Simple tangential family germs and perestroikas of their envelopes

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Abstract

A tangential family is a 1-parameter system of regular curves emanating tangentially from another regular curve. We classify simple tangential family germs up to $\mathcal{A}$-equivalence. We describe perestroikas of envelopes of simple tangential family germs of small codimension under small deformations of the germ among tangential families.

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1. Introduction

A tangential family is a system of regular curves emanating tangentially from another regular curve, called support. Tangential families naturally arise for instance in Singularity Theory, Differential Geometry, Optics and Image Treatment. For example, a family of regular curves in a neighborhood of a generic point of its envelope (the latter acting as support) is a tangential family. Another example is given by the family of geodesics tangent to a curve in a Riemannian surface. This case is discussed in [7].
The roots of the theme go back to Huygens’ study of caustics. However, the direction in which the investigation is developed here differs from the classical approach of Thom and Arnold (see [1,2] and [13]). For example, for the first time the situation when the support curve is not the only local component of the envelope is studied. In [5] we classified stable tangential family germs (under small deformations among tangential families and with respect to $\mathcal{A}$-equivalence) and we proved that their envelopes are smooth or have a second order self-tangency.

In this paper we classify simple tangential family germs, identified to smooth map germs $(\mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$. The classification up to $\mathcal{A}$-equivalence does not take into account the natural fibration of the source space: hence, $\mathcal{A}$-equivalent germs may parameterize non-diffeomorphic families of curves. However, $\mathcal{A}$-equivalence preserves the major feature of the families, their envelopes, which are the critical value sets of their parameterizations (any finer classification would have produced moduli). Moreover, we study tangential deformations, that is, deformations of tangential family germs among tangential family germs.

We prove that, in addition to the two stable singularities described in [5], there are two infinite series and four sporadic simple singularities of tangential family germs. We give their normal forms and miniversal tangential deformations, and we describe perestroikas of small codimension occurring to their envelopes under small tangential deformations. We also study envelopes of non-simple tangential family germs.

Our results are related to the classifications of simple smooth map germs (Rieger, see [11,12]) and of simple local projections of surfaces to a plane (Goryunov, see [8]). Tangential family germs are studied in the setting of Contact Geometry in [6].

2. Simple tangential family germs

Unless otherwise specified, all the objects considered below are supposed to be of class $C^\infty$; by plane curve we mean a smooth map $\mathbb{R} \to \mathbb{R}^2$. A curve is said to be regular at every point at which its first derivative is not zero. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be a smooth map of variables $\xi$ and $t$. We set $f_\xi (t) = f(\xi, t)$.

**Definition.** The one-parameter family of regular curves $\{f_\xi : \xi \in \mathbb{R}\}$ is a **tangential family** whenever the following conditions are fulfilled:

(a) the curve $\xi \mapsto f(\xi, 0)$, called support, is regular;
(b) for every $\xi \in \mathbb{R}$, the curve $f_\xi$ is tangent to the support at $f(\xi, 0)$.

These conditions imply in particular that for every $\xi \in \mathbb{R}$, $\partial_\xi f(\xi, 0)$ and $\partial_t f(\xi, 0)$ are proportional non-zero vectors.

The **graph** of a tangential family $\{f_\xi : \xi \in \mathbb{R}\}$ is the immersed surface

$$\Phi := \{(\xi, f_\xi (t)) : \xi, t \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}^2.$$

Let us denote by $\pi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ the projection $(\xi, P) \mapsto P$. The **criminant set** of the tangential family is the critical set of $\pi|_\Phi$. The **envelope** is the apparent contour of $\Phi$ in $\mathbb{R}^2$. 

(i.e. the critical value set of $\pi|_\Phi$). By the very definition, the support of the family belongs to its envelope.

Below we will study tangential family germs (TFGs for short), so we will consider their local parameterizations as map germs $(\mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$.

**Remark.** The graphs of TFGs are germs of embedded surfaces.

We will denote by $s$ and $t$ the variables of $\mathbb{R}^2$ when the space is not equipped with the fibration $\mathbb{R} \times \mathbb{R}$, leaving the variable $\xi$ for the distinguished parameter of a TFG in the fibered case. Denote by Diff$(\mathbb{R}^2, 0)$ the group of diffeomorphism germs of the plane keeping fixed the origin and by $m_{s,t}$ the ring of smooth function germs $(\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ in the variables $s$ and $t$. Then the group $\mathscr{A} := $ Diff$(\mathbb{R}^2, 0) \times $ Diff$(\mathbb{R}^2, 0)$ acts on the space $(m_{s,t})^2$ by $(\phi, \psi) \cdot f := \psi \circ f \circ \phi^{-1}$, where $(\phi, \psi) \in \mathscr{A}$ and $f \in (m_{s,t})^2$. For a survey on $\mathscr{A}$-equivalence, we refer e.g. to [3] and [10].

Two elements of $(m_{s,t})^2$ are $\mathscr{A}$-equivalent, if they belong to the same $\mathscr{A}$-orbit. In this paper, equivalence of map germs always means $\mathscr{A}$-equivalence.

Families of curves parameterized by equivalent germs may be non-diffeomorphic. In particular, a TFG may be equivalent to a map germ which is not a TFG. However, the equivalence preserves the major feature of the families of curves, namely, their envelopes.

**Proposition 1.** Critical value sets of equivalent germs are diffeomorphic.

**Proof.** Let $f$ and $g$ be two map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, such that $f = \psi \circ g \circ \phi^{-1}$, for $\phi, \psi \in $ Diff$(\mathbb{R}^2, 0)$. By the chain rule and Cauchy–Binet Theorem, we get

$$\det Df_{|(s,t)} = \det D\psi_{|(g \circ \phi^{-1})(s,t)} \cdot \det Dg_{|\phi^{-1}(s,t)} \cdot \det D\phi^{-1}_{|(s,t)},$$

where $Df$ denotes the Jacobian matrix of $f$. Since $\phi$ and $\psi$ are local diffeomorphisms, $(\det Df)(s,t) = 0$ if and only if $(\det Dg)(\phi^{-1}(s,t)) = 0$. Thus, the critical value sets of $f$ and $g$ are respectively $f(\Sigma)$ and $(g \circ \phi^{-1})(\Sigma)$, where $\Sigma = \{(s,t): \det Df(s,t) = 0\}$ is the critical set of $f$. Then the local diffeomorphism $\psi$ transforms $(g \circ \phi^{-1})(\Sigma)$ into $f(\Sigma)$. \[ \square \]

**Corollary.** Envelopes of equivalent TFGs are diffeomorphic.

**Proof.** We show that the envelope of any TFG is nothing but its critical value set. Then the statement follows from Proposition 1. Let $f$ be a TFG; the (well defined) tangent plane to its graph at $(\xi, f(\xi, t))$ is spanned by the non-zero vectors $(1, \partial_\xi f(\xi, t))$ and $(0, \partial_t f(\xi, t))$. This plane is vertical with respect to $\pi$ if and only if $\partial_\xi f(\xi, t)$ and $\partial_t f(\xi, t)$ are proportional, that is, if and only if $(\det Df)(\xi, t) = 0$. \[ \square \]

The standard definitions of deformation and versality can be translated in the setting of tangential families.

**Definition.** Let $p \in \mathbb{N} \cup \{0\}$. A $p$-parameter tangential deformation of a TFG $f$ is a map germ $F: (\mathbb{R}^2 \times \mathbb{R}^p, 0) \to (\mathbb{R}^2, 0)$, such that $F(\cdot; \lambda)$ is a tangential family for every $\lambda$ arbitrary small and $F(\cdot; 0) = f(\cdot)$. 
For instance, the tangent geodesics to a curve in a Riemannian surface form a tangential family. A perturbation of the metric induces a tangential deformation on this family.

**Definition.** A tangential deformation $F$ of a TFG $f$ is said to be *versal* if any tangential deformation of $f$ is equivalent to a tangential deformation induced from $F$.

Such a deformation is said to be *miniversal* whenever the base dimension is minimal. This minimal number $\tau$, depending only on the orbit of the TFG, is called *tangential codimension*. $F$ is stable if every versal tangential deformation is trivial; it is simple if by arbitrary sufficiently small tangential deformations of it, we can obtain only a finite number of singularities.

Recall that a map germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ defines, by the formula $f^*g := g \circ f$, a homomorphism from the ring $\mathcal{E}_{x,y}$ of the function germs in the target to the ring $\mathcal{E}_{x,t}$ of the function germs in the source. Hence, every $\mathcal{E}_{x,t}$-module is an $\mathcal{E}_{x,y}$-module via this homomorphism. The *extended tangent space* at $f$ to its orbit is the $\mathcal{E}_{x,y}$-module defined by

$$T_{e,f} := \langle \partial_x f, \partial_t f \rangle \mathcal{E}_{x,t} + f^*(\mathcal{E}_{x,y}) \cdot \mathbb{R}^2.$$  

The *extended codimension* $c$ of $f$ is the dimension of the $\mathbb{R}$-vector space $\mathcal{E}_{x,t}^2 / T_{e,f}$. Notice that $\tau \leq c$.

We state now our main result, proven in Section 4.

**Theorem 1.** Every simple tangential family germ is $\mathcal{A}$-equivalent to a tangential family germ $x = s, y = \phi(s, t)$, of support $y = 0$, where $s = \xi + t$ and $\phi$ is one of the germs listed below ($n \in \mathbb{N}$). A miniversal tangential deformation of such a normal form is obtained adding $\sum \lambda_i e_i$ to $\phi$, according to the table.

<table>
<thead>
<tr>
<th>Singularity</th>
<th>$\phi$</th>
<th>$c$</th>
<th>${e_i}$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$t^2$</td>
<td>0</td>
<td>$-$</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>$st^2 - t^3$</td>
<td>1</td>
<td>$-$</td>
<td>0</td>
</tr>
<tr>
<td>$S_{1,n}$</td>
<td>$st^2 + t^3 + t^{2n+3}$</td>
<td>$n + 1$</td>
<td>$t^3, t^5, \ldots, t^{2n+1}$</td>
<td>$n$</td>
</tr>
<tr>
<td>$S_{2,2}$</td>
<td>$st^2 + t^5 + t^6$</td>
<td>3</td>
<td>$t^3, t^4$</td>
<td>2</td>
</tr>
<tr>
<td>$S_{2,3}^{-\pm}$</td>
<td>$st^2 + t^5 \pm t^9$</td>
<td>4</td>
<td>$t^3, t^4, t^6$</td>
<td>3</td>
</tr>
<tr>
<td>$S_{2,4}$</td>
<td>$st^2 + t^5$</td>
<td>5</td>
<td>$t^3, t^4, t^6, t^9$</td>
<td>4</td>
</tr>
<tr>
<td>$T_n$</td>
<td>$s^{n+1} t^2 + t^3$</td>
<td>$2n + 1$</td>
<td>$t^2, st^2, \ldots, s^{n-1} t^2$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

**Remarks.** (1) The normal forms of this table also appear as normal forms in the classification of the simple projections of surfaces in the plane, due to V.V. Goryunov [8,9] and in the classification of simple map germs $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Due to J.H. Rieger [11, 12]. Normal forms $(s, s^{n+1} t^2 + t^3)$ correspond to Rieger’s normal forms $(s, t^3 - s^{2n+2} t)$.

(2) In [5] we proved that the only stable singularities are I and II; the corresponding envelopes are respectively smooth and have an order 2 self-tangency.

**Corollary.** A simple TFG has a singularity I or its envelope has, in addition to the support, a second local component $\delta$, tangent to the support. For $S_{1,n}$-TFGs, $\delta$ has a $(2n + 3)/2$-
cusp at the tangency point, not crossing the support; for $S_{2,2}, S^\pm_{2,3}$ or $S_{2,4}$-TFGs, $\delta$ has a $5/3$-singularity at the tangency point; for $T_n$-TFGs, $\delta$ is embedded and has a $(3n + 2)$-tangency with the support.

Envelope of simple TFGs are shown in Fig. 1.

A singularity $L$ is said to be adjacent to a singularity $K$, and we write $L \rightarrow K$, if every map germ in $L$ can be deformed into a map germ in $K$ by an arbitrary small tangential deformation. If $L \rightarrow K \rightarrow K'$, then $L$ is also adjacent to $K'$; in this case we omit the arrow $L \rightarrow K'$. It follows from Theorem 1 that adjacencies of simple singularities of TFGs are as follows.

$$
\begin{array}{cccc}
I & \leftarrow & II & \leftarrow \\
S_{1,1} & \leftarrow & S_{1,2} & \leftarrow \\
S_{1,3} & \leftarrow & S_{1,4} & \leftarrow \\
S_{1,5} & \leftarrow & \cdots & \\
S_{2,2} & \leftarrow & S^\pm_{2,3} & \leftarrow \\
S_{2,4} & \leftarrow & \cdots & \\
\end{array}
$$

The set of TFGs is naturally decomposed according to the configurations of their criminant sets. We define now this decomposition. Given a TFG, the fiber $\pi^{-1}(0,0)$ defines a vertical direction in the tangent plane to its graph at the origin. The germ is of first type if its criminant set is the germ of an embedded curve, of second type if it is the union of two non-vertical embedded local components, transversal each other. In [5] we proved that a tangential family germ has a singularity I (resp., II) if and only if it is of first type (resp., of second type).

**Definition.** Let $n \in \mathbb{N} \cup \{\infty\}$. A TFG is said to be:

1. of type $S_n$ if its criminant set is the union of two embedded local components, transversal each other, one of which has an order $n$ tangency with the vertical direction;
2. of type $T_n$ if its criminant set is the union of two non-vertical embedded local components, which have an order $n$ tangency;
3. of type $U$ if its criminant set has more than two local components or at least a singular component.

**Remark.** Equivalent TFGs are of the same type. Indeed, TFGs of different type have non-diffeomorphic envelopes.
Proposition 2. Every finitely determined TFG of type $S_1$ has a singularity $S_{1,n}$ for some $n \in \mathbb{N}$. Every TFG of type $S_2$ has one of the singularities $S_{2,2}$, $S_{2,3}^\pm$, or $S_{2,4}$. Every TFG of type $T_n$ has a singularity $T_n$.

The next result describes envelopes of TFGs of type $S_n \geq 3$.

Proposition 3. For $n \in \mathbb{N}$, $n \geq 3$, the envelope of any $S_n$-type TFG has, in addition to the support, a local component with a singularity of order $(n + 3)/(n + 1)$, tangent to the support. If $n$ is even, this singularity is a cusp, not crossing the support.

Propositions 2 and 3 are proven in Section 4.

We end this section with the description of the hierarchy of non-simple TFGs. We denote by $(L)$ and $(K)$ some classes of non-simple singularities. The arrow $(L) \to (K)$ means that there exists two singularities $L' \subset (L)$ and $K' \subset (K)$ such that $L' \to K'$. The main adjacencies of non-simple TFGs are represented in the following diagram, where $n, m, r \in \mathbb{N}$, $n \geq 3$.

$$
\begin{array}{ccc}
(S_\infty) & \to & S_{n+1} \\
\downarrow & & \downarrow \\
(U) & \to & S_{1,m} \\
\downarrow & & \downarrow \\
(T_\infty) & \to & T_r \\
\end{array}
$$

3. Bifurcation diagrams of simple tangential family germs

In this section we discuss the bifurcation diagrams of small codimension TFGs’ simple singularities, and the perestroikas of the corresponding envelopes. The discriminant of a TFG is the restriction to tangential deformations of the usual discriminant of the germ (see e.g. [3]). The bifurcation diagram of a TFG is its discriminant, together with the envelopes of the deformed germs. Below, we will consider the singularities normal forms and deformations listed in Theorem 1.

We start with singularities $S_{1,n}$. By the corollary of Theorem 1, the envelope has two branches: the first is the support, the second one has a $(2n + 3)/2$-cusp tangent to the support.

Remark. The $S_{1,n}$-discriminant contains the flag $V_{n-1} \supset \cdots \supset V_0$, where $V_i$ is defined by $\lambda_1 = \cdots = \lambda_{n-i} = 0$. A germ in the stratum $V_i \setminus V_{i-1}$ has a singularity $S_{1,i}$, according to the adjacency $S_{1,n+1} \to S_{1,n}$.

For singularity $S_{1,1}$, the envelope perestroika is shown in Fig. 2.
Remark. The envelopes of the perturbed families form a singular surface germ in $\mathbb{R}^3 = \{ x, y, \lambda \}$, diffeomorphic to a folded umbrella with a cubically tangent smooth surface; the half line of the umbrella self-intersections is tangent to the smooth surface.

The $S_{1,2}$-discriminant is represented in Fig. 3. It has been found for me experimentally by F. Aicardi. Note that it contains the flag $V_1 \supset V_0$ and a curve, corresponding to a self-tangency of the envelope second branch.

The $S_{2,2}$-discriminant in the plane $\{ \lambda_1, \lambda_2 \}$, shown in Fig. 4, is the union of the $\lambda_2$-axis and four curve germs, tangent to it at the origin, the tangency order of all these curves being 1.

Finally, we consider singularities $T_n$.

**Theorem 2.** The discriminant of the singularity $T_n$ is an $n$-dimensional swallowtail.

**Proof.** Let us denote by $\xi(s; \lambda)$ the function germ

$$s^{n+1} + \lambda_{n-1}s^{n-1} + \cdots + \lambda_0,$$

which is a miniversal deformation of $s^{n+1}$ for the Left equivalence in the space of function germs $(\mathbb{R}, 0) \to \mathbb{R}$. By Theorem 1, a miniversal tangential deformation of a $T_n$-TFG is equivalent to $(s, t^3 + t^2 \xi(s; \lambda))$. For every fixed $\lambda$ small enough, the critical set of this germ has two components, given locally by $t = 0$ and $3t + 2\xi(s; \lambda) = 0$. 

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![Fig. 2. Envelope perestroika of singularity $S_{1,1}$.](image1.png)

![Fig. 3. Bifurcation diagram of singularity $S_{1,2}$.](image2.png)
Hence, the critical value set has two components near the origin, given locally by \( y = 0 \) and \( 27y = 4\xi(s; \lambda)^3 \). The \( T_n \)-discriminant is formed by the values \( \lambda \) for which these two local components have tangency points of order greater than 2. Such a point corresponds to a root of multiplicity greater than 3 of \( \xi^3_3 \), where \( \xi_\lambda(s) = \xi(s; \lambda) \).

Thus, the parameter values for which these points exist is the set of the parameter values for which \( \xi_\lambda \) has at least one root of multiplicity greater than 1. This set is an \( n \)-dimensional swallowtail.

Bifurcation diagrams of singularities \( T_1 \), \( T_2 \) and \( T_3 \) are shown in Fig. 5.

4. Proofs

Here we prove our results. We start describing standard parameterizations of TFGs. Let us fix a TFG; one easily verifies (see [5]) that in a coordinate system in which the family support is the germ at the origin of \( y = 0 \), the TFG is of the form

\[
(\xi, t) \mapsto \left( \xi + t, t^2 \sum_{i=0}^{\infty} k_i \xi^i + t^2 R(\xi) + \alpha t^3 + t^3 \cdot \delta(0) \right),
\]

where \( \alpha, k_i \in \mathbb{R}, \delta(n) \) denotes any function of two variables with vanishing \( n \)-jet at the origin and \( R : (\mathbb{R}, 0) \to (\mathbb{R}, 0) \) is flat (i.e., its Taylor expansion is zero). In [5] we proved...
that a tangential family germ is of first type if and only if $k_0 \neq 0$; it is of second type if and only if $k_0 = 0$ and $k_1 \neq 0, \alpha$. Therefore, for any tangential family germ neither of first nor of second type, we have $k_0 = 0$ and $k_1 (k_1 - \alpha) = 0$.

Setting $s = \xi + t$ in (1), we obtain that every TFG is equivalent to $(s, \phi(s,t))$, where

$$\phi(s,t) = t^2 \sum_{i=0}^{\infty} k_is^i + t^2 \tilde{R}(s) + (\alpha - k_1)t^3 + t^2 \cdot \delta(0),$$

$\tilde{R}$ being flat. Notice that such a germ is not a TFG.

**Lemma 1.** Every TFG, neither of first nor of second type, and equivalent to $(s, \phi(s,t))$, where $\phi$ is as above, is:

- of type $S_n$ if and only if $\phi(0,t) = O(t^{n+3})$;
- of type $T_n$ if and only if $k_0 = \cdots = k_n = 0$ and $\alpha, k_{n+1} \neq 0$;
- of type $U$ if and only if $\phi(s,t) = t^3 \delta(1)$. 

**Fig. 5.** Bifurcation diagrams of singularities $T_1, T_2$ and $T_3$. 

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*Image and text.*
Fig. 6. Newton diagram of $\partial_t \phi(s,t)/t = 0$.

Similar characterizations hold for $S_\infty$ and $T_\infty$.

**Proof.** We compute the criminant set of the TFG (1). Let us define the coefficients $a_i$ such that the Taylor expansion of $\phi(0,t)$ is $t^2 \sum a_i t^i$. In particular, $a_1 = \alpha - k_1$. Since the support of (1) is the germ of $y = 0$ at the origin, $\partial_t \phi(s,t)$ is divisible by $t$; we have:

$$
\frac{\partial_t \phi(s,t)}{t} = \sum_{i \geq 1} 2k_is^i + R_1(s) + \sum_{i \geq 1} (i + 2)a_1 t^i + R_2(t) + st \cdot u(s,t),
$$

where $R_1$ and $R_2$ are flat and $u$ is some smooth function germ. Let us consider the Newton diagram, shown in Fig. 6, associated to the equation $\partial_t \phi(s,t)/t = 0$.

This equation cannot have a solution of the form $s = O(t)$, since by hypothesis at least one of the coefficients $k_1$ and $a_1$ is non-zero. We have therefore the following possibilities: (1) the equation has a solution of the form $s = O(t^{n+1})$ (iff $a_1 = \ldots = a_n = 0$ and $k_1, a_{n+1} \neq 0$); (2) it has a solution of the form $t = O(s^{n+1})$ (iff $k_1 = \ldots = k_n = 0$ and $a_1, k_{n+1} \neq 0$); (3) it has more than one solution or a singular solution (iff $k_1 = a_1 = 0$).

In the first case, $\phi(0,t) = a_{n+1}t^{n+3} + o(t^{n+3})$, and the solution is of the form

$$
s = s(t) = \frac{(n+3)a_{n+1}}{2k_1} t^{n+1} + o(t^{n+1}).
$$

Hence, the second branch of the TFG’s criminant set in the space $\{\xi; x, y\}$ is the curve $t \mapsto (s(t) - t, s(t), \phi(s(t), t))$, whose leading terms have degree respectively 1, $n + 1$ and $n + 3$. This component has a tangency order equal to $n$ with the vertical direction over the origin of the plane; in particular the TFG is of type $S_n$.

Similarly, in the second case the above equation solution is

$$
t = t(s) = -\frac{2k_1}{3a_1} s^{n+1} + o(s^{n+1}).
$$

The corresponding criminant branch in the space $\{\xi; x, y\}$ is a curve

$$(s + O(s^{n+1}), s, O(s^{3(n+1)})),$$

which has a tangency order equal to $n$ with the other criminant branch $s \mapsto (s, s, 0)$. Hence, the TFG is of type $T_n$. Finally, in the last case the criminant set has more than two component or at least a singular component. $\square$
Proposition 3 follows from Lemma 1 by an explicit computation. This lemma also provides the non-simple TFGs adjacencies, described above.

We can prove now Theorem 1 and Proposition 2. Recall that the group of $k$-jets of elements of $\mathcal{A}$ acts on the space of $k$-jets of smooth map germs. When we speak of equivalence of jets, we mean equivalence for this action.

$S_1$-type TFGs. It follows by Lemma 1 that the 4-jet of any $S_1$-TFG is equivalent to $(s, st^2 + t^4)$. It is proven in [11] (Lemma 3.2.1:1) that the only $\mathcal{A}$-finite map germs over this 4-jet are

$$f_n(s, t) = (s, st^2 + t^4 + t^{2n+3}),$$

for $n \in \mathbb{N}$, and that these germs are $(2n + 3)$-determined. These germs are not TFGs; the corresponding TFG normal forms are $\tilde{f}_n(\xi, t) := f_n(\xi + t, t)$.

We compute now the miniversal tangential deformations. Long but standard computations provide the following result (see [4] for details).

**Lemma 2.** Consider weights $\deg(s) = 2$ and $\deg(t) = 1$. Let $P$ and $Q$ be two homogeneous polynomials of weighted degree at least $2n$ and $2n + 2$ respectively. Then $(P, Q) \in T_e f_n$. Moreover, $(t^{2i}, 0)$ and $(0, t^{2i})$ belong to $T_e f_n$ for every $i \geq 0$.

The $\mathcal{E}_{s,t}$-ideal $\langle f_n \rangle$ is generated by $s$ and $t^4$. Indeed,

$$\langle s, st^2 + t^4 + t^{2n+3} \rangle = \langle s, t^4 + t^{2n+3} \rangle = \langle s, t^4 \rangle,$$

since $1 + t^{2n-1}$ is invertible in $\mathcal{E}_{s,t}$. Therefore, the $\mathbb{R}$-space $\mathcal{E}_{s,t} / (\langle f_n \rangle \cdot \mathcal{E}_{s,t})$ is generated by 1, $t$, $t^2$, $t^3$.

By the Preparation Theorem of Malgrange and Mather (see e.g. [10]), this implies that the germs 1, $t$, $t^2$, $t^3$ form a generator system of the $\mathcal{E}_{s,t}$-module $\mathcal{E}_{s,t}$. Set $e_i := (0, t^{2i+1})$.

Using Lemma 2 and equality $t^m \partial_s f_n = (t^m, t^{m+2})$, we get

$$(\mathcal{E}_{s,t})^2 = T_e f_n + f_n^*(\mathcal{E}_{s,t}) = \{e_0, e_1, e_2\}. \quad (2)$$

Now remark that the following inclusion holds for every $i \in \mathbb{N}$:

$$f_n^*(\mathcal{E}_{s,t}) \cdot \{e_i\} \subset T_{e_i} f_n + \mathbb{R} \cdot e_i + f_n^*(\mathcal{E}_{s,t}) \cdot \{e_{i+1}, e_{i+2}\}.$$

Hence we deduce from (2) that

$$(\mathcal{E}_{s,t})^2 / T_{e_i} f_n \cong \mathbb{R} e_0 \oplus \cdots \oplus \mathbb{R} e_n.$$

This shows that the extended codimension of $f_n$ is $n + 1$ and that a miniversal deformation of $f_n$ is obtained adding $\lambda_0 e_0, \ldots, \lambda_n e_n$. Since $e_i$ only depend of $t$, a miniversal deformation of the TFG normal form $\tilde{f}_n$ is obtained adding $\lambda_0 e_0, \ldots, \lambda_n e_n$ to it.

We prove now that the map germ, defined by

$$F_n(\xi, t; \lambda) := \tilde{f}_n(\xi, t) + \lambda_1 e_1 + \cdots + \lambda_n e_n,$$

is a miniversal tangential deformation of the TFGs’ normal form $\tilde{f}_n(\xi, t)$. Let $G : (\mathbb{R}^2 \times \mathbb{R}^p, 0) \rightarrow (\mathbb{R}^2, 0)$ be another tangential deformation of $\tilde{f}_n(\xi, t)$, parameterized by $\alpha$. By
implies that \( \partial_\xi G \) be two homogeneous polynomials of weighted degree \( p \).

**Lemma 3.** Let us consider the variable weights \( y \) is the germ of \( \tilde{\alpha} \) deformation of the TFG normal form \( G \). Capitanio / Bull. Sci. math. 130 (2006) 1–14

On the other hand, this deformation is clearly tangential. Hence, it is a miniversal tangential deformation of the TFG normal form \( \tilde{f}_n \).

**S2-type TFGs.** By Lemma 2, the 5-jet of every \( S2 \)-TFG is equivalent to \( (s, st^2 + t^5) \). In [11] it is proven that a map germ with 5-jet \( (s, t^5 + st^2) \) is equivalent to one of the following germs:

\[
\begin{align*}
&g_2(s,t) := (s, st^2 + t^5 + t^6), & g_3^\pm(s,t) := (s, st^2 + t^5 \pm t^9), & g_4(s,t) := (s, st^4 + t^5);
\end{align*}
\]

\( g_2 \) is 6-determined, while \( g_3^\pm \) and \( g_4 \) are 9-determined (Lemma 3.2.1:2).

We compute now their miniversal deformations. For this, we need the following results, whose proof is a direct computation (see [4]).

**Lemma 3.** Let us consider the variable weights \( \deg(s) = 3 \) and \( \deg(t) = 1 \). Let \( P \) and \( Q \) be two homogeneous polynomials of weighted degree \( p \) and \( q \). Then we have:

- if \( p \geq 3, q \geq 5 \), then \( (P, Q) \in T_e g_2 \);
- if \( p \geq 3 \) or \( p \neq 4 \) and \( q \geq 5, q \neq 6 \), then \( (P, Q) \in T_e g_3^\pm \);
- if \( p \geq 3, p \neq 4, 7 \) and \( q \geq 5, q \neq 6, 9 \), then \( (P, Q) \in T_e g_4 \).

The forthcoming computations being identical for \( g_3^+ \) and for \( g_3^- \), we omit the sign \( \pm \).

In all the four cases, the \( \mathcal{E}_{s,t} \)-ideal \( \langle g_i \rangle \) is generated by \( s \) and \( t^5 \). Hence, the quotient space \( \mathcal{E}_{s,t}/\langle g_i \rangle \) is generated by \( 1, t, t^2, t^3 \) and \( t^4 \).

By the Preparation Theorem, these germs form an \( \mathcal{E}_{x,y} \)-generator system of \( \mathcal{E}_{s,t} \). Set \( e_0 := (0,t), e_1 := (0,t^3), e_2 := (0,t^4), e_3 := (0,t^5) \) and \( e_4 := (0,t^9) \). Using Lemma 3, one obtains the following equalities:

\[
\begin{align*}
(\mathcal{E}_{s,t})^2 &= T_2 g_2 \oplus \mathbb{R} \cdot \{e_0, e_1, e_2\} \\
&= T_2 g_3 \oplus \mathbb{R} \cdot \{e_0, e_1, e_2, e_3\} \\
&= T_2 g_4 \oplus \mathbb{R} \cdot \{e_0, e_1, e_2, e_3, e_4\}.
\end{align*}
\]

Thus, the map germs \( g_i + \sum_{j=0}^{\infty} \lambda^j e_j \) are miniversal deformations of \( g_i \); in particular, the extended codimensions of \( g_2, g_3 \) and \( g_4 \) are 3, 4 and 5. As for the \( S1 \)-case, we see that a miniversal tangential deformation of the TFGs normal form \( \tilde{g}_i(\xi, t) := g_i(\xi + t,t) \) is \( \tilde{g}_i + \lambda_1 e_1 + \cdots + \lambda_t e_t \).
$T_n$-TFGs. By Lemma 2, every $T_n$-type TFG is equivalent to a map germ
\[
\left( s, t^2 \sum_{i \geq n+1} k_i s^i + t^2 \tilde{R}(s) + \alpha t^3 + t^3 \delta(0) \right),
\]
with $\alpha$ and $k_{n+1}$ both non-zero. By rescaling, we can assume $k_{n+1} = \alpha = 1$. By standard coordinate changes, one verifies that the $(2n+3)$-jet of this germ is equivalent to $h_n(s, t) = (s, s^{n+1}t^2 + t^3)$ and also equivalent to $(s, s^{3n+2} - t^3)$. In [11] it is proven that the latter germ is $(2n+3)$-determined (Lemma 3.2:2).

Therefore, every $T_n$-type TFG is equivalent to $h_n$. We compute now the miniversal deformations. For every $n$, $\langle h_n \rangle$ is generated by $s$ and $t^3$. Hence, it follows from the Preparation Theorem that the $E_{s,t}$ is generated, as $E_{x,y}$-module, by $1$, $t$ and $t^2$. We need now some computations.

**Lemma 4.** Consider weights $\deg(s) = 1$ and $\deg(t) = n + 1$. For every $P$ homogeneous polynomial of weighted degree at least $3n + 3$, we have $(0, P) \in T e h_n$. Moreover, $T e h_n$ contains $E_{s,t} \times \{0\}$.

Using this lemma we see that the quotient space is generated over $E_{x,y}$ by $(0, t)$ and $(0, t^2)$. Let us set $u_i := (0, s^i t)$ and $v_j := (0, s^j t^2)$. We remark that the $E_{x,y}$-module generated by $(0, t)$ and $(0, t^2)$ is contained in the union of the extended tangent space $T e h_n$ with the $\mathbb{R}$-vector space spanned by $u_0, \ldots, u_{2n+1}$ and $v_0, \ldots, v_n$. Since $\partial_t h_n = (0, 3t^2 + 2st^{n+1})$, these inclusions provide the equality
\[
(\mathcal{E}_{s,t})^2 = T e h_n \oplus \mathbb{R} \cdot \{u_0, \ldots, u_{n+1}, v_0, \ldots, v_n\},
\]
showing that
\[
h_n + \sum_{j=0}^{n+1} \mu_j u_j + \sum_{i=0}^n \lambda_i v_i
\]
is a miniversal deformation of $h_n$.

The corresponding TFG normal form is $\tilde{h}_n(\xi, t) := h_n(\xi + t, t)$. As in the preceding cases, we see that the a miniversal tangential deformation of the normal form $\tilde{h}_n(\xi, t)$ is obtained by taking all the $\mu_i$ equal to zero in the above miniversal deformation (evaluated at $s = \xi + t$).

The above computations end the proof of Proposition 2. To complete the proof of Theorem 1, we have to show that the TFGs of type $S_n$ for $n \geq 3$ and of type $U$ are not simple. Indeed, the TFGs of type $S_{1,\infty}$ and $T_{\infty}$ are not simple, due to adjacencies $S_{\infty} \to S_n$ and $T_{\infty} \to T_n$ for every $n$.

We prove first that no $S_{n \geq 3}$-type TFG is simple. The germ
\[
f_a(s, t) := (s, st^2 + t^6 + t^7 + at^9),
\]
$a \in \mathbb{R}$, is of type $S_3$, due to Lemma 1. By a direct computation, one verifies that the 9-jets $f_a$ and $f_b$ are not equivalent whenever $a \neq b$. Thus, the germ is not simple. Let us consider a TFG $f$ of class $S_3$. Using Lemma 1, it is easy to see that its 9-jet is equivalent to
\[
(s, st^2 + t^6 + At^7 + Bt^9)
\]
for some $A, B \in \mathbb{R}$. If $A \neq 0$, we normalize it to 1. Now the 9-jet of $f$ is of the form $f_a$ for some $a$, so $f$ is not simple. If $A = 0$, we consider the deformation $f(s, t) + (0, \varepsilon t^7)$. For every $\varepsilon \neq 0$, the deformed germ is not simple, so $f$ is also not simple. Finally, the singularities belonging to $S_{n>3}$ and $S_\infty$ are adjacent to $S_3$, so they are not simple.

We prove now that no $U$-type TFG is simple. Set

$$f_a(s, t) := (s, s^2 t^2/2 + 2a s t^3/3 + t^4/4).$$

One easily check that two map germs having 4-jet equivalent to $f_a$ and $f_b$ are not equivalent, provided that $|a| \neq |b|$. Any $U$-type TFG is equivalent to a germ of the form $f(s, t) = (s, t^3 \cdot \delta(1))$. For every $\varepsilon$ small enough, the 4-jet of $f(s, t) + (0, \varepsilon(s^2 t^2 + t^4))$ is equivalent to $f_a$ for some $a$, so $f$ is not simple. This ends the proof of Theorem 1.

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References