+1}\,(m-2\nu+1)!}{\pi}\,z^{2\nu-1} + z^{m+1}\phi^{(m-2\nu+1)}(z)\right)\right\}\cos(\pi z).$$

Since $F_{0,m}^{(m)}(z)$ is required to vanish at all the positive and negative integers the entire function

$$\left\{\sum_{\nu=0}^{[m/2]} (-1)^{\nu}\,\pi^{2\nu}\binom{m}{2\nu}\frac{(-1)^{m-2\nu}\,(m-2\nu)!}{\pi}\,z^{2\nu}\right\}\sin(\pi z)$$
$$+ \left\{\sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1}\,\pi^{2\nu-1}\binom{m}{2\nu-1}\right.$$
$$\left.\times\left(\frac{(-1)^{m-2\nu+1}\,(m-2\nu+1)!}{\pi}\,z^{2\nu-1} + z^{m+1}\phi^{(m-2\nu+1)}(z)\right)\right\}\cos(\pi z)$$

should be divisible by $\sin(\pi z)$ and its Maclaurin series expansion should be of the form $\sum_{k=m+1}^{\infty} a_k z^k$. It is easily verified that this will be the case if

$$\sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1}\,\pi^{2\nu-1}\binom{m}{2\nu-1}$$
$$\times\left\{\frac{(-1)^{m-2\nu+1}\,(m-2\nu+1)!}{\pi}\,z^{2\nu-1} + z^{m+1}\phi^{(m-2\nu+1)}(z)\right\}$$
$$= \frac{(-1)^{m+1}\,m!\,\sin(\pi z)}{\pi}. \tag{30}$$

Setting

$$H(z) = \frac{(-1)^{m+1}\,m!}{\pi z^{m+1}}\left\{\sin(\pi z) - \sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1}\frac{(\pi z)^{2\nu-1}}{(2\nu-1)!}\right\}$$

we may write this differential equation in the form

$$\sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1}\,\pi^{2\nu-1}\binom{m}{2\nu-1}\phi^{(m-2\nu+1)}(z) = H(z). \tag{31}$$

Note that $H(z)$ is an enitre function of order 1 type π and is bounded on the real axis. The general solution of the above differential equation is

$$\phi(z) = \sum_{\nu=1}^{m-1} c_\nu \exp\left(\pi z \cot \frac{\nu\pi}{m}\right) + \sum_{\nu=1}^{m-1} N_\nu \exp\left(\pi z \cot \frac{\nu\pi}{m}\right) \int_0^z \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) H(\zeta)\, d\zeta,$$

where c_ν's are arbitrary constants and N_ν's are as defined in (15). If we choose

$$c_\nu = -N_\nu \int_0^\infty \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) H(\zeta)\, d\zeta \qquad \text{for } \nu = 1, 2,..., [m/2],$$

and

$$c_\nu = N_\nu \int_{-\infty}^0 \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) H(\zeta)\, d\zeta \qquad \text{for } \nu = [m/2]+1,..., m-1,$$

then the function $\phi(z)$ becomes

$$\sum_{\nu=1}^{[m/2]} - N_\nu \exp\left(\pi z \cot \frac{\nu\pi}{m}\right) \int_z^\infty \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) H(\zeta)\, d\zeta + \sum_{\nu=[m/2]+1}^{m-1} N_\nu \exp\left(\pi z \cot \frac{\nu\pi}{m}\right) \int_{-\infty}^z \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) H(\zeta)\, d\zeta,$$

which is of order 1 type π and is bounded on the real axis. Thus the function

$$\frac{\sin(\pi z)}{\pi z} + \{\sin(\pi z)\} \left\{ \sum_{\nu=1}^{[m/2]} - N_\nu \exp\left(\pi z \cot \frac{\nu\pi}{m}\right) \int_z^\infty \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) H(\zeta)\, d\zeta + \sum_{\nu=[m/2]+1}^{m-1} N_\nu \exp\left(\pi z \cot \frac{\nu\pi}{m}\right) \int_{-\infty}^z \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) H(\zeta)\, d\zeta \right\}$$

which is $A_{0,m}(z)$ as defined in (14), possesses the required properties (29). Since $A_{\mu,m}(z)$ is simply $A_{0,m}(z-\mu)$ it satisfies conditions (17). By Theorem F any other function $F_{\mu,m}(z)$ satisfying the conditions (17) must be of the form $A_{\mu,m}(z) + C_1 \sin(\pi z) + C_2 \sin(2\pi z)$ or $A_{\mu,m}(z) + C \sin^2(\pi z)$ according as m is even or odd.

Now we will construct a function $G_{0,m}(z)$ of exponential type 2π such that

(a) $|G_{0,m}(x)|$ is bounded on the real axis

(b) $G_{0,m}(n) = 0 \qquad (n = 0, \pm 1, \pm 2,...)$ (32)

(c) $G_{0,m}^{(m)}(0) = 1, \qquad G_{0,m}^{(m)}(n) = 0 \qquad (n = \pm 1, \pm 2,...).$

Since $G_{0,m}(n) = 0$ for $n = 0, \pm 1, \pm 2, \ldots$ the function $G_{0,m}(z)$ should be of the form $\{\sin(\pi z)\}\, \phi(z)$ where $\phi(z)$ is entire and of exponential type. Again using Leibnitz's theorem we obtain

$$G_{0,m}^{(m)}(z) = \left\{ \sum_{\nu=0}^{[m/2]} (-1)^{\nu}\, \pi^{2\nu} \binom{m}{2\nu} \phi^{(m-2\nu)}(z) \right\} \sin(\pi z)$$

$$+ \left\{ \sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1}\, \pi^{2\nu-1} \binom{m}{2\nu-1} \phi^{(m-2\nu+1)}(z) \right\} \cos(\pi z).$$

Since $G_{0,m}^{(m)}(z)$ is supposed to vanish at all the positive and negative integers except at the origin where it assumes the value 1 we must have

$$\sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1}\, \pi^{2\nu-1} \binom{m}{2\nu-1} \phi^{(m-2\nu+1)}(z) = \psi(z)\, \frac{\sin(\pi z)}{\pi z} \tag{33}$$

where $\psi(z)$ is an entire function of exponential type such that $\psi(0) = 1$. Let $\psi(z)$ be identically equal to 1. Then the general solution of the differential equation (33) has the form

$$\phi(z) = \sum_{\nu=1}^{m-1} c_{\nu} \exp\left(\pi z \cot \frac{\nu\pi}{m}\right)$$

$$+ \sum_{\nu=1}^{m-1} N_{\nu} \exp\left(\pi z \cot \frac{\nu\pi}{m}\right) \int_0^z \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) \frac{\sin(\pi\zeta)}{\pi\zeta}\, d\zeta$$

where c_{ν}'s are arbitrary constants and N_{ν}'s are as in (15). Choosing

$$c_{\nu} = -N_{\nu} \int_0^{\infty} \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) \frac{\sin(\pi\zeta)}{\pi\zeta}\, d\zeta \qquad \text{for } \nu = 1, 2, \ldots, \left[\frac{m}{2}\right],$$

and

$$c_{\nu} = N_{\nu} \int_{-\infty}^{0} \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) \frac{\sin(\pi\zeta)}{\pi\zeta}\, d\zeta \qquad \text{for } \nu = \left[\frac{m}{2}\right] + 1, \ldots, m-1,$$

we obtain

$$\phi(z) = \sum_{\nu=1}^{[m/2]} - N_{\nu} \exp\left(\pi z \cot \frac{\nu\pi}{m}\right) \int_z^{\infty} \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) \frac{\sin(\pi\zeta)}{\pi\zeta}\, d\zeta$$

$$+ \sum_{\nu=[m/2]+1}^{m-1} N_{\nu} \exp\left(\pi z \cot \frac{\nu\pi}{m}\right) \int_{-\infty}^{z} \exp\left(-\pi\zeta \cot \frac{\nu\pi}{m}\right) \frac{\sin(\pi\zeta)}{\pi\zeta}\, d\zeta.$$

It is easily seen that $\phi(z)$ is an entire function of order 1 type π and is bounded on the real axis. As a function $G_{0,m}(z)$ possessing the properties (32) we obtain

$$\{\sin(\pi z)\}\left\{\sum_{\nu=1}^{[m/2]} - N_\nu \exp\left(\pi z \cot\frac{\nu\pi}{m}\right)\int_z^\infty \exp\left(-\pi\zeta\cot\frac{\nu\pi}{m}\right)\frac{\sin(\pi\zeta)}{\pi\zeta}\,d\zeta\right.$$

$$\left.+\sum_{\nu=[m/2]+1}^{m-1} N_\nu \exp\left(\pi z\cot\frac{\nu\pi}{m}\right)\int_{-\infty}^{z}\exp\left(-\pi\zeta\cot\frac{\nu\pi}{m}\right)\frac{\sin(\pi\zeta)}{\pi\zeta}\,d\zeta\right\}$$

which is $B_{0,m}(z)$ as defined in (18). Since $B_{\mu,m}(z)$ is simply $B_{0,m}(z-\mu)$ it satisfies conditions (19). By Theorem F any other function $G_{\mu,m}(z)$ satisfying the conditions (19) must be of the form $B_{\mu,m}(z) + C_1\sin(\pi z) + C_2\sin(2\pi z)$ or $B_{\mu,m}(z) + C\sin^2(\pi z)$ according as m is even or odd. With this the proof of Theorem 1 is complete.

Remark 2. Let m be a positive integer. In the study of $(0, m)$-interpolation by entire functions of exponential type it is important to know if for given $\tau_0 > 0$ there exist entire functions $r_{k,m}(z)$, $\rho_{k,m}(z)$ of exponential type τ_0 such that

$$r_{k,m}(n) = \delta_{k,n} \qquad (n = 0, \pm 1, \pm 2,\ldots)$$

$$r_{k,m}^{(m)}(n) = 0 \qquad (n = 0, \pm 1, \pm 2,\ldots)$$

whereas

$$\rho_{k,m}(n) = 0 \qquad (n = 0, \pm 1, \pm 2,\ldots)$$

$$\rho_{k,m}^{(m)}(n) = \delta_{k\,n} \qquad (n = 0, \pm 1, \pm 2,\ldots).$$

These functions may be called "fundamental functions of $(0, m)$-interpolation". From Corollary 2 it follows that there are no fundamental functions of $(0, m)$-interpolation which are of exponential type $<2\pi$. On the other hand, Theorem 1 characterizes all fundamental functions of $(0, m)$-interpolation which are of exponential type 2π and are bounded on the real axis.

Proof of Theorem 2. According to the hypothesis $f(z)$ and $f^{(m)}(z)$ vanish at all the positive integers except possibly for a sequence $\{\lambda_k\}_{k=1}^\infty$ such that $\lim_{r\to\infty} n_\lambda(r)/r = 0$. Now let

$$P(z) = \prod_{k=1}^\infty\left(1 - \frac{z^2}{\lambda_k{}^2}\right).$$

By Lemma 2,

$$P(z)\,f(z) = \phi(z)\sin(\pi z)$$

where $\phi(z)$ is regular and of exponential type in $\operatorname{Im}((z-1)\,e^{-i\alpha}) \leqslant 0$. Hence

$$f(z) = \Phi(z)\sin(\pi z)$$

where $\Phi(z) = \phi(z)/P(z)$ is regular in $\operatorname{Im}((z-1)e^{-i\alpha}) \leqslant 0$ except possibly for poles at the points λ_k. Consequently,

$$\begin{aligned}\{P(z)\}^{m+1} f^{(m)}(z) &= \left\{\sum_{\nu=0}^{[m/2]} (-1)^{\nu} \pi^{2\nu} \binom{m}{2\nu} (P(z))^{m+1} \Phi^{(m-2\nu)}(z)\right\} \sin(\pi z) \\ &+ \left\{\sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1} \pi^{2\nu-1} \binom{m}{2\nu-1} (P(z))^{m+1} \Phi^{(m-2\nu+1)}(z)\right\} \cos(\pi z).\end{aligned}$$

Since $\{P(z)\}^{m+1} f^{(m)}(z)$ vanishes at all the positive integers, it follows that the function

$$G(z) = \sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1} \pi^{2\nu-1} \binom{m}{2\nu-1} (P(z))^{m+1} \Phi^{(m-2\nu+1)}(z)$$

vanishes too at all the positive integers and by Lemma 2,

$$G(z) = \sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1} \pi^{2\nu-1} \binom{m}{2\nu-1} (P(z))^{m+1} \Phi^{(m-2\nu+1)}(z) = \psi(z) \sin(\pi z) \tag{34}$$

where $\psi(z)$ is regular and of exponential type in $\operatorname{Im}((z-2)e^{-i\alpha}) \leqslant 0$. Here $\psi(z)$ must be identically zero. Suppose, if possible, that $\psi(z) \not\equiv 0$. Then by a property of the indicator function ([1], see 5.4.4)

$$\limsup_{r\to\infty} \frac{\log|\psi(2+re^{i\alpha})|}{r} + \limsup_{r\to\infty} \frac{\log|\psi(2+re^{i(\alpha-\pi)})|}{r} \geqslant 0,$$

so that

$$\max\left\{\limsup_{r\to\infty} \frac{\log|G(2+re^{i\alpha})|}{r}, \limsup_{r\to\infty} \frac{\log|G(2+re^{i(\alpha-\pi)})|}{r}\right\} \geqslant \pi \sin \alpha.$$

With the help of Lemmas 4′ and 5 we conclude that

$$\max\left\{\limsup_{r\to\infty} \frac{\log|\phi(1+re^{i\alpha})|}{r}, \limsup_{r\to\infty} \frac{\log|\phi(1+re^{i(\alpha-\pi)})|}{r}\right\} \geqslant \pi \sin \alpha,$$

and thus $\max\{h_f(\alpha), h_f(\alpha-\pi)\} \geqslant 2\pi \sin\alpha$ which is a contradiction. Therefore $\psi(z) \equiv 0$. Consequently $\Phi(z)$ being a solution of the differential equation

$$\sum_{\nu=1}^{[(m+1)/2]} (-1)^{\nu-1} \pi^{2\nu-1} \binom{m}{2\nu-1} \Phi^{(m-2\nu+1)}(z) = 0,$$

should be of the form

$$\Phi(z) = \sum_{\nu=1}^{m-1} c_\nu \exp \left\{\pi z \cot \frac{\nu\pi}{m}\right\},$$

where $c_1, c_2, \dots, c_{m-1}$ are constants. But only the constants c_ν, $(m/2)(1 - (2\alpha/\pi)) < \nu < (m/2)(1 + (2\alpha/\pi))$, can be different from zero since $\max\{h_\Phi(\alpha), h_\Phi(\alpha - \pi)\} < \pi \sin \alpha$. Hence

$$\Phi(z) = \sum_{(m/2)(1-(2\alpha/\pi))<\nu<(m/2)(1+(2\alpha/\pi))} c_\nu \exp\left(\pi z \cot \frac{\nu\pi}{m}\right),$$

and the theorem is proved.

As promised earlier we will now give a short proof of Theorem A.

Proof of Theorem A. Let $\phi_0(z) = f(z)/\sin(\pi z)$, $\phi_j(z) = \phi_{j-1}(z)/\sin(\pi z)$, $j = 1, 2, \dots, m-1$. Applying Lemma 2 (with $\alpha = \pi/2$), successively to the functions $\phi_j(z)$, $j = 0, 1, 2, \dots, m-1$ we conclude that $f(z)/\{\sin(\pi z)\}^m = \phi_{m-1}(z)$ is of exponential type in $\mathrm{Re}(z - m) \geq 0$. If $\phi_{m-1}(z) \not\equiv 0$, then, again by 5.4.4 of [1]

$$\max\left\{\limsup_{r\to\infty} \frac{\log|\phi_{m-1}(m + re^{i(\pi/2)})|}{r}, \limsup_{r\to\infty} \frac{\log|\phi_{m-1}(m + re^{-i(\pi/2)})|}{r}\right\} \geq 0$$

and hence

$$\max\left\{\limsup_{r\to\infty} \frac{\log|f(m + re^{i(\pi/2)})|}{r}, \limsup_{r\to\infty} \frac{\log|f(m + re^{-i(\pi/2)})|}{r}\right\} \geq m\pi,$$

which in view of Lemma 4, contradicts the fact that $f(z)$ is of exponential type $< m\pi$.

References

1. R. P. Boas, Jr., "Entire Functions," Academic Press, New York, 1954.
2. F. Carlson, "Sur une classe de séries de Taylor," Thesis, Uppsala, 1914.
3. R. Gervais and Q. I. Rahman, An extension of Carlson's theorem for entire functions of exponential type, *Trans. Amer. Math. Soc.* **235** (1978), 387–394.
4. J. Korevaar, Functions of exponential type bounded on sequences of points, *Ann. Polon. Math.* **22** (1949–1950), 207–234.
5. E. C. Titchmarsh, "The Theory of Functions," 2nd ed., Oxford Univ. Press, London, 1939.