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Generalized inversion of finite rank Hankel and Toeplitz operators with rational matrix symbols

Victor M. Adukov [1]

Department of Mathematical Analysis, South-Ural State University, Lenin avenue 76, 454080 Chelyabinsk, Russian Federation

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Abstract

The goal of the paper is a generalized inversion of finite rank Hankel operators and Hankel or Toeplitz operators with block matrices having finitely many rows. To attain it a left coprime fractional factorization of a strictly proper rational matrix function and the Bezout equation are used. Generalized inverses of these operators and generating functions for the inverses are explicitly constructed in terms of the fractional factorization. © 1999 Elsevier Science Inc. All rights reserved.

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Introduction

In the works [1,2] Fuhrmann has proposed the following inversion method for finite Hankel matrices. Let $H = \|g_{i+j-1}\|_{i,j=1,2,\dots,n}$ be an invertible Hankel matrix. By the matrix H and the number $\xi = g_{2n}$ we can uniquely determine the coprime polynomials $p_\xi(z)$ and $q_\xi(z)$ (the polynomial $q_\xi(z)$ is a monic one) such that $\{g_i\}_{i=1}^{2n-1}$ are the Markov parameters of the rational function $p_\xi(z)/q_\xi(z)$. Then H^{-1} is the Bezoutian of the coprime polynomials $q_\xi(z)$ and $a(z)$, where $a(z)$ is a solution of the Bezout equation

[1] E-mail: adukov@math.tu-chel.ac.ru.

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$$a(z)p_\xi(z) + b(z)q_\xi(z) = 1.$$

Similar results have been also obtained by Heinig and Jungnickel [3]. In Ref. [4] the method was extended to the case of finite block Hankel matrices.

In the present paper we will show that the fractional representation and the Bezout equation can also be used for a generalized inversion of finite rank Hankel and Toeplitz operators. We consider Hankel operators that are determined by the infinite block Hankel matrices

$$\begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_{-3} & a_{-4} & a_{-5} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

or the block Hankel matrices with finite number of rows

$$\begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & \cdots \\ \vdots & \vdots & \vdots & \\ a_{-n} & a_{-n-1} & a_{-n-2} & \cdots \end{pmatrix},$$

and the Toeplitz operators with the matrices

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots \\ c_1 & c_0 & c_{-1} & \cdots \\ \vdots & \vdots & \vdots & \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots \end{pmatrix}.$$

We restrict our attention to the operators whose symbols $a(t) = \sum_{j=-\infty}^{-1} a_j t^j$ and $c(t) = \sum_{j=-\infty}^{n-1} c_j t^j$ are rational $p \times q$ matrix functions. We will also assume that poles of $a(t)$ and $t^{-n}c(t)$ lie into the unit disk. The operators under consideration have finite ranks. Hence their images and kernels are complemented subspaces and the operators are generalized invertible. Recall that a linear boundary operator A is called generalized invertible if there exists an operator $A^\dagger$ (a *generalized inverse* of A) such that $AA^\dagger A = A$ (see, e.g., [5]). Our goal is to obtain in explicit form generalized inverses for finite rank Hankel and Toeplitz operators. We will consider a generalized inverse having an additional property $A^\dagger A A^\dagger = A^\dagger$. In the matrix theory such generalized inverses are called $(1,2)$-inverses [6].

It turns out that the following matrix fractional representation

$$a(t) = L^{-1}(t)N(t)$$

of the strictly proper rational matrix function $a(t)$ is central to all further development. Here $L(t)$, $N(t)$ are left coprime matrix polynomials in t, and $L(t)$ is a nonsingular $p \times p$ matrix polynomial. This representation is called a *left coprime fractional matrix factorization* of $a(t)$. The coprimeness condition is equivalent to solvability of the following Bezout equation

$$L(t)U(t) + N(t)V(t) = I_p.$$

Here $U(t)$ and $V(t)$ are matrix polynomials in t. The fractional factorization of a transfer matrix function and the Bezout equation widely used in solving several important problems of linear system theory (see, e.g., [7–9]). Besides the inversion of finite Hankel matrices the fractional factorization was applied for a description of the kernel of finite Hankel matrices [3] and the kernel and image of Hankel operators with rational symbols [10,11].

In this paper we obtain formulas for a generalized inversion of finite rank Hankel and Toeplitz operators with rational matrix symbols in terms of the factor $L(t)$ from the fractional factorization for their symbols and the polynomial $V(t)$ from the solution $(U(t), V(t))$ of the Bezout equation.

1. Notation and usual definitions

Let $\mathbb{C}^{p\times q}$ be the set of complex $p \times q$ matrices.

By $W_{p\times q}$ we denote the set of $p \times q$ matrices with entries in the Wiener algebra W. If $a(t) \in W_{p\times q}$, then

$$a(t) = \sum_{k=-\infty}^{\infty} a_k t^k, \quad a_k \in \mathbb{C}^{p\times q}, \ |t| = 1,$$

where $\sum_{k=-\infty}^{\infty} |a_k| < \infty$. Here $|\cdot|$ is any norm in $\mathbb{C}^{p\times q}$. The set $W_{p\times q}$ endowed with the norm $\|a(t)\| = \sum_{k=-\infty}^{\infty} |a_k|$ is a Banach space. We will denote by $W_{p\times q}(\Omega)$, where $\Omega \subset \mathbb{Z}$, the subspace of $W_{p\times q}$ consisting of matrix functions of the form

$$a(t) = \sum_{k\in\Omega} a_k t^k, \quad |t| = 1.$$

For brevity we will use the notation $W^{\pm}_{p\times q}$ for $W_{p\times q}(\mathbb{Z}_{\pm})$, where $\mathbb{Z}_{\pm}$ is the set of all nonnegative/nonpositive integers. It is obvious that matrix functions in $W^{+}_{p\times q}$ $(W^{-}_{p\times q})$ are analytic in $D_+ = \{z \in \mathbb{C} \mid |z| < 1\}$ $(D_- = \{z \in \mathbb{C} \cup \{\infty\} \mid |z| > 1\})$.

By $E_{q\times 1}$ we denote any of the following Banach spaces of double infinite sequences $\{x_k\}_{k=-\infty}^{\infty}$ $(x_k \in \mathbb{C}^{q\times 1})$:

$$l^s_{q\times 1} (1 \leqslant s < \infty), \ c^0_{q\times 1}, \ c_{q\times 1}, \ m_{q\times 1}.$$

We will denote by $E_{q\times 1}(\Omega)$, where $\Omega \subset \mathbb{Z}$, the subspace of $E_{q\times 1}$ consisting of sequences $\{x_k\}_{k=-\infty}^{\infty}$ for which $x_k = 0$ if $k \bar{\in} \Omega$. It is evident that

$$E_{q\times 1} = E_{q\times 1}(\mathbb{Z}_+)\dot{+}E_{q\times 1}(\mathbb{Z}_-^*).$$

Here $\mathbb{Z}_-^*$ is the set of all negative integers.

Let Q_+ be the projector from $E_{q\times 1}$ onto $E_{q\times 1}(\mathbb{Z}_+)$ along $E_{q\times 1}(\mathbb{Z}_-^*)$ and let Q_- be the complementary projector. Similarly, we denote by P_+ the projector from $E_{p\times 1}$ onto $E_{p\times 1}(\mathbb{Z}_+)$ along $E_{p\times 1}(\mathbb{Z}_-^*)$ and by P_- the complementary projector.

If $a(t) = \sum_{k=-\infty}^{\infty} a_k t^k \in W_{p\times q}$, then we will use the notation $\hat{a}$ for the convolution operator acting from $E_{q\times 1}$ into $E_{p\times 1}$ according to the formula

$$(\hat{a}x)_i = \sum_{j=-\infty}^{\infty} a_{i-j}x_j, \quad i \in \mathbb{Z}.$$

A *Toeplitz (or a discrete Wiener–Hopf) operator with the symbol* $a(t) \in W_{p\times q}$ is defined by the formula

$$\mathbb{T}_a = P_+\hat{a}Q_+|E_{q\times 1}(\mathbb{Z}_+), \tag{1.1}$$

that is,

$$(\mathbb{T}_a x)_i = \sum_{j=0}^{\infty} a_{i-j}x_j, \quad i = 0, 1, 2, \ldots.$$

Hence the operator $\mathbb{T}_a$ from $E_{q\times 1}(\mathbb{Z}_+)$ into $E_{p\times 1}(\mathbb{Z}_+)$ is determined by the infinite block Toeplitz matrix

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \ldots \\ a_1 & a_0 & a_{-1} & \ldots \\ a_2 & a_1 & a_0 & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

For Toeplitz operators there is the following property of partial multiplicativity:

$$\mathbb{T}_{aa_+} = \mathbb{T}_a\mathbb{T}_{a_+}, \quad \mathbb{T}_{a_-a} = \mathbb{T}_{a_-}\mathbb{T}_a. \tag{1.2}$$

Here $a_+(t) \in W_{q\times k}^+$, $a_-(t) \in W_{l\times p}^-$.

We will also use the Wiener–Hopf operators with respect to the space $E_{q\times 1}(\mathbb{Z}_-^*)$

$$\mathbb{T}_a' = P_-\hat{a}Q_-|E_{q\times 1}(\mathbb{Z}_-^*).$$

For these operators we have

$$\mathbb{T}_{aa_-}' = \mathbb{T}_a'\mathbb{T}_{a_-}', \quad \mathbb{T}_{a_+a}' = \mathbb{T}_{a_+}'\mathbb{T}_a'. \tag{1.3}$$

A *Hankel operator with the symbol* $a(t) \in W_{p\times q}$ is the operator acting from $E_{q\times 1}(\mathbb{Z}_+)$ into $E_{p\times 1}(\mathbb{Z}_-^*)$ by the formula

$$\mathbb{H}_a = P_-\hat{a}Q_+|E_{q\times 1}(\mathbb{Z}_+).$$

Hence

$$(\mathbb{H}_a x)_i = \sum_{j=0}^{\infty} a_{i-j}x_j, \quad i = -1, -2, -3, \ldots,$$

and this operator is determined by the infinite block Hankel matrix

$$\begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \ldots \\ a_{-2} & a_{-3} & a_{-4} & \ldots \\ a_{-3} & a_{-4} & a_{-5} & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{1.4}$$

Also we will use the Hankel operator $\mathbb{H}'_a = P_+\hat{a}Q_-|E_{q\times 1}(\mathbb{Z}^*_-)$ determined by the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & \ldots \\ a_2 & a_3 & a_4 & \ldots \\ a_3 & a_4 & a_5 & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

2. Generalized inversion of finite rank Hankel operators

In this paper we will assume that the symbol $a(t) = \sum_{j=-\infty}^{-1} a_j t^j$ of the Hankel operator $\mathbb{H}_a$ is the strictly proper rational $p \times q$ matrix function having poles in D_+ only.

In the system theory the following matrix fractional representation

$$a(t) = L^{-1}(t)N(t), \tag{2.1}$$

of the strictly proper rational matrix function $a(t)$ plays an important role (see, e.g., [7–9]). Here $L(t)$, $N(t)$ are left coprime matrix polynomials in t, and $L(t)$ is a nonsingular $p \times p$ matrix polynomial. This representation is called a *left coprime fractional matrix factorization* of $a(t)$. The coprimeness condition is equivalent to solvability of the following Bezout equation

$$L(t)U(t) + N(t)V(t) = I_p. \tag{2.2}$$

Here $U(t)$ and $V(t)$ are matrix polynomials in t. (Algorithms of an effective construction of representation (2.1) and effective solving of Eq. (2.2) by elementary row operations see, e.g., [12].) Since poles of $a(t)$ lie in D_+, $\det L(t) \neq 0$ if $|t| \geqslant 1$. Hence $L^{-1}(t) \in W_{p\times p}$.

We will show that for a generalized inversion of the Hankel operator $\mathbb{H}_a$ representation (2.1) plays a role of the Wiener–Hopf factorization.

Theorem 2.1. *Let $\mathbb{H}_a$ be a Hankel operator with the rational matrix symbol $a(t)$ and*

$$a(t) = L^{-1}(t)N(t)$$

its fractional matrix factorization. Let $(U(t), V(t))$ be an arbitrary solution of the Bezout Eq. (2.2).

Then the operator $\mathbb{H}_a^\dagger = \mathbb{T}_V\mathbb{H}'_L$ is a *generalized inverse of $\mathbb{H}_a$. Moreover, $\mathbb{H}_a^\dagger\mathbb{H}_a\mathbb{H}_a^\dagger = \mathbb{H}_a^\dagger$.*

Proof. It follows from the fractional factorization and the definition of the operator $\mathbb{H}_a$ that

$$\mathbb{H}_a = \mathbb{H}_{L^{-1}}\mathbb{T}_N.$$

Hence, using the partial multiplicativity (1.2) and the Bezout Eq. (2.2), we have

$$\begin{aligned}\mathbb{H}_a\mathbb{H}_a^\dagger\mathbb{H}_a &= \mathbb{H}_{L^{-1}}\mathbb{T}_N\mathbb{T}_V\mathbb{H}'_L\mathbb{H}_{L^{-1}}\mathbb{T}_N \\ &= \mathbb{H}_{L^{-1}}\mathbb{H}'_L\mathbb{H}_{L^{-1}}\mathbb{T}_N - \mathbb{H}_{L^{-1}}\mathbb{T}_L\mathbb{T}_U\mathbb{H}'_L\mathbb{H}_{L^{-1}}\mathbb{T}_N.\end{aligned}$$

Since

$$\begin{aligned}\mathbb{H}_{L^{-1}}\mathbb{T}_L &= P_-\widehat{L^{-1}}P_+\hat{L}P_+|E_{p\times 1}(\mathbb{Z}_+) \\ &= P_-\widehat{L^{-1}L}P_+|E_{p\times 1}(\mathbb{Z}_+) = 0,\end{aligned}$$

$$\begin{aligned}\mathbb{H}_{L^{-1}}\mathbb{H}'_L\mathbb{H}_{L^{-1}}P_+ &= P_-\widehat{L^{-1}}P_+\hat{L}P_-\widehat{L^{-1}}P_+ \\ &= P_-\widehat{L^{-1}}P_+\widehat{LL^{-1}}P_+ - P_-\widehat{L^{-1}}P_+\hat{L}P_+\widehat{L^{-1}}P_+ \\ &= P_-\widehat{L^{-1}}P_+ - P_-\widehat{L^{-1}}\hat{L}P_+\widehat{L^{-1}}P_+ = P_-\widehat{L^{-1}}P_+ = \mathbb{H}_{L^{-1}}P_+,\end{aligned}$$

we obtain

$$\mathbb{H}_a\mathbb{H}_a^\dagger\mathbb{H}_a = \mathbb{H}_a.$$

In a similar manner we can prove that $\mathbb{H}_a^\dagger\mathbb{H}_a\mathbb{H}_a^\dagger = \mathbb{H}_a^\dagger$ □

Remark 2.1. A similar theorem holds for integral Hankel operators with rational matrix symbols. Moreover, if $a(t)$ is an arbitrary strictly proper rational matrix function and

$$a(t) = \sum_{j=-\infty}^{-1} a_j t^j$$

is the Laurent-series expansion of $a(t)$ at infinity, then it is not difficult to see that the infinite block matrix

$$\mathbb{H}_a^\dagger = \begin{pmatrix} V_0 & 0 & 0 & \dots \\ V_1 & V_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} L_1 & L_2 & L_3 & \dots \\ L_2 & L_3 & L_4 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

is a generalized inverse of the infinite block Hankel matrix

$$\begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \dots \\ a_{-2} & a_{-3} & a_{-4} & \dots \\ a_{-3} & a_{-4} & a_{-5} & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Here V_j and L_j are the coefficients of the matrix polynomials $V(t)$, $L(t)$.

In the system theory a description of the kernel and image of a Hankel operator with a rational symbol plays an important role (see, e.g., [13]). It turns out that the spaces $\ker \mathbb{H}_a$ and $\operatorname{Im} \mathbb{H}_a$ are directly related to the coprime fractional factorization of the symbol $a(t)$ [10,11,13].

Using the generalized inverse $\mathbb{H}_a^\dagger$, now we can describe $\ker \mathbb{H}_a$, $\operatorname{Im} \mathbb{H}_a$ in terms of the factorization of $a(t)$ for the matrix case. We will need the fractional factorization of $a(t)$ with an additional property. Let

$$a(t) = \bar{L}^{-1}(t)\bar{N}(t)$$

be an arbitrary left coprime fractional matrix factorization of $a(t)$. It is well known (see, e.g., [8]) that for the nonsingular $p \times p$ matrix polynomial $\bar{L}(t)$ there exists a unimodular matrix polynomial $S(t)$ such that $L(t) = S(t)\bar{L}(t)$ is a row proper matrix polynomial, i.e. the constant $p \times p$ matrix L^{row} consisting of the coefficients of the highest degrees in each row of $L(t)$ is nonsingular. Hence any left coprime factorization $a(t) = \bar{L}^{-1}(t)\bar{N}(t)$ can be reduced to the row proper form $a(t) = L^{-1}(t)N(t)$. In the system theory it is shown that the row degrees $\rho_1, \dots, \rho_p$ coincide with *observability indices* of the system with the transfer matrix function $a(t)$. The sum κ of the indices is the *McMillan degree* of the system (and the transfer matrix function $a(t)$). We can assume that $\rho_1 \leqslant \dots \leqslant \rho_p$.

Let $d(t) = \operatorname{diag}[t^{\rho_1}, \dots, t^{\rho_p}]$ and $L_-(t) = d^{-1}(t)L(t)$. Since $\det L(t) \neq 0$ for $|t| \geqslant 1$ and $\det L_-(\infty) = \det L^{\text{row}} \neq 0$, $L_-(t)$ is an invertible element of $W^-_{p\times p}$.

Theorem 2.2. *Let $a(t) = L^{-1}(t)N(t)$ be an arbitrary left coprime fractional matrix factorization of $a(t)$. The space* $\ker \mathbb{H}_a$ *consists of the vectors of the form*

$$u - \mathbb{T}_V(\mathbb{I} - \mathbb{T}_L\mathbb{T}_{L^{-1}})\mathbb{T}_N u,$$

where $u \in E_{q\times 1}(\mathbb{Z}_+)$.

Let, in addition, $L(t)$ be the row proper matrix polynomial. Then the space $\operatorname{Im} \mathbb{H}_a$ *is κ-dimensional and the vector functions*

$$\{t^{-1}[L_-^{-1}(t)]^j, \dots, t^{-\rho_j}[L_-^{-1}(t)]^j\}_{j=1}^p \tag{2.3}$$

are the generating functions for the elements of a basis for $\operatorname{Im} \mathbb{H}_a$ *in any space $E_{p\times 1}(\mathbb{Z}_-^*)$. Here $L_-(t) = d^{-1}(t)L(t)$, $d(t) = \operatorname{diag}[t^{\rho_1}, \dots, t^{\rho_p}]$ and $\kappa = \sum_{j=1}^p \rho_j$ is the McMillan degree of $a(t)$.*

Proof. Since $\mathbb{H}_a^\dagger$ is a generalized inverse of $\mathbb{H}_a$ and $\mathbb{H}_a^\dagger\mathbb{H}_a\mathbb{H}_a^\dagger = \mathbb{H}_a^\dagger$, the operator $\mathbb{H}_a^\dagger\mathbb{H}_a$ is the projector onto $\operatorname{Im} \mathbb{H}_a^\dagger$ along $\ker \mathbb{H}_a$. It is easily seen that

$$\mathbb{H}_a^\dagger\mathbb{H}_a = \mathbb{T}_V(\mathbb{I} - \mathbb{T}_L\mathbb{T}_{L^{-1}})\mathbb{T}_N.$$

Hence $\ker \mathbb{H}_a = \operatorname{Im}(\mathbb{I} - \mathbb{T}_V(\mathbb{I} - \mathbb{T}_L\mathbb{T}_{L^{-1}})\mathbb{T}_N)$ and we get the first assertion of the theorem.

The operator $\mathbb{H}_a\mathbb{H}_a^\dagger$ is the projector onto $\operatorname{Im} \mathbb{H}_a$ along $\ker\mathbb{H}_a^\dagger$ and

$$\mathbb{H}_a\mathbb{H}_a^\dagger = \mathbb{I} - P_-\widehat{L^{-1}}P_-\hat{L}P_-|E_{p\times 1}(\mathbb{Z}_-^*) = \mathbb{I} - \mathbb{T}'_{L^{-1}}\mathbb{T}'_L.$$

Here $\mathbb{T}'_L = P_-\hat{L}P_-|E_{p\times 1}(\mathbb{Z}_-^*)$ is the Wiener–Hopf operator with respect to the space $E_{p\times 1}(\mathbb{Z}_-^*)$. Hence $\operatorname{Im} \mathbb{H}_a = \ker\mathbb{T}'_{L^{-1}}\mathbb{T}'_L$.

Since $L(t) = d(t)L_-(t)$ and $L_-(t)$, $d^{-1}(t)$ are elements of $W_{p\times p}^-$, we have, by Eq. (1.3), $\mathbb{T}'_{L^{-1}}\mathbb{T}'_L = \mathbb{T}'_{L_-^{-1}}\mathbb{T}'_{d^{-1}}\mathbb{T}'_d\mathbb{T}'_{L_-}$. Thus $\ker\mathbb{T}'_{L^{-1}}\mathbb{T}'_L = \mathbb{T}'_{L_-^{-1}}\ker\mathbb{T}'_d$. It is easily seen that the vector functions $\{t^{-1}e_j, \dots, t^{-\rho_j}e_j\}_{j=1}^p$, where $\{e_j\}_{j=1}^p$ is the standard basis of $\mathbb{C}^{p\times 1}$, are the generating functions for elements of a basis of the space $\ker\mathbb{T}'_d$ in any space $E_{p\times 1}(\mathbb{Z}_-^*)$. It follows from this that vector functions (2.3) are the generating functions for elements of a basis of $\ker\mathbb{T}'_{L^{-1}}\mathbb{T}'_L$. □

Remark 2.2. We can also obtain a generalized inverse of $\mathbb{H}_a$ in terms of a right fractional factorization

$$a(t) = M(t)R^{-1}(t)$$

of the symbol $a(t)$. In this case we have a more simple description of the kernel:

$$\ker \mathbb{H}_a = \operatorname{Im} \mathbb{T}_R\mathbb{T}_{R^{-1}},$$

but a more complicated description of the image $\operatorname{Im} \mathbb{H}_a$

Thus the Hankel operator $\mathbb{H}_a$ with the rational matrix symbol $a(t)$ is a finite rank operator and its rank coincides with the McMillan degree of $a(t)$. It is easily seen that if the matrix (1.4) has finite rank then its symbol $a(t)$ is a rational matrix function. Hence we arrive to the following well-known result (a block analog of the Kronecker theorem): the rank of a block Hankel matrix is finite iff the symbol of this matrix is a rational matrix function.

We can use Theorems 2.1 and 2.2 for solving of an infinity system of linear equations with a Hankel matrix. The equation $\mathbb{H}_a x = y$ has a solution iff the generating function of y is a linear combination of the functions (2.3). If this condition is fulfilled, then $x = \mathbb{T}_V \mathbb{H}'_L y$ is a solution of the equation. The general solution can be found by the formula (see, e.g., [6])

$$x = \mathbb{T}_V \mathbb{H}'_L y + u - \mathbb{T}_V(\mathbb{I} - \mathbb{T}_L \mathbb{T}_L^{-1})\mathbb{T}_N u,$$

where u is an arbitrary element of $E_{p\times 1}(\mathbb{Z}_+)$.

Example 2.1. *Let us consider as an example the following system of equations*

$$\begin{array}{lllllll}
\frac{1}{4}x_1 + & \frac{4}{8}x_2 + & \frac{7}{16}x_3 + & \frac{10}{32}x_4 + & \cdots & = & 0,\\
\frac{4}{8}x_1 + & \frac{7}{16}x_2 + & \frac{10}{32}x_3 + & \frac{13}{64}x_4 + & \cdots & = & \frac{1}{4},\\
\frac{7}{16}x_1 + & \frac{10}{32}x_2 + & \frac{13}{64}x_3 + & \frac{16}{128}x_4 + & \cdots & = & \frac{2}{8},\\
\frac{10}{32}x_1 + & \frac{13}{64}x_2 + & \frac{16}{128}x_3 + & \frac{19}{256}x_4 + & \cdots & = & \frac{3}{16},\\
 & & & & \cdots & & .
\end{array}$$

The symbol of the Hankel operator $\mathbb{H}_a$ in the left-hand side of the system is $a(t) = \sum_{n=0}^{\infty}(3n+1)/2^{n+2}t^{n+1})$. It is easily seen that

$$a(t) = \frac{t+1}{(2t-1)^2}$$

is the fractional factorization of $a(t)$. Hence $L(t) = (2t-1)^2$, $N(t) = t+1$. By Euclidean algorithm we have $U(t) = 1$, $V(t) = -\frac{4}{9}t + \frac{8}{9}$. Then

$$\mathbb{H}_a^{\dagger} = \mathbb{T}_V \mathbb{H}'_L = \frac{16}{9}\begin{pmatrix} 2 & 0 & 0 & 0 & \dots \\ -1 & 2 & 0 & 0 & \dots \\ 0 & -1 & 2 & 0 & \dots \\ 0 & 0 & -1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

$$= \frac{16}{9}\begin{pmatrix} -2 & 2 & 0 & 0 & \dots \\ 3 & -1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Since

$$t^{-1}L_{-}^{-1}(t)=\sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}t^{n+1}},\quad t^{-2}L_{-}^{-1}(t)=\sum_{n=0}^{\infty}\frac{n+1}{2^{n+2}t^{n+2}},$$

the vectors

$$\left(\frac{1}{4},\frac{2}{8},\frac{3}{16},\ldots\right),\quad\left(0,\frac{1}{4},\frac{2}{8},\frac{3}{16},\ldots\right)$$

form basis of Im $\mathbb{H}_a$. Hence the right-hand side of the system belongs to Im $\mathbb{H}_a$ and the system is solvable. The vector

$$\mathbb{H}_a^{\dagger}y=\left(\frac{8}{9},\ -\frac{4}{9},\ 0,\ 0,\ \ldots\right)$$

is a solution of the system. Now the general solution of the system has the following form

$$\left(u_0+\frac{4}{3}\alpha-\frac{8}{9}\beta+\frac{8}{9},\ u_1-\frac{10}{3}\alpha-\frac{4}{9}\beta-\frac{4}{9},\ u_2+\frac{4}{3}\alpha+\frac{4}{9}\beta,\ u_3,\ u_4,\ \ldots\right).$$

Here $(u_0,\ u_1,\ u_2,\ \ldots)$ is an arbitrary vector in $E(\mathbb{Z}_+)$ and $\alpha=\sum_{n=0}^{\infty}(nu_n/2^n)$, $\beta=\sum_{n=0}^{\infty}(u_n/2^n)$.

In conclusion of the section we find the generating matrix function $\mathbb{G}(t,s)$ for the matrix $\|g_{ij}\|_{i,j=0}^{\infty}$ of the operator $\mathbb{H}_a^{\dagger}=\mathbb{T}_V\mathbb{H}_L'$. It is evident that $g_{ij}=0$ for sufficiently large i,j. Hence

$$\mathbb{G}(t,s)=\sum_{i,j=0}^{\infty}g_{ij}t^is^j$$

is a matrix polynomial in t,s.

Proposition 2.1. *The generating matrix function of the generalized inverse* $\mathbb{H}_a^{\dagger}$ *from Theorem* 2.1 *is found by the formula*

$$\mathbb{G}(t,s)=V(t)\frac{L(t)-L(s)}{t-s}.$$

Proof. Apply the operator $\mathbb{H}_a^{\dagger}=\mathbb{T}_V\mathbb{H}_L'$ to the sequence

$$E=(I_p,sI_p,s^2I_p,\ldots).$$

Let $0<|s|<1$. Then the sequence belongs to $l_{p\times 1}^1(\mathbb{Z}_-^*)$ and the symbol (i.e. the Fourier transform) of the sequence $\mathbb{H}_a^{\dagger}E$ coincides with the generating matrix function $\mathbb{G}(t,s)$. Let us find the symbol of the sequence

$$\mathbb{H}_L' E = \left(\frac{L(s) - L_0}{s}, \frac{L(s) - L_0 - L_1 s}{s^2}, \ldots \right).$$

Here $L_0, L_1, \ldots$ are the coefficients of the matrix polynomial $L(t)$. Denote by λ the degree of $L(t)$.

Then the generating function of $\mathbb{H}_L' E$ is

$$\frac{L(s) - L_0}{s} + \frac{L(s) - L_0 - L_1 s}{s^2} t + \cdots + \frac{L(s) - L_0 - L_1 s - \cdots - L_{\lambda-1} s^{\lambda-1}}{s^{\lambda}} t^{\lambda-1}$$

$$= \frac{L(s)}{s}\left(1 + ts^{-1} + \cdots + t^{\lambda-1}s^{-\lambda+1}\right) - \frac{L_0}{s}\left(1 + ts^{-1} + \cdots + t^{\lambda-1}s^{-\lambda+1}\right)$$

$$- \frac{L_1 t}{s}\left(1 + ts^{-1} + \cdots + t^{\lambda-2}s^{-\lambda-2}\right) - \cdots - \frac{L_{\lambda-2}t^{\lambda-2}}{s}\left(1 + ts^{-1}\right) - \frac{L_{\lambda-1}}{s} t^{\lambda-1}$$

$$= \frac{L(s)(1 - t^{\lambda}s^{-\lambda})}{s - t} - \frac{L_0(1 - t^{\lambda}s^{-\lambda})}{s - t} - \frac{L_1 t(1 - t^{\lambda-1}s^{-\lambda+1})}{s - t} - \cdots$$

$$- \frac{L_{\lambda-2}t^{\lambda-2}(1 - t^2 s^{-2})}{s - t} - \frac{L_{\lambda-1}t^{\lambda-1}(1 - ts^{-1})}{s - t}$$

$$= \frac{1}{s - t}\left[L(s) - t^{\lambda}s^{-\lambda}L(s) - L(t) + L_{\lambda}t^{\lambda} + t^{\lambda}s^{-\lambda}L(s) - L_{\lambda}t^{\lambda}\right]$$

$$= \frac{L(s) - L(t)}{s - t}.$$

Since the matrix polynomial $(L(t) - L(s))/(t - s)$ in t belongs to $W^+_{p\times p}$, and $V(t) \in W^+_{q\times p}$, the symbol of $\mathbb{T}_V \mathbb{H}_L' E$ coincides with $V(t)(L(t) - L(s))/(t - s)$. The condition $0 < |s| < 1$ can be omitted because $(L(t) - L(s))/(t - s)$ is a matrix polynomial in s. The proposition is proved. □

3. Generalized inversion of block Toeplitz and Hankel matrices with finite number of rows

In this section we will consider a generalized inversion of Toeplitz and Hankel operators acting from the space $E_{q\times 1}(\mathbb{Z}_+)$ into the finite-dimensional space $\mathbb{C}^{np\times 1}$. These operators are determinated by the block Toeplitz or Hankel matrices

$$T_c = \begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots \\ c_1 & c_0 & c_{-1} & \cdots \\ \vdots & \vdots & \vdots & \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots \end{pmatrix}, \quad H_a = \begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & \cdots \\ \vdots & \vdots & \vdots & \\ a_{-n} & a_{-n-1} & a_{-n-2} & \cdots \end{pmatrix}. \tag{3.1}$$

having finitely many rows. We will assume that the symbols of these operators $c(t) = \sum_{-\infty}^{n-1} c_k t^k$ and $a(t) = \sum_{-\infty}^{-1} a_k t^k$ are rational matrix functions and $t^{-n}c(t)$, $a(t)$ have poles in D_+ only.

Denote by P_i the projector from $E_{p\times 1}(\mathbb{Z}_+)$ onto the first i coordinates. It is easily seen that $P_i = \mathbb{I} - \mathbb{T}_{t^i I_p}\mathbb{T}_{t^{-i} I_p}$. Here $\mathbb{I}$ is the identity operator and I_p is the identity $p \times p$ matrix. Obviously, now we have

$$T_c = P_n \mathbb{T}_c.$$

Similarly, if $P_i' = \mathbb{I} - \mathbb{T}'_{t^{-i} I_p}\mathbb{T}'_{t^i I_p}$ is the projector from $E_{p\times 1}(\mathbb{Z}_-^*)$ onto the first i coordinates, then

$$H_a = P_n' \mathbb{H}_a.$$

Let us now show that the fractional factorization of the symbols $c(t)$ and $a(t)$ allows to obtain a generalized inversion of the operators T_c and H_a. Denote by $a(t)$ the strictly proper rational matrix function $t^{-n}c(t)$. Let

$$a(t) = L^{-1}(t)N(t) \tag{3.2}$$

be its left coprime fractional factorization with the row proper matrix polynomial $L(t)$. Let $(U(t), V(t))$ be an arbitrary solution of the Bezout equation

$$L(t)U(t) + N(t)V(t) = I_p.$$

Denote by $\rho_1, \dots, \rho_p$ the degrees of the rows of $L(t)$. Let

$$d(t) = \operatorname{diag}[t^{\rho_1}, \dots, t^{\rho_p}], \quad L_-(t) = d^{-1}(t)L(t).$$

As we show in Section 1, $L_-(t)$ is an invertible element of the algebra $W^-_{p\times p}$. The fractional factorization (3.2) gives now the following factorization of $c(t)$

$$c(t) = L_-^{-1} t^n d^{-1}(t)N(t). \tag{3.3}$$

Theorem 4.1. *The operator*

$$T_c^\dagger = \mathbb{T}_V \mathbb{T}_{t^{-n}d} P_n \mathbb{T}_{L_-} P_n | \operatorname{Im} P_n \tag{3.4}$$

is a generalized inverse of the operator $T_c = P_n \mathbb{T}_c$. *Moreover,* $T_c^\dagger T_c T_c^\dagger = T_c^\dagger$. *If* $\rho_j \geqslant n$ *for* $j = 1, \dots, p$, *then* $T_c^\dagger$ *is a right inverse of* T_c.

Proof. It follows from Eq. (1.2) that

$$P_n \mathbb{T}_{L_-} P_n = P_n \mathbb{T}_{L_-}.$$

Hence

$$T_c^\dagger = \mathbb{T}_V \mathbb{T}_{t^{-n}d} \mathbb{T}_{L_-} P_n | \operatorname{Im} P_n.$$

Moreover, in virtue of Eq. (3.3), we have

$$T_c = P_n \mathbb{T}_{L_-^{-1}} \mathbb{T}_{t^n d^{-1}} \mathbb{T}_N.$$

Since

$$\mathbb{T}_N \mathbb{T}_V = \mathbb{T}_{NV} = \mathbb{I} - \mathbb{T}_L \mathbb{T}_U,$$

we obtain

$$T_c T_c^\dagger T_c = P_n \mathbb{T}_{L_-^{-1}} \mathbb{T}_{t^n d^{-1}} \mathbb{T}_{t^{-n} d} \mathbb{T}_{L_-} P_n \mathbb{T}_c - P_n \mathbb{T}_{L_-^{-1}} \mathbb{T}_{t^n d^{-1}} \mathbb{T}_L \mathbb{T}_U \mathbb{T}_{t^{-n} d} \mathbb{T}_{L_-} P_n \mathbb{T}_c.$$

Taking into account the relations

$$P_n \mathbb{T}_{L_-^{-1}} \mathbb{T}_{t^n d^{-1}} \mathbb{T}_L = P_n \mathbb{T}_{L_-^{-1} t^n d^{-1} L} = P_n \mathbb{T}_{t^n I_p} = 0, \tag{3.5}$$

$$P_n \mathbb{T}_c = \mathbb{T}_{L_-^{-1}} \mathbb{T}_{t^n d^{-1}} \mathbb{T}_N - \mathbb{T}_{t^n I_p} \mathbb{T}_{t^{-n} c},$$

$$\mathbb{T}_{L_-} \mathbb{T}_{L_-^{-1}} = \mathbb{I}, \quad \mathbb{T}_{t^n d^{-1}} \mathbb{T}_{t^{-n} d} \mathbb{T}_{t^n d^{-1}} = \mathbb{T}_{t^n d^{-1}},$$

we have

$$T_c T_c^\dagger T_c = P_n \mathbb{T}_{L_-^{-1}} \mathbb{T}_{t^n d^{-1}} \mathbb{T}_N - P_n \mathbb{T}_{L_-^{-1}} \mathbb{T}_{t^n d^{-1}} \mathbb{T}_{t^{-n} d} \mathbb{T}_{t^n L_-} \mathbb{T}_{t^{-n} c}.$$

Using Eq. (3.5) and

$$\mathbb{T}_{t^n L_-} = \mathbb{T}_{t^n d^{-1}} \mathbb{T}_L,$$

we finally obtain

$$T_c T_c^\dagger T_c = P_n \mathbb{T}_{L_-^{-1} t^n d^{-1} N} = P_n \mathbb{T}_c = T_c.$$

Similarly we can prove the relation

$$T_c^\dagger T_c T_c^\dagger = T_c^\dagger.$$

If $\rho_j \geqslant n$, then $t^n d^{-1}(t) \in W_{p\times p}^-$ and it is easily seen that

$$T_c T_c^\dagger = \mathbb{I}.$$

The theorem is proved. □

Obviously, the matrix $JT_{t^n a}$, where

$$J = \begin{pmatrix} 0 & \dots & I_p \\ \vdots & & \vdots \\ I_p & \dots & 0 \end{pmatrix},$$

coincides with the matrix H_a. Hence $T^{\dagger}_{t^n a}J$ is a generalized inverse of H_a. Since J is the matrix of the operator $P_n\mathbb{H}'_{t^n I_p}P'_n|\operatorname{Im} P'_n$ and

$$\mathbb{T}_{L_-}\mathbb{H}'_{t^n I_p} = \mathbb{H}'_{t^n L_-};$$

we arrive to the following proposition on a generalized inversion of the block Hankel matrix H_a with finite number of rows.

Proposition 3.1. *The operator $H^{\dagger}_a = \mathbb{T}_V\mathbb{T}_{t^{-n}d}P_n\mathbb{H}'_{t^n L_-}P'_n|\operatorname{Im} P'_n$ is a generalized inverse of the operator $H_a = P'_n\mathbb{H}_a$. Moreover, $H^{\dagger}_a H_a H^{\dagger}_a = H^{\dagger}_a$. If $\rho_j \geqslant n$ for $j = 1, \ldots, p$, then $H^{\dagger}_a$ is a right inverse of H_a.*

Now we find generating functions for the matrices of the operators $T^{\dagger}_c$, $H^{\dagger}_a$. If τ_{ij} (h_{ij}) are the entries of the matrix $T^{\dagger}_c$ ($H^{\dagger}_a$), then, by definition, the generating matrix function of this matrix is

$$\mathscr{T}(t,s) = \sum_{i=0}^{\infty}\sum_{j=0}^{n-1}\tau_{ij}t^i s^{-j}, \quad \left(\mathscr{H}(t,s) = \sum_{i=0}^{\infty}\sum_{j=0}^{n-1}h_{ij}t^i s^j\right).$$

Let us introduce the operator $\mathbb{B} = \mathbb{T}_V\mathbb{T}_{t^{-n}d}\mathbb{T}_{L_-}$ acting from $l^1_{p\times 1}(\mathbb{Z}_+)$ into $l^1_{q\times 1}(\mathbb{Z}_+)$. It is easily seen that $\mathbb{B}$ is determined by the infinity block matrix $B = \|b_{ij}\|^{\infty}_{i,j=0}$ ($b_{ij} \in \mathbb{C}^{q\times p}$), having absolutely summable block columns and rows. Hence we can define the generating function of B:

$$\mathscr{B}(t,s) = \sum_{i,j=0}^{\infty} b_{ij}t^i s^{-j}, \quad |t| \leqslant 1, \ |s| > 1.$$

Then the generating function for the matrix of the operator $\mathbb{T}_V\mathbb{T}_{t^{-n}d}\mathbb{T}_{L_-}|\operatorname{Im} P_n$ is

$$\mathscr{T}(t,s) = \mathscr{P}_s(-n+1,0)\mathscr{B}(t,s),$$

where $\mathscr{P}_s(-n+1,0)$ is the projector acting by the rule

$$\mathscr{P}_s(-n+1,0)\mathscr{B}(t,s) = \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} b_{ij}t^i s^{-j}.$$

Proposition 3.2. *The generating matrix function of the generalized inverse $T^{\dagger}_c$ from Theorem 3.1 is found by the formula*

$$\mathscr{T}(t,s) = \mathscr{P}_s(-n+1,0)\frac{V(t)d_\sigma(t,s)L_-(s)}{1-ts^{-1}}, \quad |t| \leqslant 1.$$

Here

$$d_\sigma(t,s) = \operatorname{diag}[s^{\rho_1-n}, \ldots, s^{\rho_\sigma-n}, \ t^{\rho_{\sigma+1}-n}, \ldots, t^{\rho_p-n}]$$

and the integer σ *is found from the condition*

$$\rho_1 \leqslant \cdots \leqslant \rho_\sigma \leqslant n < \rho_{\sigma+1} \leqslant \cdots \leqslant \rho_p.$$

Proof. Apply the operator $\mathbb{B}$ to the sequence $E = (I_p, s^{-1}I_p, s^{-2}I_p, \ldots)$. For $|s| > 1$ the sequence belongs to $l^1_{p\times 1}$ and has the symbol (the Fourier transform)

$$\sum_{j=0}^{\infty} t^j s^{-j} I_p = \frac{1}{1 - ts^{-1}} I_p, \quad |t| \leqslant 1.$$

The symbol of the sequence $\mathbb{B}E$ is the function $\sum_{i,j=0}^{\infty} b_{ij} t^i s^{-j}$, that is the generating function of B.

On the other hand, the sequence $\mathbb{T}_{L_-}E = (L_-(s), s^{-1}L_-(s), s^{-2}L_-(s), \ldots)$ has the symbol $L_-(s)/(1 - ts^{-1})$. Hence the symbol of the sequence $\mathbb{T}_{t^{-n}d}\mathbb{T}_{L_-}E$ is $P_+(t^{-n}d(t)L_-(s)/(1 - ts^{-1}))$. Since

$$P_+\frac{t^k}{1 - ts^{-1}} = \begin{cases} \frac{t^k}{1 - ts^{-1}}, & k \geqslant 0, \\ x \\ \frac{s^k}{1 - ts^{-1}}, & k \leqslant 0, \end{cases}$$

we have

$$P_+\frac{t^{-n}d(t)L_-(s)}{1 - ts^{-1}} = \frac{d_\sigma(t,s)L_-(s)}{1 - ts^{-1}}.$$

Thus the symbol of the sequence $\mathbb{T}_V\mathbb{T}_{t^{-n}d}\mathbb{T}_{L_-}E$ is the function

$$P_+\frac{V(t)d_\sigma(t,s)L_-(s)}{1 - ts^{-1}} = \frac{V(t)d_\sigma(t,s)L_-(s)}{1 - ts^{-1}} = \mathscr{B}(t,s).$$

Hence $\mathscr{B}(t,s)$ is the generating function of the operator $\mathbb{B}$:

$$\mathscr{B}(t,s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij}t^i s^{-j}, \quad |t| \leqslant 1, \ |s| > 1.$$

Then the generating function of the matrix of the operator $T_c^\dagger$ coincides with the matrix function $\mathscr{P}_s(-n+1, 0)\mathscr{B}(t,s)$. Since this function is a polynomial in s^{-1}, we can omit the condition $|s| > 1$. The proposition is proved. □

From this proposition it follows at once the formula for the generating function of $H_a^\dagger$.

Proposition 3.3. *The generating matrix function of the generalized inverse* $H_a^\dagger$ *from Proposition* 3.1 *is found by the formula*

$$\mathscr{H}(t,s) = \mathscr{P}_s(0,n-1)\frac{V(t)\hat{d}_\sigma(t,s)L(s)}{s-t}, \quad |t| \leqslant 1.$$

Here

$$\hat{d}_\sigma(t,s) = \operatorname{diag}\left[1,\ldots,1,\ (ts^{-1})^{\rho_{\sigma+1}-n},\ldots,(ts^{-1})^{\rho_p-n}\right]$$

and the integer σ is found from the condition

$$\rho_1 \leqslant \cdots \leqslant \rho_\sigma \leqslant n < \rho_{\sigma+1} \leqslant \cdots \leqslant \rho_p.$$

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