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Wielandt type theorem for Cartesian product of digraphs ☆

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Abstract

We show that $mn - 1$ is an upper bound of the exponent of the Cartesian product $D \times E$ of two digraphs D and E on m, n vertices, respectively and we prove our upper bound is extremal when $(m, n) = 1$. We also find all D and E when the exponent of $D \times E$ is $mn - 1$. In addition, when $m = n$, we prove that the extremal upper bound of $\exp(D \times E)$ is $n^2 - n + 1$ and only the Cartesian product, $Z_n \times W_n$, of the directed cycle and Wielandt digraph has exponent equals to this bound.

Keywords: Exponent; Cartesian product; Digraphs

1. Introduction

Let $D = (V, A)$ be a digraph on n vertices. Throughout this paper, we assume that D has no loops and multiple arcs. For each pair (u, v) of vertices on D, we define a $u \to v$ *walk*, or a walk from u to v, in D by a sequence of vertices on D, $u = u_0, u_1, \ldots, u_p = v$, and a sequence of arcs of D, $(u, u_1), (u_1, u_2), \ldots, (u_{p-1}, v)$ where the vertices (arcs) are not necessarily distinct. The *length* of a $u \to v$ walk is the length of the sequence of arcs within it. A digraph $D = (V, A)$ is *primitive* if there is a positive integer, k, such that for any given pair of vertices, u, v, there is a $u \to v$ walk of length k. We say that the smallest such value of k is the *exponent* of D, denoted

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by $\exp(D)$. The exponent of a primitive digraph D is equal to the minimal k such that all entries of A^k is positive for the adjacency matrix A of D. See book [2] for more details. Wielandt [10] found that the maximum exponent of a primitive digraph on n vertices is $n^2 - 2n + 2$. Motivated by Wielandt, many results have been appeared on the upper and lower bounds for the exponents of digraphs [3–6,8,9,11].

Let $D = (V_D, A_D)$ and $E = (V_E, A_E)$ be digraphs such that $|V_D| = n$, $|V_E| = m$. The Cartesian product $D \times E = (V, A)$ of D and E is defined as $V = V_D \times V_E$ and $A = \{((u_1, v_1), (u_2, v_2))|((u_1, u_2) \in A_D \text{ and } v_1 = v_2) \text{ or } (u_1 = u_2 \text{ and } (v_1, v_2) \in A_E)\}$. Lamprey and Barnes [7] showed

$$\exp(D \times E) \leqslant (n + m)^2 - 4(n + m) + 5.$$

We improve this upper bound more sharply. That is,

$$\exp(D \times E) \leqslant mn - 1. \tag{1}$$

We also show that the upper bound (1) is extremal when $(m, n) = 1$. And we characterize all digraphs D and E which satisfy (1). Moreover, we prove that

$$\exp(D \times E) \leqslant n^2 - n + 1$$

when D and E are digraphs on n vertices.

2. Some lemmas

From now on, D and E are digraphs on n, m vertices ($3 \leqslant n \leqslant m$), respectively. And we assume that $D \times E$ is primitive. We use the notation $u_0 \xrightarrow{k_1} u_1 \xrightarrow{k_2} u_2 \xrightarrow{k_3} \cdots \xrightarrow{k_{p-1}} u_{p-1} \xrightarrow{k_p} u_p$ when there are $u_{i-1} \to u_i$ walks of length k_i for all $i = 1, 2, \ldots, p$. If $k_1 = \cdots = k_p = 1$, we write $u_0 \to u_1 \to \cdots \to u_p$.

Definition 1. (1) The directed n-cycle $Z_n = (V, A)$ is defined by $V = \{z_0, z_1, \ldots, z_{n-1}\}$ and $A = \{(z_i, z_j)|j \equiv i + 1 \pmod n\}$.

(2) For $m > n$, define $Z_{m,n} = (V_1, A_1)$ by $V_1 = \{z_0, z_1, \ldots, z_{m-1}\}$ and $A_1 = \{(z_i, z_j)|j \equiv i + 1 \pmod m$ or $i > j,\ i \equiv j - 1 \pmod n\}$.

Definition 2. The Wielandt digraph $W_n = (V, A)$ on n vertices is defined by $V = \{\omega_0, \omega_1, \ldots, \omega_{n-1}\}$ and $A = \{(\omega_i, \omega_{i+1})|0 \leqslant i \leqslant n - 2\} \cup \{(\omega_{n-1}, \omega_0), (\omega_{n-1}, \omega_1)\}$.

The Frobenius number $g(l_1, l_2, \ldots, l_t)$ of the relatively prime positive numbers $l_1, \ldots, l_t$ is the largest number G such that the equation $l_1x_1 + \cdots + l_tx_t = G$ is not solvable for nonnegative integers $x_1, \ldots, x_t$. If H is a subgraph of a digraph F, we write $H \prec F$.

Lemma 1. *Let C_i $(i = 1, \ldots, t)$ be the directed l_i-cycles in $D \cup E$. Let k be the number of C_i such that $C_i \prec D$ and $(l_1, l_2, \ldots, l_t) = 1$. Then we have*

$$\exp(D \times E) \leqslant g(l_1, l_2, \ldots, l_t) - l_1 - \cdots - l_t + (k + 1)n + (t - k + 1)m - 1.$$

Proof. Assume $C_1, C_2, \ldots, C_k \prec D$. Let $(u, v), (z, w) \in D \times E$. Since $D \times E$ is primitive, D and E are strongly connected. So there is a shortest path from u to a vertex of C_1. Thus there is a vertex u_1 of C_1 such that $u = u_0 \xrightarrow{s_1} u_1$ for some $s_1 \leqslant n - l_1$. Similarly, $u = u_0 \xrightarrow{s_1} u_1 \xrightarrow{s_2} u_2 \xrightarrow{s_3}$

$\cdots \xrightarrow{s_k} u_k \xrightarrow{r_1} z$ and $v = v_0 \xrightarrow{s_{k+1}} v_1 \xrightarrow{s_{k+2}} v_2 \xrightarrow{s_{k+3}} \cdots \xrightarrow{s_t} v_{t-k} \xrightarrow{r_2} w$ where u_i is a vertex of C_i and $s_i \leqslant n - l_i$, v_i is a vertex of C_{k+i} and $s_{k+i} \leqslant m - l_{k+i}$, $r_1 \leqslant n-1$ and $r_2 \leqslant m-1$. If $\alpha > g(l_1, l_2, \ldots, l_t) - l_1 - \cdots - l_t + (k+1)n + (t-k+1)m - 1$, then

$$\begin{aligned}&\alpha - s_1 - \cdots - s_t - r_1 - r_2\\ &\quad \geqslant \alpha - (n-l_1) - \cdots - (n-l_k) - (m-l_{k+1}) - \cdots - (m-l_t)\\ &\quad -(n-1)-(m-1)\\ &\quad = \alpha + l_1 + \cdots + l_t - (k+1)n - (t-k+1)m + 2\\ &\quad > g(l_1, l_2, \ldots, l_t).\end{aligned}$$

So there are nonnegative integers $x_1, \ldots x_t$ such that $l_1x_1 + \cdots + l_tx_t = \alpha - s_1 - \cdots - s_t - r_1 - r_2$. Then $(u, v) = (u_0, v_0) \xrightarrow{s_1} (u_1, v_0) \xrightarrow{l_1x_1} (u_1, v_0) \xrightarrow{s_2} \cdots \xrightarrow{s_k} (u_k, v_0) \xrightarrow{l_kx_k} (u_k, v_0) \xrightarrow{s_{k+1}} (u_k, v_1) \to l_{k+1}x_{k+1}(u_k, v_1) \xrightarrow{s_{k+2}} \cdots \xrightarrow{s_t} (u_k, v_{t-k}) \xrightarrow{l_tx_t} (u_k, v_{t-k}) \xrightarrow{r_1} (z, v_{t-k}) \xrightarrow{r_2} (z, w)$. Since $s_1 + l_1x_1 + s_2 + l_2x_2 + \cdots + s_t + l_tx_t + r_1 + r_2 = \alpha$, $(u, v) \xrightarrow{\alpha} (z, w)$. Thus

$$\exp(D \times E) \leqslant g(l_1, l_2, \ldots, l_t) - l_1 - \cdots - l_t + (k+1)n + (t-k+1)m - 1. \qquad \square$$

Lemma 2. *If* $Z_m \prec E \prec Z_{m,n}$, *then* $\exp(Z_n \times E) \geqslant mn - 1$.

Proof. Suppose $(z_0, z_0) \xrightarrow{mn-2} (z_{n-1}, z_{m-1})$. Then, $(z_0, z_0) = (u_0, v_0) \longrightarrow (u_1, v_1) \longrightarrow \cdots \longrightarrow (u_{mn-2}, v_{mn-2}) = (z_{n-1}, z_{m-1})$ for some vertices (u_i, v_i) of $Z_n \times E$. Let

$$A = \{i \mid 1 \leqslant i \leqslant mn-2, u_{i-1} \neq u_i\} \quad \text{and} \quad B = \{i \mid 1 \leqslant i \leqslant mn-2, v_{i-1} \neq v_i\}.$$

Then,

$$A \cup B = \{i \mid 1 \leqslant i \leqslant mn-2\} \quad \text{and} \quad A \cap B = \phi.$$

If $A = \{i_1 < i_2 < \cdots < i_s\}$ and $B = \{j_1 < j_2 < \cdots < j_t\}$, $s + t = mn - 2$. Let $i_0 = j_0 = 0$. Then, for each $0 \leqslant h \leqslant s-1$, $u_{i_h} = u_{i_h+1} = \cdots = u_{i_{h+1}-1} \to u_{i_{h+1}}$. Thus, $z_0 = u_{i_0} \to u_{i_1} \to \cdots \to u_{i_s} = u_{i_s+1} = \cdots = u_{mn-2} = z_{n-1}$. Similarly, $z_0 = v_{j_0} \to v_{j_1} \to \cdots \to v_{j_t} = z_{m-1}$. For any h $(0 \leqslant h \leqslant s)$, $u_{i_h} = t$ for some $t \equiv h \pmod n$. Since $z_{n-1} = u_{i_s}$, $s = n - 1 + nx$ for some nonnegative integer x. If $z_0 = v_{j_p} \to v_{j_{p+1}} \to \cdots \to v_{j_q} = z_0$ and $v_{j_h} \neq z_0$ for all $p < h < q$, we can show that if $v_{j_h} = z_t$, $h - p \geqslant t$ and $h - p \equiv t \pmod n$. Since $v_{j_{q-1}} \to v_{j_q} = z_0$, $v_{j_{q-1}} = z_{m-1}$. If $h = q-1$, then $q - p - 1 \geqslant m - 1$, $v_{j_{q-1}} = z_{m-1}$ and $m - 1 \equiv q - p - 1 \pmod n$. Thus, $q - p = m + ny$ for some nonnegative integer y. If $e = |\{h \mid 1 \leqslant h \leqslant t, v_{j_h} = z_0\}|$, since $v_t = z_{m-1}$, $t = em + nz + m - 1$ for some nonnegative integer z. From $mn - 2 = s + t = em + n(x+z) + n - 1 + m - 1$, we have $mn - n - m = em + (x+z)n$, which is impossible. Thus, $\exp(Z_n \times E) \geqslant mn - 1$. $\square$

Lemma 3 [1]. *If* $(l_1, l_2, \ldots, l_t) = 1$, *then*

$$g(l_1, l_2, \ldots, l_t) \leqslant l_2\frac{d_1}{d_2} + l_3\frac{d_2}{d_3} + \cdots + l_t\frac{d_{t-1}}{d_t} - l_1 - l_2 - \cdots - l_t,$$

where $d_1 = l_1$, $d_i = (l_1, \ldots, l_i)$.

Lemma 4. *If* $a_1, \ldots, a_t \geqslant 2$, *then*

(1) $a_1 + \cdots + a_t \leqslant a_1 \cdots a_t$,
(2) $a_1 + \cdots + a_t \leqslant \frac{a_1 \cdots a_t}{2} + 2$.

Proof. This can be shown easily. We omit the proof. □

3. Main theorems

Theorem 1. *Let D and E be digraphs on n, m vertices $(n \leqslant m)$, respectively. Let $D \times E$ be primitive. Then*

$$\exp(D \times E) \leqslant mn - 1.$$

Proof. Let C_1 be a directed l_1-cycle of D such that l_1 is the smallest among the lengths of all directed cycles of D. Let $d_1 = l_1$ and construct $C_2, C_3, \ldots, C_k$ in D such that C_i is a directed cycle of D whose length l_i is the smallest among the lengths of all directed cycles of D which are not multiples of $d_{i-1} = (l_1, \ldots, l_{i-1})$ for $i \geqslant 2$ and $d_k = (l_1, \ldots, l_k)$ is the greatest common divisor of lengths of all directed cycles of D. Similarly, construct $C_{k+1}, C_{k+2}, \ldots, C_t$ in E such that for $1 \leqslant i \leqslant t-k$, C_{k+i} is a directed cycle of E whose length l_{k+i} is the smallest among the lengths of all directed cycles of E which are not multiples of $d_{k+i-1} = (l_1, \ldots, l_{k+i-1})$. Since $D \times E$ is primitive, $d_t = 1$. Then, $\frac{d_1}{d_2}, \frac{d_2}{d_3}, \ldots, \frac{d_{t-1}}{d_t} \geqslant 2$. By Lemmas 1 and 3,

$$\begin{aligned}
\exp(D \times E) &\leqslant g(l_1, l_2, \ldots, l_t) - l_1 - \cdots - l_t + (k+1)n + (t-k+1)m - 1 \\
&\leqslant l_2 \frac{d_1}{d_2} + l_3 \frac{d_2}{d_3} + \cdots + l_t \frac{d_{t-1}}{d_t} - 2l_1 - 2l_2 - \cdots \\
&\quad -2l_t + (k+1)n + (t-k+1)m - 1 \\
&\leqslant l_2 \left(\frac{d_1}{d_2} - 2\right) + l_3 \left(\frac{d_2}{d_3} - 2\right) + \cdots + l_t \left(\frac{d_{t-1}}{d_t} - 2\right) \\
&\quad -2l_1 + (k+1)n + (t-k+1)m - 1.
\end{aligned}$$

Since $n \leqslant m$, $l_i \leqslant m$ for all i. Thus,

$$\begin{aligned}
\exp(D \times E) &\leqslant m \left(\frac{d_1}{d_2} - 2\right) + m \left(\frac{d_2}{d_3} - 2\right) + \cdots + m \left(\frac{d_{t-1}}{d_t} - 2\right) \\
&\quad -2d_1 + (k+1)n + (t-k+1)m - 1 \\
&= m \left(\frac{d_1}{d_2} + \cdots + \frac{d_{t-1}}{d_t}\right) - 2d_1 + (k+1)n - (t+k-3)m - 1.
\end{aligned}$$

By Lemma 4(1), we have

$$\begin{aligned}
\exp(D \times E) &\leqslant md_1 - 2d_1 + (k+1)n - (t+k-3)m - 1 \\
&\leqslant (m-2)n + (k+1)n - (t+k-3)m - 1 \\
&= mn + (k-1)n - (t+k-3)m - 1 \\
&\leqslant mn + (k-1)m - (t+k-3)m - 1 \\
&= mn - (t-2)m - 1 \\
&\leqslant mn - 1. \qquad \square
\end{aligned}$$

Corollary 1. *If $Z_m \prec E \prec Z_{m,n}$, then $\exp(Z_n \times E) = mn - 1$.*

Proof. This follows from Lemma 2 and Theorem 1. □

Lemma 5. *If* $t \geqslant 3$, *we have*

$$\exp(D \times E) \leqslant \frac{mn}{2} + m - 1.$$

Proof. Let $C_1, \ldots, C_t$ and $l_1, \ldots, l_t$ be the same notations as in the proof of Theorem 1. From the proof of Theorem 1 and Lemma 4(2), we have

$$\begin{aligned}\exp(D \times E) &\leqslant m\left(\frac{d_1}{d_2} + \cdots + \frac{d_{t-1}}{d_t}\right) - 2l_1 + (k+1)n - (t+k-3)m - 1\\ &\leqslant m\left(\frac{d_1}{2} + 2\right) - 2d_1 + 2n + (k-1)n - (t+k-3)m - 1\\ &\leqslant d_1\left(\frac{m}{2} - 2\right) + 2n - (t-4)m - 1\\ &\leqslant \frac{mn}{2} + (4-t)m - 1\\ &\leqslant \frac{mn}{2} + m - 1. \qquad \square\end{aligned}$$

Theorem 2. *Let* D *and* E *be digraphs on* n, m *vertices* $(n \leqslant m)$, *respectively. Let* $D \times E$ *be primitive. Then*

$$\exp(D \times E) = mn - 1 \textit{ if and only if } (m, n) = 1,\ D \cong Z_n \quad \textit{and} \quad Z_m \prec E \prec Z_{m,n}.$$

Proof. Let $C_1, \ldots, C_t$ and $l_1, \ldots, l_t$ be the same notations as in the proof of Theorem 1. If $\exp(D \times E) = mn - 1$ and $t \geqslant 3$, $mn - 1 \leqslant \frac{mn}{2} + n - 1$ by Lemma 5, which is impossible. So $t = 2$. Thus

$$\begin{aligned}mn - 1 &= \exp(D \times E)\\ &\leqslant g(l_1, l_2) - l_1 - l_2 + 2n + 2m - 1\\ &= l_1 l_2 - 2l_1 - 2l_2 + 2n + 2m - 1.\end{aligned}$$

Therefore,

$$\begin{aligned}(m-2)(n-2) &= mn - 2m - 2n + 4\\ &\leqslant l_1 l_2 - 2l_1 - 2l_2 + 4\\ &= (l_1 - 2)(l_2 - 2)\\ &\leqslant (m-2)(n-2).\end{aligned}$$

This implies $l_1 = n$ and $l_2 = m$. Since $C_1 : u_0 \to u_1 \to \cdots \to u_{n-1} \to v_0$ is a directed cycle of D, $V_D = \{u_0, \ldots, u_{n-1}\}$. If $u_i \to u_j$ and $j > i + 1$, $u_i \to u_j \to u_{j+1} \to \cdots \to u_{n-1} \to u_0 \to \cdots \to u_i$ is a cycle of D with length $n + i - j + 1 \leqslant n - 1$. This gives rise to a contradiction. If $u_i \to u_j$, $i > j$ and $(i, j) \neq (n-1, 0)$, $u_i \to u_{i+1} \to \cdots \to u_j \to u_i$ is a cycle of D with length $j - i + 1 \leqslant n - 1$. This gives rise to a contradiction. Thus $D = C_1 \simeq Z_n$. Let $C_2 : v_0 \to v_1 \to \cdots \to v_{m-1} \to v_0$ be a directed cycle of E. Then $V_E = \{v_0, \ldots, v_{m-1}\}$. Since $\exp(D \times E) = mn - 1$, there are $(u, v), (z, w) \in D \times E$ such that $(u, v) \overset{mn-2}{\nrightarrow} (z, w)$. We may assume that $v = v_0$. If $u \overset{r}{\longrightarrow} z$ and $r \leqslant n - 2$ or $v \overset{s}{\longrightarrow} w$ and $s \leqslant m - 2$, since $r + s \leqslant n + m - 3$,

$$\begin{aligned} mn-2-r-s &\geqslant mn-m-n+1 \\ &> mn-m-n=g(m,n). \end{aligned}$$

So there are x, y such that $nx+my=mn-2-r-s$. Then, $(u,v)\xrightarrow{nx}(u,v)\xrightarrow{my}(u,v)\xrightarrow{r}(z,v)\xrightarrow{s}(z,w)$. Thus $(u,v)\xrightarrow{mn-2}(z,w)$. This gives rise to a contradiction. Therefore $w=v_{m-1}$. If $v_i\to v_j$ for some $j>i+1$, $v_0\to v_1\to\cdots\to v_i\to v_j\to v_{j+1}\to\cdots\to v_{m-1}$. So $v_0\xrightarrow{m+i-j}v_{m-1}$ and $m+i-j\leqslant m-2$. This gives rise to a contradiction. If $v_i\to v_j$ for some $i>j$ and $(i,j)\neq(n-1,0)$, $v_i\to v_j\to v_{j+1}\to\cdots\to v_i$ is a directed cycle of E with length $i-j+1$. Since the length of a directed cycle in E is a multiple of n or equals to m, $n|i-j+1$ or $i-j+1=m$. If $i-j+1=m$, $i=m-1$ and $j=0$. Thus,

$$A_{C_2}\subset A_E\subset A_{C_2}\cup\{(v_i,v_j)|i>j \text{ and } n|i-j+1\}.$$

Therefore,

$$Z_n\prec E\prec Z_{m,n}. \qquad \square$$

Lemma 6. $\exp(Z_n\times W_n)\geqslant n^2-n+1$.

Proof. Suppose $(z_0,\omega_0)\xrightarrow{n^2-n}(z_{n-1},\omega_0)$. By a similar method used in Lemma 2, there are α and β such that $z_0\xrightarrow{\alpha}z_{n-1}$ and $\omega_0\xrightarrow{\beta}\omega_0$ where $\alpha+\beta=n^2-n$. In the proof of Lemma 2, it is proved that $\alpha\equiv n-1 \pmod n$. So $\alpha=nx+n-1$ for some nonnegative integer x. Since $\omega_0\xrightarrow{\beta}\omega_0$, let $\omega_0=v_0\to v_1\cdots\to v_\beta=\omega_0$. Let $i_1<i_2<\cdots<i_k$ be all i such that $0\leqslant i\leqslant\beta$ and $v_i=\omega_{n-1}$. Since $v_0=\omega_0$, $v_1=\omega_1$. We can prove $v_t=\omega_t$ for all $1\leqslant t<i_1$ by induction. Since $v_{i_1-1}=\omega_{n-2}$, $i_1=n-1$. For all $1\leqslant s\leqslant k-1$, since $v_{i_s}=\omega_{n-1}$, v_{i_s+1} is ω_0 or ω_1. If $v_{i_s+1}=\omega_0$, $i_{s+1}=i_s+n$ and if $v_{i_s+1}=\omega_1$, $i_{s+1}=i_s+n-1$. Since $v_{i_k+1}=\omega_0=v_\beta$, $i_k=\beta-1$. Since $i_{s+1}-i_s$ is $n-1$ or n,

$$\beta=n-1+\sum_{s=1}^{k-1}(i_{s+1}-i_s)+1=n+ny+(n-1)z$$

for some nonnegative integers y and z. We have $n^2-3n+1=n^2-n+(-2n+1)=\alpha+\beta+(-2n+1)=n(x+y)+(n-1)z$. So $(n-1)z\equiv 1 \pmod n$. Thus $z\equiv -1 \pmod n$. Therefore $z\geqslant n-1$. Then $(n-1)^2\leqslant(n-1)z\leqslant n^2-3n+1<(n-1)^2$. This is a contradiction. Thus $\exp(Z_n\times W_n)\geqslant n^2-n+1$. $\square$

Theorem 3. *Let D, E be digraphs on n vertices and $D\times E$ be primitive. Then,*

$$\exp(D\times E)\leqslant n^2-n+1.$$

Moreover, $\exp(D\times E)=n^2-n+1$ *if and only if $D\times E$ is isomorphic to $Z_n\times W_n$.*

Proof. Let $C_1,\ldots,C_t$ and $l_1,\ldots,l_t$ be the same notations as in the proof of Theorem 1. Suppose $\exp(D\times E)>n^2-n+1$. If $t\geqslant 3$, from Lemma 5,

$$\exp(D\times E)\leqslant\frac{n^2}{2}+n-1<n^2-n+1.$$

This is a contradiction. So $t=2$. We may assume that $l_1<l_2$. If $l_1\leqslant n-2$, from Lemma 1,

$$\begin{aligned}\exp(D \times E) &\leqslant g(l_1, l_2) - l_1 - l_2 + 4n - 1\\ &= l_1 l_2 - 2l_1 - 2l_2 + 4n - 1 = (l_1 - 2)(l_2 - 2) + 4n - 5\\ &\leqslant (n-4)(n-2) + 4n - 5 = n^2 - 2n + 3 < n^2 - n + 1.\end{aligned}$$

This is a contradiction. So $l_1 = n - 1$ and $l_2 = n$. From Lemma 1, $\exp(D \times E) \leqslant n^2 - n + 1$. So $\exp(D \times E) = n^2 - n + 1$. Thus there are vertices (x_1, y_1) and (x_2, y_2) on $D \times E$ such that $(x_1, y_1) \overset{n^2-n}{\nrightarrow} (x_2, y_2)$. Assume C_1 is a directed cycle $u_0 \to u_1 \to \cdots \to u_{n-2} \to u_0$ in D and C_2 is a directed cycle $v_0 \to v_1 \to \cdots \to v_{n-1} \to v_0$. Since D and E are strongly connected, without loss of generality, we may assume that E has a directed cycle of length n. Let w be a vertex of D different from $u_0, u_1, \ldots, u_{n-2}$. Since D is strongly connected, there are $0 \leqslant i, j \leqslant n-2$ such that $u_i \to w \to u_j$. We may assume $i = 0$. If $j \geqslant 3$, $w \to u_j \to u_{j+1} \to \cdots u_0 \to w$ is a directed cycle of length $n - i + 1$ ($\leqslant n - 2$). This is a contradiction. If $j = 0$, $w \to u_0 \to w$ is a cycle of length 2. This is a contradiction. If $j = 2$, $v \overset{n-1}{\longrightarrow} v$ for any vertex v of D. There are integers s_1 and s_2 such that $x_1 \overset{s_1}{\longrightarrow} x_2$, $y_1 \overset{s_2}{\longrightarrow} y_2$ and $0 \leqslant s_1, s_2 \leqslant n-1$. Since

$$\begin{aligned}n^2 - n - s_1 - s_2 &\geqslant n^2 - n - (n-1) - (n-1)\\ &= n^2 - 3n + 2 > (n-1)(n-2) - 1 = g(n-1, n),\end{aligned}$$

there are nonnegative integers p and q such that $(n-1)p + nq = n^2 - n - s_1 - s_2$. Since

$$(x_1, y_1) \overset{(n-1)p}{\longrightarrow} (x_1, y_1) \overset{nq}{\longrightarrow} (x_1, y_1) \overset{s_1}{\longrightarrow} (x_2, y_1) \overset{s_2}{\longrightarrow} (x_2, y_2)$$

and $p(n-1) + qn + s_1 + s_2 = n^2 - n$, $(x_1, y_1) \overset{n^2-n}{\longrightarrow} (x_2, y_2)$. This is a contradiction. So $j = 1$ and there is no other arc starting from w. Similarly there is no arc ending at w except (u_0, w). If F is the union of C_1 and a path $u_0 \to w \to u_0$, F is a subgraph of D and isomorphic to Wielandt digraph W_n on n-vertices.

If $D \neq F$, there is an arc (u_i, u_j) of D which is not an arc of F. If $j > i$, $j \geqslant i + 2$. So there is a directed cycle $u_i \to u_j \to u_{j+1} \to \cdots \to u_{n-1} \to u_0 \to u_1 \to \cdots \to u_i$ of length $n - j + i \leqslant n - 2$. This is a contradiction. If $j < i$, there is a directed cycle $u_i \to u_j \to u_{j+1} \to \cdots \to u_i$ of length $i - j + 1$. If $i - j + 1 \geqslant n - 1$, $i = n - 1$ and $j = 0$. But this is an arc of F. This is a contradiction. So $D = F$.

If $E \neq C_2$, there is an arc (v_i, v_j) of E which is not an arc of C_2. We may assume $i = 0$. If $j \geqslant 3$, there is a directed cycle $v_0 \to v_j \to v_{j+1} \to \cdots \to v_{n-1} \to v_0$ of length $n - j + 1 \leqslant n - 2$. This is a contradiction. If $j = 2$, there are integers r_1, r_2 such that $x_1 \overset{r_1}{\longrightarrow} x_2$, $y_1 \overset{r_2}{\longrightarrow} y_2$ and $0 \leqslant r_1, r_2 \leqslant n - 1$. Since

$$\begin{aligned}n^2 - n - r_1 - r_2 &\geqslant n^2 - n - (n-1) - (n-1)\\ &= n^2 - 3n + 2 > (n-1)(n-2) - 1 = g(n-1, n),\end{aligned}$$

there are nonnegative integers p and q such that $(n-1)p + nq = n^2 - n - r_1 - r_2$. Note that $y_1 \overset{n}{\longrightarrow} y_1$ and $y_1 \overset{n-1}{\longrightarrow} y_1$ except $y_1 = v_1$. So there is a vertex y' of E such that $y_1 \overset{h}{\longrightarrow} y' \overset{n-h}{\longrightarrow} y_1$ and $y' \overset{n-1}{\longrightarrow} y'$. If $q = 0$, since

$$(x_1, y_1) \overset{(n-1)p}{\longrightarrow} (x_1, y_1) \overset{s_1}{\longrightarrow} (x_2, y_1) \overset{s_2}{\longrightarrow} (x_2, y_2)$$

and $p(n-1) + s_1 + s_2 = n^2 - n$, $(x_1, y_1) \overset{n^2-n}{\longrightarrow} (x_2, y_2)$. This is a contradiction. If $q \geqslant 1$, since

$$(x_1, y_1) \stackrel{(n-1)p}{\longrightarrow} (x_1, y_1) \stackrel{h}{\longrightarrow} (x_1, y') \stackrel{n(q-1)}{\longrightarrow} (x_1, y') \stackrel{n-h}{\longrightarrow} (x_1, y_1) \stackrel{r_1}{\longrightarrow} (x_2, y_1) \stackrel{r_2}{\longrightarrow} (x_2, y_2)$$

and $p(n-1) + h + n(q-1) + n - h + r_1 + r_2 = n^2 - n$, $(x_1, y_1) \stackrel{n^2-n}{\longrightarrow} (x_2, y_2)$. This is a contradiction. So $E = C_2$. Thus $D \times E$ is isomorphic to $W_n \times Z_n$. From Lemma 6, $\exp(D \times E) = n^2 - n + 1$. □

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