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Cohomologies and extensions in monoidal categories*

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Introduction

In this paper we construct some new cohomologies and extensions in a symmetric monoidal category **A**, and investigate the connection between them.

In Section 0 we give some preliminaries and notations.

In Section 1 we construct cohomology groups $H^n(B, M)$ of a cocommutative Hopf monoid B with the coefficients in a left B-object commutative monoid M in the category **A**, and cohomology groups $H^n_h(B, A)$ of an abelian matched pair of Hopf monoids (B, A) in the same category.

In Section 2 we define the notion of extensions in the category **A**. More precisely, there appear three kinds of extensions. We call them $\mathscr{M}$-extension, $\mathscr{C}$-extension and $\mathscr{H}$-extension. In fact, $\mathscr{C}$-extensions are dual to $\mathscr{M}$-extensions, and the notion of $\mathscr{H}$-extension is somehow the intersection of both above-mentioned notions.

For example, let M be a commutative monoid and B a cocommutative Hopf monoid (both in **A**). Then the subject of $\mathscr{M}$-extension theory is the following: How 'many' structures of a monoid exist on the product $M \otimes B$, such that the arrows $M \xrightarrow{M\otimes\eta_B} M \otimes B \xrightarrow{M\otimes\psi_B} M \otimes B \otimes B$ are morphisms of monoids? We investigate some properties of such extensions. Namely, similar to the classical group-extension theory, for an $\mathscr{M}$-extension there are induced an 'action' of B on M (i.e. an arrow $\sigma_M : B \otimes M \to M$) and a 'twisting function' $\tau_M : B \otimes B \to M$.

* This paper is a translation into English of [10], with very few and only stylistic changes.

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Main identities for such arrows (σ_M, τ_M) are obtained (see Theorem 2.3 and Proposition 2.6). Unfortunately, these identities are not yet (on this level of generality) like the 'cocycle condition' for τ_M. For $\mathcal{H}$-extensions the same is done.

In Section 3 we put some additional conditions on extensions. For example, these assumptions on an $\mathcal{M}$-extension imply that it becomes a diagram of a cotensor product of B-coobjects; the notion of cotensor product is dual to the one from [12, p. 198]. These conditions enable us to transform the main identities for (σ_M, τ_M) in such a way that they become like the 'cocycle condition'. As a consequence, we get a bijection between a set of equivalence classes of $\mathcal{M}$-extensions and the second group of cohomology $H^2(B, M)$. Similarly, a bijection between a set of equivalence classes of $\mathcal{H}$-extensions and the first group of cohomology $H^1_h(B, A)$ is obtained.

In Section 4 we show that our cohomologies and extensions give in particular cases several well-known theories, such as: group cohomology [2, 6], cohomology of a group in an arbitrary category by Pareigis [11], Sweedler's cohomology of cocommutative Hopf algebras [14], Singer's cohomology of an abelian matched pair of Hopf algebras [13] and Doi's cohomologies over commutative Hopf algebras [3].

0. Preliminaries

Here we briefly recall the relevant definitions and facts from [7].

If **B** is a category, then the class of objects of **B** will be denoted by $|\mathbf{B}|$ and if $A, B \in |\mathbf{B}|$, then the set of arrows (morphisms) of **B** from A to B will be denoted by $\mathbf{B}(A, B)$.

A monoidal category **A** consists of a category **A**, a bifunctor $\otimes : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$, which is associative up to a natural isomorphism and an object $e \in |\mathbf{A}|$, which is a unit object up to natural isomorphisms of the bifunctor $\otimes$ (i.e. $e \otimes A \xrightarrow{\sim} A \xleftarrow{\sim} A \otimes e$, $A \in |\mathbf{A}|$), plus several axioms on these isomorphisms [7]. If, additionally, **A** is equipped with the natural (in both variables) isomorphisms $A \otimes B \xrightarrow{\sim} B \otimes A$, which, together with the above-mentioned ones satisfy several axioms, then **A** is called a symmetric monoidal category [7]. All diagrams in this paper are of the kind to which the coherence results of [7] can be applied. That is, we can omit all parentheses in big products; writing, for example, $A \otimes B \otimes C \otimes D$ instead of $A \otimes (B \otimes (C \otimes D))$. Similarly, we can and will omit explicit references to isomorphisms derived from the associative, unitary, and permutation axioms.

In what follows, $(\mathbf{A}, \otimes, e)$ denotes the fixed symmetric monoidal category. All objects and arrows considered in this paper, are assumed to be from **A**.

We write permutation isomorphisms of the factors in the product of objects of **A** as in [5]. For example,

$$(2, 1, 4, 5, 3) : A \otimes B \otimes C \otimes D \otimes E \xrightarrow{\sim} B \otimes A \otimes D \otimes E \otimes C\,.$$

The symbol $A^{\otimes n}$ denotes the nth power of A, i.e. $A^{\otimes n} = A \otimes (n\text{-times}) \otimes A$, $A^{\otimes 1} = A$ and $A^{\otimes 0} = e$.

0.1. Definition. A *monoid* (in $\mathbf{A}$) is a triple

$$(A, \mu_A : A \otimes A \to A, \eta_A : e \to A),$$

such that $\mu_A(A \otimes \mu_A) = \mu_A(\mu_A \otimes A)$ and $\mu_A(\eta_A \otimes A) = 1_A = \mu_A(A \otimes \eta_A)$.

The arrows μ_A and η_A are called a *multiplication* and a *unit*, respectively.

A set with an associative binary operation and with a unit element (i.e. a real monoid) will be called a *set-monoid* in order to distinguish this notion from the one of monoid objects in $\mathbf{A}$.

0.2. Definition. If (A, μ_A, η_A) and $(A', \mu_{A'}, \eta_{A'})$ are monoids (in $\mathbf{A}$), then the arrow $f : A \to A'$ is called a *morphism of monoids* (or a *monoid morphism*), iff $\mu_{A'}(f \otimes f) = f\mu_A$ and $\eta_{A'} = f\eta_A$.

0.3. Remark. If (A, μ_A, η_A) and $(A', \mu_{A'}, \eta_{A'})$ are monoids, then, when we speak of the monoid structure on $A \otimes A'$, we shall mean a multiplication

$$(\mu_A \otimes \mu_{A'})(1, 3, 2, 4) : A \otimes A' \otimes A \otimes A' \to A \otimes A'$$

and a unit $\eta_A \otimes \eta_{A'} : e \to A \otimes A'$.

0.4. Definition. Let (A, μ_A, η_A) be a monoid. We shall call it *commutative*, if one of the following equivalent conditions occurs:

(i) $\mu_A(2, 1) = \mu_A$,

(ii) the arrow μ_A is a morphism of monoids.

0.5. Remark. Let (A, μ_A, η_A) be a monoid, a symbol $\mu_A^n : A^{\otimes n} \to A$, $n \geq 2$, denotes any chain of arrows beginning at $A^{\otimes n}$ and going to A, built of μ_A and its $\otimes$-products with itself or with A's. Proposition 1 from [7, p. 167] guarantees that our description of μ_A^n is unambiguous. The symbols μ_A^1 and μ_A^0 denote the arrows $1_A = A$ and η_A, respectively. Moreover, if A is commutative, we have $\mu_A^n = \mu_A^n \circ t$, where t is any of the permutation isomorphisms (of factors in A's nth power). If A and A' are commutative monoids, the monoid $A \otimes A'$ (see 0.3) is commutative too.

The category of monoids (in $\mathbf{A}$) and monoid morphisms is denoted by $\mathcal{M}(\mathbf{A})$, while the category of commutative monoids is denoted by $\mathcal{M}\mathcal{M}(\mathbf{A})$.

0.6. Definition. Let (A, μ_A, η_A) be a monoid, a *left A-object* is a pair $(M, \sigma_M : A \otimes M \to M)$, such that $\sigma_M(A \otimes \sigma_M) = \sigma_M(\mu_A \otimes M)$ and $\sigma_M(\eta_A \otimes M) =$

M. If (M, σ_M) and $(M', \sigma_{M'})$ are the left A-objects, we call $f: M \to M'$ a *morphism* of the left A-objects, iff $f\sigma_M = \sigma_{M'}(A \otimes f)$.

The category of the left A-objects, and their morphisms is denoted by ${}_A\mathbf{A}$.

0.7. Proposition. (Mac Lane [7]). *Let (A, μ_A, η_A) be a monoid. Then there exists an adjunction*

$$\langle T, U\rangle : \mathbf{A} \to {}_A\mathbf{A},$$

where U is the forgetful functor and

$$T(X) = (A \otimes X,\ \mu_A \otimes X : A \otimes A \otimes X \to A \otimes X),$$

$$T(f) = A \otimes f. \quad \square$$

0.8. Definition. A *comonoid* (in $\mathbf{A}$) is a triple

$$(B,\ \psi_B : B \to B \otimes B,\ \varepsilon_B : B \to e),$$

such that $(\psi_B \otimes B)\psi_B = (B \otimes \psi_B)\psi_B$ and $(\varepsilon_B \otimes B)\psi_B = B = (B \otimes \varepsilon_B)\psi_B$.

The arrows ψ_B and ε_B are called a *comultiplication* and a *counit*, respectively. Definition of *morphisms of comonoids* (or *comonoid morphisms*) is similar to 0.2. If $(B, \psi_B, \varepsilon_B)$ and $(B', \psi_{B'}, \varepsilon_{B'})$ are comonoids, then, when we speak of the comonoid structure on $B \otimes B'$, we mean the duals to the formulas from 0.3. A comonoid $(B, \psi_B, \varepsilon_B)$ is called *cocommutative*, iff ψ_B is a comonoid morphism, or equivalently, if $\psi_B = (2, 1)\psi_B$. We assume that we have made considerations dual to those from 0.5. The category of comonoids and comonoid morphisms is denoted by $\mathscr{C}(\mathbf{A})$, while the category of cocommutative comonoids is denoted by $\mathscr{C}\mathscr{C}(\mathbf{A})$.

If B is a comonoid, then the notions of a right B-coobject $(N, \rho_N : N \to N \otimes B)$ and of a morphism of right B-coobjects are defined dually to 0.6 and 0.7. The category of right B-coobjects and their morphisms is denoted by $\mathbf{A}^B$.

Now we define a notion dual to that of a tensor product in [12, p. 198].

0.9. Definition. Let B be a comonoid and $(N, \rho_N : N \to N \otimes B)$ and $(N', \rho_{N'} : N' \to B \otimes N')$ be a right and a left B-coobject, respectively. The *equalizer* (see [7]) of the pair of arrows $(\rho_N \otimes N', N \otimes \rho_{N'})$ in $\mathbf{A}$ (if it does exist) is a cotensor product of B-coobjects N and N' and is denoted by $N \otimes^B N'$. Thus we have the equalizer diagram in $\mathbf{A}$

$$N \otimes^B N' \xrightarrow{\nu} N \otimes N' \underset{N \otimes \rho_{N'}}{\overset{\rho_N \otimes N'}{\rightrightarrows}} N \otimes B \otimes N'.$$

All the following notions and propositions (0.10–0.22) are straightforward generalizations of well-known ones and are inspired by [8, 13, 14].

0.10. Remark. If A is a monoid and B is a comonoid, then the set $\mathbf{A}(B, A)$ is equipped with a set-monoidal structure (denoted additively), which is determined by the following formulas:

$$f + g = \mu_A(f \otimes g)\psi_B, \quad f, g \in \mathbf{A}(B, A) \quad \text{and} \quad 0 = \eta_A \varepsilon_B .$$

The subgroup of all those elements of the set-monoid $\mathbf{A}(B, A)$, which have an inverse, is called a subgroup of *regular arrows* from B to A and is denoted by

$$\operatorname{Reg}(B, A) = \{f \in \mathbf{A}(B, A) \mid \exists g \in \mathbf{A}(B, A), f + g = 0 = g + f\} .$$

The inverse to $f \in \operatorname{Reg}(B, A)$ is denoted by f^*, so $f + f^* = 0 = f^* + f$.

0.11. Remark. Let $p: A \to A'$ be a monoid morphism and $q: B' \to B$ a comonoid morphism. Then for any $f, g \in \mathbf{A}(B, A)$ we have the identities

$$p(f + g)q = (pf + pg)q = pfq + pgq = p(fq + gq) .$$

If $h \in \operatorname{Reg}(B, A)$, then we have the identities

$$ph^*q = (ph)^*q = (phq)^* = p(hq)^* .$$

Hence the mapping

$$\mathbf{A}(q, p): \mathbf{A}(B, A) \to \mathbf{A}(B', A'), \qquad \mathbf{A}(q, p)(f) = pfq ,$$

is a homomorphism of set-monoids, and determines the homomorphism of the corresponding groups

$$\operatorname{Reg}(q, p): \operatorname{Reg}(B, A) \to \operatorname{Reg}(B', A'), \qquad \operatorname{Reg}(q, p)(f) = pfq .$$

If A is a commutative monoid and B is a cocommutative comonoid, then the set-monoid $\mathbf{A}(B, A)$ and the group $\operatorname{Reg}(B, A)$ are abelian.

0.12. Definition. A *Hopf monoid* (in $\mathbf{A}$) is a 5-tuple $(H, \mu_H, \psi_H, \eta_H, \varepsilon_H)$, such that

(i) (H, μ_H, η_H) is a monoid,
(ii) $(H, \psi_H, \varepsilon_H)$ is a comonoid,
(iii) the arrows ψ_H and ε_H are the monoid morphisms.

The latter implies the identity

$$\psi_H \mu_H = (\mu_H \otimes \mu_H)(1,3,2,4)(\psi_H \otimes \psi_H),$$

which is called a *Hopf condition* by many authors.

The notion of a *morphism* of Hopf monoids (Hopf monoid morphism) is defined obviously. A Hopf monoid H is called *commutative*, iff μ_H is commutative, and it is *cocommutative*, iff ψ_H is of the same kind. The categories of Hopf monoids, of commutative Hopf monoids and of cocommutative Hopf monoids are denoted by $\mathscr{H}(\mathbf{A})$, $\mathscr{M}\mathscr{H}(\mathbf{A})$ and $\mathscr{C}\mathscr{H}(\mathbf{A})$, respectively.

0.13. Definition. An *antipode of* $H \in |\mathscr{H}(\mathbf{A})|$ is an arrow (if it does exist) $S_H : H \to H$, such that $S_H + 1_H = 0 = 1_H + S_H$ in the set-monoid $\mathbf{A}(H, H)$. Thus, $S_H = (1_H)^*$.

0.14. Remark. (i) If H is a Hopf monoid and M, M' are the left H-objects, then, when we speak of the left H-object structure on $M \otimes M'$, we mean the action

$$(\sigma_M \otimes \sigma_{M'})(1,3,2,4)(\psi_H \otimes M \otimes M') : H \otimes M \otimes M' \to M \otimes M'.$$

(ii) If H is a Hopf monoid and e is a unit object of $\mathbf{A}$, then, when we speak of the left H-object or the right H-coobject structure on e, we mean the action $\varepsilon_H \otimes e : H \otimes e \to e$ or coaction $e \otimes \eta_H : e \to e \otimes H$.

0.15. Remark. Let H be a Hopf monoid. A *left H-object monoid* is a 4-tuple $(C, \sigma_C, \mu_C, \eta_C)$, such that

(i) (C, σ_C) is a left H-object,

(ii) (C, μ_C, η_C) is a monoid,

(iii) the arrows μ_C and η_C are the left H-object morphisms.

A morphism $f : C \to C'$ of the left H-object monoids is an arrow $f : C \to C'$, such that f is a morphism of the left H-objects and of monoids, simultaneously. The category of left H-object monoids and their morphisms is denoted by $\mathscr{M}({}_H\mathbf{A})$. The full subcategory of the left H-object monoids which are commutative as monoids, is denoted by $\mathscr{M}\mathscr{M}({}_H\mathbf{A})$.

0.16. Remark. Similarly defined are (H is a Hopf monoid):

(i) The notion of a right H-coobject monoid

$$(D, \rho_D : D \to D \otimes H, \mu_D : D \otimes D \to D, \eta_D : e \to D),$$

here the coaction of H on $D \otimes D$ is defined dually to 0.14(i); and the respective categories $\mathscr{M}(\mathbf{A}^H)$ and $\mathscr{M}\mathscr{M}(\mathbf{A}^H)$.

(ii) The notion of a left H-object comonoid

$$(X, \sigma_X : H \otimes X \to X, \psi_X : X \to X \otimes X, \varepsilon_X : X \to e),$$

and the respective categories $\mathscr{C}({}_H\mathbf{A})$, $\mathscr{C}\mathscr{C}({}_H\mathbf{A})$.

(iii) The notion of a right H-coobject comonoid

$$(Y, \rho_Y : Y \to Y \otimes H, \psi_Y : Y \to Y \otimes Y, \varepsilon_Y : Y \to e),$$

and the respective categories $\mathscr{C}(\mathbf{A}^H)$, $\mathscr{C}\mathscr{C}(\mathbf{A}^H)$.

0.17. Remark. As the reader has already noticed, every time considering the dual to any notion, we change not only the direction of arrows but the ordering of factors in $\otimes$-products too, for example: a left H-object $(M, \sigma_M : H \otimes M \to M)$ and its 'dual', a right H-coobject $(N, \rho_N : N \to N \otimes H)$.

In what follows, every time, while speaking of the dual to any notion or consideration, we shall reverse all arrows and, simultaneously, reverse the ordering of factors in $\otimes$-products, i.e. $A \otimes B \otimes C \to D \otimes E$ becomes $E \otimes D \to C \otimes B \otimes A$.

0.18. Definition. (Compare Definition 3.1 of [13].) An *abelian matched pair* of Hopf monoids (*abelian pair*) is a 4-tuple (B, A, σ_A, ρ_B), such that

(i) A is a commutative Hopf monoid and B is a cocommutative Hopf monoid,

(ii) (A, σ_A) is a left B-object monoid and (B, ρ_B) is a right A-coobject comonoid,

(iii) $\varepsilon_A \sigma_A = \varepsilon_B \otimes \varepsilon_A$, $\rho_B \eta_B = \eta_B \otimes \eta_A$,

(iv) $\psi_A \sigma_A = (A \otimes \mu_A)(\sigma_A \otimes A \otimes \sigma_A)(1, 4, 2, 3, 5)(\rho_B \otimes B \otimes A \otimes A)(\psi_B \otimes \psi_A)$,

(v) $\rho_B \mu_B = (\mu_B \otimes \mu_A)(B \otimes B \otimes A \otimes \sigma_A)(1, 4, 2, 3, 5)(\rho_B \otimes B \otimes \rho_B)(\psi_B \otimes B)$.

0.19. Remark. let (B, A, σ_A, ρ_B) be an abelian pair, $(M, \sigma_M) \in |{}_B\mathbf{A}|$ and $(N, \rho_N) \in |\mathbf{A}^A|$. Then we denote:

$$\bar{\sigma}_{M \otimes A} = (M \otimes \mu_A)(\sigma_M \otimes A \otimes \sigma_A)(1, 4, 2, 3, 5) \circ (\rho_B \otimes B \otimes M \otimes A)(\psi_B \otimes M \otimes A),$$

$$\underline{\rho}_{B \otimes N} = (B \otimes N \otimes \mu_A)(B \otimes N \otimes A \otimes \sigma_A)(1, 4, 2, 3, 5) \circ (\rho_B \otimes B \otimes \rho_N)(\psi_B \otimes N).$$

0.20. Proposition. *Let* (B, A, σ_A, ρ_B) *be an abelian pair and* $(M, \sigma_M) \in |{}_B\mathbf{A}|$. *Then* $(M \otimes A, \bar{\sigma}_{M \otimes A})$ *is a left B-object.*

Proof. Using the facts that (A, σ_A) is the left B-object monoid and (M, σ_M) is the left B-object, one can show that

$$
\begin{aligned}
&\bar{\sigma}_{M\otimes A}(B\otimes\bar{\sigma}_{M\otimes A})\\
&\quad=(M\otimes\mu_A^3)(\sigma_M\otimes A\otimes\sigma_A\otimes\sigma_A)\\
&\qquad\circ(\mu_B\otimes M\otimes A\otimes B\otimes A\otimes\mu_B\otimes A)\\
&\qquad\circ(1,5,8,2,3,6,4,7,9)\\
&\qquad\circ(\rho_B\otimes B\otimes B\otimes\rho_B\otimes B\otimes M\otimes A)\\
&\qquad\circ(\psi_B^3\otimes\psi_B\otimes M\otimes A)\,.
\end{aligned}
$$

On the other hand, using the Hopf condition on B and (v) from 0.18, one can show that the arrow $\bar{\sigma}_{M\otimes A}(\mu_B\otimes M\otimes A)$ is equal to the right-hand composition of the identity above. Thus we conclude that $\bar{\sigma}_{M\otimes A}(B\otimes\bar{\sigma}_{M\otimes A})=\bar{\sigma}_{M\otimes A}(\mu_B\otimes M\otimes A)$.

The verification of the identity $\bar{\sigma}_{M\otimes A}(\eta_B\otimes M\otimes A)=M\otimes A$ can be done similarly. □

We would like to mention that 0.20 is the generalization of Proposition 3.2 from [13], and our proof follows step by step the proof of that original one.

0.21. Definition. Let (B, A, σ_A, ρ_B) be an abelian pair. A *(B, A)-biobject* is a triple (M, σ_M, ρ_M), such that

(i) (M, σ_M) is a left B-object and (M, ρ_M) is a right A-coobject,

(ii) $\rho_M\sigma_M=\bar{\sigma}_{M\otimes A}(B\otimes\rho_M)=(\sigma_M\otimes A)\rho_{B\otimes M}$.

0.22. Definition. Let (B, A, σ_A, ρ_B) be an abelian pair. A *(B, A)-biobject monoid* is a 5-tuple $(M, \sigma_M, \rho_M, \mu_M, \eta_M)$, such that

(i) (M, σ_M, ρ_M) is a (B, A)-biobject,

(ii) the arrows μ_M and η_M are both morphisms of left B-objects and right A-coobjects.

The category of (B, A)-biobject monoids will be denoted by $\mathcal{M}({}_{(B,A)}\mathbf{A})$ and the full subcategory of (B, A)-biobjects which are commutative as monoids, will be denoted by $\mathcal{MM}({}_{(B,A)}\mathbf{A})$.

1. Cohomologies

The adjunction from 0.7 induces a new one.

1.1. Proposition. *Let B be a cocommutative Hopf monoid. Then there exists an adjunction*

$$\langle T, U\rangle:\mathscr{CC}(\mathbf{A})\to\mathscr{CC}({}_B\mathbf{A})\,,$$

where U is a forgetful functor and T is determined by the formulas

$$T(C, \psi_C, \varepsilon_C) = (B \otimes C, \sigma_{T(C)} = (\mu_B \otimes c),$$
$$\psi_{T(C)} = (1, 3, 2, 4)(\psi_B \otimes \psi_C), \varepsilon_{T(C)} = \varepsilon_B \otimes \varepsilon_C),$$
$$T(f) = B \otimes f.$$

Proof. If $(C, \psi_C, \varepsilon_C) \in |\mathscr{C}\mathscr{C}(\mathbf{A})|$, then the identities

$$\sigma_{T(C)}(B \otimes \sigma_{T(C)}) = (\mu_B^3 \otimes C) = \sigma_{T(C)}(\mu_B \otimes T(C)),$$
$$\psi_{T(C)}\sigma_{T(C)} = (\mu_B \otimes C \otimes \mu_B \otimes C)(1, 3, 5, 2, 4, 6)$$
$$\circ(\psi_B \otimes \psi_C \otimes \psi_B \otimes \psi_C)$$
$$= (\sigma_{T(C)} \otimes \sigma_{T(C)})(1, 3, 4, 2, 5, 6)(\psi_B \otimes \psi_{T(C)}),$$
$$\varepsilon_{T(C)}\sigma_{T(C)} = \varepsilon_B \otimes \varepsilon_{T(C)},$$

show that $T(C) \in |\mathscr{C}\mathscr{C}({}_B\mathbf{A})|$.

If $f : C \to C'$ is an arrow in $\mathscr{C}\mathscr{C}(\mathbf{A})$, then the identities

$$T(f)\sigma_{T(C)} = \mu_B \otimes f = \sigma_{T(C')}(B \otimes T(f)),$$
$$\psi_{T(C')}T(f) = (B \otimes f \otimes B \otimes f)(1, 3, 2, 4)(\psi_B \otimes \psi_C)$$
$$= (T(f) \otimes T(f))\psi_{T(C)},$$

show that $T(f)$ is from $\mathscr{C}\mathscr{C}({}_B\mathbf{A})$.

A bijection of the adjunction

$$\vartheta : \mathscr{C}\mathscr{C}({}_B\mathbf{A})(B \otimes C, D) \tilde{\to} \mathscr{C}\mathscr{C}(\mathbf{A})(C, D)$$

is determined as $\vartheta(f) = f(\eta_B \otimes C)$ and $\vartheta^{-1}(g) = \sigma_D(B \otimes g)$. □

1.2. Remark. Any adjunction induces a comonad [7]; the comonad induced by the adjunction from 1.1 is denoted by

$$\mathbb{T} = (\mathbb{T} = T\mathscr{U} : \mathscr{C}\mathscr{C}({}_B\mathbf{A}) \to \mathscr{C}\mathscr{C}({}_B\mathbf{A}), \alpha : \mathbb{T} \to 1, \delta : \mathbb{T} \to \mathbb{T}^2),$$

where $\alpha_C = \sigma_C : B \otimes C \to C$ and $\delta_C = (B \otimes \eta_B \otimes C) : B \otimes C \to B \otimes B \otimes C$.

Let $(M, \sigma_M, \mu_M, \eta_M)$ be a fixed left B-object commutative monoid. For any $(C, \sigma_C, \psi_C, \varepsilon_C) \in |\mathscr{C}\mathscr{C}({}_B\mathbf{A})|$ and $f, g \in {}_B\mathbf{A}(C, M)$, the sum $f + g = \mu_M(f \otimes g)\psi_C$ is a morphism of B-objects, i.e. $f + g \in {}_B\mathbf{A}(C, M)$. The reason is that the arrows ψ_C and μ_M are morphisms of B-objects. Thus one can obtain the abelian group

${}_B\mathrm{Reg}(C, M) = {}_B\mathbf{A}(C, M) \cap \mathrm{Reg}(C, M)$. Certainly, the correspondence of C and the abelian group ${}_B\mathrm{Reg}(C, M)$, determines the cotravariant functor

$$ {}_B\mathrm{Reg}(-, M): \mathscr{C}\mathscr{C}({}_B\mathbf{A})^{\mathrm{op}} \to \mathbf{Ab}. $$

1.3. Definition. Let us consider the right derived functors [1] of the functor ${}_B\mathrm{Reg}(-, M)$, relative to the comonad $\mathbb{T}$ from 1.2. The values of these derived functors on the unit object e (regarded, in an obvious sense, as an object of $\mathscr{C}\mathscr{C}({}_B\mathbf{A})$) shall be called the *cohomology groups* of the cocommutative Hopf monoid B with the coefficients in the left B-object commutative monoid M and be denoted by $H^n(B, M)$, $n \geq 0$.

The bijection of the adjunction from 0.7 gives the isomorphism of abelian groups (recall, that the group structure is described by means of 0.3 and 0.10):

$$ {}_B\mathbf{A}(B^{\otimes n+1}, M) \cap \mathrm{Reg}(B^{\otimes n+1}, M) \tilde{\to} \mathrm{Reg}(B^{\otimes n}, M), \quad n \geq 0. $$

The last ones enable us to identify the cohomology groups $H^n(B, M)$ with the homology groups of the complex of abelian groups:

$$ \begin{aligned} &\mathbb{C}^n = \mathrm{Reg}(B^{\otimes n}, M), \qquad d^n: \mathbb{C}^n \to \mathbb{C}^{n+1}, \\ &d^n(f) = \sigma_M(B \otimes f) + \sum_{i=1}^{n} (-1)^i f(B^{\otimes i-1} \otimes \mu_B \otimes B^{\otimes n-i}) \\ &\qquad + (-1)^{n+1} f(B^{\otimes n} \otimes \varepsilon_B), \qquad n \geq 0. \end{aligned} \tag{1} $$

Let us observe that here arises one interesting subgroup of the set-monoid $\mathbf{A}(B, M)$, namely the group of one-dimensional cocycles of the complex (1), i.e

$$ \mathbb{Z}^1(B, M) = \{ f \in \mathrm{Reg}(B, M) \mid d^1(f) = \sigma_M(B \otimes f) - f\mu_B + f(B \otimes \varepsilon_B) = 0 \}. $$

We call them *regular derivations* and denote them by

$$ \mathrm{Der}(B, M) = \mathbb{Z}^1(B, M). $$

By the normalization theorem for the cosimplicial abelian groups, the cohomology groups $H^n(B, M)$ also can be identified with the homology groups of the normalized subcomplex of the complex (1):

$$ \begin{aligned} &\mathbb{N}^n = \{ f \in \mathbb{C}^n \mid f(B^{\otimes i-1} \otimes \eta_B \otimes B^{\otimes n-i}) = 0, 1 \leq i \leq n \}, \quad n \geq 1, \\ &\mathbb{N}^0 = \mathbb{C}^0. \end{aligned} \tag{2} $$

We would like to mention that complex (1) and its normalized subcomplex (2) are straightforward generalizations of well-known ones (see 4.1–4.3) and are inspired by [2, 4, 11, 14].

1.4. Remark. Dually to 1.1–1.3, the cohomology groups $H^n_C(N, A)$ can be defined, where A is a commutative Hopf monoid and N is a right A-coobject cocommutative comonoid. The cohomology groups $H^n_C(N, A)$ can be identified with the homology groups of the complex of abelian groups:

$$\begin{aligned} &\mathbb{C}^n_C = \mathrm{Reg}(N, A^{\otimes n}), \qquad d^n_C : \mathbb{C}^n \to \mathbb{C}^{n+1}, \\ &d^n_C(g) = (\eta_A \otimes A^{\otimes n})g + \sum_{i=1}^{n} (-1)^i (A^{\otimes i-1} \otimes \psi_A \otimes A^{\otimes n-i})g \\ &\qquad + (-1)^{n+1}(g \otimes A)\rho_N, \quad n \geq 0, \end{aligned} \tag{3}$$

or with the homology groups of the normalized subcomplex of (3):

$$\begin{aligned} &\mathbb{N}^n_C = \{g \in \mathbb{C}^n_C \mid (A^{\otimes i-1} \otimes \varepsilon_A \otimes A^{\otimes n-i})g = 0, \ 1 \leq i \leq n\}, \quad n \geq 1, \\ &\mathbb{N}^0_C = \mathbb{C}^0_C. \end{aligned} \tag{4}$$

We would like to mention that complex (3) and its normalized subcomplex (4) are straightforward generalizations of well-known ones (see 4.4 and 4.5) and are inspired by [3].

1.5. Remark. In most of the proofs of our propositions we have to show several identities. We do this by the very simple machinery of commutative diagrams in the category **A**, i.e. both sides of an identity are transformed (identically) to one and the same expression. But, in this process, there appear tremendous calculations, and to save space we shall show for every identity only the form of the expression to which both sides of the identity are transformed.

Now, we shall translate Proposition 3.8 from [13] into the context of monoidal category.

1.6. Proposition. *Let (B, A, σ_A, ρ_B) be an abelian pair. Then there exists an adjunction*

$$\langle \mathcal{U}, G \rangle : \mathcal{M}\mathcal{M}({}_{(B,A)}\mathbf{A}) \to \mathcal{M}\mathcal{M}({}_B\mathbf{A}),$$

where the left adjoint $\mathcal{U}$ is the forgetful functor and G is determined by the formulas

$$\begin{aligned} &G(M, \sigma_M, \mu_M, \eta_M) \\ &\quad = (M \otimes A, \sigma_{G(M)} = \bar{\sigma}_{M \otimes A}, \rho_{G(M)} = M \otimes \psi_A, \end{aligned}$$

$$\mu_{G(M)} = (\mu_M \otimes \mu_A)(1, 3, 2, 4)\,,\ \eta_{G(M)} = \eta_M \otimes \eta_A)\,,$$

$$G(f) = f \otimes A$$

(*for* $\bar{\sigma}_{M\otimes A}$ *see* 0.19).

Proof. First, let us show that G has values in $\mathcal{MM}({}_{(B,A)}\mathbf{A})$. If $M \in |\mathcal{MM}({}_B\mathbf{A})|$, then we have that $(M \otimes A, \rho_{G(M)}) \in |\mathbf{A}^A|$ and $(M \otimes A, \mu_{G(M)}, \eta_{G(M)}) \in |\mathcal{MM}(\mathbf{A})|$. It remains then to show that:

(i) $(M \otimes A, \bar{\sigma}_{M\otimes A}) \in |{}_B\mathbf{A}|$,

(ii) $\rho_{G(M)}\sigma_{G(M)} = \bar{\sigma}_{G(M)\otimes A}(B \otimes \rho_{G(M)}) = (\sigma_{G(M)} \otimes A)\underline{\rho}_{B\otimes G(M)}$,

(iii) $(M \otimes A, \sigma_{G(M)}, \mu_{G(M)}, \eta_{G(M)}) \in |\mathcal{MM}({}_B\mathbf{A})|$,

(iv) $(M \otimes A, \rho_{G(M)}, \mu_{G(M)}, \eta_{G(M)}) \in |\mathcal{MM}(\mathbf{A}^A)|$.

For (i) observe 0.20.

For (ii) we have

$$\begin{aligned}\rho_{G(M)}\sigma_{G(M)} &= (M \otimes \mu_A \otimes \mu_A^3)(\sigma_M \otimes A \otimes \sigma_A \otimes A \otimes A \otimes \sigma_A)\\ &\circ(1, 7, 2, 4, 8, 3, 5, 6, 9)\\ &\circ(B \otimes \psi_A \otimes B \otimes A \otimes B \otimes M \otimes \psi_A)\\ &\circ(\rho_B \otimes \rho_B \otimes B \otimes M \otimes A)(\psi_B^3 \otimes M \otimes A)\\ &= \bar{\sigma}_{G(M)\otimes A}(B \otimes \rho_{G(M)}) = (\sigma_{G(M)} \otimes A)\underline{\rho}_{B\otimes G(M)}\,.\end{aligned}$$

For (iii) it remains to mention that

$$\begin{aligned}\sigma_{G(M)}(B \otimes \mu_{G(M)}) &= (\mu_M \otimes \mu_A^4)(\sigma_M \otimes \sigma_M \otimes A \otimes A \otimes \sigma_A \otimes \sigma_A)\\ &\circ(1, 7, 3, 9, 2, 4, 5, 8, 6, 10)\\ &\circ(\rho_B \otimes \rho_B \otimes B \otimes B \otimes M \otimes A \otimes M \otimes A)\\ &\circ(\psi_B^4 \otimes M \otimes A \otimes M \otimes A)\\ &= \mu_{G(M)}(\sigma_{G(M)} \otimes \sigma_{G(M)})(1, 3, 4, 2, 5, 6)\\ &\circ(\psi_B \otimes G(M) \otimes G(M))\,,\end{aligned}$$

and $\sigma_{G(M)}(B \otimes \eta_{G(M)}) = (\eta_M \otimes \eta_A)\varepsilon_B = \eta_{G(M)}\varepsilon_B$.

For (iv) it remains to observe that

$$\begin{aligned}\rho_{G(M)}\mu_{G(M)} &= (\mu_M \otimes \mu_A \otimes \mu_A)(1, 4, 2, 5, 3, 6)\\ &\circ(M \otimes \psi_A \otimes M \otimes \psi_A)\\ &= (\mu_{G(M)} \otimes A)(G(M) \otimes G(M) \otimes \mu_A)\\ &\circ(1, 2, 4, 5, 3, 6)(\rho_{G(M)} \otimes \rho_{G(M)})\,,\end{aligned}$$

and

$$\rho_{G(M)}\eta_{G(M)} = (\eta_M \otimes \eta_A \otimes \eta_A) = (\eta_{G(M)} \otimes \eta_A)\,.$$

Behaviour of G on the arrows of $\mathcal{MM}({}_B\mathbf{A})$ is described similarly.
Now let us define a natural (in M and N) bijection

$$\vartheta : \mathcal{MM}({}_B\mathbf{A})(N, M) \tilde{\to} \mathcal{MM}({}_{(B,A)}\mathbf{A})(N, M \otimes A)\,,$$

by putting $\vartheta(f) = (f \otimes A)\rho_N$ and $\vartheta^{-1}(g) = (M \otimes \varepsilon_A)g$. □

The adjunction from 1.6 induces the monad

$$\begin{aligned} \bar{G} = (\bar{G} = G\mathcal{U} : \mathcal{MM}({}_{(B,A)}\mathbf{A}) \to \mathcal{MM}({}_{(B,A)}\mathbf{A})\,, \\ \beta : 1 \to \bar{G}, \gamma : \bar{G}^2 \to \bar{G})\,, \end{aligned} \tag{5}$$

where $\beta_M = \rho_M$ and $\gamma_M = (M \otimes \varepsilon_A \otimes A)$.
Dually, the existence of an adjunction

$$\langle S, \mathcal{U} \rangle : \mathcal{CC}(\mathbf{A}^A) \to \mathcal{CC}({}_{(B,A)}\mathbf{A})$$

is proved, where the right adjoint $\mathcal{U}$ is the forgetful functor and S is determined by the formulas

$$\begin{aligned} S(N, \rho_N, \psi_N, \varepsilon_N) = (B \otimes N,\ \sigma_{S(N)} = \mu_B \otimes N,\ \rho_{S(N)} = \underline{\rho}_{B\otimes N}\,, \\ \psi_{S(N)} = (1,3,2,4)(\psi_B \otimes \psi_N),\ \varepsilon_{S(N)} = \varepsilon_B \otimes \varepsilon_N), \\ S(f) = B \otimes f\,. \end{aligned}$$

The last adjunction induces the comonad

$$\begin{aligned} \bar{S} = (\bar{S} = S\mathcal{U} : \mathcal{CC}({}_{(B,A)}\mathbf{A}) \to \mathcal{CC}({}_{(B,A)}\mathbf{A})\,, \\ \alpha : \bar{S} \to 1, \delta : \bar{S} \to \bar{S}^2)\,, \end{aligned} \tag{6}$$

where $\alpha_N = \sigma_N$ and $\delta_N = (B \otimes \eta_B \otimes N)$.

1.7. Proposition. *Let (B, A) be an abelian pair (we omit here σ_A, ρ_B), $M \in |\mathcal{MM}({}_{(B,A)}\mathbf{A})|$ and $N \in |\mathcal{CC}({}_{(B,A)}\mathbf{A})|$, then the correspondence of f and the composition $\bar{\sigma}_{M\otimes A}(B \otimes f \otimes A)(B \otimes \rho_N)$ determines the natural (in M and N) isomorphism of abelian set-monoids*

$$\varphi : \mathbf{A}(N, M) \tilde{\to} {}_{(B,A)}\mathbf{A}(S(N), G(M))\,.$$

Proof. The agreement of $\varphi(f)$ with B-action and A-coaction follows from the identities

$$\begin{aligned}\sigma_{G(M)}(B\otimes\varphi(f)) &= \bar{\sigma}_{M\otimes A}(\mu_B\otimes M\otimes A)(B\otimes B\otimes f\otimes A)\\ &\quad\circ(B\otimes B\otimes\rho_N)\\ &= \varphi(f)\sigma_{S(N)},\\ \rho_{G(M)}\varphi(f) &= (M\otimes\mu_A\otimes\mu_A^3)(\sigma_M\otimes A\otimes\sigma_A\otimes A\otimes A\otimes\sigma_A)\\ &\quad\circ(1,7,2,4,8,3,5,6,9)\\ &\quad\circ(B\otimes\psi_A\otimes B\otimes A\otimes B\otimes M\otimes\psi_A)\\ &\quad\circ(\rho_B\otimes\rho_B\otimes B\otimes M\otimes A)\\ &\quad\circ(\psi_B^3\otimes f\otimes A)(B\otimes\rho_N)\\ &= (\varphi(f)\otimes A)\rho_{S(N)}.\end{aligned}$$

Preservation of sums:

$$\begin{aligned}\varphi(f)+\varphi(g) &= (\mu_M\otimes\mu_A^4)(\sigma_M\otimes\sigma_M\otimes A\otimes A\otimes\sigma_A\otimes\sigma_A)\\ &\quad\circ(B\otimes f\otimes B\otimes g\otimes A\otimes A\otimes B\otimes A\otimes B\otimes A)\\ &\quad\circ(1,7,3,9,2,4,5,8,6,10)\\ &\quad\circ(\rho_B\otimes\rho_B\otimes B\otimes B\otimes\rho_N\otimes\rho_N)(\psi_B^4\otimes\psi_N)\\ &= \varphi(f+g).\end{aligned}$$

The inverse to φ is defined as

$$\varphi^{-1}(g) = (M\otimes\varepsilon_A)g(\eta_B\otimes N). \qquad \square$$

1.8. Remark. As is shown in [1], every monad on a category induces a functor from the same category to the category of cosimplicial objects of the main category. Thus, for the monad $\bar{G}$ (see (5)) we have the functor

$$\bar{G}_*: \mathcal{MM}({}_{(B,A)}\mathbf{A}) \to \mathrm{Cosimpl}(\mathcal{MM}({}_{(B,A)}\mathbf{A})).$$

Dually, the comonad $\bar{S}$ (see (6)) induces the functor to the category of simplicial objects,

$$\bar{S}^*: \mathcal{CC}({}_{(B,A)}\mathbf{A}) \to \mathrm{Simpl}(\mathcal{CC}({}_{(B,A)}\mathbf{A})).$$

Let us consider the bicosimplicial abelian group

$$_{(B,A)}\mathbf{A}(\bar{S}^*(e), \bar{G}_*(e)) \cap \mathrm{Reg}(\bar{S}^*(e), \bar{G}_*(e)),$$

where e is the unit object of $\mathbf{A}$, considered, in the obvious sense, as the object of both categories. From this bicosimplicial abelian group we induce the bicocomplex

$$\begin{aligned}
&\bar{\mathbb{C}}^{m,n} = {}_{(B,A)}\mathbf{A}(B^{\otimes m+1}, A^{\otimes n+1}) \cap \mathrm{Reg}(B^{\otimes m+1}, A^{\otimes n+1}),\\
&\bar{d}_1^{m,n}:\bar{\mathbb{C}}^{m,n}\to\bar{\mathbb{C}}^{m+1,n}, \quad \bar{d}_2^{m,n}:\bar{\mathbb{C}}^{m,n}\to\bar{\mathbb{C}}^{m,n+1}, \qquad m,n\geq 0,\\
&\bar{d}_1^{m,n}(f) = \sum_{i=0}^{m}(-1)^i f(B^{\otimes i}\otimes\mu_B\otimes B^{\otimes m-i})\\
&\qquad\qquad + (-1)^{m+1}f(B^{\otimes m+1}\otimes\varepsilon_B),\\
&\bar{d}_2^{m,n}(f) = (-1)^m((\eta_A\otimes A^{\otimes n+1})f\\
&\qquad\qquad + \sum_{j=1}^{n+1}(-1)^j(A^{\otimes j-i}\otimes\psi_A^{\otimes}\otimes A^{\otimes n-j+1})f).
\end{aligned} \tag{7}$$

Now, let us delete from (7) the first vertical and horizontal lines and make a dimensional shift of 1. We get the bicomplex

$$\tilde{\mathbb{C}}^{m,n} = \bar{\mathbb{C}}^{m+1,n+1}, \quad \tilde{d}_i^{m,n} = \bar{d}_i^{m+1,n+1}, \qquad i=1,2,\ m,n\geq 0. \tag{8}$$

1.9. Definition. The *cohomology groups* $H_h^n(B, A)$ of an abelian matched pair of Hopf monoids (B, A, σ_A, ρ_B) are the homology groups of the total complex associated with the bicomplex of abelian groups (8).

The restriction of the isomorphism from 1.7 induces the natural (in M and N) isomorphism of abelian groups

$$\varphi:\mathrm{Reg}(N, M)\tilde{\to}{}_{(B,A)}\mathbf{A}(B\otimes N, M\otimes A)\cap\mathrm{Reg}(B\otimes N, M\otimes A). \tag{9}$$

1.10. Remark. The isomorphism (9) enables us to identify the cohomology groups $H_h^n(B, A)$ with the homology groups of the total complex associated with the bicomplex

$$\begin{aligned}
&\mathbb{C}^{m,n} = \mathrm{Reg}(B^{\otimes m+1}, A^{\otimes n+1}),\\
&d_1^{m,n}:\mathbb{C}^{m,n}\to\mathbb{C}^{m+1,n}, \quad d_2^{m,n}:\mathbb{C}^{m,n}\to\mathbb{C}^{m,n+1}, \qquad m,n\geq 0,\\
&d_1^{m,n}(f) = \bar{\sigma}_{A^{\otimes n+1}}(B\otimes f)\\
&\qquad\qquad + \sum_{i=1}^{m+1}(-1)^i f(B^{\otimes i-1}\otimes\mu_B\otimes B^{\otimes m-i+1})\\
&\qquad\qquad + (-1)^{m+2}f(B^{\otimes m+1}\otimes\varepsilon_B),
\end{aligned}$$

$$d_2^{m,n}(f) = (-1)^{m+1}((\eta_A \otimes A^{\otimes n+1})f + \sum_{j=1}^{n+1} (-1)^j (A^{\otimes j-1} \otimes \psi_A \otimes A^{\otimes n-j+1})f + (-1)^{n+2}(f \otimes A)\underline{\rho}_{B^{\otimes m+1}}), \tag{10}$$

where the actions $\bar{\sigma}_{A^{\otimes n+1}} : B \otimes (A^{\otimes n}) \otimes A \to (A^{\otimes n}) \otimes A$ and the coactions $\underline{\rho}_{B^{\otimes m+1}} : B \otimes (B^{\otimes m}) \to B \otimes (B^{\otimes m}) \otimes A$ are defined by induction:

$$\bar{\sigma}_A = \sigma_A, \qquad \underline{\rho}_B = \rho_B,$$

$$\bar{\sigma}_{A^{\otimes n+1}} = ((A^{\otimes n}) \otimes \mu_A)(\bar{\sigma}_{A^{\otimes n}} \otimes A \otimes \sigma_A)(1,4,2,3,5) \circ (\rho_B \otimes B \otimes (A^{\otimes n}) \otimes A)(\psi_B \otimes (A^{\otimes n}) \otimes A),$$

$$\underline{\rho}_{B^{\otimes m+1}} = (B \otimes (B^{\otimes m}) \otimes \mu_A)(B \otimes (B^{\otimes m}) \otimes A \otimes \sigma_A) \circ (1,4,2,3,5)(\rho_B \otimes B \otimes \underline{\rho}_{B^{\otimes m}})(\psi_B \otimes (B^{\otimes m})).$$

From the theory of spectral sequences and the theorem on the normalization of cosimplicial abelian groups, one can deduce that the cohomology groups $H_R^n(B, A)$ (in the reduction (10)) can also be identified with the homology groups of the total complex associated with the subbicomplex of the last one in (10):

$$\begin{aligned}
&\mathbb{N}^{0,0} = \mathbb{C}^{0,0}, \\
&\mathbb{N}^{m,0} = \{f \in \mathbb{C}^{m,0} \mid f(B^{\otimes i} \otimes \eta_B \otimes B^{\otimes m-i}) = 0,\ 0 \le i \le m\}, \\
&\mathbb{N}^{0,n} = \{f \in \mathbb{C}^{0,n} \mid (A^{\otimes j} \otimes \varepsilon_A \otimes A^{\otimes n-j})f = 0,\ 0 \le j \le n\}, \\
&\mathbb{N}^{m,n} = \{f \in \mathbb{C}^{m,n} \mid f(B^{\otimes i} \otimes \eta_B \otimes B^{\otimes m-i}) = 0,\ 0 \le i \le m, \\
&\qquad (A^{\otimes j} \otimes \varepsilon_A \otimes A^{\otimes n-j})f = 0,\ 0 \le j \le n\}, \\
&m, n \ge 0.
\end{aligned} \tag{11}$$

We would like to mention that the bicomplex (10) and its normalized subbicomplex (11) are the straightforward generalizations (see § 6) of the bicomplex (4.1) from [13] and are inspired by that one.

2. Extensions

2.1. Definition. An *$\mathcal{M}$-extension* $\mathcal{E}$ of a cocommutative Hopf monoid B by a commutative monoid M is a commutative diagram in the category **A**

$$\begin{array}{ccccc} E = M & \xrightarrow{\alpha} & C & \xrightarrow{\rho} & C \otimes B \\ & \searrow^{M \otimes \eta_B} & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda \otimes B} \\ & & M \otimes B & \xrightarrow{M \otimes \psi_B} & M \otimes B \otimes B\,, \end{array}$$

such that

(i) C is a monoid,

(ii) α and ρ are monoid morphisms,

(iii) λ is an isomorphism in the category **A** and $(\mu_M \otimes B)(M \otimes \lambda) = \lambda\mu_C(\alpha \otimes C)$.

The assumptions of 2.1 imply that C and $M \otimes B$ are the left M-objects via the structural arrows $\mu_C(\alpha \otimes C)$ and $(\mu_M \otimes B)$, respectively. C and $M \otimes B$ are the right B-coobjects via the structural arrows ρ and $(M \otimes \psi_B)$, respectively. Moreover, C is the right B-coobject monoid, because ρ is the monoid morphism. Now, the arrow λ becomes the isomorphism of both left M-objects and right B-coobjects (recall the commutative quadrangle of the diagram and (iii) from 2.1).

The idea of an arrow being simultaneously an isomorphism of left modules over some algebra and of right comodules over some coalgebra is due to Milnor and Moore [8].

2.2. Remark. Every time, while speaking of the system of arrows induced from the $\mathcal{M}$-extension E, we shall mean the arrows

$$\delta = (M \otimes \varepsilon_B)\lambda : C \to M\,, \qquad \gamma = \lambda^{-1}(\eta_M \otimes B) : B \to C\,,$$
$$\sigma_M = \delta\mu_C(\gamma \otimes \alpha) : B \otimes M \to M\,, \quad \tau_M = \delta\mu_C(\gamma \otimes \gamma) : B \otimes B \to M\,.$$

2.3. Theorem. *Let E be an $\mathcal{M}$-extension and $(\delta, \gamma, \sigma_M, \tau_M)$ be the system induced from it. Then we have the following identities:*

(i) $\lambda\mu_C(\gamma \otimes \alpha) = (\sigma_M \otimes B)(1, 3, 2)(\psi_B \otimes M)$.

(ii) $\lambda\mu_C(\gamma \otimes \gamma) = (\tau_M \otimes \mu_B)(1, 3, 2, 4)(\psi_B \otimes \psi_B)$.

(iii) $\lambda\mu_C(\lambda^{-1} \otimes \lambda^{-1}) = (\mu_M^3 \otimes \mu_B)(M \otimes \sigma_M \otimes \tau_M \otimes B \otimes B)$
$\circ(1, 2, 5, 3, 6, 4, 7)(M \otimes \psi_B^3 \otimes M \otimes \psi_B)$.

(iv) $\sigma_M(\eta_B \otimes M) = M$, $\sigma_M(B \otimes \eta_M) = \eta_M\varepsilon_B$,
$\tau_M(B \otimes \eta_B) = \eta_M\varepsilon_B = \tau_M(\eta_B \otimes B)$.

(v) $\sigma_M(B \otimes \mu_M) = (\sigma_M \otimes \sigma_M)(1, 3, 2, 4)(\psi_B \otimes M \otimes M)$.

(vi) $$\mu^3_M(\sigma_M \otimes \sigma_M \otimes \tau_M)(B \otimes \sigma_M \otimes B \otimes \tau_M \otimes B \otimes \mu_B)$$
$$\circ(1,4,7,2,5,8,3,6,9)(\psi^3_B \otimes \psi^3_B \otimes M \otimes \psi_B)$$
$$= \mu^3_M(M \otimes \sigma_M \otimes \tau_M)(\tau_M \otimes \mu_B \otimes M \otimes \mu_B \otimes B)$$
$$\circ(1,4,2,5,7,3,6,8)(\psi^3_B \otimes \psi^3_B \otimes M \otimes B).$$

Proof. The key to the whole proof is the identity

$$\lambda = (M \otimes \varepsilon_B \otimes B)(\lambda \otimes B)\rho. \tag{12}$$

(i) Using (12) it is not difficult to show that

$$(M \otimes \psi_B)\lambda\mu_C(\gamma \otimes \alpha)$$
$$= (\lambda \otimes B)(\mu_C \otimes B)$$
$$\circ(\lambda^{-1} \otimes \lambda^{-1} \otimes B)(\eta_M \otimes B \otimes M \otimes \eta_B \otimes B)(1,3,2)(\psi_B \otimes M)$$
$$= (M \otimes \varepsilon_B \otimes B \otimes B)(\lambda \otimes B \otimes B)(\mu_C \otimes B \otimes B)$$
$$\circ(\lambda^{-1} \otimes \lambda^{-1} \otimes B \otimes B)(\eta_M \otimes B \otimes M \otimes \eta_B \otimes B \otimes B)$$
$$\circ(1,4,2,3)(\psi^3_B \otimes M)$$
$$= (M \otimes \psi_B)(\sigma_M \otimes B)(1,3,2,4)(\psi_B \otimes M). \tag{13}$$

Then, multiplying the first and the last compositions of (13) by $(M \otimes \varepsilon_B \otimes B)$ on the left, we get (i).

(ii) Using (12), we have

$$(M \otimes \psi_B)\lambda\mu_C(\gamma \otimes \gamma)$$
$$= (\lambda \otimes B)(\mu_C \otimes \mu_B)(\lambda^{-1} \otimes \lambda^{-1} \otimes B \otimes B)$$
$$\circ(\eta_M \otimes B \otimes \eta_M \otimes B \otimes B \otimes B)(1,3,2,4)(\Psi_B \otimes \Psi_B)$$
$$= (M \otimes \varepsilon_B \otimes B \otimes B)(\lambda \otimes B \otimes B)(\mu_C \otimes \mu_B \otimes \mu_B)$$
$$\circ(\lambda^{-1} \otimes \lambda^{-1} \otimes B \otimes B \otimes B \otimes B)$$
$$\circ(\eta_M \otimes B \otimes \eta_M \otimes B \otimes B \otimes B \otimes B \otimes B)$$
$$\circ(1,4,2,5,3,6)(\Psi^3_B \otimes \Psi^3_B)$$
$$= (M \otimes \Psi_B)(\tau_M \otimes \mu_B)(1,3,2,4)(\Psi_B \otimes \Psi_B). \tag{14}$$

Then, multiplying the first and the last compositions of (14) by $(M \otimes \varepsilon_B \otimes B)$ on the left, we get (ii).

(iii) From (i) and (ii), associativity of μ_C and the identity $\lambda^{-1} = \mu_C(\alpha \otimes \gamma)$ we get (iii).

(iv) We have

$$\begin{aligned}\sigma_M(\eta_B \otimes M) &= (M \otimes \varepsilon_B)\lambda\mu_C(\gamma \otimes \alpha)(\eta_B \otimes M)\\ &= (M \otimes \varepsilon_B)\lambda\mu_C(\eta_C \otimes C)\alpha\\ &= (M \otimes \varepsilon_B)\lambda\alpha = (M \otimes \varepsilon_B)(M \otimes \eta_B) = M\,.\end{aligned}$$

The other identities of (iv) are verified similarly. Before doing (v) and (vi), let us mention that the associativity of μ_C implies

$$\begin{aligned}&\lambda\mu_C(C \otimes \mu_C)(\lambda^{-1} \otimes \lambda^{-1} \otimes \lambda^{-1})\\ &\quad = \lambda\mu_C(\mu_C \otimes C)(\lambda^{-1} \otimes \lambda^{-1} \otimes \lambda^{-1})\,. \end{aligned} \tag{15}$$

Decomposing both sides of (15) and using (iii), we get that

$$\begin{aligned}&(\mu_M^3 \otimes \mu_B^3)(M \otimes \sigma_M \otimes M \otimes B \otimes B \otimes B)\\ &\circ(M \otimes B \otimes \mu_M^3 \otimes \tau_M \otimes B \otimes B \otimes B)\\ &\circ(M \otimes B \otimes M \otimes \sigma_M \otimes \tau_M \otimes B \otimes \mu_B \otimes B \otimes B \otimes B)\\ &\circ(1, 2, 5, 6, 10, 7, 11, 3, 8, 4, 9, 13)\\ &\circ(M \otimes \psi_B^3 \otimes M \otimes \psi_B^4 \otimes M \otimes \psi_B^3)\\ &\quad = (\mu_M^3 \otimes \mu_B^3)(\mu_M^3 \otimes \sigma_M \otimes \tau_M \otimes B \otimes B \otimes B)\\ &\quad\quad \circ(M \otimes \sigma_M \otimes \tau_M \otimes \mu_B \otimes M \otimes \mu_B \otimes B \otimes B \otimes B \otimes B)\\ &\quad\quad \circ(1, 2, 7, 3, 8, 4, 9, 12, 5, 10, 13, 6, 11, 14)\\ &\quad\quad \circ(M \otimes \psi_B^5 \otimes M \otimes \psi_B^4 \otimes M \otimes \psi_B)\,. \end{aligned} \tag{16}$$

(v) Multiplying both sides of (16) by $(\eta_M \otimes B \otimes M \otimes \eta_B \otimes M \otimes \eta_B)$ on the right and by $(M \otimes \varepsilon_B)$ on the left and using (iv), we get (v).

(vi) Multiplying both sides of (16) by $(\eta_M \otimes B \otimes \eta_M \otimes B \otimes M \otimes B)$ on the right and by $(M \otimes \varepsilon_B)$ on the left and using (iv) and (v), we get (vi). □

2.4. Remark. The previous theorem shows that

(iii) The monoid structure on $M \otimes B$ induced from C by the isomorphism λ is fully described in terms of 'action' σ_M and 'twisting function' τ_M.

(iv), (v). The pair (M, σ_M) is 'like' a left B-object monoid; 'like' because all axioms are satisfied, except the identity $\sigma_M(B \otimes \sigma_M) = \sigma_M(\mu_B \otimes M)$.

(vi) This identity can be transformed into a 'cocycle condition' for τ_M by multiplying both sides of (vi) by $(B \otimes B \otimes \eta_M \otimes B)$ on the right. But still, τ_M is not regular (i.e. an element of $\mathrm{Reg}(B \otimes B, M)$) and what is more important, there is no possibility to obtain the identity $\sigma_M(B \otimes \sigma_M) = \sigma_M(\mu_B \otimes M)$. The last identity is a very important part of (vi). In Section 3, putting additional conditions on extensions, we shall overcome these difficulties.

2.5. Remark. Let B be a cocommutative Hopf monoid and M a commutative monoid, and let $\sigma_M : B \otimes M \to M$ and $\tau_M : B \otimes B \to M$ be arrows of the category **A**. Then these data determine the commutative diagram in **A**,

$$
\begin{array}{ccccc}
E = M & \xrightarrow{\alpha} & C & \xrightarrow{\rho} & C \otimes B \\
 & {\scriptstyle M\otimes\eta_B}\searrow & \Vert\ {\scriptstyle \lambda = 1} & & \Vert \\
 & & M \otimes B & \xrightarrow{M \otimes \psi_B} & M \otimes B \otimes B ,
\end{array}
$$

and arrows

$$\eta : e \to C , \quad \mu : C \otimes C \to C , \quad \eta = \eta_M \otimes \eta_B ,$$
$$\mu = (\mu_M^3 \otimes \mu_B)(M \otimes \sigma_M \otimes \tau_M \otimes B \otimes B)$$
$$\circ (1, 2, 5, 3, 6, 4, 7)(M \otimes \psi_B^3 \otimes M \otimes \psi_B) .$$

2.6. Proposition. *Let B, M, σ_M, τ_M, E, C, α, ρ, η, μ, $\lambda = 1$ be as in* 2.5. *Then E is an $\mathcal{M}$-extension, if and only if the arrows σ_M and τ_M satisfy conditions* (iv), (v) *and* (vi) *of* 2.3.

Proof. Let E be an $\mathcal{M}$-extension. Then the system of arrows induced from E is $(M \otimes \varepsilon_B, \eta_M \otimes B, \sigma_M, \tau_M)$; the last two are the original ones from 2.5. Then 2.3 gives the necessity condition.

Conversely, let us assume that the pair (σ_M, τ_M) satisfies identities (iv), (v) and (vi) from 2.3. The agreement of μ with η is implied from (iv). The identities (v) and (vi) imply the identities (16) and (15), and this means that $(C = M \otimes B, \mu, \eta)$ is a monoid. The Hopf condition on B implies that ρ is the morphism of right B-coobjects. $\square$

2.7. Definition. Let

$$E_i = (M \xrightarrow{\alpha_i} C_i \xrightarrow{\rho_i} C_i \otimes B , \ \lambda_i : C_i \xrightarrow{\sim} M \otimes B) , \quad i = 1, 2 ,$$

be $\mathcal{M}$-extensions. A morphism of monoids $f : C_1 \to C_2$ is called a *morphism of $\mathcal{M}$-extensions* $f : E_1 \to E_2$, iff the following diagram commutes:

$$\begin{array}{ccccc} E_1 = M & \xrightarrow{\alpha_1} & C_1 & \longrightarrow & C_1 \otimes B \\ {\scriptstyle f}\downarrow \quad \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\otimes B} \\ E_2 = M & \xrightarrow{\alpha_2} & C_2 & \xrightarrow{\rho_2} & C_2 \otimes B\,. \end{array}$$

Note, that λ_1 and λ_2 are not involved.

2.8. Proposition. *Let $f: E_1 \to E_2$ be a morphism of $\mathcal{M}$-extensions, $(\delta', \gamma', \sigma'_M, \tau'_M)$ and $(\delta'', \gamma'', \sigma''_M, \tau''_M)$ be the systems of arrows induced from E_1 and E_2, respectively (see 2.2). Let us put $g = \delta'' f \gamma' : B \to M$. Then we have the following identities:*

(i) $\lambda_2 f \lambda_1^{-1} = (\mu_M \otimes B)(M \otimes g \otimes B)(M \otimes \psi_B)$,

(ii) $f\gamma' = \alpha_2 g + \gamma''$, *the sum is taken in the set-monoid* $\mathbf{A}(B, C_2)$,

(iii)
$$\mu^3_M(M \otimes M \otimes g)(\sigma'_M \otimes \tau'_M \otimes \mu_B)(1,4,2,5,3,6)(\psi^3_B \otimes M \otimes \psi_B)$$
$$= \mu^4_M(M \otimes M \otimes \sigma''_M \otimes M)(g \otimes \sigma''_M \otimes B \otimes g \otimes \tau''_M)$$
$$\circ (1,2,5,3,6,4,7)(\psi^4_B \otimes M \otimes \psi_B)\,.$$

Proof. (i) The arrow $\lambda_2 f \lambda_1^{-1}$ is the morphism of 'extended' left M-objects and right B-coobjects and this fact implies (i).

(ii) Multiplying both sides of (i) by γ' on the left, we get (ii).

(iii) Multiplying both sides of the identity

$$f\mu_{C_1} = \mu_{C_2}(f \otimes f)$$

by λ_2 on the left and by $\lambda_1^{-1} \otimes \lambda_1^{-1}$ on the right and using (iii) and (iv) from 2.3 we get the identity

$$(\mu^4_M \otimes \mu_B)(M \otimes M \otimes M \otimes g \otimes B \otimes B)(M \otimes \sigma'_M \otimes \tau'_M \otimes \mu_B \otimes B \otimes B)$$
$$\circ (1,2,6,3,7,4,8,5,9)(M \otimes \psi^4_B \otimes M \otimes \psi^3_B)$$
$$= (\mu^5_M \otimes \mu_B)(M \otimes M \otimes \sigma''_M \otimes \tau''_M \otimes B \otimes B)$$
$$\circ (M \otimes g \otimes B \otimes M \otimes B \otimes g \otimes B \otimes B \otimes B \otimes B)$$
$$\circ (1,2,3,7,4,8,5,9,6,10)(M \otimes \psi^5_B \otimes M \otimes \psi^3_B)\,. \tag{17}$$

Multiplying both sides of (17) by $(\eta_M \otimes B \otimes M \otimes B)$ on the right and by $(M \otimes \varepsilon_B)$ on the left, we get (iii). □

Unfortunately, we cannot prove that every morphism of $\mathcal{M}$-extensions is an isomorphism; as we shall see in Section 3, the last property is shortly connected with the regularity (of arrow g) condition.

Arguments similar to 2.6 enable us to prove the following:

2.9. Proposition. *Let* B *be a cocommutative Hopf monoid,* M *a commutative monoid,* (σ'_M, τ'_M) *and* (σ''_M, τ''_M) *the pairs of arrows which satisfy the conditions of* 2.6 *and thus, determine the* $\mathcal{M}$*-extensions* E_1 *and* E_2 *(see* 2.5), *respectively. Let* $g: B \to M$ *be an arrow from* **A** *and put* $f = (\mu_M \otimes B)(M \otimes g \otimes B)(M \otimes \psi_B)$. *Then* f *is a morphism of* $\mathcal{M}$*-extensions* $f: E_1 \to E_2$, *if and only if* g *satisfies* (iii) *from* 2.8. □

Considerations dual to 2.1, 2.2 and 2.3 give the following:

2.10. Definition. A $\mathscr{C}$*-extension* E of a cocommutative comonoid N by a commutative Hopf monoid A is a commutative diagram in **A**,

$$E = \begin{array}{ccccc} A \otimes C & \xrightarrow{\sigma} & C & \xrightarrow{\beta} & N \\ \downarrow{\scriptstyle A \otimes \lambda} & & \downarrow{\scriptstyle \lambda} & \nearrow{\scriptstyle \varepsilon_A \otimes N} & \\ A \otimes A \otimes N & \xrightarrow{\mu_A \otimes N} & A \otimes N, & & \end{array}$$

such that

(i) C is a comonoid,

(ii) σ and β are comonoid morphisms,

(iii) λ is an isomorphism in the category **A** and

$$(A \otimes \psi_N)\lambda = (\lambda \otimes N)(C \otimes \beta)\psi_C .$$

2.11. Theorem. *Let* E *be a* $\mathscr{C}$*-extension and put*

$$\delta = (A \otimes \varepsilon_N)\lambda : C \to A , \qquad \gamma = \lambda^{-1}(\eta_A \otimes N): N \to C ,$$

$$\rho_N = (\beta \otimes \delta)\psi_C \gamma : N \to N \otimes A , \quad \varphi_N = (\delta \otimes \delta)\psi_C \gamma : N \to A \otimes A .$$

Then we have the following identities:

(i) $(\beta \otimes \delta)\psi_C \lambda^{-1} = (N \otimes \mu_A)(2, 1, 3)(A \otimes \rho_N)$.

(ii) $(\delta \otimes \delta)\psi_C \lambda^{-1} = (\mu_A \otimes \mu_A)(1, 3, 2, 4)(\psi_A \otimes \varphi_A)$.

(iii) $(\lambda \otimes \lambda)\psi_C \lambda^{-1} = (\mu_A \otimes N \otimes \mu^3_A \otimes N)(1, 3, 5, 2, 4, 6, 7) \circ (A \otimes A \otimes \varphi_N \otimes \rho_N \otimes N)(\psi_A \otimes \psi^3_N)$.

(iv) $(N \otimes \varepsilon_A)\rho_N = N$, $(\varepsilon_N \otimes A)\rho_N = \eta_A \varepsilon_N$, $(A \otimes \varepsilon_A)\varphi_N = \eta_A \varepsilon_N = (\varepsilon_A \otimes A)\varphi_N$.

(v) $(\psi_N \otimes A)\rho_N = (N \otimes N \otimes \mu_A)(1, 3, 2, 4)(\rho_N \otimes \rho_N)\psi_N$.

(vi) $(\mu_A \otimes N \otimes \mu^3_A \otimes \mu^3_A)(1, 4, 7, 2, 5, 8, 3, 6, 9) \circ (\psi_A \otimes A \otimes \varphi_N \otimes A \otimes \rho_N \otimes A)(\varphi_N \otimes \rho_N \otimes \rho_N)\psi^3_N$

$$= (A \otimes N \otimes \mu_A^3 \otimes \mu_N^3)(1,4,2,5,7,3,6,8)$$
$$\circ (A \otimes \psi_A \otimes N \otimes \psi_A \otimes \varphi_N)(\varphi_N \otimes \rho_N \otimes N)\psi_N^3 .$$

Proof. Dual to 2.3. □

Certainly, one can develop dual to the whole theory of $\mathcal{M}$-extensions. We shall assume that this is done and use facts from $\mathcal{C}$-extension theory.

There appears an interesting notion obtained from the joining of the notions of $\mathcal{M}$-extensions and $\mathcal{C}$-extensions.

2.12. Definition. An *$\mathcal{H}$-extension* of a cocommutative Hopf monoid B by a commutative Hopf monoid A is a commutative diagram in the category $\mathbf{A}$,

$$\begin{array}{ccccc} E = A & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & B , \\ & {\scriptstyle A\otimes\eta_B}\searrow & \downarrow{\scriptstyle \lambda} & \nearrow{\scriptstyle \varepsilon_A\otimes B} & \\ & & A \otimes B & & \end{array} \qquad (18)$$

such that

(i) C is a Hopf monoid,
(ii) α and β are Hopf monoid morphisms,
(iii) λ is an isomorphism in $\mathbf{A}$ and

$$\lambda\mu_C(\alpha \otimes C) = (\mu_A \otimes B)(A \otimes \lambda) ,$$
$$(A \otimes \psi_B)\lambda = (\lambda \otimes B)(C \otimes \beta)\psi_C .$$

2.13. Proposition. *Let A be a commutative and B a cocommutative Hopf monoid. A commutative diagram like* (18) *is an $\mathcal{H}$-extension, if and only if we have the following*:

(i) *C is a Hopf monoid, α and β are Hopf monoid morphisms,*
(ii) *$E' = (A \xrightarrow{\alpha} C \xrightarrow{(C\otimes\beta)\psi_C} C \otimes B, \lambda)$ is an $\mathcal{M}$-extension,*
(iii) *$E'' = (A \otimes C \xrightarrow{\mu_C(\alpha\otimes C)} C \xrightarrow{\beta} B, \lambda)$ is a $\mathcal{C}$-extension.*

Proof. Obvious. □

2.14. Theorem. *Let E be an $\mathcal{H}$-extension and put*:

$$\delta = (A \otimes \varepsilon_B)\lambda : C \to A , \qquad \gamma = \lambda^{-1}(\eta_A \otimes B) : B \to C ,$$
$$\sigma_A = \delta\mu_C(\gamma \otimes \alpha) : B \otimes A \to A , \qquad \rho_B = (\beta \otimes \delta)\psi_C\gamma : B \to B \otimes A ,$$
$$\tau_A = \delta\mu_C(\gamma \otimes \gamma) : B \otimes B \to A , \qquad \varphi_B = (\delta \otimes \delta)\psi_C\gamma : B \to A \otimes A .$$

Then we have the following:

(i) *We have* (i)–(vi) *from* 2.3 *only one must make the substitution* $M = A$.

(ii) *We have* (i)–(vi) *from* 2.11, *only one must make the substitution* $N = B$.

(iii) *We have the following identity* (*sum is taken in the set-monoid* $\mathbf{A}(B\otimes A\otimes B, A\otimes B\otimes A)$):

$$\begin{aligned}
&(A\otimes\eta_B\otimes A)\psi_A\sigma_A(B\otimes A\otimes\varepsilon_B)\\
&+(A\otimes\eta_B\otimes A)\psi_A\tau_A(B\otimes\varepsilon_A\otimes B)\\
&+(A\otimes\eta_B\otimes A)\varphi_B\mu_B(B\otimes\varepsilon_A\otimes B)\\
&+(\eta_A\otimes B\otimes A)\rho_B\mu_B(B\otimes\varepsilon_A\otimes B)\\
&\quad=(A\otimes\eta_B\otimes A)\bar{\sigma}_{A\otimes A}(B\otimes\psi_A)(B\otimes A\otimes\varepsilon_B)\\
&\quad\quad+(\eta_A\otimes B\otimes A)(\mu_A\otimes A)\underline{\rho}_{B\otimes B}(B\otimes\varepsilon_A\otimes B)\\
&\quad\quad+(A\otimes\eta_B\otimes A)\varphi_B(B\otimes\varepsilon_A\otimes\varepsilon_B)\\
&\quad\quad+(A\otimes\eta_B\otimes A)\bar{\sigma}_{A\otimes A}(B\otimes\varphi_B)(B\otimes\varepsilon_A\otimes B)\\
&\quad\quad+(A\otimes\eta_B\otimes A)(\tau_A\otimes A)\underline{\rho}_{B\otimes B}(B\otimes\varepsilon_A\otimes B)\\
&\quad\quad+(\eta_A\otimes\eta_B\otimes A)\tau_A(B\otimes\varepsilon_A\otimes B)\,.
\end{aligned}$$

(*Expressions for* $\bar{\sigma}_{A\otimes A}$ *and* $\underline{\rho}_{B\otimes B}$ *are given in* 0.19).

Proof. Recalling 2.13, 2.3 and 2.11, it remains to prove only (iii). The Hopf condition on C implies the identity

$$\begin{aligned}
&(\lambda\otimes\lambda)\psi_C\mu_C(\lambda^{-1}\otimes\lambda^{-1})\\
&\quad=(\lambda\otimes\lambda)(\mu_C\otimes\mu_C)(1,3,2,4)(\psi_C\otimes\psi_C)(\lambda^{-1}\otimes\lambda^{-1})\,.
\end{aligned}$$

Using 2.3 and 2.11, we get from the above identity

$$\begin{aligned}
&(\mu_A^4\otimes B\otimes\mu_A^5\otimes\mu_B)(1,3,5,7,9,2,4,6,8,10,11,12)\\
&\circ(A\otimes A\otimes\Psi_A\otimes\Psi_A\otimes\varphi_B\otimes\rho_B\otimes B\otimes B)\\
&\circ(A\otimes A\otimes\sigma_A\otimes\tau_A\otimes\mu_B\otimes\mu_B\otimes B\otimes B)\\
&\circ(1,2,3,8,4,9,5,10,6,11,7,12)(\psi_A\otimes\psi_B^5\otimes A\otimes\psi_B^4)\\
&\quad=(\mu_A^5\otimes\mu_B\otimes\mu_A^{11}\otimes\mu_B)\\
&\quad\quad\circ(A\otimes A\otimes\sigma_A\otimes\sigma_A\otimes\tau_A\otimes B\otimes B\otimes A\\
&\quad\quad\quad\otimes A\otimes A\otimes A\otimes A\otimes A\otimes\sigma_A\otimes\sigma_A\otimes\sigma_A\otimes\sigma_A\otimes\tau_A\otimes B\otimes B)
\end{aligned}$$

$$\circ(1, 3, 5, 19, 7, 21, 9, 23, 11, 25, 2, 4, 6, 8, 10, 12, 13, 20, 14, 22, 15, 24, 16, 26, 17, 27, 18, 28)$$
$$\circ(A\otimes A\otimes \varphi_B\otimes \rho_B\otimes \rho_B\otimes \rho_B\otimes \rho_B$$
$$\otimes B\otimes B\otimes B\otimes B\otimes B\otimes B\otimes A\otimes A\otimes \varphi_B\otimes \rho_B\otimes \rho_B\otimes B\otimes B)$$
$$\circ(\psi_A\otimes \psi_B^{11}\otimes \psi_A\otimes \psi_B^{5}) . \tag{19}$$

Multiplying both sides of (19) by $(\eta_A\otimes B\otimes A\otimes B)$ on the right and by $(A\otimes B\otimes A\otimes \varepsilon_B)$ on the left, we get identity (iii). □

2.15. Proposition. *Let A be a commutative and B a cocommutative Hopf monoid and let*

$$\sigma_A : B\otimes A\to A , \quad \rho_B : B\to B\otimes A ,$$
$$\tau_A : B\otimes B\to A , \quad \varphi_B : B\to A\otimes A$$

be arrows from the category **A**. *The commutative diagram*

$$\begin{array}{ccccc} E = A & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & B , \\ & \searrow^{A\otimes\eta_B} & \Big\|{\scriptstyle \lambda = 1} & \nearrow_{\varepsilon_A\otimes B} & \\ & & A\otimes B & & \end{array}$$

and the data

$$\eta = \eta_A\otimes\eta_B , \qquad \varepsilon = \varepsilon_A\otimes\varepsilon_B ,$$
$$\mu = (\mu_A^3\otimes\mu_B)(A\otimes\sigma_A\otimes\tau_A\otimes B\otimes B)$$
$$\circ(1, 2, 5, 3, 6, 4, 7)(A\otimes\psi_B^3\otimes A\otimes\psi_B) ,$$
$$\psi = (\mu_A\otimes B\otimes\mu_A^3\otimes B)$$
$$\circ(1, 3, 5, 2, 4, 6, 7)(A\otimes A\otimes\varphi_B\otimes\rho_B\otimes B)(\psi_A\otimes\psi_B^3)$$

determine an $\mathcal{H}$*-extension, if and only if we have the following*:

(i) *We have* (iv)–(vi) *of* 2.3, *only one must make the substitution* $M = A$.
(ii) *We have* (iv)–(vi) *of* 2.11, *only one must make the substitution* $N = B$.
(iii) *We have* (iii) *of* 2.14.

Proof. Recalling 2.6, its dual consideration and 2.13, it remains to verify the Hopf condition for C. But this condition comes from a straightforward calculation, which uses the Hopf conditions for A and B and the identity (iii) from 2.14. □

Certainly, one can develop the whole theory of $\mathcal{H}$-extensions, similar to the theory of $\mathcal{M}$-extensions. We shall assume that this is done and use facts from the $\mathcal{H}$-extension theory.

3. Connections between cohomologies and extensions

3.1. Proposition. *Let B be a cocommutative Hopf monoid and let $(X, \rho_X: X \to X \otimes B, \mu_X, \eta_X)$ and $(Y, \rho_Y: Y \to B \otimes Y, \mu_Y, \eta_Y)$ be B-coobject monoids. If there exists a cotensor product $(X \otimes^B Y, \nu)$ of B-coobjects (see* 0.9), *then there exist unique arrows $\bar{\mu}$ and $\bar{\eta}$, such that they define the monoid structure on $X \otimes^B Y$ and the arrow $\nu: X \otimes^B Y \to X \otimes Y$ is a monoid morphism (the monoid structure μ on $X \otimes Y$ is given as in* 0.3). *Thus, $(X \otimes^B Y, \nu)$ is the equalizer of a pair in the category of monoids, too.*

Proof. Let us consider an arrow $\mu_1 = \mu(\nu \otimes \nu) = (\mu_X \otimes \mu_Y)(1, 3, 2, 4)(\nu \otimes \nu)$. For this arrow we have $(\rho_X \otimes Y)\mu_1 = (X \otimes \rho_Y)\mu_1$ and then, by the universal property of the equalizer, there exists an arrow $\bar{\mu}$, such that $\mu_1 = \mu(\nu \otimes \nu) = \nu\bar{\mu}$. $\bar{\eta}$ is defined similarly. □

3.2. Definition. An $\mathcal{M}$-extension $E = (M \xrightarrow{\alpha} C \xrightarrow{\rho} C \otimes B, \lambda: C \tilde{\to} M \otimes B)$ is called a *regular* $\mathcal{M}$-extension, iff $\gamma = \lambda^{-1}(\eta_M \otimes B) \in \mathrm{Reg}(B, C)$, i.e. if the arrow γ is the invertible element of the set-monoid $\mathbf{A}(B, C)$.

3.3. Theorem. *Let M be a commutative monoid and B a cocommutative Hopf monoid with an antipode S_B (see* 0.13). *Let $(C, \rho_C, \mu_C, \eta_C)$ be a right B-coobject monoid. Then the following two statements are equivalent:*

(i) *There exist arrows α and λ, such that the diagram*

$$\begin{array}{ccccc} E = M & \xrightarrow{\alpha} & C & \xrightarrow{\rho_C} & C \otimes B \\ & \searrow^{M \otimes \eta_B} & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda \otimes B} \\ & & M \otimes B & \xrightarrow{M \otimes \psi_B} & M \otimes B \otimes B \end{array}$$

is a regular $\mathcal{M}$-extension.

(ii) *There exist an arrow $\gamma \in \mathrm{Reg}(B, C) \cap \mathbf{A}^B(B, C)$ and an isomorphism of monoids $\omega: M \tilde{\to} C \otimes^B e$, where e is the unit object of the category* $\mathbf{A}$ *(see* 0.14(ii)).

Proof. Let us assume (i), then we have

$$\begin{aligned} \rho_C \alpha &= (\lambda^{-1} \otimes B)(M \otimes \psi_B)\lambda\lambda^{-1}(M \otimes \eta_B) \\ &= (\lambda^{-1} \otimes B)(M \otimes \psi_B)(M \otimes \eta_B) \\ &= (\lambda^{-1} \otimes B)(M \otimes B \otimes \eta_B)\lambda\lambda^{-1}(M \otimes \eta_B) = (C \otimes \eta_B)\alpha . \end{aligned}$$

Now let $h: D \to C$ be an arrow in **A**, such that $\rho_C h = (C \otimes \eta_B)h$. Then we have

$$\begin{aligned} h &= (C \otimes \varepsilon_B)\rho_C h = (C \otimes \varepsilon_B)(\lambda^{-1} \otimes B)(M \otimes \psi_B)\lambda h \\ &= \lambda^{-1}(M \otimes B \otimes \varepsilon_B)(M \otimes \psi_B)\lambda h \\ &= \lambda^{-1}(M \otimes \varepsilon_B \otimes B)(\lambda \otimes B)\rho_C h \\ &= \lambda^{-1}(M \otimes \varepsilon_B \otimes B)(\lambda \otimes B)(C \otimes \eta_B)h \\ &= \lambda^{-1}(M \otimes \eta_B)(M \otimes \varepsilon_B)\lambda h \\ &= \alpha(M \otimes \varepsilon_B)\lambda h . \end{aligned}$$

Thus, M is the equalizer of the pair $(\rho_C, (C \otimes \eta_B))$ in the category **A**, but by 3.1 it is the equalizer in the category of monoids, too.

Conversely, let us assume (ii). We can assume that $\gamma\eta_B = \eta_C$; if this is not so, then we can consider a new arrow $\gamma_1 = \gamma + \gamma^*\eta_B\varepsilon_B \in \mathrm{Reg}(B, C)$, here γ^* is the inverse element to γ (see 0.10). Then the fact that γ is an element of $\mathbf{A}^B(B, C)$ and the identity $\rho_C\gamma^* = (\gamma^* \otimes S_B)\psi_B$ imply that $\gamma_1 \in \mathbf{A}^B(B, C)$; but we must prove that the identity $\rho_C\gamma^* = (\gamma^* \otimes S_B)\psi_B$ is true. For this let us consider the sums

$$\rho_C\gamma + (\gamma^* \otimes S_B)\psi_B \quad \text{and} \quad (\gamma^* \otimes S_B)\psi_B + \rho_C\gamma .$$

We shall show that both are equal to the zero element of the set-monoid $\mathbf{A}(B, C \otimes B)$. The first sum is equal to

$$\rho_C\gamma + (\gamma^* \otimes S_B)\psi_B = (\gamma \otimes B)\psi_B + (\gamma^* \otimes S_B)\psi_B;$$

but B is cocommutative, and thus ψ_B is a comonoid morphism

$$\begin{aligned} (\gamma \otimes B)\psi_B + (\gamma^* \otimes S_B)\psi_B &= ((\gamma \otimes B) + (\gamma^* S_B))\psi_B \\ &= (\eta_C \otimes \eta_B)(\varepsilon_B \otimes \varepsilon_B)\psi_B \\ &= (\eta_C \otimes \eta_B)\varepsilon_B = 0 . \end{aligned}$$

Arguments for the second sum are similar.

Thus, we assume that $\gamma\eta_B = \eta_C$. Let us consider the composition $\tilde{\delta} = \mu_C(C \otimes \gamma^*)\rho_C : C \to C$, using the identity $\rho_C\gamma^* = (\gamma^* \otimes S_B)\psi_B$ one can easily verify that $\rho_C\tilde{\delta} = (C \otimes \eta_B)\tilde{\delta}$. Then, by the universality of $w: M \tilde{\to} C \otimes^B e$ there exists an arrow $\delta: C \to A$, such that $\tilde{\delta} = \alpha\delta$, where $\alpha = \nu w$.

Let us consider the compositions

$$\lambda = (\delta \otimes B)\rho_C : C \to M \otimes B , \qquad \lambda^{-1} = \mu_C(\alpha \otimes \gamma) : M \otimes B \to C .$$

It can be easily verified that $\lambda^{-1}\lambda = 1$.

We have the identity $\alpha\delta\mu_C(\alpha\otimes\gamma)=\alpha(M\otimes\varepsilon_B)$ and then, because $\alpha=\nu w$ is a monomorphism, we get $\delta\mu_C(\alpha\otimes\gamma)=(M\otimes\varepsilon_B)$. Using this last identity, it can be easily verified that $\lambda\lambda^{-1}=1$. □

3.4. Lemma. *Let E be an $\mathcal{M}$-extension and $(\delta,\gamma,\sigma_M,\tau_M)$ the system of arrows induced from it (see 2.2).*

(i) *If E is the regular $\mathcal{M}$-extension and B has an antipode S_B, then:*
 (1) $\sigma_M=\delta\mu_C^3(\gamma\otimes\alpha\otimes\gamma^*)(1,3,2)(\psi_B\otimes M)$,
 (2) $\tau_M=\delta(\gamma(B\otimes\varepsilon_B)+\gamma(\varepsilon_B\otimes B)+\gamma^*\mu_B)\in\operatorname{Reg}(B\otimes B,M)$.

(ii) *If $\tau_M\in\operatorname{Reg}(B\otimes B,M)$, then:*
 (1) $(M,\sigma_M,\mu_M,\eta_M)\in|\mathcal{MM}(_B\mathrm{A})|$,
 (2) $\sigma_M(B\otimes\tau_M)+\tau_M^*(\mu_B\otimes B)+\tau_M(B\otimes\mu_B)+\tau_M^*(B\otimes B\otimes\varepsilon_B)=0$.

(iii) *If $\tau_M\in\operatorname{Reg}(B,C)$ and B has an antipode S_B, then $\gamma\in\operatorname{Reg}(B,C)$ and*

$$\gamma^*=\alpha\sigma_M(B\otimes\tau_M^*)(S_B\otimes B\otimes S_B)\psi_B^3+\gamma S_B\,.$$

Proof. (i) Assume that E is regular and B has an antipode S_B.

(1) Using (i) from 2.3 we have

$$\begin{aligned}
&\delta\mu_C^3(\gamma\otimes\alpha\otimes\gamma^*)(1,3,2)(\psi_B\otimes M)\\
&\quad=\delta\mu_C(\mu_C\otimes C)(\gamma\otimes\alpha\otimes\gamma^*)(1,3,2)(\psi_B\otimes M)\\
&\quad=\delta\mu_C(\mu_C\otimes C)(\alpha\otimes\gamma\otimes C)(\sigma_M\otimes B\otimes C)\\
&\qquad\circ(1,3,2,4)(\psi_B\otimes M\otimes C)(B\otimes M\otimes\gamma^*)(1,3,2)(\psi_B\otimes M)\\
&\quad=\delta\mu_C(C\otimes\mu_C)(C\otimes\gamma\otimes\gamma^*)(C\otimes\psi_B)\\
&\qquad\circ(\alpha\otimes B)(\sigma_M\otimes B)(1,3,2)(\psi_B\otimes M)\\
&\quad=\delta\mu_C(\alpha\otimes\eta_C)\sigma_M=\delta\alpha\sigma_M=\sigma_M\,.
\end{aligned}$$

(2) Using (ii) from 2.3 we have

$$\alpha\tau_M+\gamma\mu_B=\mu_C(\gamma\otimes\gamma)=\gamma(B\otimes\varepsilon_B)+\gamma(\varepsilon_B\otimes B)$$

and, recalling that $(\gamma\mu_B)^*=\gamma^*\mu_B$, we conclude that

$$\alpha\tau_M=\gamma(B\otimes\varepsilon_B)+\gamma(\varepsilon_B\otimes B)+\gamma^*\mu_B\,.$$

Multiplying both sides of the last identity by δ on the left, we get the formula for τ_M.

Now we must construct an inverse of τ_M. Let us consider the arrow

$$h=\gamma\mu_B+\mu_C(\gamma^*\otimes\gamma^*)(2,1)=\gamma\mu_B+\gamma^*(\varepsilon_B\otimes B)+\gamma^*(B\otimes\varepsilon_B)\,. \tag{20}$$

It is easily verified that $\rho h = (C \otimes \eta_B)h$. Then by the universality of $\nu : M \to C \otimes^B e$ we have $h = \alpha\delta h$ (see the proof of 3.3). Then we have that $\alpha(\delta h + \tau_M) = \alpha\delta h + \alpha\tau_M = h + \tau_M = 0 = \alpha\eta_M(\varepsilon_B \otimes \varepsilon_B)$. The arrow α is mono, and thus $\delta h + \tau_M = \eta_M(\varepsilon_B \otimes \varepsilon_B) = 0$, i.e. $\delta h = \tau_M^*$.

(ii) Now let τ_M be regular arrow with inverse τ_M^*.

(1) Recalling the identities (iv) and (v) from 2.3, it remains to show that $\sigma_M(\mu_B \otimes M) = \sigma_M(B \otimes \sigma_M)$.

By straightforward calculation we get that

$$\sigma_M(\mu_B \otimes M) = \mu_M^3(\sigma_M \otimes M \otimes M)(\mu_B \otimes M \otimes \tau_M \otimes \tau_M^*) \circ (1,4,7,2,5,3,6)(\psi_B^3 \otimes \psi_B^3 \otimes M). \tag{21}$$

Multiplying both sides of (vi) from 2.3 by $(B \otimes B \otimes M \otimes \eta_B)$ on the right, we get

$$\mu_M(\sigma_M \otimes M)(\mu_B \otimes M \otimes \tau_M)(1,3,5,2,4)(\psi_B \otimes \psi_B \otimes M)$$
$$= \mu_M(\sigma_M \otimes M)(\mu_B \otimes \sigma_M \otimes \tau_M)(1,3,5,2,4)(\psi_B \otimes \psi_B \otimes M).$$

Using this last identity, the right-hand side of (21) can be transformed to $\sigma_M(B \otimes \sigma_M)$.

(2) Multiplying both sides of (vi) from 2.3 by $(B \otimes B \otimes \eta_M \otimes B)$ on the right, we get that

$$\sigma_M(B \otimes \tau_M) + \tau_M(B \otimes \mu_B) = \tau_M(\mu_B \otimes B) + \tau_M(B \otimes B \otimes \varepsilon_B).$$

Now it remains to notice that

$$\tau_M^*(\mu_B \otimes B) = (\tau_M(\mu_B \otimes B))^* \quad \text{and}$$
$$\tau_M^*(B \otimes B \otimes \varepsilon_B) = (\tau_M(B \otimes B \otimes \varepsilon_B))^*.$$

(iii) This is verified by a straightforward calculation which uses (20). □

3.5. Lemma. *Let $f : E_1 \to E_2$ be a morphism of $\mathcal{M}$-extensions, $(\delta', \gamma', \sigma_M', \tau_M')$ and $(\delta'', \gamma'', \sigma_M'', \tau_M'')$ the systems of arrows induced from E_1 and E_2, respectively, and $g = \delta'' f \gamma'$.*

(i) *If E_1 is a regular $\mathcal{M}$-extension and B has an antipode S_B, then:*

(1) *$g \in \mathrm{Reg}(B, M)$ and $g^* = \delta''(\gamma'' + f(\gamma')^*)$.*

(2) *$\gamma'' \in \mathrm{Reg}(B, C_2)$ and $(\gamma'')^* = f(\gamma')^* + \alpha_2 g$, and thus E_2 is a regular $\mathcal{M}$-extension, too.*

(ii) *The morphism $f : C_1 \to C_2$ is an isomorphism, if and only if g is a regular arrow (i.e. $g \in \mathrm{Reg}(B, M)$).*

(iii) *If* $g \in \operatorname{Reg}(B, M)$, *then* $\sigma'_M = \sigma''_M$ (*and we denote both of them by* σ_M), *and*

$$\tau'_M + g\mu_B = g(B \otimes \varepsilon_B) + \sigma_M(B \otimes g) + \tau''_M .$$

Proof. (i) (1) Using the identity $\rho_1(\gamma')^* = ((\gamma')^* \otimes S_B)\psi_B$, one can easily see that

$$\rho_2(\gamma'' + f(\gamma')^*) = (C_2 \otimes \eta_B)(\gamma'' + f(\gamma')^*) .$$

Let us denote $h = \gamma'' + f(\gamma')^*$; then (22) implies that $\rho_2 h = (C_2 \otimes \eta_B)h$. Using this last identity, we have

$$\begin{aligned} h &= (C_2 \otimes \varepsilon_B)\rho_2 h = (C_2 \otimes \varepsilon_B)(\lambda_2^{-1} \otimes B)(M \otimes \psi_B)\lambda_2 h \\ &= \lambda_2^{-1}(M \otimes B \otimes \varepsilon_B)(M \otimes \psi_B)\lambda_2 h \\ &= \lambda_2^{-1}(M \otimes \varepsilon_B \otimes B)(\lambda_2 \otimes B)\rho_2 h \\ &= \lambda_2^{-1}(M \otimes \varepsilon_B \otimes B)(\lambda_2 \otimes B)(C_2 \otimes \eta_B)h \\ &= \lambda_2^{-1}(M \otimes \eta_B)(M \otimes \varepsilon_B)\lambda_2 h \\ &= \alpha_2(M \otimes \varepsilon_B)h . \end{aligned}$$

Thus we obtain

$$\alpha_2\delta''(\gamma'' + f(\gamma')^*) = \gamma'' + f(\gamma')^* . \tag{23}$$

From (ii) of 2.8 it is known that $f\gamma' = \alpha_2 g + \gamma''$ and that α_2 and f are the monoid morphisms. Then, using (23), we have

$$\begin{aligned} \alpha_2(g + \delta''(\gamma'' + f(\gamma')^*)) &= \alpha_2 g + \alpha_2\delta''(\gamma'' + f(\gamma')^*) \\ &= \alpha_2 g + \gamma'' + f(\gamma')^* = 0 . \end{aligned}$$

The arrow α_2 is mono, hence we obtain $g + \delta''(\gamma'' + f(\gamma')^*) = 0$ and so $g^* = \delta''(\gamma'' + f(\gamma')^*)$.

(2) From the identity $f\gamma' = \alpha_2 g + \gamma''$ we obtain $\gamma'' = \alpha_2 g^* + f\gamma'$ and then $(\gamma'')^* = f(\gamma')^* + \alpha_2 g$.

(ii) Let $f: C_1 \to C_2$ be an isomorphism. Then $f^{-1}: C_2 \to C_1$ is a morphism of $\mathcal{M}$-extensions too, and then, for the arrow $\tilde{g} = \delta' f^{-1}\gamma'': B \to M$ we have

$$f^{-1} = (\mu_M \otimes B)(M \otimes \tilde{g} \otimes B)(M \otimes \psi_B) .$$

Now, if we make a decomposition of the identity $\lambda_1 f^{-1}\lambda_2^{-1}\lambda_2 f\lambda_1^{-1} = M \otimes B$, we see that

$$(\mu_M \otimes B)(M \otimes (\tilde{g} + g) \otimes B)(M \otimes \psi_B) = M \otimes B\,. \tag{24}$$

Multiplying both sides of (24) by $(M \otimes \varepsilon_B)$ on the left and by $(\eta_M \otimes B)$ on the right, we obtain $g + \tilde{g} = 0$, i.e. $g^* = \tilde{g} = \delta' f^{-1}\gamma''$.

Conversely, let $g \in \mathrm{Reg}(B, M)$, then one can put

$$f^{-1} = \lambda_1^{-1}(\mu_M \otimes B)(M \otimes g^* \otimes B)(M \otimes \psi_B)\lambda_2\,.$$

(iii) Let $g \in \mathrm{Reg}(B, M)$. It is easily verified that

$$\sigma'_M = \mu^3_M(\sigma'_M \otimes g \otimes g^*)(1, 4, 2, 3)(\psi^3_B \otimes M)\,. \tag{25}$$

Multiplying both sides of (iii) from 2.8 by $(B \otimes M \otimes \eta_B)$ on the right, we obtain

$$\mu_M(\sigma'_M \otimes g)(1, 3, 2)(\psi_B \otimes M) = \mu_M(g \otimes \sigma''_M)(\psi_B \otimes M)\,. \tag{26}$$

Using (26) one can easily transform the right-hand side of (25) to σ''_M. Hence, $\sigma'_M = \sigma''_M$, and we denote both by σ_M.

Multiplying both sides of (iii) from 2.8 by $(B \otimes \eta_M \otimes B)$ on the right and recalling that $\sigma'_M = \sigma''_M = \sigma_M$, we obtain

$$\tau'_M + g\mu_B = g(B \otimes \varepsilon_B) + \sigma_M(B \otimes g) + \tau''_M\,. \quad \square$$

3.6. Remark. Let E be an $\mathcal{M}$-extension, we shall make our considerations in two (somehow parallel) situations: the first one is when E is regular and B has an antipode and the second one is when $\mathbf{A}(B^{\otimes n}, M) = \mathrm{Reg}(B^{\otimes n}, M)$, $n = 1, 2$.

In both cases 3.4 shows that (M, σ_M) is a left B-object commutative monoid and $\tau_M : B \otimes B \to M$ is a two-dimensional cocycle of the respective normalized complex (2) from Section 1. Lemma 3.5 shows that if $f : E_1 \to E_2$ is a morphism of $\mathcal{M}$-extensions, then, if E_1 is regular, E_2 is also regular (in the assumption that B has an antipode), and that f is an isomorphism in both above-mentioned situations. Moreover, the actions of B on M induced from E_1 and E_2 coincide and the difference between τ'_M and τ''_M is an element of $\mathrm{Reg}(B, M)$, i.e. if $g = \delta'' f\gamma'$, then $\tau'_M - \tau''_M = \tau'_M + (\tau''_M)^* = \sigma_M(B \otimes g) + g^*\mu_B + g(B \otimes \varepsilon_B) = d^1(g)$ (for d^1 see (1)) and this fact holds in both situations.

We can make considerations in the inverse direction. Let (M, σ_M) be a left B-object commutative monoid and τ_M be a two-dimensional cocycle of (2) from Section 1. Recalling 2.6 we have to show that we have the identities (iv)–(vi) of 2.3. Parts (iv) and (v) are obviously satisfied. Let us put a cocycle condition for τ_M in the following form:

$$\sigma_M(B \otimes \tau_M) + \tau_M(B \otimes \mu_B) = \tau_M(\mu_B \otimes B) + \tau_M(B \otimes B \otimes \varepsilon_B)\,.$$

Now, decomposing both sides of this last identity, we obtain

$$\begin{aligned}&\mu_M(\sigma_M \otimes M)(B \otimes \tau_M \otimes \tau_M)(B \otimes B \otimes B \otimes B \otimes \mu_B)\\ &\circ(1,3,5,2,4,6)(\psi_B \otimes \psi_B \otimes \psi_B)\\ &\quad= \mu_M(\tau_M \otimes \tau_M)(B \otimes B \otimes \mu_B \otimes B)(1,3,2,4,5)(\psi_B \otimes \psi_B \otimes B)\end{aligned}$$

Using this last identity and the fact that M is a left B-object commutative monoid, we have

$$\begin{aligned}&\mu_M^3(\sigma_M \otimes \sigma_M \otimes \tau_M)(B \otimes \sigma_M \otimes B \otimes \tau_M \otimes B \otimes \mu_B)\\ &\circ(1,4,7,2,5,8,3,6,9)(\psi_B^3 \otimes \psi_B^3 \otimes M \otimes \psi_B)\\ &\quad= \mu_M^3(\sigma_M \otimes M \otimes M)(B \otimes \sigma_M \otimes M \otimes M)(B \otimes B \otimes M \otimes \sigma_M \otimes M)\\ &\qquad\circ(B \otimes B \otimes M \otimes B \otimes \tau_M \otimes \tau_M)\\ &\qquad\circ(B \otimes B \otimes M \otimes B \otimes B \otimes B \otimes B \otimes \mu_B)\\ &\qquad\circ(1,4,7,2,5,8,3,6,9)(\psi_B^3 \otimes \psi_B^3 \otimes M \otimes \psi_B)\\ &\quad= \mu_M^3(\sigma_M \otimes M \otimes M)(\mu_B \otimes M \otimes M)(B \otimes B \otimes M \otimes \tau_M \otimes \tau_M)\\ &\qquad\circ(B \otimes B \otimes M \otimes B \otimes B \otimes \mu_B \otimes B)(1,4,7,2,5,3,6,8)\\ &\qquad\circ(\psi_B^3 \otimes \psi_B^3 \otimes M \otimes B)\\ &\quad= \mu_M^3(M \otimes \sigma_M \otimes \tau_M)(\tau_M \otimes \mu_B \otimes M \otimes \mu_B \otimes B)\\ &\qquad\circ(1,4,2,5,7,3,6,8)(\psi_B^3 \otimes \psi_B^3 \otimes M \otimes B)\,.\end{aligned}$$

And thus, by 2.6 we see that τ_M determines the $\mathcal{M}$-extension. If, additionally, B has an antipode, then 3.4 shows that the induced $\mathcal{M}$-extension is a regular one.

Let τ'_M and τ''_M be two cocycles representing one and the same cohomology classes from $H^2(B, M)$, i.e. there exists $g \in \mathrm{Reg}(B, M)$, such that $\tau'_M - \tau''_M = d^1(g) = \sigma_M(B \otimes g) - g\mu^B + g(B \otimes \varepsilon_B)$. Let us put this condition in the following form:

$$\tau'_M + g\mu_B = g(B \otimes \varepsilon_B) + \sigma_M(B \otimes g) + \tau''_M\,.$$

Decomposing both sides of the last identity, we obtain

$$\begin{aligned}&\mu_M(M \otimes g)(\tau'_M \otimes \mu_B)(1,3,2,4)(\psi_B \otimes \psi_B)\\ &\quad= \mu_M^3(M \otimes \sigma_M \otimes M)(g \otimes B \otimes g \otimes \tau''_M)(1,2,4,3,5)(\psi_B^3 \otimes \psi_B)\,.\end{aligned}$$

Now we wish to check the assumption of 2.9. Using the last identity, the observation that in our case $\sigma''_M = \sigma'_M = \sigma_M$ and the facts that M is commutative and B is cocommutative, we obtain

$$\begin{aligned}
&\mu^3_M(M\otimes M\otimes g)(\sigma_M\otimes\tau'_M\otimes\mu_B)(1,4,2,5,3,6)(\psi^3_B\otimes M\otimes\psi_B)\\
&\quad=\mu_M(M\otimes\mu_M)(M\otimes M\otimes g)(M\otimes\tau'_M\otimes\mu_B)(1,2,4,3,5)\\
&\qquad\circ(M\otimes\psi_B\otimes\psi_B)(\sigma_M\otimes B\otimes B)(1,3,2,4)(\psi_B\otimes M\otimes B)\\
&\quad=\mu_M(M\otimes\mu^3_M)(M\otimes M\otimes\sigma_M\otimes M)(M\otimes g\otimes B\otimes g\otimes\tau''_M)\\
&\qquad\circ(1,2,3,5,4,6)(M\otimes\psi^3_B\otimes\psi_B)(\sigma_M\otimes B\otimes B)\\
&\qquad\circ(1,3,2,4)(\psi_B\otimes M\otimes B)\\
&\quad=\mu^4_M(M\otimes M\otimes\sigma_M\otimes M)(g\otimes\sigma_M\otimes B\otimes g\otimes\tau''_M)\\
&\qquad\circ(1,2,5,3,6,4,7)(\psi^4_B\otimes M\otimes\psi_B)\,.
\end{aligned}$$

Thus, 2.9 implies that $f=(\mu_M\otimes B)(M\otimes g\otimes B)(M\otimes\psi_B)$ is the morphism of $\mathcal{M}$-extensions $f: E_1\to E_2$ and (ii) from 3.3 implies that f is the isomorphism.

Thus we have proved the following:

3.7. Theorem. *Let B be a cocommutative Hopf monoid and M a commutative monoid.*

(i) *If B has an antipode, then any regular $\mathcal{M}$-extension E induces a left B-object commutative monoid structure (M,σ_M) on M, and the correspondence $E\mapsto\tau_M$ determines the bijection*

$$\mathcal{M}_{\mathrm{reg}}(B,M)\xrightarrow{\sim}H^2(B,M)\,,$$

between the set of equivalence classes of regular $\mathcal{M}$-extensions and the (*respectively by σ_M*) *second cohomology group from* 1.3.

(ii) *If* $\mathrm{A}(B^{\otimes n},M)=\mathrm{Reg}(B^{\otimes n},M)$, $n=1,2$, *then any $\mathcal{M}$-extension E induces a left B-object commutative monoid structure (M,σ_M) on M and the correspondence $E\mapsto\tau_M$ determines the bijection*

$$\mathcal{M}(B,M)\xrightarrow{\sim}H^2(B,M),$$

between the set of equivalence classes of $\mathcal{M}$-extensions and the (*respectively by σ_M*) *second cohomology group from* 1.3. □

We would like to mention that 3.7 generalizes Theorem 8.6 from [14] and is inspired by that one (see 4.3).

3.8. Remark. The dual considerations in the original category **A** (or the original considerations in the dual category $\mathbf{A}^{op}$) give the connection between $\mathscr{C}$-extensions and the second cohomology group from 1.4 in both situations (see 3.6).

3.9. Definition. A $\mathscr{H}$-extension $E=(A\xrightarrow{\alpha}C\xrightarrow{\beta}B,\ \lambda: C\tilde{\to}A\otimes B)$ is called a *regular* $\mathscr{H}$-extension, iff $(A\otimes\varepsilon_B)\lambda\in \mathrm{Reg}(C,A)$ and $\lambda^{-1}(\eta_A\otimes B)\in \mathrm{Reg}(B,C)$.

3.10. Theorem. *Let A be a commutative and B a cocommutative Hopf monoid.*

(i) *If A and B have antipodes, then any regular $\mathscr{H}$-extension E induces an abelian matched pair of Hopf monoids (B, A, σ_A, ρ_B) and the one-dimensional cocycle (τ_A, φ_B) of the (respectively by (σ_A, ρ_B)) normalized subbicomplex* (10) *from Section* 1; *the correspondence $E\mapsto(\tau_A, \varphi_B)$ determines the bijection*

$$\mathscr{H}_{\mathrm{reg}}(B,A)\tilde{\to}H_h^1(B,A),$$

between the set of equivalence classes of regular $\mathscr{H}$-extensions and the (respectively by (σ_A, ρ_B)) first cohomology group from 1.9.

(ii) *If* $\mathbf{A}(B^{\otimes m}, A^{\otimes n})=\mathrm{Reg}(B^{\otimes m}, A^{\otimes n})$, $m,n=1,2$, *then any $\mathscr{H}$-extension E induces an abelian matched pair of Hopf monoids (B, A, σ_A, ρ_B) and the one-dimensional cocycle (τ_A, φ_B) from* (10); *the correspondence $E\mapsto(\tau_A, \varphi_B)$ determines the bijection*

$$\mathscr{H}(B,A)\tilde{\to}H_h^1(B,A)$$

between the set of equivalence classes of $\mathscr{H}$-extensions and the (respectively by (σ_A, ρ_B)) first cohomology group from 1.9.

Proof. As we see, we have two different situations to discuss. Most of our argument is valid in both situations. If this is the case, we do not mention in which situation we are. But, if there is any difference, we explain what we are doing in each case.

Propositions 2.13 and 2.15, Theorem 2.14 and Remarks 3.6 and 3.8 insure us that the only things which are left to be proved, are the following:

(i) We have (iii)–(v) from 0.18 for the pair (σ_A, ρ_B) induced from E.

(ii) The pair (τ_A, φ_B) induced from E satisfies the condition $d_2^{1,0}(\tau_A)+d_1^{0,1}(\varphi_B)=0$ in the normalized bicomplex (11) from Section 1.

(iii) If $f: E_1\to E_2$ is a morphism of $\mathscr{H}$-extensions and $(\sigma'_A, \rho'_B, \tau'_A, \varphi'_B)$ and $(\sigma''_A, \rho''_B, \tau''_A, \varphi''_B)$ are systems of arrows induced by E_1 and E_2, respectively, then the pairs (σ'_A, ρ'_B) and (σ''_A, ρ''_B) coincide, the arrow f is the isomorphism and the pairs (τ'_A, φ'_B) and (τ''_A, φ''_B) represent the equal cohomology classes.

(iv) if (τ_A, φ_B) is a one-dimensional cocycle from (11) in Section 1, then we have the identity (iii) from 2.14.

(i) Now, part (iii) of Definition 0.18 is evident. Multiplying both sides of (iii) from 2.14 by $(B \otimes A \otimes \eta_B)$ on the right and by $(A \otimes \varepsilon_B \otimes A)$ on the left and using the fact that both these arrows are Hopf monoid morphisms, we see that

$$\begin{aligned}&\psi_A \sigma_A + 0 + \varphi_B(B \otimes \varepsilon_B) + 0 \\ &\quad = \bar{\sigma}_{A\otimes A}(B \otimes \psi_A) + 0 + \varphi_B(B \otimes \varepsilon_B) + 0 + 0 + 0 .\end{aligned}$$

The arrow φ_B is regular in both situations ((i) and (ii)), and then the composition $\varphi_B(B \otimes \varepsilon_B)$ is also regular. Thus, from the last identity we get $\psi_A \sigma_A = \bar{\sigma}_{A\otimes A}(B \otimes \psi_A)$ and this means exactly (iv) from 0.18.

Similarly we prove part (v) from 0.18.

(ii) Multiplying both sides of (iii) from 2.14 by $(B \otimes \eta_A \otimes B)$ on the right and by $(A \otimes \varepsilon_B \otimes A)$ on the left, we see that

$$\begin{aligned}0 + \psi_A \tau_A + \varphi_B \mu_B + 0 = 0 + \varphi_B(B \otimes \varepsilon_B) + \bar{\sigma}_{A\otimes A}(B \otimes \varphi_B) \\ + (\tau_A \otimes A)\underline{\rho}_{B\otimes B} + (\eta_A \otimes A)\tau_A . \qquad (27)\end{aligned}$$

Using the fact that $\tau_A \in \mathrm{Reg}(B \otimes B, A)$ and $\varphi_B \in \mathrm{Reg}(B, A \otimes A)$ in both situations ((i) and (ii)), we obtain from (27) that

$$\begin{aligned}&d_2^{1,0}(\tau_A) + d_1^{0,1}(\varphi_B) \\ &\quad = (\eta_A \otimes A)\tau_A + \psi_A \tau_A^* + (\tau_A \otimes A)\underline{\rho}_{B\otimes B} \\ &\qquad + \bar{\sigma}_{A\otimes A}(B \otimes \varphi_B) + \varphi_B^* \mu_B + \varphi_B(B \otimes \varepsilon_B) = 0 .\end{aligned}$$

(iii) Assume that $f: E_1 \to E_2$ is a morphism of $\mathscr{H}$-extensions. We can assume that f is the morphism of $\mathscr{M}$-extensions, which are induced from E_1 and E_2 (by the scheme of 2.13), respectively. Then 3.6 implies that $\sigma_A' = \sigma_A''$ and if $g = (A \otimes \varepsilon_B)\lambda_2 f \lambda_1^{-1}(n_A \otimes B)$, then g is regular. Thus f becomes an isomorphism and $\tau_A' - \tau_A'' = \sigma_A(B \otimes g) - g\mu_B + g(B \otimes \varepsilon_B)$.

Similarly, one can assume that f is the morphism of $\mathscr{C}$-extensions, which are induced from E_1 and E_2 (by the scheme of 2.13), and then the arguments dual to those from 3.6 imply that $\rho_B' = \rho_B''$ and $\varphi_B - \varphi_B' = (\eta_A \otimes A)g - \psi_A g + (g \otimes A)\rho_B$, for the same g. Thus, f is the isomorphism, pairs (σ_A', ρ_B') and (σ_A'', ρ_B'') coincide, and

$$(\tau_A', \varphi_B') - (\tau_A'', \varphi_B'') = d_1^{0,0}(g) + d_2^{0,0}(g) .$$

(iv) So we have that (B, A, σ_A, ρ_B) is a matched pair and (τ_A, φ_B) is a normalized one-dimensional cocycle from (11). From (iv) and (v) of 0.18 we obtain

$$(A \otimes \eta_B \otimes A)\psi_A \sigma_A (B \otimes A \otimes \varepsilon_B)$$
$$= (A \otimes \eta_B \otimes A)\bar{\sigma}_{A\otimes A}(B \otimes \psi_A)(B \otimes A \otimes \varepsilon_B)\,, \tag{28}$$
$$(\eta_A \otimes B \otimes A)\rho_B \mu_B (B \otimes \varepsilon_A \otimes B)$$
$$= (\eta_A \otimes B \otimes A)(\mu_B \otimes A)\underline{\rho}_{B\otimes B}(B \otimes \varepsilon_A \otimes B)\,. \tag{29}$$

From the condition $d_2^{1,0}(\tau_A) + d_1^{0,1}(\varphi_B) = 0$, we obtain

$$(A \otimes \eta_B \otimes A)(\varphi_B \mu_B + \psi_A \tau_A)(B \otimes \varepsilon_A \otimes B)$$
$$= (A \otimes \eta_B \otimes A)(\bar{\sigma}_{A\otimes A}(B \otimes \varphi_B) + \varphi_B(B \otimes \varepsilon_B)$$
$$+ (\eta_A \otimes A)\tau_A + (\tau_A \otimes A)\underline{\rho}_{B\otimes B})(B \otimes \varepsilon_B \otimes B)\,. \tag{30}$$

Now, taking the sum of (28), (29) and (30) and opening the parentheses in (30), we arrive at the identity (iii) from (2.14) □

We would like to mention that 3.10 generalizes Proposition (5.1) from [13] and is inspired by that one (see 4.6).

4. Examples

Examples, which we are going to give, are obtained by application of our considerations in the main part of this paper to different particular cases of monoidal categories, i.e. we make substitutions $\mathbf{A} = \mathbf{Set}$, $\mathbf{A} = K\text{-}\mathbf{mod}$, etc.

4.1. $\mathbf{A} = \mathbf{Set}$

Let $\mathbf{A}$ be the category of sets and mappings, $\otimes$ be the bifunctor of the direct product of two sets, $\times$, and e be the (terminal) one-point set $e = \{*\}$.

There is a unique way to equip each set B with a natural comonoid structure. The comultiplication $\psi_B : B \to B \times B$ must be the diagonal mapping, and the counit $\varepsilon_B : B \to e$ is the unique mapping to the one-point set. So, we identify the following three categories:

$$\mathscr{C}\mathscr{C}(\mathbf{Set}) = \mathscr{C}(\mathbf{Set}) = \mathbf{Set}\,.$$

After this, the list of structures from Section 0 (evidently) gives the well-known notions of set-monoid, group and modules (in the case of existence of antipodes), etc.

Let B be a set-monoid and M a left B-module. Then we can apply 1.3 and we get the cohomology groups $H^n(B, M)$ from [2], only there the set-monoid is called a semi-group. If additionally B has an antipode, then 1.3 gives the well-known cohomology of group B with coefficients in the left B-module M from [2, 6].

If B is a set-monoid and M is an abelian group, then (ii) of 3.7 gives the bijection between the set of equivalence classes of extensions of the set-monoid B by the abelian group M and the (respective) second cohomology group $H^2(B, M)$ from [9]. If, additionally, B has an antipode, then (i) of 3.7 coincides with (ii) of 3.7 and (taking into account 3.3) both give the well-known bijection between the set of equivalence classes of group extensions of the group B by the abelian group M and the (respective) second cohomology group from [2, 6]. Here we must explain, that if M and B have antipodes, then $M \times B$ (with the set-monoid structure described by σ_M and τ_M) also has an antipode, namely $-(m, b) = (-b^{-1}m - b^{-1}\tau_M(b, b^{-1}), b^{-1})$, structures on M and $M \times B$ are written additively and on B multiplicatively.

Let A be an abelian group and N be a set. There exists a unique right A-coobject comonoid structure on N, namely $\rho_N : N \to N \times A$, $\rho_N(n) = (n, 0)$. In this case we can apply 1.4 and we get the cohomology groups $H^m_C(N, A)$. But in this case the complex (3) from Section 1 has a contracting homotopy and we have $H^0_C(N, A) = \mathbf{Set}(N, A)$ and $H^m_C(N, A) = 0$, $m \geq 1$.

Let B be a group and A a left B-module. These data can be considered as the assumptions of 1.9 and then we obtain the cohomology groups $H^n_h(B, A)$. In such case all vertical complexes $(\mathbb{C}^{m,*}, d_2^{m,*})$, $m \geq 0$, of the bicomplex (10) from Section 1 have the contracting homotopies and we get $H^n_h(B, A) = H^{n+1}(B, A)$, $n \geq 1$, while in zero dimension we have the epimorphism

$$H^0_h(B, A) = \mathbf{Set}(B, A) \to H^1(B, A)\,,$$

where $H^n(B, A)$ are the usual group cohomologies.

4.2. $\otimes$ = *direct product*

Let $\mathbf{A}$ be a category with finite direct products (and the terminal object), $\otimes$ the bifunctor of the direct product (in $\mathbf{A}$) and e the terminal object of $\mathbf{A}$.

In this case arguments similar to those in Section 4.1 are true, and we identify $\mathscr{C}\mathscr{C}(\mathbf{A}) = \mathscr{C}(\mathbf{A}) = \mathbf{A}$. The data, B is a monoid (in $\mathbf{A}$) and M is a left B-object commutative monoid with an antipode S_M, can be considered as the assumptions of 1.3 and we get $H^n(B, M)$ of [11, Proposition 3.2]. In [11] the groups $H^n(B, M)$ are called the cohomology of the semi-group B with the coefficients in the left unitary B-module M. Then 3.7 (ii) makes it possible to describe $H^2(B, M)$ by extensions.

4.3. $\mathbf{A} = K$-mod

Let K be a commutative ring (with unit) and $\mathbf{A}$ the category of K-modules, $\otimes$ the tensor product over K and e equal to K.

In this case 1.3 gives the cohomology groups $H^n(B, M)$ of a cocommutative Hopf algebra B with the coefficients in the left B-module commutative algebra M from [14, p. 208], only there K is assumed to be a field.

A regular $\mathcal{M}$-extension $E = (M \xrightarrow{\alpha} C \xrightarrow{\rho} C \otimes B\ ,\ \lambda : C \to M \otimes B)$ from 3.2 coincides with a cleft extension in the sense of [14, p. 229]. Then 3.7(i) gives the bijection of [14, Theorem 8.6] between the cleft extensions and $H^2(B, M)$.

4.4. $\mathbf{A} = (K\text{-mod})^{op}$

Let $\mathbf{A}$ be dual to the category of K-modules and a new product $X \otimes Y$ (in $\mathbf{A}$) be the old $Y \otimes X$ one (see 0.17).

In this case 1.3 (or, equivalently, 1.4 in the case of $\mathbf{A} = K$-mod) gives the cohomology groups $H^n(B, M)$, where M is a commutative Hopf algebra and B is a right M-comodule cocommutative coalgebra. These groups coincide with the groups Coalg-$H^n(B, M)$ from [3, p. 684], only there M coacts on B from the left side.

A regular $\mathcal{M}$-extension (or, equivalently, a regular $\mathcal{C}$-extension in the case of $\mathbf{A} = K$-mod) coincides with the cleft coalgebra extension from [3, p. 694]. Then 3.7(i) gives the bijection from [3, Theorem 5.5] between the cleft coalgebra extensions and Coalg-$H^2(B, M)$.

4.5. $\mathbf{A} = (K\text{-alg})^{op}$

Let K-alg denote the category of K-algebras, $\otimes$ the tensor product over K, and if A and B are K-algebras, then $A \otimes B$ is a K-algebra with multiplication $(\mu_A \otimes \mu_B)(1, 3, 2, 4)$ and unit $\eta_A \otimes \eta_B$. Thus we obtain the monoidal category $(K\text{-alg}, \otimes, K)$.

Let us put $\mathbf{A} = (K\text{-alg})^{op}$, dual to the above-described category.

In this case 1.3 (or, what is equivalent, 1.4 in the case of $\mathbf{A} = K$-alg) gives the cohomology groups $H^n(B, M)$, where M is a commutative Hopf algebra and B is a right M-comodule cocommutative Hopf algebra [3, p. 683]. These groups coincide with the groups Hopf-$H^n(B, M)$ from [3, p. 684] only there M coacts on B from the left side.

A regular $\mathcal{M}$-extension (or what is equivalent, a regular $\mathcal{C}$-extension in the case $\mathbf{A} = K$-alg) coincides with a cleft Hopf extension [3, p. 694]. Then, 3.7(i) gives the bijection from [3, Theorem 5.6] between the cleft Hopf extensions and Hopf-$H^2(B, M)$.

4.6. $\mathbf{A} =$ G.C. K-mod

Let $\mathbf{A}$ be the category of the graded connected K-modules [8, 13].

In this case 1.9 gives the cohomology groups $H^n_h(B, A)$ of an abelian matched pair of graded connected Hopf algebras. In [13] the cohomology groups of the same pair are constructed and we denote them by Singer-$H^n(B, A)$. There we have the following identities:

$$\text{Singer-}H^n(B, A) = \begin{cases} 0, & n = 0, 1, \\ H_h^{n-2}(B, A), & n \geq 2. \end{cases}$$

A regular $\mathscr{H}$-extension coincides with a $\mathscr{H}$-extension and coincides with an extension of the graded connected Hopf algebras from [13]. Then, 3.10(i) coincides with 3.10(ii) and both give the bijection from [13, Proposition 5.1] between the Hopf extensions and $H_h^1(B, A) = \text{Singer-}H^3(B, A)$.

The particular case of $\mathscr{H}$-extensions in this category, with trivial action and coaction has been considered in [5].

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