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Formulas for the Reidemeister, Lefschetz and Nielsen coincidence number of maps between infra-nilmanifolds

Ku Yong Ha[1], Jong Bum Lee[1*] and Pieter Penninckx[2]

* Correspondence: jlee@sogang.ac.kr
[1]Department of Mathematics, Sogang University, Seoul 121-742, Korea
Full list of author information is available at the end of the article

Abstract

We prove practical formulas for the Reidemeister coincidence number, the Lefschetz coincidence number and the Nielsen coincidence number of continuous maps between oriented infra-nilmanifolds of equal dimension. In order to obtain these formulas, we use the averaging formulas for the Lefschetz coincidence number and for the Nielsen coincidence number and we develop an averaging formula for the Reidemeister coincidence number. We also give a simple proof of the averaging formula for the Lefschetz coincidence number.

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1. Introduction

In order to study the number of fixed points of a continuous selfmap $f : M \to M$ on a closed, connected manifold M, three homotopy invariant numbers are associated to f: the Reidemeister number $R(f)$, the Lefschetz number $L(f)$ and the Nielsen number $N(f)$. The non-vanishing of the Lefschetz number of f implies the existence of a fixed point, while the Nielsen number is a lower bound for the number of fixed points. The Nielsen number is of particular interest since a classical result by Wecken [1] states that the Nielsen number coincides with the minimal number of fixed point in the homotopy class of the map when the dimension of M is at least three.

Because of the mixture of geometry and algebra that occur in the definition of the Nielsen number, its computation is very hard in general and it is up to now the subject of a great deal of research. Simple and practical formulas have only been obtained in specific cases (for an overview, see for instance [2,3]) and one often turns the attention to comparing the Nielsen number to other numbers that are relatively more easy to compute, such as the Lefschetz number and the Reidemeister number (see for instance [4] for an overview).

Closely related to fixed point theory is coincidence theory. A point $x \in M_1$ is a coincidence of a pair of continuous maps $f, g : M_1 \to M_2$ when $f(x) = g(x)$. In the case where M_1 and M_2 are both oriented and of equal dimension, the Reidemeister coincidence number $R(f, g)$, the Lefschetz coincidence number $L(f, g)$ and the Nielsen coincidence number $N(f, g)$ are defined. The Nielsen coincidence number is particularly

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interesting since it is, just as in the fixed point case, a strong lower bound for the number of coincidences. Unfortunately, the Nielsen coincidence number is usually at least as hard to compute as the Nielsen number in fixed point theory.

In 1985, Anosov [5] shows that for nilmanifolds, the Nielsen number can easily be computed via the Lefschetz number since $N(f) = |L(f)|$ for any continuous selfmap f on a nilmanifold. The result of Anosov was also proved by Fadell and Husseini [6]. This is the beginning of the fruitful study of fixed point theory and coincidence theory for larger classes of manifolds, such as infra-nilmanifolds [7,8], solvmanifolds (see for instance [9]) and infra-solvmanifolds (see for example [10]). In [11,12], formulas for the Lefschetz (fixed point) number and the Nielsen (fixed point) number of continuous selfmaps on infra-nilmanifolds have been proved. In coincidence theory however, only recently a formula has been proved for the Lefschetz coincidence number and the Nielsen coincidence number of a pair of continuous maps between solvmanifolds of type (R) [13]. For infra-nilmanifolds however, formulas for the Lefschetz coincidence number and the Nielsen coincidence number are still open for study.

In this article, we address this problem and we prove in Theorem 6.11 explicit and practical formulas for the Reidemeister coincidence number, the Lefschetz coincidence number and the Nielsen coincidence number of a pair of continuous maps between oriented infra-nilmanifolds of equal dimension, generalizing [12] from fixed point theory to coincidence theory and generalizing [13] from nilmanifolds to infra-nilmanifolds. In order to prove these formulas, we use the averaging formulas for the Nielsen coincidence number [14] and for the Lefschetz coincidence number (see [[9], p. 88] and [10]) and we develop an averaging formula for the Reidemeister coincidence number. We also formulate a simple proof for the averaging formula for the Lefschetz coincidence number.

2. Preliminaries
In this section, we fix notation and we give some definitions that will be needed to prove our results.

Definition 2.1. If $\widetilde{M} \to M$ is a covering map, then we use $A\left(\widetilde{M}, p\right)$ or simply $A\left(\widetilde{M}\right)$ to denote the *covering transformation group*. If $p_1 : \widetilde{M}_1 \to M_1$ and $p_2 : \widetilde{M}_2 \to M_2$ are covering maps, then we say that a continuous map $\tilde{f} : \widetilde{M}_1 \to \widetilde{M}_2$ is a *homotopy lift* of a continuous map $f : M_1 \to M_2$ when $\tilde{f}$ is the lift of a map homotopic to f.

Definition 2.2. Let G be a group and $g \in G$. Then we use $\tau_g : G \to G : g' \mapsto gg'g^{-1}$ to denote the conjugation map $\lambda_g : G \to G : g' \mapsto gg'$ to denote the left multiplication map. If $\phi, \psi : G \to H$ are morphisms of groups, then we define $\mathrm{coin}(\phi, \psi) = \{g \in G \mid \phi(g) = \psi(g)\}$.

2.1. Coincidence theory
In this section, we introduce basic notions concerning coincidence theory. A reference on coincidence theory is [15].

Let M_1 and M_2 be oriented, closed, connected manifolds of equal dimension. In order to study the coincidence set $\mathrm{Coin}(f, g) = \{x \in M_1 \mid f(x) = g(x)\}$ of a pair of continuous maps $f, g : M_1 \to M_2$, one splits the coincidence set into so-called coincidence classes. In this article, we define coincidence classes by fixing lifts $\tilde{f}, \tilde{g} : \widetilde{M}_1 \to \widetilde{M}_2$ of f

and g, where $p_1 : \widetilde{M}_1 \to M_1$ and $p_2 : \widetilde{M}_2 \to M_2$ are universal covers. Remark that by fixing a lift $\tilde{f} : \widetilde{M}_1 \to \widetilde{M}_2$ of $f : M_1 \to M_2$, the continuous map $f : M_1 \to M_2$ induces a morphism $f_\times : A\left(\widetilde{M}_1, p_1\right) \to A\left(\widetilde{M}_2, p_2\right)$ between the covering transformation groups as follows.

Definition 2.3. Let $A\left(\widetilde{M}_1, p_1\right)$ be the covering transformation group of the cover $p_1 : \widetilde{M}_1 \to M_1$ and $A\left(\widetilde{M}_2, p_2\right)$ the covering transformation group of the cover $p_2 : \widetilde{M}_2 \to M_2$. For every $\alpha \in A\left(\widetilde{M}_1, p_1\right)$, let $f_\times(\alpha)$ be the unique covering transformation in $A\left(\widetilde{M}_2, p_2\right)$ that satisfies $\tilde{f}\alpha = f_\times(\alpha)\tilde{f}$.

Fix a lift $\tilde{f} : \widetilde{M}_1 \to \widetilde{M}_2$ of f and a lift $\tilde{g} : \widetilde{M}_1 \to \widetilde{M}_2$ of g. Then for any $\alpha \in A\left(\widetilde{M}_2, p_2\right)$, the set $p_1\left(\text{Coin}\left(\alpha\tilde{f}, \tilde{g}\right)\right)$ is by definition a coincidence class of f, g. Now for any pair of covering transformations $\alpha, \beta \in A\left(\widetilde{M}_2, p_2\right)$, if $p_1\left(\text{Coin}\left(\alpha\tilde{f}, \tilde{g}\right)\right) \cap p_1\left(\text{Coin}\left(\beta\tilde{f}, \tilde{g}\right)\right) \neq \emptyset$, then one can show that $p_1\left(\text{Coin}\left(\alpha\tilde{f}, \tilde{g}\right)\right) = p_1\left(\text{Coin}\left(\beta\tilde{f}, \tilde{g}\right)\right)$ and that there exists $\gamma \in A\left(\widetilde{M}_1, p_1\right)$ such that $\beta = g_\times(\gamma)\alpha f_\times(\gamma)^{-1}$. This is the motivation for the following definitions.

Definition 2.4. Let G_1 and G_2 be groups and $\phi, \psi : G_1 \to G_2$ morphisms of groups. Define an equivalence relation $\sim$ on G_2 by

$$\alpha \sim \beta \Leftrightarrow \exists \gamma \in G_1 : \beta = \psi(\gamma)\alpha\varphi(\gamma)^{-1}.$$

The equivalence classes are called coincidence Reidemeister classes or (doubly) twisted conjugacy classes and $\mathcal{R}[\varphi, \psi]$ denotes the set of coincidence Reidemeister classes. For any $\alpha \in G_2$, we use $[\alpha]$ to denote the coincidence Reidemeister class containing α. The Reidemeister coincidence number $R(\phi, \psi)$ of ϕ, ψ is defined as the cardinality of $\mathcal{R}[\varphi, \psi]$.

Definition 2.5. The Reidemeister coincidence number $R(f, g)$ of the continuous maps f and g is defined as the Reidemeister coincidence number $R(f_\times, g_\times)$ of the induced morphisms $f_\times$ and $g_\times$. For any coincidence Reidemeister class $[\alpha] \in \mathcal{R}\left[f_\times, g_\times\right]$, the set $p_1\left(\text{Coin}\left(\alpha\tilde{f}, \tilde{g}\right)\right)$ does not depend on the particular choice of the representative α of the coincidence Reidemeister class and we call $p_1\left(\text{Coin}\left(\alpha\tilde{f}, \tilde{g}\right)\right)$ the coincidence class of f, g corresponding to $[\alpha]$.

Note that the Reidemeister coincidence number $R(f, g)$ does not depend on the particular choice of the lifts $\tilde{f}, \tilde{g}$. Also the coincidence classes do not depend on the choice of lifts, although the corresponding coincidence Reidemeister classes may.

To each isolated subset C of $\text{Coin}(f, g)$, one associates an integer $\text{ind}(f, g; C)$, called the coincidence index, which generalizes the well-known fixed point index to Nielsen coincidence theory in the setting of maps between oriented manifolds of the equal dimension by Schirmer [16] (see also [17]). Each coincidence class is an isolated subset of $\text{Coin}(f, g)$. If the coincidence index of a coincidence class is non-zero, then we call the coincidence class essential. One can prove that the number of essential coincidence classes is finite. The Lefschetz coincidence number $L(f, g)$ is by definition the sum of the coincidence indices of the coincidence classes. If $L(f, g) \neq 0$, then f and g have a coincidence. The Nielsen coincidence number $N(f, g)$ is defined as the number of

essential coincidence classes of f, g. This number plays a central role in coincidence theory since $N(f, g)$ is a lower bound for the cardinality of Coin(f, g). The Nielsen coincidence number, the Lefschetz coincidence number and the Reidemeister coincidence numbers are homotopy invariants: if f' is homotopic to f and g' is homotopic to g, then $N(f', g') = N(f, g)$, $L(f', g') = L(f, g)$ and $R(f', g') = R(f, g)$. Schirmer [16] shows that when the dimension of M_1 and M_2 is at least three, then for any pair of continuous maps f, $g : M_1 \rightarrow M_2$, there exist maps f' homotopic to f and g' homotopic to g such that the cardinality of Coin(f', g') is precisely $N(f, g)$. In other words: when the dimension is at least three, then the Nielsen number coincides with the minimal number of coincidence points in the homotopy class of the map.

2.2. Infra-nilmanifolds

In this section, we shortly review infra-nilmanifolds. A reference is [18]. Let G be a connected, simply connected, nilpotent Lie group and let C be a maximal compact subgroup of Aut(G). A discrete and cocompact subgroup Π of $G \rtimes C \subseteq$ Aff(G) = $G \rtimes$ Aut(G) is called an *almost-crystallographic group*. Moreover, if Π is torsion free, then Π is called an *almost-Bieberbach group* and the quotient space Π\G is a closed manifold that we call an *infra-nilmanifold*. In particular, if Π ⊂ G, then Π\G is called a *nilmanifold*. Recall from [19] that Γ = Π ∩ G is the maximal normal nilpotent subgroup of Π with finite quotient group Π/Γ. The quotient Π/Γ is isomorphic to the group {A ∈ Aut(G) | ∃a ∈ G such that $(a, A) \in$ Π} which we call the *holonomy group* of the infra-nilmanifold Π\G or of the almost-Bieberbach group Π.

Let us recall an important result on maps between infra-nilmanifolds.

Theorem 2.6. [[8], Corollary 1.2] *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups and M_1 and M_2 infra-nilmanifolds modeled on G_1 and G_2 respectively. Let $f : M_1 \rightarrow M_2$ be a continuous map, then there exist $d \in G_2$ and a morphism of Lie groups $D : G_1 \rightarrow G_2$ such that $\lambda_d \circ D : G_1 \rightarrow G_2$ is a homotopy lift of f.*

In fact, the result in [8] is only formulated for selfmaps, but it is straight forward to generalize the proof to the setting in Theorem 2.6. There is a similar result for maps between nilmanifolds, cf. [[6], Proposition 3.2] or [[9], Lemma 2.7]:

Theorem 2.7. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups and N_1 and N_2 nilmanifolds modeled on G_1 and G_2 respectively. Let $f : N_1 \rightarrow N_2$ be a continuous map. Then f has a homotopy lift $D : G_1 \rightarrow G_2$ that is a morphism of Lie groups.*

When $D : G_1 \rightarrow G_2$ is a morphism of Lie groups, then we will use D_* to denote the corresponding morphism of Lie algebras.

3. The Reidemeister coincidence number

Suppose we have a commutative diagram of groups:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_1 & \xrightarrow{i_1} & \Pi_1 & \xrightarrow{u_1} & \Pi_1/\Gamma_1 & \longrightarrow & 1 \\
& & \varphi' \downarrow \psi' & & \varphi \downarrow \psi & & \bar{\varphi} \downarrow \bar{\psi} & & \\
1 & \longrightarrow & \Gamma_2 & \xrightarrow{i_2} & \Pi_2 & \xrightarrow{u_2} & \Pi_2/\Gamma_2 & \longrightarrow & 1
\end{array}
\tag{3.1}
$$

where the top and bottom sequences are exact and where the quotient groups Π_1/Γ_1 and Π_2/Γ_2 are finite. For each $\bar{\alpha} \in \Pi_2/\Gamma_2$ and $\alpha \in u_2^{-1}(\bar{\alpha})$, we have a commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_1 & \xrightarrow{\;i_1\;} & \Pi_1 & \xrightarrow{\;u_1\;} & \Pi_1/\Gamma_1 & \longrightarrow & 1 \\
& & {\scriptstyle \tau_\alpha\varphi'}\Big\downarrow{\scriptstyle \psi'} & & {\scriptstyle \tau_\alpha\varphi}\Big\downarrow{\scriptstyle \psi} & & {\scriptstyle \tau_{\bar\alpha}\bar\varphi}\Big\downarrow{\scriptstyle \bar\psi} & & \\
1 & \longrightarrow & \Gamma_2 & \xrightarrow{\;i_2\;} & \Pi_2 & \xrightarrow{\;u_2\;} & \Pi_2/\Gamma_2 & \longrightarrow & 1
\end{array}
\tag{3.2}
$$

Moreover the following sequence of groups

$$
1 \to \operatorname{coin}\left(\tau_\alpha\varphi', \psi'\right) \xrightarrow{i_1} \operatorname{coin}\left(\tau_\alpha\varphi, \psi\right) \xrightarrow{u_1} \operatorname{coin}\left(\tau_{\bar\alpha}\bar\varphi, \bar\psi\right)
$$

is exact. Remark that $i_2 : \Gamma_2 \to \Pi_2$ and $u_2 : \Pi_2 \to \Pi_2/\Gamma_2$ induce maps $\hat{i}_2^\alpha : \mathcal{R}\left[\tau_\alpha\varphi', \psi'\right] \to \mathcal{R}\left[\tau_\alpha\varphi, \psi\right]$ and $\hat{u}_2^\alpha : \mathcal{R}\left[\tau_\alpha\varphi, \psi\right] \to \mathcal{R}\left[\tau_{\bar\alpha}\bar\varphi, \bar\psi\right]$ such that $\hat{u}_2^\alpha$ is surjective and $\left(\hat{u}_2^\alpha\right)^{-1}\left(\left[\bar{1}\right]\right) = \operatorname{im}\left(\hat{i}_2^\alpha\right)$. Define $\hat{i}_2 = \hat{i}_2^{\mathrm{Id}}$ and $\hat{u}_2 = \hat{u}_2^{\mathrm{Id}}$, where $\mathrm{Id} \in \Pi_2$ is the identity element.

With this notation:

Lemma 3.1. [[14], Lemma 2.1] *Given the commutative diagram* (3.1), *we have for each* $\alpha \in \Pi_2$ *that*

$$
\#\hat{u}_2^{-1}\left(\left[\bar{\alpha}\right]\right) = \#\left(\hat{u}_2^\alpha\right)^{-1}\left(\left[\bar{1}\right]\right),
\tag{1}
$$

$$
\begin{aligned}
R\left(\varphi, \psi\right) &= \sum_{[\bar\alpha] \in \mathcal{R}[\bar\varphi, \bar\psi]} \#\hat{u}_2^{-1}\left(\left[\bar{\alpha}\right]\right) = \sum_{[\bar\alpha] \in \mathcal{R}[\bar\varphi, \bar\psi]} \#\left(\hat{u}_2^\alpha\right)^{-1}\left(\left[\bar{1}\right]\right) \\
&= \sum_{[\bar\alpha] \in \mathcal{R}[\bar\varphi, \bar\psi]} \#\operatorname{im}\left(\hat{i}_2^\alpha\right),
\end{aligned}
\tag{2}
$$

$$
R\left(\tau_\alpha\varphi', \psi'\right) = \sum_{[\gamma] \in \operatorname{im}\left(\hat{i}_2^\alpha\right)} \#\left(\hat{i}_2^\alpha\right)^{-1}\left(\left[\gamma\right]\right),
\tag{3}
$$

$$
\#\left(\hat{i}_2^\alpha\right)^{-1}\left(\left[\gamma\right]\right) = \left[\operatorname{coin}\left(\tau_{\bar\alpha}\bar\varphi, \bar\psi\right) : u_1\left(\operatorname{coin}\left(\tau_{\gamma\alpha}\varphi, \psi\right)\right)\right] \quad \textit{for each } \gamma \in \Gamma_2,
\tag{4}
$$

$$
R\left(\tau_\alpha\varphi', \psi'\right) = \sum_{[\gamma] \in \operatorname{im}\left(\hat{i}_2^\alpha\right)} \left[\operatorname{coin}\left(\tau_{\bar\alpha}\bar\varphi, \bar\psi\right) : u_1\left(\operatorname{coin}\left(\tau_{\gamma\alpha}\varphi, \psi\right)\right)\right].
\tag{5}
$$

This lemma is stated in [[14], Lemma 2.1] as a straightforward extension of [[11], Lemma 2.1] in which a topological proof of (4) is given. We will translate the topological proof to algebra.

Proof. Choose arbitrary $\alpha \in \Pi_2$.

(1) Choose arbitrary $\gamma \in \Pi_2$ and define $\bar{\gamma} = u_2(\gamma)$. Then

$$
\begin{aligned}
\left[\gamma\right] \in \hat{u}_2^{-1}\left(\left[\bar{\alpha}\right]\right) &\Leftrightarrow \left[\bar{\gamma}\right] = \left[\bar{\alpha}\right] \ \text{in } \mathcal{R}[\bar\varphi, \bar\psi] \\
&\Leftrightarrow \exists \bar{\delta} \in \Pi_1/\Gamma_1 \text{ such that } \bar{\gamma}\bar{\alpha}^{-1} = \bar{\psi}(\bar\delta)\bar{\alpha}\bar{\varphi}(\bar\delta)^{-1}\bar{\alpha}^{-1} \\
&\qquad\qquad = \bar{\varphi}(\bar\delta)\bar{1}\left(\tau_{\bar\alpha}\bar\varphi\right)(\bar\delta)^{-1} \\
&\Leftrightarrow \hat{u}_2^\alpha\left(\left[\gamma\alpha^{-1}\right]\right) = \left[\bar{\gamma}\bar{\alpha}^{-1}\right] = \left[\bar{1}\right] \text{ in } \mathcal{R}\left[\tau_{\bar\alpha}\bar\varphi, \bar\psi\right] \\
&\Leftrightarrow \left[\gamma\alpha^{-1}\right] \in \left(\hat{u}_2^\alpha\right)^{-1}\left(\left[\bar{1}\right]\right).
\end{aligned}
$$

(2) Because $\hat{u}_2$ is surjective, $\mathcal{R}[\varphi, \psi]$ equals the disjoint union

$$\mathcal{R}[\varphi, \psi] = \coprod_{[\bar{\alpha}] \in \mathcal{R}[\bar{\varphi}, \bar{\psi}]} \hat{u}_2^{-1}([\bar{\alpha}]).$$

Hence the first equality follows. The second equality follows from (1) and the last equality follows from the fact that $\left(\hat{u}_2^{\alpha}\right)^{-1}\left([\bar{1}]\right) = \operatorname{im}\left(\hat{i}_2^{\alpha}\right)$.

(3) This follows from the fact that $\mathcal{R}\left[\tau_{\alpha}\varphi', \psi'\right]$ equals the disjoint union

$$\mathcal{R}[\tau_{\alpha}\varphi', \psi'] = \coprod_{[\gamma] \in \operatorname{im}\left(\hat{i}_3^{\alpha}\right)} \left(\hat{i}_2^{\alpha}\right)^{-1}([\gamma]).$$

(4) Choose arbitrary $\gamma \in \Gamma_2$. Define

$$A : \operatorname{coin}\left(\tau_{\bar{\alpha}}\bar{\varphi}, \bar{\psi}\right) \to \left(\hat{i}_2^{\alpha}\right)^{-1}([\gamma]) : \bar{\delta} \mapsto A\left(\bar{\delta}\right),$$

where

$$A(\bar{\delta}) = \left[\psi(\delta)\gamma\left(\tau_{\alpha}\varphi\right)(\delta)^{-1}\right] \in \left(\hat{i}_2^{\alpha}\right)^{-1}([\gamma]) \subset \mathcal{R}\left[\tau_{\alpha}\varphi', \psi'\right],$$

and where $\delta \in \Pi_1$ is chosen such that $u_1(\delta) = \bar{\delta}$.

First we prove that A is well defined:

$\psi(\delta)\gamma(\tau_{\alpha}\phi)(\delta)^{-1}$ **belongs to** Γ_2: This follows from the fact that $u_2\left(\psi(\delta)\gamma(\tau_{\alpha}\varphi)(\delta)^{-1}\right) = \bar{\psi}(\bar{\delta})\left(\tau_{\bar{\alpha}}\bar{\varphi}\right)(\bar{\delta})^{-1} = 1$ because $\bar{\delta} \in \operatorname{coin}\left(\tau_{\bar{\alpha}}\bar{\varphi}, \bar{\psi}\right)$.

$[\psi(\delta)\gamma(\tau_{\alpha}\phi)(\delta)^{-1}]$ **does not depend on the choice of** δ:

Suppose that $u_1(\delta) = u_1(\delta\prime) = \bar{\delta}$. Then there exists $\beta \in \Gamma_1$ such that $\delta' = \beta\delta$. Then

$$\psi(\delta')\gamma\left(\tau_{\alpha}\varphi\right)(\delta')^{-1} = \psi'(\beta)\left(\psi(\delta)\gamma(\tau_{\alpha}\varphi)(\delta)^{-1}\right)\left(\tau_{\alpha}\varphi'\right)(\beta)^{-1}$$

so that

$$\left[\psi(\delta')\gamma\left(\tau_{\alpha}\varphi\right)(\delta')^{-1}\right] = \left[\psi(\delta)\gamma\left(\tau_{\alpha}\varphi\right)(\delta)^{-1}\right].$$

Let us now prove that A is surjective. Choose arbitrary $[\beta] \in \left(\hat{i}_2^{\alpha}\right)^{-1}([\gamma])$. Because $\left(\hat{i}_2^{\alpha}\right)([\beta]) = [\gamma]$, there exists $\delta \in \Pi_1$ such that $\beta = \psi(\delta)\gamma(\tau_{\alpha}\phi)(\delta)^{-1}$. Because $\beta, \gamma \in \Gamma_2$, we have that $1 = u_2(\beta) = \bar{\psi}(\bar{\delta})\left(\tau_{\bar{\alpha}}\bar{\varphi}\right)(\bar{\delta})^{-1}$, where $\bar{\delta} = u_2(\delta)$ so that $\bar{\delta} \in \operatorname{coin}\left(\tau_{\bar{\alpha}}\bar{\varphi}, \bar{\psi}\right)$. Additionally, $A(\bar{\delta}) = [\beta]$.

In order to prove statement (4), it is left to prove that for each $\bar{\delta}, \bar{\delta}' \in \operatorname{coin}\left(\tau_{\bar{\alpha}}\bar{\varphi}, \bar{\psi}\right)$, we have that $A(\bar{\delta}) = A(\bar{\delta}\prime)$ if and only if there exists $\beta \in \operatorname{coin}(\tau_{\gamma\alpha}\phi, \psi)$ such that $\bar{\delta}' = \bar{\delta}u_1(\beta)$. Choose arbitrary $\bar{\delta}, \bar{\delta}' \in \operatorname{coin}\left(\tau_{\bar{\alpha}}\bar{\varphi}, \bar{\psi}\right)$.

First assume that there exists $\beta \in \operatorname{coin}(\tau_{\gamma\alpha}\phi, \psi)$ such that $\bar{\delta}' = \bar{\delta}u_1(\beta)$. Choose $\delta \in \Pi_1$ such that $u_1(\delta) = \bar{\delta}$ and define $\delta' = \delta\beta$, then $u_1(\delta') = \bar{\delta}'$. Remark that because $\beta \in \operatorname{coin}(\tau_{\gamma\alpha}\phi, \psi)$ we have that $\psi(\beta) = \gamma(\tau_{\alpha}\phi)(\beta)\gamma^{-1}$ so that $\psi(\beta)\gamma(\tau_{\alpha}\phi)(\beta)^{-1} = \gamma$. Hence

$$A(\bar{\delta}') = \left[\psi(\delta')\gamma(\tau_\alpha\varphi)(\delta')^{-1}\right] = \left[\psi(\delta\beta)\gamma(\tau_\alpha\varphi)(\delta\beta)^{-1}\right]$$

$$= \left[\psi(\delta)\psi(\beta)\gamma(\tau_\alpha\varphi)(\beta)^{-1}(\tau_\alpha\varphi)(\delta)^{-1}\right]$$

$$= \left[\psi(\delta)\gamma(\tau_\alpha\varphi)(\delta)^{-1}\right] = A(\bar{\delta}).$$

Conversely suppose that $A(\bar{\delta}) = A(\bar{\delta}\prime)$. It then suffices to prove that $\bar{\delta}^{-1}\bar{\delta}' \in u_1\left(\text{coin}\left(\tau_{\gamma\alpha}\varphi, \psi\right)\right)$. Choose $\delta, \delta' \in \Pi_1$ such that $u_1(\delta) = \bar{\delta}$ and $\bar{\delta}' = \bar{\delta}u_1(\beta)$. Then because $A(\bar{\delta}') = A(\bar{\delta})$, there exists $\eta \in \Gamma_1$ such that

$$\psi(\delta')\gamma(\tau_\alpha\varphi)(\delta')^{-1} = \psi(\eta)\psi(\delta)\gamma(\tau_\alpha\varphi)(\delta)^{-1}(\tau_\alpha\varphi)(\eta)^{-1},$$

$$= \psi(\eta\delta)\gamma(\tau_\alpha\varphi)(\eta\delta)^{-1}$$

or

$$\psi\left(\delta^{-1}\eta^{-1}\delta'\right)\gamma = \gamma(\tau_\alpha\varphi)\left(\delta^{-1}\eta^{-1}\delta'\right).$$

Hence $\delta^{-1}\eta^{-1}\delta\prime \in \text{coin}\left(\tau_{\gamma\alpha}\varphi, \psi\right)$ and $\bar{\delta}^{-1}\bar{\delta}' = u_1\left(\delta^{-1}\eta^{-1}\delta'\right) \in \left(\text{coin}\left(\tau_{\gamma\alpha}\varphi, \psi\right)\right).$

(5) This follows from (3) and (4).

Corollary 3.2. [[20], Corollary 2.1] *Given the commutative diagram* (3.1), $\text{coin}\left(\tau_{\bar{\alpha}}\bar{\varphi}, \bar{\psi}\right) = \{\bar{1}\}$ *for all* $\alpha \in \Pi_2$ *then*

$$R(\varphi, \psi) = \sum_{[\bar{\alpha}]\in\mathcal{R}[\bar{\varphi},\bar{\psi}]} R\left(\tau_\alpha\varphi', \psi'\right).$$

Proof. By (2) of Lemma 3.1,

$$R(\varphi, \psi) = \sum_{[\bar{\alpha}]\in\mathcal{R}[\bar{\varphi},\bar{\psi}]} \#\text{im}\left(\hat{i}_2^\alpha\right) = \sum_{[\bar{\alpha}]\in\mathcal{R}[\bar{\varphi},\bar{\psi}]} \#\hat{i}_2^\alpha\left(\mathcal{R}\left[\tau_\alpha\varphi', \psi'\right]\right).$$

By (4) of Lemma 3.1, $\hat{i}_2^\alpha$ is injective and hence the conclusion follows.

Remark 3.3. The group Π_1/Γ_1 acts on the set $[\bar{\alpha}] \in \mathcal{R}[\bar{\varphi}, \bar{\psi}]$ by the rule $\bar{\alpha} \mapsto \bar{\psi}\left(\bar{\beta}\right)\bar{\alpha}\bar{\varphi}(\bar{\beta})^{-1}$. This action is transitive. The isotropy subgroup is

$$\left\{\bar{\beta}|\bar{\psi}\left(\bar{\beta}\right)\bar{\alpha}\bar{\varphi}(\bar{\beta})^{-1} = \bar{\alpha}\right\} = \text{coin}\left(\tau_{\bar{\alpha}}\bar{\varphi}, \bar{\psi}\right).$$

Hence $[\Pi_1 : \Gamma_1] = \#[\bar{\alpha}] \cdot \#\text{coin}\left(\tau_{\bar{\alpha}}\bar{\varphi}, \bar{\psi}\right).$

Corollary 3.4. [[21], Proposition 3.8] *Suppose we are given the commutative diagram* (3.1). *Then $R(\phi, \psi)$ is finite if and only if $R(\tau_\alpha\phi', \psi')$ is finite for every $\alpha \in \Pi_2$.*

Proof. Suppose that $R(\phi, \psi)$ is finite. Choose arbitrary $\alpha \in \Pi_2$. By (2) of Lemma 3.1, $\text{im}\left(\hat{i}_2^\alpha\right)$ is finite. Because also $\left(\tau_{\bar{\alpha}}\bar{\varphi}, \bar{\psi}\right)$ is finite, by (5) of Lemma 3.1, $R(\tau_\alpha\phi', \psi')$ is finite. The converse is also easy to check.

Furthermore, the Reidemeister coincidence numbers $R(\phi, \psi)$ and $R(\tau_\alpha\phi', \psi')$ are directly related as follows:

Theorem 3.5. *Suppose we have a commutative diagram of groups:*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Gamma_1 & \xrightarrow{i_1} & \Pi_1 & \xrightarrow{u_1} & \Pi_1/\Gamma_1 & \longrightarrow & 1 \\
& & \varphi'\downarrow\psi' & & \varphi\downarrow\psi & & \bar{\varphi}\downarrow\bar{\psi} & & \\
1 & \longrightarrow & \Gamma_2 & \xrightarrow{i_2} & \Pi_2 & \xrightarrow{u_2} & \Pi_2/\Gamma_2 & \longrightarrow & 1
\end{array}
$$

where the top and bottom sequences are exact and the quotient groups Π_1/Γ_1 and Π_2/Γ_2 are finite. Then

$$R(\varphi, \psi) \geq \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar{\alpha} \in \Pi_2/\Gamma_2} R\left(\tau_\alpha \varphi', \psi'\right).$$

When either side of the inequality is finite, then equality occurs if and only if coin $(\tau_\alpha \phi, \psi)) \subset \Gamma_1$ for each $\alpha \in \Pi_2$.

Remark that $R(\tau_\alpha \phi', \psi')$ depends only on $\bar{\alpha} = u_2(\alpha)$. Indeed, suppose that $u_2(\alpha) = u_2(\beta)$, then $\alpha\beta^{-1} \in \Gamma_2$ and $\beta\alpha^{-1} \in \Gamma_2$ and we can define

$$A : \mathcal{R}\left[\tau_\alpha \varphi', \psi'\right] \to \mathcal{R}\left[\tau_\beta \varphi', \psi'\right] : [\gamma] \mapsto [\gamma\alpha\beta^{-1}]$$

and

$$B : \mathcal{R}\left[\tau_\beta \varphi', \psi'\right] \to \mathcal{R}\left[\tau_\alpha \varphi', \psi'\right] : [\gamma] \mapsto [\gamma\beta\alpha^{-1}].$$

A and B are bijections that are each other's inverse, hence $R(\tau_\alpha \phi', \psi') = R(\tau_\beta \phi', \psi')$.

Proof of Theorem 3.5. This follows from the following observations:

$$\sum_{\bar{\alpha} \in \Pi_2/\Gamma_2} R\left(\tau_\alpha \varphi', \psi'\right)$$

$$= \sum_{\bar{\alpha} \in \Pi_2/\Gamma_2} \sum_{[\gamma] \in \mathrm{im}\left(\hat{i}_2^\alpha\right)} \left[\mathrm{coin}\left(\tau_{\bar{\alpha}} \bar{\varphi}, \bar{\psi}\right) : u_1\left(\mathrm{coin}\left(\tau_{\gamma\alpha} \varphi, \psi\right)\right)\right]$$

$$\text{by (5) of Lemma 3.1}$$

$$= [\Pi_1 : \Gamma_1] \sum_{\bar{\alpha} \in \Pi_2/\Gamma_2} \sum_{[\gamma] \in \mathrm{im}\left(\hat{i}_2^\alpha\right)} \frac{\left[\mathrm{coin}\left(\tau_{\bar{\alpha}} \bar{\varphi}, \bar{\psi}\right) : u_1\left(\mathrm{coin}\left(\tau_{\gamma\alpha} \varphi, \psi\right)\right)\right]}{\#[\bar{\alpha}] \cdot \#\mathrm{coin}\left(\tau_{\bar{\alpha}} \bar{\varphi}, \bar{\psi}\right)}$$

$$\text{by Remark 3.3}$$

$$= [\Pi_1 : \Gamma_1] \sum_{\bar{\alpha} \in \Pi_2/\Gamma_2} \frac{1}{\#[\bar{\alpha}]} \sum_{[\gamma] \in im(i_2^\alpha)} \frac{1}{\#u_1\left(\mathrm{coin}\left(\tau_{\gamma\alpha} \varphi, \psi\right)\right)}$$

$$\leq [\Pi_1 : \Gamma_1] \sum_{\bar{\alpha} \in \Pi_2/\Gamma_2} \frac{1}{\#[\bar{\alpha}]} \sum_{[\gamma] \in im\left(\hat{i}_2^\alpha\right)} 1$$

$$= [\Pi_1 : \Gamma_1] \sum_{\bar{\alpha} \in \mathcal{R}[\bar{\varphi}, \bar{\psi}]} \#\mathrm{im}\left(\hat{i}_2^\alpha\right)$$

$$= [\Pi_1 : \Gamma_1] \cdot R(\varphi, \psi)$$

$$\text{by (2) of Lemma 3.1.}$$

Moreover, when either side of the inequality is finite, then equality holds if and only if for each $\alpha \in \Pi_2$, $u_1(\mathrm{coin}(\tau_\alpha \phi, \psi))$ is the trivial group.

4. Reidemeister coincidence numbers for covering spaces

Now we can translate the previous results on algebraic Reidemeister coincidence numbers to results on topological Reidemeister coincidence numbers.

Theorem 4.1. *Let M_1 and M_2 be closed and connected manifolds with universal covers $p_1 : \widetilde{M}_1 \to M_1$ and $p_2 : \widetilde{M}_2 \to M_2$. Let $(f, g) : M_1 \to M_2$ be a pair of maps and let $\left(\tilde{f}, \tilde{g}\right) : \widetilde{M}_1 \to \widetilde{M}_2$ be a pair of lifts of (f, g). Let $\Pi_1 = A\left(\widetilde{M}_1, p_1\right)$ and $\Pi_2 = A\left(\widetilde{M}_2, p_2\right)$ be*

the covering transformation groups and let $f_\times, g_\times : \Pi_1 \to \Pi_2$ *be the morphisms of groups induced by* $\left(\tilde{f}, \tilde{g}\right)$. *Suppose* Γ_1 *is a finite index normal subgroup of* Π_1 *and* Γ_2 *is a finite index normal subgroup of* Π_2 *such that* $f_\times(\Gamma_1) \subseteq \Gamma_2$ *and* $g_\times(\Gamma_1) \subseteq \Gamma_2$. *Let* $\left(\bar{f}, \bar{g}\right) : \Gamma_1 \backslash \widetilde{M}_1 \to \Gamma_2 \backslash \widetilde{M}_2$ *be a pair of lifts of* (f, g) *so that the following diagram commutes*

$$
\begin{array}{ccc}
\widetilde{M}_1 & \xrightarrow[\tilde{g}]{\tilde{f}} & \widetilde{M}_2 \\
\downarrow{\scriptstyle p'_1} & & \downarrow{\scriptstyle p'_2} \\
\Gamma_1 / \widetilde{M}_1 & \xrightarrow[\bar{g}]{\bar{f}} & \Gamma_2 / \widetilde{M}_2 \\
\downarrow{\scriptstyle \bar{p}_1} & & \downarrow{\scriptstyle \bar{p}_2} \\
M & \xrightarrow[g]{f} & M
\end{array}
$$

Then:

(1) *If* $\mathrm{coin}\left(\tau_{\bar{\alpha}}\bar{f}_\times, \bar{g}_\times\right) = \{\bar{1}\}$ *for all* $\alpha \in \Pi_2$, *then*

$$
R(f, g) = \sum_{[\bar{\alpha}] \in \mathcal{R}[\bar{f}_\times, \bar{g}_\times]} R\left(\bar{\alpha}\bar{f}, \bar{g}\right).
$$

(2) R(f, g) *is finite if and only if* $R\left(\bar{\alpha}\bar{f}, \bar{g}\right)$ *is finite for every* $\alpha \in \Pi_2$.

(3) *We have*

$$
R(f, g) \geq \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar{\alpha} \in A\left(\Gamma_2 \backslash \widetilde{M}_2, \bar{p}_2\right)} R\left(\bar{\alpha}\bar{f}, \bar{g}\right).
$$

When either side of the inequality is finite, then equality occurs if and only if coin $(\tau_\alpha f_\times, g_\times) \subseteq \Gamma_1$ *for each* $\alpha \in \Pi_2$.

Theorem 4.2 (Averaging Formula for the Reidemeister Coincidence Number). *Let* M_1 *and* M_2 *be orientable infra-nilmanifolds of equal dimension modeled on connected, simply connected, nilpotent Lie groups* G_1 *and* G_2 *respectively. Let* (f, g) : $M_1 \to M_2$ *be a pair of maps. Let* $\Pi_1 = A(G_1, p_1)$ *and* $\Pi_2 = A(G_2, p_2)$ *be the covering transformation groups and let* $f_\times, g_\times : \Pi_1 \to \Pi_2$ *be the morphisms of groups induced by f, g respectively. Let* Γ_1 *and* Γ_2 *be finite index normal subgroups of* Π_1 *and* Π_2 *respectively such that* $f_\times(\Gamma_1) \subseteq \Gamma_2$ *and* $g_\times(\Gamma_1) \subseteq \Gamma_2$ *and such that* $N_1 = \Gamma_1 \backslash G_1$ *and* $N_2 = \Gamma_2 \backslash G_2$ *are nilmanifolds. If* $\left(\bar{f}, \bar{g}\right) : N_1 \to N_2$ *is a pair of lifts of* (f, g), *then*

$$
R(f, g) = \frac{1}{[\Pi_1 : \Gamma_1]} \sum_{\bar{\alpha} \in A(N_2, \bar{p}_2)} R\left(\bar{\alpha}\bar{f}, \bar{g}\right).
$$

Proof. Suppose *f* and *g* have an inessential coincidence class. Then from the proof of [[21], Theorem 5.1], it follows that R(f, g) = ∞ and that there exists $\bar{\alpha} \in A\left(N_2, \bar{p}_2\right)$ such that $R\left(\bar{\alpha}\bar{f}, \bar{g}\right) = \infty$. So without loss of generality, we may assume that all coincidence

classes are essential. Choose arbitrary $\alpha \in \Pi_2$. Then by assumption, $p_1\left(\text{Coin}\left(\alpha\tilde{f}, \tilde{g}\right)\right)$ is an essential coincidence class, where $p_1 : G_1 \rightarrow M_1$ is the natural covering projection. By Theorem 4.1 (3), it suffices to show that $\text{coin}(\tau_\alpha\phi, \psi) \subseteq \Gamma_1$. In fact, $\text{coin}(\tau_\alpha\phi, \psi) = \{1\}$ by [[14], Lemma 4.8] together with the proof of [[14], Theorem 4.9].

5. Averaging formula for the Lefschetz coincidence number

The averaging formula for the Lefschetz coincidence number relates the Lefschetz coincidence number of a pair of continuous maps $f, g : M_1 \rightarrow M_2$ to the Lefschetz coincidence numbers of lifts of f and g to finite sheeted regular covers of M_1 and M_2. The averaging formula for the Lefschetz coincidence number has been proved in [10] (see also [[9], p. 88]). In this section, we formulate a simple proof.

Remark 5.1 (See [[21], Lemma 3.9], [10] or [[22], p. 37]). Let $f, g : M_1 \rightarrow M_2$ be continuous maps between closed oriented manifolds M_1, M_2 of equal dimension. Let $\overline{M}_1$ and $\overline{M}_2$ be covers of M_1 and M_2 respectively and suppose that the covering projections $\bar{p}_1 : \overline{M}_1 \rightarrow M_1$ and $\bar{p}_2 : \overline{M}_2 \rightarrow M_2$ are orientation preserving. Let $(\bar{f}, \bar{g}) : \overline{M}_1 \rightarrow \overline{M}_2$ be a lifting pair of (f, g). Let $\bar{x} \in \text{Coin}(\bar{f}, \bar{g})$. Then $x = p_1(\bar{x}) \in p_1\left(\text{Coin}(\bar{f}, \bar{g})\right) \subset \text{Coin}(f, g)$. Because the covering projections are orientation-preserving local diffeomorphisms,

$$\text{ind}\left(f, g; \{x\}\right) = \text{ind}\left(\bar{f}, \bar{g}; \{\bar{x}\}\right).$$

Theorem 5.2 (Averaging formula for the Lefschetz coincidence number). *Let* $p_1 : \widetilde{M}_1 \rightarrow M_1$ *and* $p_2 : \widetilde{M}_2 \rightarrow M_2$ *be universal covers, let* $\bar{p}_2 : \overline{M}_2 \rightarrow M_2$ *and* $(\bar{f}, \bar{g}) : \overline{M}_1 \rightarrow \overline{M}_2$ *be finite sheeted, regular covers and let* $p'_1 : \widetilde{M}_1 \rightarrow \overline{M}_1$ *and* $p'_2 : \widetilde{M}_2 \rightarrow \overline{M}_2$ *be covers such that the diagrams*
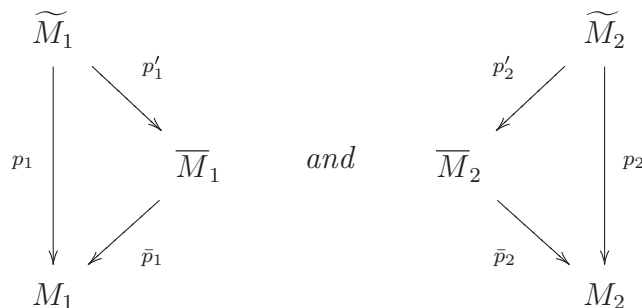
commute, where all spaces are connected oriented manifolds, where $\overline{M}_1$ *and* $\overline{M}_2$ *are closed manifolds of equal dimension and where all covering projections are orientation preserving local diffeomorphisms. Suppose that* f *has a lift* $\bar{f} : \overline{M}_1 \rightarrow \overline{M}_2$ *and* g *has a lift* $\bar{g} : \overline{M}_1 \rightarrow \overline{M}_2$. *Then*

$$L(f, g) = \frac{1}{\#A(\overline{M}_1, \bar{p}_1)} \sum_{\bar{\alpha} \in A(\overline{M}_2, \bar{p}_2)} L\left(\bar{\alpha}, \bar{f}, \bar{g}\right).$$

Proof. It is well-known (cf. [[23], Theorem 3.1]) that there exists a pair of maps (f', g') such that (f, g) is homotopic to (f', g') and $\text{Coin}(f', g')$ is finite. Since the Lefschetz coincidence number is a homotopy invariant, we may assume that $\text{Coin}(f, g)$ is finite.

Because every $x \in \text{Coin}(f, g)$ has $\#A\left(\overline{M}_1, \bar{p}_1\right)$ preimages under $\bar{p}_1$,

$$
\begin{aligned}
L(f, g) &= \sum_{x \in \text{Coin}(f, g)} \text{ind}(f, g; \{x\}) \\
&= \frac{1}{\#A\left(\overline{M}_1, \bar{p}_1\right)} \sum_{\bar{x} \in p_1^{-1}(\text{Coin}(f, g))} \text{ind}(f, g; \{\bar{p}_1(\bar{x})\}).
\end{aligned}
$$

Now

$$
\bar{p}_1^{-1}\left(\text{Coin}(f, g)\right) = \coprod_{\bar{\alpha} \in A(\overline{M}_2, \bar{p}_2)} \text{Coin}\left(\bar{\alpha}\bar{f}, \bar{g}\right).
$$

Indeed, $\bar{x} \in \text{Coin}(\bar{\alpha}\bar{f}, \bar{g}) \cap \text{Coin}(\bar{\beta}\bar{f}, \bar{g})$ implies that $\bar{\alpha}(\bar{f}(\bar{x})) = \bar{g}(\bar{x}) = \bar{\beta}(\bar{f}(\bar{x}))$, such that $\bar{\alpha} = \bar{\beta}$. Hence

$$
\begin{aligned}
L(f, g) &= \frac{1}{\#A\left(\overline{M}_1, \bar{p}_1\right)} \sum_{\bar{\alpha} \in A(\overline{M}_2, \bar{p}_2)} \sum_{\bar{x} \in \text{Coin}(\bar{\alpha}\bar{f}, \bar{g})} \text{ind}\left(f, g; \{\bar{p}_1(\bar{x})\}\right) \\
&= \frac{1}{\#A\left(\overline{M}_1, \bar{p}_1\right)} \sum_{\bar{\alpha} \in A(\overline{M}_2, \bar{p}_2)} \sum_{\bar{x} \in \text{Coin}(\bar{\alpha}\bar{f}, \bar{g})} \text{ind}\left(\bar{\alpha}\bar{f}, \bar{g}; \{\bar{x}\}\right)
\end{aligned}
$$

because of Remark 5.1. Hence

$$
\begin{aligned}
L(f, g) &= \frac{1}{\#A\left(\overline{M}_1, \bar{p}_1\right)} \sum_{\bar{\alpha} \in A(\overline{M}_2, \bar{p}_2)} \text{ind}\left(\bar{\alpha}\bar{f}, \bar{g}; \text{Coin}\left(\bar{\alpha}\bar{f}, \bar{g}\right)\right) \\
&= \frac{1}{\#A\left(\overline{M}_1, \bar{p}_1\right)} \sum_{\bar{\alpha} \in A(\overline{M}_2, \bar{p}_2)} L(\bar{\alpha}\bar{f}, \bar{g}).
\end{aligned}
$$

6. Formulas for the Reidemeister, Lefschetz and Nielsen co-incidence number of maps between infra-nilmanifolds

In this section, we give practical formulas for the Reidemeister coincidence number, the Lefschetz coincidence number and the Nielsen coincidence number of a pair of continuous maps between oriented infra-nilmanifolds of equal dimension. First we give some definitions and recall some useful results.

Definition 6.1. Let G be an n-dimensional oriented, connected, simply connected, nilpotent Lie group and Λ a uniform lattice in G. Let $\{a_1, ..., a_n\}$ be a set of generators of Λ and write $b_i = \log a_i$, then $\{b_1, ..., b_n\}$ is a basis for the Lie algebra $\mathfrak{g}$ corresponding to G. Suppose additionally that the basis $\{b_1, ..., b_n\}$ is positively oriented. Then we refer to $\{b_1, ..., b_n\}$ as a preferred basis of Λ.

Definition 6.2. Let V and W be finite dimensional vector spaces and $L : V \to W$ a linear map. Let β_V be a basis for V and β_W a basis for W, then we use $L_{\beta_W}^{\beta_V}$ to denote the matrix corresponding to L, where this matrix is expressed with respect to the bases β_V and β_W.

Let G_1 and G_2 be oriented, connected, simply connected, nilpotent Lie groups of equal dimension with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Let Λ_1 be a uniform lattice in G_1 and Λ_2 a uniform lattice in G_2. Let $L : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a linear map. Let β_1 be a preferred basis of Λ_1 and β_2 a preferred basis of Λ_2. Then $\det\left(L_{\beta_2}^{\beta_1}\right)$ does not depend on

the particular choice of the preferred bases β_1 of Λ_1 and β_2 of Λ_2 (see [[24], p. 253]). Hence we can define

$$\overset{\Lambda_1}{\underset{\Lambda_2}{\det}}(L) = \det\left(L_{\beta_2}^{\beta_1}\right)$$

for any choice of preferred bases β_1 of Λ_1 and β_2 of Λ_2.

The following theorem gives a formula for the Lefschetz coincidence number of a pair of continuous maps between nilmanifolds of equal dimension.

Theorem 6.3. [[13], Theorem 3.1] *Let G_1, G_2 be oriented, connected, simply connected, nilpotent Lie groups of equal dimension. Let Λ_1 be a uniform lattice in G_1 and Λ_2 a uniform lattice in G_2. Let $N_1 = \Lambda_1\backslash G_1$ and $N_2 = \Lambda_2\backslash G_2$ be the corresponding nilmanifolds. Let $f, f' : N_1 \to N_2$ be continuous maps. Let $D, D' : G_1 \to G_2$ be morphisms of Lie groups such that D is a homotopy lift of f and D' is a homotopy lift of f. Then*

$$L(f, f') = \overset{\Lambda_1}{\underset{\Lambda_2}{\det}}\left(D'_* - D_*\right).$$

Remark that Theorem 2.7 guarantees the existence of morphisms of Lie groups $D, D' : G_1 \to G_2$ that are homotopy lifts of $f, f' : N_1 \to N_2$.

Lemma 6.4. [[24], Lemma 3.2] *Let G be an oriented, connected, simply connected, nilpotent Lie group and Λ a uniform lattice in G. Let $\hat{\Lambda}$ be a finite index subgroup of Λ. Then*

$$\overset{\hat{\Lambda}}{\underset{\Lambda}{\det}}(\mathrm{Id}) = \left[\Lambda : \hat{\Lambda}\right].$$

Proof. Define $N = \Lambda\backslash G$ and $\hat{N} = \hat{\Lambda}\backslash G$ and give N and $\hat{N}$ orientations so that the covering projections $G \to N$ and $G \to \hat{N}$ are orientation preserving. Then $\mathrm{Id} : G \to G : g \mapsto g$ induces an orientation preserving map $f' : \hat{N} \to N$ and $O : G \to G : g \mapsto 1_G$ induces a constant map $f : \hat{N} \to N$. By Theorem 6.3,

$$L(f, f') = \overset{\hat{\Lambda}}{\underset{\Lambda}{\det}}(\mathrm{Id}).$$

On the other hand, by [[7], Lemma 3.10], every essential coincidence class is a singleton and has coincidence index $\mathrm{sign}\left(\det_{\Lambda}^{\hat{\Lambda}}(\mathrm{Id})\right)$. Remark that sign $\mathrm{sign}\left(\det_{\Lambda}^{\hat{\Lambda}}(\mathrm{Id})\right) = 1$. So it suffices to show that $\#\mathrm{Coin}(f, f') = \left[\Lambda : \hat{\Lambda}\right]$. Now

$$\begin{aligned}
\mathrm{Coin}(f, f') &= \left\{\hat{\Lambda}g \in \hat{N} | f(\hat{\Lambda}g) = f'(\hat{\Lambda}g)\right\}\\
&= \left\{\hat{\Lambda}g \in \hat{N} | \Lambda g = \Lambda\right\}\\
&= \left\{\hat{\Lambda}g \in \hat{N} | g = \Lambda\right\}.
\end{aligned}$$

Hence $\#\mathrm{Coin}(f, f') = \left[\Lambda : \hat{\Lambda}\right]$.

Corollary 6.5. *Let G be an oriented, connected, simply connected, nilpotent Lie group and Λ a uniform lattice in G. Let $\hat{\Lambda}$ be a finite index subgroup of Λ. Then*

$$\overset{\hat{\Lambda}}{\underset{\Lambda}{\det}}(\mathrm{Id}) = \frac{1}{\left[\Lambda : \hat{\Lambda}\right]}.$$

Proof. Let β be a preferred basis of Λ and $\hat{\beta}$ a preferred basis of $\hat{\Lambda}$. Then

$$\overset{\hat{\Lambda}}{\underset{\Lambda}{\det}}(\mathrm{Id}) \overset{\Lambda}{\underset{\hat{\Lambda}}{\det}}(\mathrm{Id}) = \det\left(\mathrm{Id}_{\beta}^{\hat{\beta}} \mathrm{Id}_{\hat{\beta}}^{\beta}\right) = \det\left(\mathrm{Id}_{\beta}^{\beta}\right) = 1$$

and the corollary follows from the previous lemma.

Corollary 6.6. *Let G_1 and G_2 be oriented, connected, simply connected, nilpotent Lie groups with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Let Λ_1 be a uniform lattice of G_1 and Λ_2 a uniform lattice of G_2. Let $\hat{\Lambda}_1$ be a finite index subgroup of Λ_1 and $\hat{\Lambda}_2$ a finite index subgroup of Λ_2. Then for any linear map $L : \mathfrak{g}_1 \to \mathfrak{g}_2$,*

$$\overset{\hat{\Lambda}_1}{\underset{\hat{\Lambda}_2}{\det}}(L) = \frac{\left[\Lambda_1 : \hat{\Lambda}_1\right]}{\left[\Lambda_2 : \hat{\Lambda}_2\right]} \overset{\Lambda_1}{\underset{\Lambda_2}{\det}}(L).$$

Proof. This follows from a short calculation:

$$\overset{\hat{\Lambda}_1}{\underset{\hat{\Lambda}_2}{\det}}(L) = \overset{\Lambda_2}{\underset{\hat{\Lambda}_2}{\det}}(\mathrm{Id}) \overset{\Lambda_1}{\underset{\Lambda_2}{\det}}(L) \overset{\hat{\Lambda}_1}{\underset{\Lambda_1}{\det}}(\mathrm{Id}) = \frac{\left[\Lambda_1 : \hat{\Lambda}_1\right]}{\left[\Lambda_2 : \hat{\Lambda}_2\right]} \overset{\Lambda_1}{\underset{\Lambda_2}{\det}}(L).$$

In order to prove the main formulas of this article, we will prove the following technical theorem, which generalizes [[12], Lemma 3.2].

Theorem 6.7. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups of equal dimension with associated Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Let $D, D' : G_1 \to G_2$ be morphisms of Lie groups inducing morphisms of Lie algebras $D_*, D'_* : \mathfrak{g}_1 \to \mathfrak{g}_2$. Then for any $g \in G_2$,*

$$\det\left(D'_* - D_*\right) = \det\left(D'_* - \mathrm{Ad}(g)D_*\right).$$

The proof of this theorem expresses the right hand side of the equality as a polynomial. Then we use the following lemma to prove that either this polynomial is the zero polynomial or it has no roots.

Lemma 6.8. [[21], Lemma 3.5] *Let G_1 and G_2 be connected, simply connected, nilpotent (complex) Lie groups with associated (complex) Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ respectively. Let $D, D' : G_1 \to G_2$ be morphisms of Lie groups inducing morphisms of Lie algebras $D_*, D'_* : \mathfrak{g}_1 \to \mathfrak{g}_2$. Then for any $g \in G_2$, $D'_* - D_*$ is surjective if and only if $D'_* - \mathrm{Ad}(g)D_*$ is surjective.*

In fact, this lemma was only proved in the real case, but its proof is also valid in the complex case. Now we can prove Theorem 6.7.

Proof of Theorem 6.7. Let $G_1^{\mathbb{C}}$ and $G_2^{\mathbb{C}}$ be the complexifications of the real Lie groups G_1 and G_2 respectively with canonical maps $\alpha_i : G_i \to G_i^{\mathbb{C}}$. Since G_i is simply connected, α_i is injective. Let $\mathfrak{g}_i$ and $\mathfrak{g}_i^{\mathbb{C}}$ denote the respective Lie algebras. Then $\mathfrak{g}_i^{\mathbb{C}} \cong \mathfrak{g}_i \otimes_{\mathbb{R}} \mathbb{C}$.

The morphism $D : G_1 \to G_2$ extends uniquely to a morphism $D^{\mathbb{C}} : G_1^{\mathbb{C}} \to G_2^{\mathbb{C}}$ of complex Lie groups so that $\left(D^{\mathbb{C}}\right)_* = D_*^{\mathbb{C}}$. Similarly, we define $D'^{\mathbb{C}} : G_1^{\mathbb{C}} \to G_2^{\mathbb{C}}$, $D'^{\mathbb{C}}_* : \mathfrak{g}_1^{\mathbb{C}} \to \mathfrak{g}_2^{\mathbb{C}}$ and $\mathrm{Ad}(g)^{\mathbb{C}}$ for any $g \in G_2$.

Let $\{X_1^1, ..., X_1^n\}$ be a basis for $\mathfrak{g}_1$ and $\{X_2^1, ..., X_2^n\}$ a basis for $\mathfrak{g}_2$. Write $Y_1^i = (\alpha_1)_* \left(X_1^i\right)$ and $Y_2^i = (\alpha_2)_* \left(X_2^i\right)$, then $\{Y_1^1 ..., Y_1^n\}$ is a basis for the complex vector space $\mathfrak{g}_1^{\mathbb{C}}$ and $\{Y_2^1 ..., Y_2^n\}$ is a basis for the complex vector space $\mathfrak{g}_2^{\mathbb{C}}$. Now

$$\det\left(D'_* - \mathrm{Ad}(g)D_*\right) = \det\left(D'^{\mathbb{C}}_* - \mathrm{Ad}\left(\alpha_2(g)\right)D_*^{\mathbb{C}}\right),$$

where on the left hand side, the matrices are expressed with respect to the bases $\{X_1^1, ..., X_1^n\}$ and $\{X_2^1, ..., X_2^n\}$ and on the right hand side, the matrices are expressed with respect to the bases $\{Y_1^1 ..., Y_1^n\}$ and $\{Y_2^1 ..., Y_2^n\}$.

Now we consider the function $f : \mathfrak{g}_2^{\mathbb{C}} \to \mathbb{C}$ defined by

$$f(Y) = \det\left(D'^{\mathbb{C}}_* - \mathrm{Ad}\left(\exp(Y)\right)D_*^{\mathbb{C}}\right).$$

By Lemma 6.8, either f is the zero map or f has no roots.

Choose arbitrary $Y \in \mathfrak{g}_2^{\mathbb{C}}$. With respect to the complex basis $\{Y_2^1 ..., Y_2^n\}$ of $\mathfrak{g}_2^{\mathbb{C}}$, $\mathrm{ad}(Y)$ is regarded as a matrix $[z_{ij}(Y)]$ where all $z_{ij} : \mathfrak{g}_2^{\mathbb{C}} \to \mathbb{C}$ are complex-valued functions. Now there exist complex numbers c_i^{kj} such that

$$\left[Y_2^k, Y_2^j\right] = \sum_i c_i^{kj} Y_2^i.$$

Writing $Y = \sum_k \lambda_k Y_2^k$, we remark that

$$\sum_i z_{ij}(Y) Y_2^i = \mathrm{ad}(Y)\left(Y_2^j\right) = \left[Y, Y_2^j\right] = \sum_k \lambda_k \left[Y_2^k, Y_2^j\right]$$

$$= \sum_k \sum_i \lambda_k c_i^{kj} Y_2^i = \sum_i \left(\sum_k \lambda_k c_i^{kj}\right) Y_2^i.$$

So the entries $z_{ij}(Y) = \sum_k c_i^{kj} \lambda_k$ of $\mathrm{ad}(Y)$ depend linearly on the λ_k, and hence the entries of $\mathrm{Ad}(\exp(Y)) = \exp(\mathrm{ad}(Y))$ depend polynomially on the λ_k. Consequently,

$$f(Y) = \det\left(D'^{\mathbb{C}}_* - \exp\left(\mathrm{ad}(Y)\right)D_*^{\mathbb{C}}\right)$$

is a polynomial in Y.

If f is not the zero map, then f has no roots and from the fundamental theorem of algebra, it follows that f is a constant polynomial. Hence, regardless of whether f is the zero map or not, f is constant and for any $g \in G_2$,

$$\det\left(D'_* - \mathrm{Ad}(g)D_*\right) = \det\left(D'^{\mathbb{C}}_* - \mathrm{Ad}\left(\alpha_2(g)\right)D_*^{\mathbb{C}}\right) = f\left(\log\left(\alpha_2(g)\right)\right)$$

does not depend on g. This proves the theorem.

Now we generalize [[12], Lemma 3.1], in which the existence of a fully invariant subgroup of finite index in an almost-Bieberbach group is proved. The proof consists

merely of a straightforward adaptation of that of [[12], Lemma 3.1] to this more general, but very analogous situation.

Lemma 6.9. *Let Π_1 and Π_2 be almost-crystallographic groups and let Γ_i be the maximal normal nilpotent subgroup of Π_i or, more generally, let $\Pi_i \subset S_i \rtimes \mathrm{Aut}(S_i)$ be a finite extension of the lattice Γ_i of a connected, and simply connected solvable Lie group S_i. Then there exist fully invariant subgroups $\Lambda_i \subset \Gamma_i$ of Π_i, which are of finite index, so that any morphism $\Pi_1 \to \Pi_2$ maps Λ_1 into Λ_2.*

Proof. Let k be the least common multiple of the orders of the holonomy groups Π_1/Γ_1 and Π_2/Γ_2. Let Λ_1 be the subgroup of Π_1 generated by the set

$$\left\{ x^k \mid x \in \prod_1 \right\}.$$

Clearly, the generating set is a subset of Γ_1 so that Λ_1 is a subgroup of Γ_1. Similarly, the subgroup Λ_2 of Π_2 generated by $\{ y^k \mid y \in \Pi_2 \}$ is a subgroup of Γ_2. Obviously, any morphism $\theta : \Pi_1 \to \Pi_2$ sends the generating set $\{ x^k \mid x \in \Pi_1 \}$ of Λ_1 into the generating set $\{ y^k \mid y \in \Pi_2 \}$ of Λ_2. Thus θ maps Λ_1 into Λ_2.

We claim that Λ_1 has finite index in Γ_1 (and hence in Π_1). Consider the subgroup $\Gamma(k)$ generated by the set $\{ x^k \mid x \in \Gamma_1 \}$. Since Γ_1 is a lattice in the connected and simply connected solvable Lie group S_1, it is a (strongly) polycyclic group. Then $\Gamma(k)$ has finite index in Γ_1, see [[25], Lemma 4.4]. Since $\Gamma(k) \subset \Lambda_1$, we find that Λ_1 has finite index in Γ_1. Similarly, Λ_2 has finite index in Γ_2 and hence in Π_2.

By taking $S_1 = S_2$ and $\Pi_1 = \Pi_2$, we see that any morphism on Π_i maps Λ_i into Λ_i itself. Hence Λ_i is a fully invariant subgroup of Π_i.

Let us recall the averaging formula for the Nielsen coincidence number in the special case of maps between infra-nilmanifolds.

Theorem 6.10. *[[14], Theorem 4.9] Let M_1 and M_2 be closed oriented infra-nilmanifolds of equal dimension and $f, g : M_1 \to M_2$ continuous maps. Suppose there exist finite sheeted regular covers $\bar{p}_1 : N_1 \to M_1$ and $\bar{p}_2 : N_2 \to M_2$, where N_1 and N_2 are nilmanifolds. Suppose that $\bar{f} : N_1 \to N_2$ is a lift of f and $\bar{g} : N_1 \to N_2$ is a lift of g. Then*

$$N(f,g) = \frac{1}{\#A\left(N_1, \bar{p}_1\right)} \sum_{\bar{\alpha} \in A\left(N_2, \bar{p}_2\right)} N\left(\bar{\alpha}\bar{f}, \bar{g}\right).$$

Now we prove practical formulas for the Reidemeister coincidence number, the Lefschetz coincidence number and the Nielsen coincidence number of a pair of continuous maps between oriented infra-nilmanifolds of equal dimension.

Theorem 6.11. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups of equal dimension. Let Π_1 and Π_2 be almost-Bieberbach groups modeled on G_1 and G_2, respectively and suppose that the corresponding infra-nilmanifolds $M_1 = \Pi_1 \backslash G_1$ and $M_2 = \Pi_2 \backslash G_2$ are oriented. Let $F_1 \subset \mathrm{Aut}(G_1)$ be the holonomy group of M_1 and $F_2 \subset \mathrm{Aut}(G_2)$ the holonomy group of M_2. Let $f, g : M_1 \to M_2$ be continuous maps. Let $D, D' : G_1 \to G_2$ be morphisms of Lie groups and $d, d' \in G_2$ such that $\lambda_d \circ D : G_1 \to G_2$ is a homotopy lift of f and $\lambda_{d'} \circ D' : G_1 \to G_2$ is a homotopy lift of g. (Recall that $\lambda_d : G_2 \to G_2 : g' \mapsto dg'$ is the left multiplication map.) Then*

$$L(f, g) = \frac{1}{\#F_1} \sum_{A \in F_2} \det \left(D'_* - A_* D_* \right),$$

$$N(f, g) = \frac{1}{\#F_1} \sum_{A \in F_2} \left| \det \left(D'_* - A_* D_* \right) \right|,$$

and

$$R(f, g) = \frac{1}{\#F_1} \sum_{A \in F_2} \sigma \left(\det \left(D'_* - A_* D_* \right) \right),$$

where the morphisms of Lie groups D_*, D'_* and A_* induced by D, D' and A are expressed with respect to preferred bases of $\Pi_1 \cap G_1$ and $\Pi_2 \cap G_2$ and where $\sigma : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is defined by $\sigma(0) = \infty$ and $\sigma(x) = |x|$ for all $x \neq 0$.

Remark that by Theorem 2.6, there exist $d, d' \in G_2$ and morphisms of Lie groups D, $D' : G_1 \to G_2$ such that $\lambda_d \circ D : G_1 \to G_2$ is a homotopy lift of f and $\lambda_{d'} \circ D' : G_1 \to G_2$ is a homotopy lift of g.

Proof of Theorem 6.11. Without loss of generality, we may assume that $\lambda_d \circ D$ is a lift of f and $\lambda_{d'} \circ D'$ is a lift of g. Define $\Gamma_1 = \Pi_1 \cap G_1$ and $\Gamma_2 = \Pi_2 \cap G_2$. By Lemma 6.9, there exist uniform lattices $\Lambda_1 \subset G_1$ and $\Lambda_2 \subset G_2$ such that Λ_i is a fully invariant subgroup of Π_i, Λ_i is a finite index subgroup of Γ_i and such that $\theta(\Lambda_1) \subset \Lambda_2 \subset \Gamma_2$ for every morphism of groups $\theta : \Pi_1 \to \Pi_2$. Define $N_1 = \Lambda_1 \backslash G_1$ and $N_2 = \Gamma_2 \backslash G_2$. Let $\bar{p}_2 : N_2 \to M_2$, $\bar{p}_2 : N_2 \to M_2$, $p'_1 : G_1 \to N_1$ and $p'_2 : G_2 \to N_2$ be natural projections. Then f, g lift to continuous maps $\bar{f}, \bar{g} : N_1 \to N_2$ so that the following diagram commutes up to homotopy
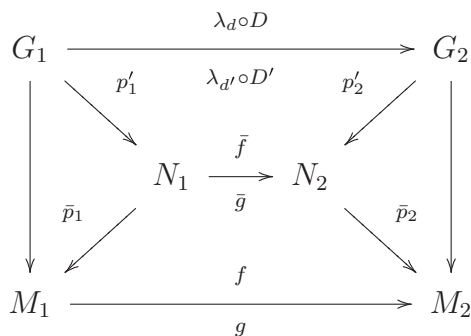
By Theorem 5.2, the averaging formula for the Lefschetz coincidence number,

$$L(f, g) = \frac{1}{\left[\prod_1 : \Lambda_1 \right]} \sum_{\bar{\alpha} \in A(N_2, \bar{p}_2)} L \left(\bar{\alpha} \bar{f}, \bar{g} \right).$$

By [[9], Theorem 1.1], $N \left(\bar{\alpha} \bar{f}, \bar{g} \right) = \left| L \left(\bar{\alpha} \bar{f}, \bar{g} \right) \right|$ for every $\bar{\alpha} \in A \left(N_2, \bar{p}_2 \right)$. Hence by Theorem 6.10, the averaging formula for the Nielsen coincidence number,

$$N(f, g) = \frac{1}{\left[\prod_1 : \Lambda_1 \right]} \sum_{\bar{\alpha} \in A(N_2, \bar{p}_2)} N \left(\bar{\alpha} \bar{f}, \bar{g} \right)$$

$$= \frac{1}{\left[\prod_1 : \Lambda_1 \right]} \sum_{\bar{\alpha} \in A(N_2, \bar{p}_2)} \left| L \left(\bar{\alpha} \bar{f}, \bar{g} \right) \right|.$$

By the main result of [26], $R\left(\bar{\alpha}\bar{f}, \bar{g}\right) = \sigma\left(L(\bar{\alpha}\bar{f}, \bar{g})\right)$ for every $\bar{\alpha} \in A\left(N_2, \bar{p}_2\right)$. Hence by Theorem 4.2, the averaging formula for the Reidemeister coincidence number,

$$
\begin{aligned}
R(f, g) &= \frac{1}{[\prod_1 : \Lambda_1]} \sum_{\bar{\alpha} \in A(N_2, \bar{p}_2)} R\left(\bar{\alpha}\bar{f}, \bar{g}\right) \\
&= \frac{1}{[\prod_1 : \Lambda_1]} \sum_{\bar{\alpha} \in A(N_2, \bar{p}_2)} \sigma\left(L\left(\bar{\alpha}\bar{f}, \bar{g}\right)\right).
\end{aligned}
$$

Choose arbitrary $\bar{\alpha} \in A\left(N_2, \bar{p}_2\right)$. Then there exist $a \in G_2$ and A in the holonomy group $F_2 \subseteq \mathrm{Aut}(G_2)$ such that $\lambda_a \circ A : G_2 \rightarrow G_2$ is a lift of $\bar{\alpha}$. We now claim that

$$
L\left(\bar{\alpha}\bar{f}, \bar{g}\right) = [\Gamma_1 : \Lambda_1] \det_{\Gamma_2}^{\Gamma_1} \left(D'_* - A_* D_*\right).
$$

First remark that $\bar{f} : N_1 \rightarrow N_2$ induces a morphism $\bar{f}_\times : A\left(G_1, p'_i\right) \rightarrow A\left(G_2, p'_2\right)$ defined by

$$
\bar{f}_\times(\lambda) \circ (\lambda_d \circ D) = (\lambda_d \circ D) \circ \lambda \quad \text{for every } \lambda \in A\left(G_1, p'_1\right).
$$

If we apply both sides of this equality to the identity element 1_{G_1} of G_1, then we see that $\bar{f}_\times(\lambda) = (\tau_d \circ D)(\lambda)$. Hence $(\tau_d \circ D)(\Lambda_1) \subseteq \Gamma_2$ and $\tau_d \circ D : G_1 \rightarrow G_2$ induces a continuous map $h : N_1 \rightarrow N_2$. A similar calculation shows that $h_\times = \tau_d \circ D = \bar{f}_\times$. Because N_1 and N_2 are aspherical, $\bar{f}$ and h are homotopic and we see that $\tau_d \circ D$ is a homotopy lift of $\bar{f}$.

Similarly, one can show that $\tau_{d'} \circ D'$ is a homotopy lift of $\bar{g}$ and that $\tau_a \circ A$ is a homotopy lift of $\bar{\alpha}$. Then by Theorem 6.3,

$$
\begin{aligned}
L(\bar{\alpha}\bar{f}, \bar{g}) &= \det_{\Gamma_2}^{\Lambda_1} \left((\tau_{d'} \circ D')_* - (\tau_a \circ A \circ \tau_d \circ D)_*\right) \\
&= \det_{\Gamma_2}^{\Lambda_1} \left((\tau_{d'})_* D'_* - (\tau_{aA(d)})_* A_* D_*\right).
\end{aligned}
$$

By applying Theorem 6.7 twice, we see that

$$
L\left(\bar{\alpha}\bar{f}, \bar{g}\right) = \det_{\Gamma_2}^{\Lambda_1} \left((\tau_{d'})_* D'_* - (\tau_{aA(d)})_* A_* D_*\right) = \det_{\Gamma_2}^{\Lambda_1} \left(D'_* - A_* D_*\right).
$$

By Corollary 6.6,

$$
L\left(\bar{\alpha}\bar{f}, \bar{g}\right) = \det_{\Gamma_2}^{\Lambda_1} \left(D'_* - A_* D_*\right) = [\Gamma_1 : \Lambda_1] \det_{\Gamma_2}^{\Gamma_1} \left(D'_* - A_* D_*\right).
$$

Since $\bar{\alpha}$ was chosen arbitrarily, this equality holds for every $\bar{\alpha}$, where $A \in F_2$ is such that $\lambda_a \circ A$ is a lift of $\bar{\alpha}$ for some $a \in G_2$. Hence

$$
\begin{aligned}
L(f, g) &= \frac{1}{[\prod_1 : \Lambda_1]} \sum_{\bar{\alpha} \in A(N_2, \bar{p}_2)} L\left(\bar{\alpha}, \bar{f}, \bar{g}\right) \\
&= \frac{1}{[\prod_1 : \Lambda_1]} \sum_{A \in F_2} [\Gamma_1 : \Lambda_1] \det_{\Gamma_2}^{\Gamma_1} \left(D'_* - A_* D_*\right) \\
&= \frac{1}{[\prod_1 : \Gamma_1]} \sum_{A \in F_2} \det_{\Gamma_2}^{\Gamma_1} \left(D'_* - A_* D_*\right) \\
&= \frac{1}{\#F_1} \sum_{A \in F_2} \det_{\Gamma_2}^{\Gamma_1} \left(D'_* - A_* D_*\right).
\end{aligned}
$$

Similar calculations show that

$$N(f, g) = \frac{1}{\#F_1} \sum_{A \in F_2} \left| \det_{\Gamma_2}^{\Gamma_1} \left(D'_* - A_* D_* \right) \right|$$

and that

$$R(f, g) = \frac{1}{\#F_1} \sum_{A \in F_2} \sigma \left(\det_{\Gamma_2}^{\Gamma_1} \left(D'_* - A_* D_* \right) \right).$$

The following generalizes [[8], Theorem 2.2] from the fixed point version to the coincidence version.

Corollary 6.12. *Let G_1 and G_2 be connected, simply connected, nilpotent Lie groups of equal dimension. Let M_1 and M_2 be oriented infra-nilmanifolds modeled on G_1 and G_2 respectively. Let $F_2 \subseteq \mathrm{Aut}(G_2)$ be the holonomy group of M_2. Let f, $g : M_1 \to M_2$ be continuous maps. Let D, $D' : G_1 \to G_2$ be morphisms of Lie groups and d, $d' \in G_2$ such that $\lambda_d \circ D : G_1 \to G_2$ is a homotopy lift of f and $\lambda_{d'} \circ D' : G_1 \to G_2$ is a homotopy lift of g. Then $N(f, g) = L(f, g)$ if and only if $\det \left(D'_* - A_* D_* \right) \geq 0$ for every $A \in F_2$ and $N(f, g) = -L(f, g)$ if and only if $\det \left(D'_* - A_* D_* \right) \leq 0$ for every $A \in F_2$, where D_*, D'_* and A_* are the morphisms of Lie algebras induced by D, D' and A respectively, expressed with respect to positively oriented bases of the Lie algebras associated to G_1 and G_2.*

7. Examples

In this section we illustrate, by some examples, how practical the averaging formulas on infra-nilmanifolds are. For this purpose we will consider maps from a 3-dimensional flat Riemannian manifold to an infra-nilmanifold modeled on the Heisenberg group Nil.

Let Nil be the 3-dimensional Heisenberg group defined by

$$\mathrm{Nil} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \Big| x, y, z \in \mathbb{R} \right\}.$$

Then it is a connected and simply connected 2-step nilpotent Lie group. The corresponding Lie algebra is

$$\mathfrak{nil} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \Big| x, y, z \in \mathbb{R} \right\}.$$

It is easy to see that (cf. [[27], Proposition 2.2])

$$\mathrm{Aut}(\mathrm{Nil}) = \left\{ \begin{bmatrix} \det(A) & p \\ 0 & A \end{bmatrix} \Big| A \in \mathrm{GL}(2, \mathbb{R}), p \in \mathbb{R}^2 \text{ is a row vector} \right\}$$

and an element

$$\begin{bmatrix} \det(A) & p \\ 0 & A \end{bmatrix} \in \mathrm{Aut}(\mathrm{Nil}) \text{ with } p = \begin{bmatrix} u & v \end{bmatrix}, A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

acts on Nil as follows:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & px + qy & z' \\ 0 & 1 & rx + sy \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$z' = (ps - qr)z + \frac{1}{2}\left(prx^2 + 2qrxy + qsy^2\right) + ux + vy.$$

Example 7.1. Let Π_1 be the Bieberbach group generated by the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$. Let $M_1 = \Pi_1\backslash\mathbb{R}^3$ be the corresponding infra-nilmanifold, the 3-dimensional flat torus, with the trivial holonomy group.

Consider the almost Bieberbach group Π_2 given by

$$\Pi_2 = \left\langle s_1, s_2, s_3, \alpha \,\middle|\, \begin{array}{l} [s_2, s_1] = s_3^2, [s_3, s_1] = [s_3, s_2] = 1, \\ \alpha^2 = s_1 \alpha s_1 \alpha^{-1} = s_1, \alpha s_2 = s_2^{-1}\alpha s_3^{-1} \end{array} \right\rangle.$$

This is a 3-dimensional orientable almost Bieberbach group π_3 with Seifert bundle type 3 ([[28], Proposition 6.1], or the list of [[29], p. 800]). We can embed Π_2 into Aff (Nil) = Nil $\rtimes$ Aut(Nil) by taking

$$s_1 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I\right), s_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, I\right),$$

$$s_3 = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I\right), \quad \alpha = \left(\begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right).$$

Then the translation lattice is

$$\Gamma_2 = \Pi_2 \cap \text{Nil} = \langle s_1, s_2, s_3 \rangle = \left\{ \begin{bmatrix} 1 & p & \frac{r}{2} \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, p, q, r \in \mathbb{Z} \right\}$$

and the holonomy group of Π_2 is $F_2 = \Pi_2/\Gamma_2 \cong \mathbb{Z}_2$, which is generated by the image A of α under the natural map Aff(Nil) $\to$ Aut(Nil). Thus, A is the automorphism on Nil defined by

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & x & -z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $M_2 = \Pi_2\backslash$Nil be the corresponding infra-nilmanifold.

Define $\phi : \Pi_1 \to \Pi_2$ by

$$\varphi(\mathbf{e}_1) = \alpha \quad \text{and} \quad \varphi(\mathbf{e}_2) = \varphi(\mathbf{e}_3) = 1_{\Pi_2}.$$

Define the morphism of Lie groups $D : \mathbb{R}^3 \to$ Nil by

$$D(x, y, z) = \begin{bmatrix} 1 & \frac{x}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define $\mathbf{d} = 1_{\mathrm{Nil}}$, then one can verify that $\phi(\gamma) \circ \lambda_{\mathbf{d}} \circ D = \lambda_{\mathbf{d}} \circ D \circ \gamma$ for $\gamma = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and hence for all $\gamma \in \Pi_1$. Thus $\lambda_{\mathbf{d}} \circ D : \mathbb{R}^3 \to \mathrm{Nil}$ induces a map $f : M_1 \to M_2$ so that $f_\times = \phi$.

Define $\psi : \Pi_1 \to \Pi_2$ by

$$\psi(\mathbf{e}_1) = 1_{\mathrm{Nil}}, \quad \psi(\mathbf{e}_2) = s_2 \ \text{ and } \ \psi(\mathbf{e}_3) = s_3^{-1}.$$

Define $\mathbf{d'} = 1_{\mathrm{Nil}} \in \mathrm{Nil}$ and define the morphism of Lie groups $D' : \mathbb{R}^3 \to \mathrm{Nil}$ by

$$D'(x, y, z) = \begin{bmatrix} 1 & 0 & -\frac{z}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Then one can verify that $\psi(\gamma) \circ \lambda_{\mathbf{d'}} \circ D' = \lambda_{\mathbf{d'}} \circ D' \circ \gamma$ for $\gamma = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and hence for all $\gamma \in \Pi_1$. This implies that $\lambda_{\mathbf{d'}} \circ D' : \mathbb{R}^3 \to \mathrm{Nil}$ induces a map $f' : M_1 \to M_2$ so that $f'_\times = \psi$.

Since $\Gamma_1 = \Pi_1 \cap \mathbb{R}^3$ is generated by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\Gamma_2 = \Pi_2 \cap \mathrm{Nil}$ is generated by $\{s_1, s_2, s_3\}$, the basis $\{\log(\mathbf{e}_1), \log(\mathbf{e}_2), \log(\mathbf{e}_3)\}$ is a preferred basis for Γ_1 and $\{\log(s_1), \log(s_2), \log(s_3)\}$ is a preferred basis for Γ_2. With respect to these preferred bases, the matrices corresponding to the induced morphisms of Lie algebras $D_*, D'_* : \mathbb{R}^3 \to \mathfrak{nil}$ and $A_* : \mathfrak{nil} \to \mathfrak{nil}$ are

$$D_* = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D'_* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Hence by Theorem 6.11

$$L(f, f') = \det(D'_* - D_*) + \det(D'_* - A_* D_*) = -\frac{1}{2} - \frac{1}{2} = -1,$$

$$N(f, f') = \left| \det(D'_* - D_*) \right| + \left| \det(D'_* - A_* D_*) \right| = 1,$$

$$R(f, f') = \sigma\left(\det(D'_* - D_*) \right) + \sigma\left(\det(D'_* - A_* D_*) \right) = 1.$$

Example 7.2. In this example we will consider a 3-dimensional orientable flat Riemannian manifold M_1 and a 3-dimensional orientable infra-nilmanifold M_2.

We consider first a 3-dimensional orientable flat Riemannian manifold $M_1 = \Pi_1 \backslash \mathbb{R}^3$ where Π_1 is the 3-dimensional orientable Bieberbach group $\mathfrak{G}_2$ [[30], Theorem 3.5.5]:

$$\Pi_1 = \langle t_1, t_2, t_3, \alpha \,|\, [t_i, t_j] = 1, \quad \alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1} \rangle.$$

We can embed this group into $\mathrm{Aff}(\mathbb{R}^3)$ by taking $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as the standard basis for $\mathbb{R}^3$ and

$$t_i = (\mathbf{e}_i, I)(i = 1, 2, 3), \quad \alpha = \left(\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right).$$

Then the translation lattice is

$$\Gamma_1 = \Pi_1 \cap \mathbb{R}^3 = \langle t_1, t_2, t_3 \rangle = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \rangle = \mathbb{Z}^3$$

and the holonomy group is $F_1 = \Pi_1 / \Gamma_1 \cong \mathbb{Z}_2$. The holonomy group $F_1 \subset \mathrm{Aut}(\mathbb{R}^3)$ is generated by $A : \mathbb{R}^3 \to \mathbb{R}^3 : (x, y, z) \mapsto (x, -y, -z)$. With respect to the preferred basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the differential $A_* : \mathbb{R}^3 \to \mathbb{R}^3$ can be expressed as a matrix as follows:

$$A_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Next we consider an infra-nilmanifold $M_2 = \Pi_2 \backslash \text{Nil}$ where Π_2 is a 3-dimensional orientable almost Bieberbach group $\pi_{5,3}$ with Seifert bundle type 5 ([[28], Proposition 6.1], or the list of [[29], p. 800]):

$$\Pi_2 = \left\langle s_1, s_2, s_3, \beta \,\middle|\, \begin{array}{l} [s_2, s_1] = s_3^4, [s_3, s_1] = [s_3, s_2] = 1, \\ \beta^4 = s_3 \beta s_1 \beta^{-1} = s_2, \beta s_2 \beta^{-1} = s_1^{-1} \end{array} \right\rangle.$$

Now we can embed Π_2 into $\text{Aff}(\text{Nil}) = \text{Nil} \rtimes \text{Aut}(\text{Nil})$ by taking

$$s_1 = \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right), \quad s_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, I \right),$$

$$s_3 = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I \right), \quad \beta = \left(\begin{bmatrix} 1 & 0 & -\frac{1}{16} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right).$$

Then the translation lattice is

$$\Gamma_2 = \Pi_2 \cap \text{Nil} = \langle s_1, s_2, s_3 \rangle = \left\{ \begin{bmatrix} 1 & p & \frac{r}{4} \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix} \middle| p, q, r \in Z \right\}$$

and the holonomy group of Π_2 is $F_2 = \Pi_2 / \Gamma_2 \cong \mathbb{Z}_4$. The holonomy group $F_2 \subset \text{Aut}(\text{Nil})$ is generated by the image B of β under the natural map $\text{Aff}(\text{Nil}) \to \text{Aut}(\text{Nil})$. Thus, B is the automorphism on Nil defined by

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -y & -xz \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence the differential B_* of B is given by

$$B_* : \begin{bmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -v & w \\ 0 & 0 & u \\ 0 & 0 & 0 \end{bmatrix}.$$

With respect to the preferred basis $\{\log s_1, \log s_2, \log s_3\}$ of nil, the differential B_* can be written as a matrix as follows:

$$B_* = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now define a morphism of groups $\phi : \Pi_1 \to \Pi_2$ by

$$\varphi(t_1) = s_3, \varphi(t_2) = s_2, \varphi(t_3) = 1_{\text{Nil}}, \varphi(\alpha) = \beta^2.$$

Also define $\mathbf{d} = 1_{\text{Nil}}$ and define the morphism of Lie groups D by

$$D : \mathbb{R}^3 \to \text{Nil} : (x, y, z) \mapsto \begin{bmatrix} 1 & 0 & -x/4 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\phi(\gamma) \circ \lambda_{\mathbf{d}} \circ D = \lambda_{\mathbf{d}} \circ D \circ \gamma$ for $\gamma = t_1, t_2, t_3, \alpha$. Hence this equation holds for every $\gamma \in \Pi_1$ and $\lambda_{\mathbf{d}} \circ D : \mathbb{R}^3 \to$ Nil induces a continuous map $f : M_1 \to M_2$ so that $f_\times = \phi$. With respect to the preferred bases chosen above, the differential D_* can be written as a matrix as follows:

$$D_* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Similarly, by defining the morphism of groups $\psi : \Pi_1 \to \Pi_2$ by

$$\psi(t_1) = s_3^{-1}, \quad \psi(t_2) = 1, \quad \psi(t_3) = s_1, \quad \psi(\alpha) = \beta^{-2},$$

one can show that $\lambda_{\mathbf{d}'} \circ D' : \mathbb{R}^3 \to$ Nil induces a continuous map $f' : M_1 \to M_2$ so that $f'_\times = \psi$, where

$$\mathbf{d}' = 1_{\text{Nil}},$$

$$D' = \mathbb{R}^3 \to \text{Nil} : (x, \gamma, z) \mapsto \begin{bmatrix} 1 & z & x/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

With respect to the same preferred bases, D'_* can be written as a matrix as follows:

$$D'_* = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Hence $\det(D'_* - D_*) = -2$, $\det(D'_* - B_* D_*) = 0$, $\det(D'_* - B_*^2 D_*) = 2$ and $\det(D'_* - B_*^3 D_*) = 0$. From the formulas in Theorem 6.11, it follows that

$$L(f, f') = 0, \quad N(f, f') = 2, \quad R(f, f') = \infty.$$

Author details
[1]Department of Mathematics, Sogang University, Seoul 121-742, Korea [2]Department of Mathematics, Katholieke Universiteit Leuven, Campus Kortrijk, B-8500 Kortrijk, Belgium

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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