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The topological fundamental group and free topological groups

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ABSTRACT

The topological fundamental group π_1^{top} is a homotopy invariant finer than the usual fundamental group. It assigns to each space a quasitopological group and is discrete on spaces which admit universal covers. For an arbitrary space X, we compute the topological fundamental group of the suspension space $\Sigma(X_+)$ and find that $\pi_1^{top}(\Sigma(X_+))$ either fails to be a topological group or is the free topological group on the path component space of X. Using this computation, we provide an abundance of counterexamples to the assertion that all topological fundamental groups are topological groups. A relation to free topological groups allows us to reduce the problem of characterizing Hausdorff spaces X for which $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group to some well-known classification problems in topology.

1. Introduction

The fact that classical homotopy theory is insufficient for studying spaces with homotopy type other than that of a CW-complex has motivated the introduction of a number of invariants useful for studying spaces with complex local structure. For instance, in Čech theory, one typically approximates complicated spaces with "nice" spaces and takes the limit or colimit of an algebraic invariant evaluated on the approximating spaces. Another approach is to directly transfer topological data to algebraic invariants such as homotopy or (co)homology groups by endowing them with natural topologies that behave nicely with respect to the algebraic structure. While this second approach does not yield purely algebraic objects, it does have the advantage of allowing direct application of the rich theory of topological algebra. The notion of "topologized" homotopy invariant seems to have been introduced by Hurewicz in [1] and studied subsequently by Dugundji in [2]. Whereas these early methods focused on "finite step homotopies" through open covers of spaces, we are primarily interested in the properties of a topologized version of the usual fundamental group.

The topological fundamental group $\pi_1^{top}(X,x)$ of a based space (X,x), as first specified by Biss [3], is the fundamental group $\pi_1(X,x)$ endowed with the natural topology that arises from viewing it as a quotient space of the space of loops based at x. This choice of topological structure, makes π_1^{top} particularly useful for studying the homotopy of spaces that lack universal covers, i.e. that fail to be locally path connected or semilocally simply connected. Previously, some authors asserted that topological fundamental groups are always topological groups [3–6], overlooking the fact that products of quotient maps are not always quotient maps.[1] The first objective of this paper is to produce counterexamples to this claim. Recently, Fabel [8] has shown that the Hawaiian earring group $\pi_1^{top}(\mathbb{HE})$ fails to be a topological group. In this paper, we find an abundance of spaces whose topological fundamental group fails to be a topological group independent of Fabel's example. Surprisingly, multiplication can fail to be continuous even for a space as nice as a locally simply connected subset of $\mathbb{R}^2$ (see Example 4.27).

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[1] One may similarly define the topological fundamental groupoid [7] of an unbased space, however, the same care must be taken with respect to pullbacks of quotient maps.

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The existence of such spaces then begs the question: What type of object is $\pi_1^{top}(X)$? In Section 2 we find that topological fundamental groups are quasitopological groups in the sense of [9]. Here, we also provide preliminaries and some of the basic theory of topological fundamental groups.

The counterexamples mentioned above come from a class of spaces studied in Section 3. This class consists of reduced suspensions of arbitrary spaces with isolated basepoint (written $\Sigma(X_+)$). The suspension-loop adjunction provides an unexpected relation to the notorious free (Markov) topological groups. The second objective of this paper is to "compute" $\pi_1^{top}(\Sigma(X_+))$, that is, provide a simple description of the isomorphism class of $\pi_1^{top}(\Sigma(X_+))$ in the category of quasitopological groups. Specifically, we prove that $\pi_1^{top}(\Sigma(X_+))$ is the quotient of the free topological monoid on two disjoint copies of X.

Theorem 1.1. *For an arbitrary topological space X, $\pi_1^{top}(\Sigma(X_+))$ is naturally isomorphic as a quasitopological group to the free group $F(\pi_0(X))$ with the quotient topology with respect to the canonical map*

$$\coprod_{n\geqslant 0}\left(X\sqcup X^{-1}\right)^n\to F\left(\pi_0(X)\right).$$

A direct consequence is that:

Corollary 1.2. *$\pi_1^{top}(\Sigma(X_+))$ either fails to be a topological group or is the free topological group $F_M(\pi_0^{top}(X))$ on the path component space $\pi_0^{top}(X)$.*

This new connection to free topological groups is particularly surprising since, in general, it is difficult to describe the topological structure of both topological fundamental groups and free topological groups. The realization of many free topological groups as homotopy invariants may offer a geometric approach to their study.

In Section 4, we provide a detailed study of the topology of $\pi_1^{top}(\Sigma(X_+))$. Our third objective is to characterize the spaces X for which $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group using techniques and results from topological algebra. Theorem 1.1 and Corollary 1.2 motivate our interest in the free topological groups described in the following statement.

Fact 1.3. *([10, Statement 5.1]) For a Tychonoff space Y the canonical map*

$$\coprod_{n\geqslant 0}\left(Y\sqcup Y^{-1}\right)^n\to F_M(Y)$$

reducing words in the free topological monoid on two disjoint copies of Y is quotient if and only if the following two conditions hold:

1. *$F_M(Y)$ has the inductive limit topology of the subspaces $F_M(Y)_n$ consisting of words of length $\leqslant n$.*
2. *For every $n\geqslant 1$, the canonical multiplication map $\coprod_{i=0}^{n}(Y\sqcup Y^{-1})^i\to F_M(Y)_n$ is a quotient map.*

This approach to free topological groups was initiated by A.I. Mal'tsev [11] and Sipacheva [10] remarks that it is "immeasurably more convenient" to work with free topological groups having this type of quotient structure but that this convenience occurs "fairly rarely." Full characterizations of the spaces Y for which 1. and 2. are held individually remain important open problems in the study of free topological groups. A number of references for recent results appear in Sections 5–8 of [10]. For the sake of simplicity, we restrict ourselves to knowing that when Y is a k_ω-space,[2] both of the above conditions 1. and 2. hold [12]. The problem of determining when $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group reduces to these two characterization problems, a separation property, and the well-studied problem of products of quotient maps being quotient.

Theorem 1.4. *For a Hausdorff space X, $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group if and only if all four of the following conditions hold*:

1. *$\pi_0^{top}(X)$ is Tychonoff.*
2. *Every finite power of the canonical quotient map $\pi_X : X\to\pi_0^{top}(X)$ is a quotient map.*
3. *The free topological group $F_M(\pi_0^{top}(X))$ has the inductive limit topology of subspaces $F_M(\pi_0^{top}(X))_n$ consisting of words of length $\leqslant n$.*
4. *For every $n\geqslant 1$, the canonical multiplication map $\mathbf{i}_n:\coprod_{i=0}^{n}(\pi_0^{top}(X)\sqcup\pi_0^{top}(X)^{-1})^i\to F(\pi_0^{top}(X))_n$ is a quotient map.*

[2] A space Y is a k_ω-space if it is the inductive limit of a sequence of compact subspaces.

2. Preliminaries

2.1. Path component spaces

The *path component space* of a topological space X is the set of path components $\pi_0(X)$ of X with the quotient topology with respect to the canonical map $\pi_X : X \to \pi_0(X)$. We denote this space as $\pi_0^{top}(X)$ and remove or change the subscript of the map π_X when convenient. A map of $f : X \to Y$ induces a map $f_* : \pi_0^{top}(X) \to \pi_0^{top}(Y)$ taking the path component of x in X to the path component of $f(x)$ in Y, which is continuous by the universal property of quotient spaces. If X has basepoint x, we take the basepoint of $\pi_0^{top}(X)$ to be the path component of x in X. We may write π_0^{top} as an endofunctor of both **Top** and $\mathbf{Top}_*$, however, the presence of basepoint in $\pi_0^{top}(X)$ will be clear from context. The following remarks illustrate some of the basic properties of path component spaces.

Definition 2.1. A space X is *semilocally 0-connected* if for each point $x \in X$, there is an open neighborhood U of x such that the inclusion $i : U \hookrightarrow X$ induces the constant map $i_* : \pi_0^{top}(U) \to \pi_0^{top}(X)$.

Remark 2.2. X is semilocally 0-connected if and only if $\pi_0^{top}(X)$ has the discrete topology.

Remark 2.3. π_0^{top} preserves coproducts and quotients, but does not preserve products. Since the non-topological functor $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ does preserve products, the projections maps of a product $X = \prod_{\lambda \in \Lambda} X_\lambda$ induce a continuous bijection $\psi : \pi_0^{top}(X) \to \prod_{\lambda \in \Lambda} \pi_0^{top}(X_\lambda)$ such that $\psi \circ \pi_X = \prod_{\lambda \in \Lambda} \pi_{X_\lambda}$.

Remark 2.4. Every topological space Y is homeomorphic to the path component space of some paracompact Hausdorff space $\mathcal{H}(Y)$. Some properties and variants of the functor $\mathcal{H}$ are included in [13] where it is first introduced.

2.2. The compact-open topology

For spaces X, Y, let $M(X, Y)$ be the space of unbased maps $X \to Y$ with the compact-open topology. A subbasis for this topology consists of neighborhoods of the form $\langle C, U \rangle = \{f \mid f(C) \subset U\}$ where $C \subseteq X$ is compact and U is open in Y. If $A \subseteq X$ and $B \subseteq Y$, then $M(X, A; Y, B) \subseteq M(X, Y)$ is the subspace of relative maps (such that $f(A) \subseteq B$) and if X and Y have basepoints x and y respectively, $M_*(X, x; Y, y)$ (or just $M_*(X, Y)$) is the subspace of basepoint preserving maps. In particular, $\Omega(X, x) = M_*(S^1, (1, 0); X, x)$ is the space of based loops. When X is path connected and the basepoint is clear, we just write $\Omega(X)$. For convenience, we will often replace $\Omega(X, x)$ by the homeomorphic relative mapping space $M(I, \{0, 1\}; X, \{x\})$ where $I = [0, 1]$ is the closed unit interval. The constant path at $x \in X$ is denoted c_x.

For any fixed, closed subinterval $A \subseteq I$, let $H_A : I \to A$ be the unique, increasing, linear homeomorphism. For a path $p : I \to X$, the *restricted path of* p *to* A is the composite $p_A = p|_A \circ H_A : I \to A \to X$. As a convention, if $A = \{t\} \subseteq I$ is a singleton, p_A will denote the constant path at $p(t)$. Note that if $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = 1$, knowing the paths $p_{[t_{i-1}, t_i]}$ for $i = 1, \dots, n$ uniquely determines p. For integer $n \geqslant 1$ and $j = 1, \dots, n$, let K_n^j be the closed subinterval $[\frac{j-1}{n}, \frac{j}{n}] \subseteq I$. If $p_j : I \to X$, $j = 1, \dots, n$ are paths such that $p_j(1) = p_{j+1}(0)$ for each $j = 1, \dots, n-1$, then the *n-fold concatenation* of these paths is the unique path $q = *_{j=1}^n p_i = p_1 * p_2 * \cdots * p_n$ such that $q_{K_n^j} = p_j$ for each j. If $\alpha \in M(I, X)$, then $\alpha^{-1}(t) = \alpha(1-t)$ is the *reverse* of α and for a set $A \subseteq M(I, X)$, $A^{-1} = \{\alpha^{-1} \mid \alpha \in A\}$. It is a well-known fact of algebraic topology that concatenation of loops $\Omega(X) \times \Omega(X) \to \Omega(X)$, $(\alpha, \beta) \mapsto \alpha * \beta$ and loop inversion $\Omega(X) \to \Omega(X)$, $\alpha \mapsto \alpha^{-1}$ are continuous operations [14, Chapter 2.8].

Let $\mathscr{U} = \bigcap_{j=1}^n \langle C_j, U_j \rangle$ be a basic open neighborhood of a path p in $M(I, X)$. Then $\mathscr{U}_A = \bigcap_{A \cap C_j \neq \emptyset} \langle H_A^{-1}(A \cap C_j), U_j \rangle$ is a basic open neighborhood of p_A called the *restricted neighborhood* of $\mathscr{U}$ to A. If $A = \{t\}$ is a singleton, then $\mathscr{U}_A = \bigcap_{t \in C_j} \langle I, U_j \rangle = \langle I, \bigcap_{t \in C_j} U_j \rangle$. On the other hand, if $p = q_A$ for some path $q \in M(I, X)$, then $\mathscr{U}^A = \bigcap_{j=1}^n \langle H_A(C_j), U_j \rangle$ is a basic open neighborhood of q called the *induced neighborhood* of $\mathscr{U}$ on A. If A is a singleton so that p_A is a constant map, we let $\mathscr{U}^A = \bigcap_{j=1}^n \langle \{t\}, U_j \rangle$. The next remark illustrates some elementary but useful properties of restricted and induced neighborhoods.

Remark 2.5. For every closed interval $A \subseteq I$ and open neighborhood $\mathscr{U} = \bigcap_{j=1}^n \langle C_j, U_j \rangle$ in $M(I, X)$ such that $\bigcup_{j=1}^n C_j = I$, we have:

1. $(\mathscr{U}^A)_A = \mathscr{U} \subseteq (\mathscr{U}_A)^A$.
2. If $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n = 1$, then $\mathscr{U} = \bigcap_{i=1}^n (\mathscr{U}_{[t_{i-1}, t_i]})^{[t_{i-1}, t_i]}$.

The following lemma is a basic property of free path spaces and allows us to intuit basic open neighborhoods of paths and loops as finite, ordered sets of “instructions.” For details see [15].

Lemma 2.6. *Let $\mathscr{B}_X$ be a basis for the topology of X which is closed under finite intersection. The collection of open neighborhoods of the form $\bigcap_{j=1}^{n}\langle K_n^j, U_j\rangle$ where $U_j \in \mathscr{B}_X$ is a basis for the compact-open topology of the free path space $M(I, X)$. Moreover, this basis is closed under finite intersection.*

2.3. The topological fundamental group

The topological fundamental group of a based space (X, x) is the path component space $\pi_1^{top}(X, x) = \pi_0^{top}(\Omega(X, x))$. Let $I_+ = I \sqcup \{*\}$ denote the unit interval with added isolated basepoint and $\Sigma(I_+)$ its reduced suspension. The natural homeomorphisms $M_*(\Sigma(I_+), X) \cong M(I, \Omega(X, x))$ indicate that homotopy classes of based loops in X are the same as path components in $\Omega(X, x)$. Consequently, $\pi_1^{top}(X, x)$ may be described as the fundamental group of (X, x) with the final topology with respect to the canonical map $\pi : \Omega(X, x) \to \pi_1(X, x)$ identifying homotopy classes of loops.

Remark 2.7. The topology of $\pi_1^{top}(X, x)$ is the finest topology on $\pi_1(X, x)$ such that the canonical map $\pi : \Omega(X, x) \to \pi_1(X, x)$ is continuous.

A *quasitopological group* G is a group with topology such that inversion $G \to G$, $g \mapsto g^{-1}$ is continuous and multiplication $G \times G \to G$ is continuous in each variable (all translations are continuous). A morphism of quasitopological groups is a continuous homomorphism and the category of quasitopological groups is denoted **qTopGrp**. A basic account of the theory of quasitopological groups may be found in [9]. Let $\mathbf{hTop}_*$ be the homotopy category of based topological spaces. We obtain the following two lemmas by combining results and proofs from [3] and [16].

Lemma 2.8. $\pi_1^{top} : \mathbf{hTop}_* \to \mathbf{qTopGrp}$ *is a functor.*

Lemma 2.9. *If x and y lie in the same path component of X, then $\pi_1^{top}(X, x) \cong \pi_1^{top}(X, y)$ as quasitopological groups.*

Since choice of basepoint within path components is irrelevant, we suppress the basepoint and simply write $\pi_1^{top}(X)$ whenever X is path connected.

Proposition 2.10. *For each integer $n \geqslant 1$, the n-th power map $p_n : \pi_1^{top}(X) \to \pi_1^{top}(X)$, $p_n([\alpha]) = [\alpha]^n$ is continuous.*

Proof. The diagonal map $\Delta_n : \Omega(X) \to \Omega(X)^n$, $\Delta_n(\alpha) = (\alpha, \dots, \alpha)$ and the n-fold concatenation map $m_n : \Omega(X)^n \to \Omega(X)$, $m_n(\alpha_1, \dots, \alpha_n) = \alpha_1 * \cdots * \alpha_n$ are continuous. The power map $p_n = \pi_0^{top}(m_n \circ \Delta_n) : \pi_1^{top}(X) \to \pi_1^{top}(X)$ is continuous by the functorality of π_0^{top}. $\square$

There are quasitopological groups with discontinuous power maps. For instance, consider the subset $K = \{\frac{\epsilon}{3^n} \mid n \geqslant 1, \epsilon = \pm 1\}$ of the additive group $\mathbb{R}$ of real numbers. The subbasis consisting of all translates of the sets $(-\delta, \delta) - K$, $\delta > 0$, generates a topology which makes $\mathbb{R}$ a quasitopological group. With this topology, the square map $s : \mathbb{R} \to \mathbb{R}$, $s(t) = 2t$ is discontinuous. In particular, the sequence $\frac{1}{2(3^n)}$ converges to 0 but $s(\frac{1}{2(3^n)}) = \frac{1}{3^n}$ does not. The existence of such groups illustrates another complication: Not every quasitopological group is a topological fundamental group.

In general, it is difficult to determine if the topological fundamental group of a space is a topological group. There are, however, instances when it is easy to answer in the affirmative, namely those spaces X for which $\pi_1^{top}(X)$ has the discrete topology and the added topological structure provides no new information.

Proposition 2.11. *For any path connected based space X, the following are equivalent*:

1. *$\pi_1^{top}(X)$ has the discrete topology.*
2. *$\Omega(X)$ is semilocally 0-connected.*
3. *The singleton $\{[c_x]\}$ containing the identity is open in $\pi_1^{top}(X)$.*
4. *Each null-homotopic loop $\alpha \in \Omega(X)$ lies in an open neighborhood in $\Omega(X)$ containing only null-homotopic loops.*

Proof. 1. ⇔ 2. follows from Remark 2.2. 1. ⇔ 3. is true for all quasitopological groups. 3. ⇔ 4. follows directly from the definition of the quotient topology. $\square$

These obvious characterizations are inconvenient in that they do not characterize discreteness in terms of the topological properties of X itself. The next theorem, proven in [16], is much more useful. Recall that a space is *semilocally simply connected* if for each $x \in X$ there is an open neighborhood U of x such that the inclusion $i : U \hookrightarrow X$ induces the trivial homomorphism $i_* : \pi_1(U, x) \to \pi_1(X, x)$.

Theorem 2.12. *Suppose X is path connected. If $\pi_1^{top}(X)$ is discrete, then X is semilocally simply connected. If X is locally path connected and semilocally simply connected, then $\pi_1^{top}(X)$ is discrete.*

Corollary 2.13. *If X is path connected and has the homotopy type of a polyhedron or manifold, then $\pi_1^{top}(X)$ is a discrete group.*

In [3] and [17], the harmonic archipelago (a non-compact subspace of $\mathbb{R}^3$) is shown to have uncountable, indiscrete topological fundamental group. The next example is a metric space with topological fundamental group isomorphic to the indiscrete group of integers.

Example 2.14. Let $S^1 = \{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ be the unit circle in the xy-plane of $\mathbb{R}^3$. For all integers $n \geqslant 1$, we let

$$C_n = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, \left(x - \frac{1}{n}\right)^2 + y^2 + z^2 = \left(1 + \frac{1}{n}\right)^2\right\}.$$

The space $X = S^1 \cup (\bigcup_{n \geqslant 1} C_n)$ with basepoint $(-1, 0, 0)$ is weakly equivalent to the wedge of spheres $S^1 \vee (\bigvee_{n \geqslant 1} S^2)$. The inclusion $S^1 \hookrightarrow X$ induces a group isomorphism $\mathbb{Z} \cong \pi_1(S^1) \cong \pi_1(X)$, however, every open neighborhood of every loop $\alpha : S^1 \to S^1 \subset X$ contains a loop $\beta : S^1 \to \bigcup_{n \geqslant 1} C_n \subset X$ which is null-homotopic. Therefore every open neighborhood of the class $[\alpha]$ in $\pi_1^{top}(X)$ contains the identity. From this, one may easily use the properties of quasitopological groups to show $\pi_1^{top}(X)$ has the indiscrete topology.

Since there are simple metric spaces with non-trivial, indiscrete topological fundamental group, we cannot take any separation properties for granted.

Proposition 2.15. *Suppose X is path connected. The following are equivalent*:

1. *For each loop $\alpha \in \Omega(X)$ which is not null-homotopic, there is an open neighborhood V of α such that V contains no null-homotopic loops.*
2. *The singleton containing the identity is closed in $\pi_1^{top}(X)$.*
3. *$\pi_1^{top}(X)$ is T_0.*
4. *$\pi_1^{top}(X)$ is T_1.*

Proof. 1. $\Leftrightarrow$ 2. follows from the definition of the quotient topology and 2. $\Leftrightarrow$ 3. $\Leftrightarrow$ 4. holds for all quasitopological groups. □

As the continuity of multiplication is critical in proving that every T_0 topological group is Tychonoff, it can be difficult to prove the existence of separation properties T_i, $i \geqslant 2$ in quasitopological groups. Additionally, the complex nature of homotopy as an equivalence relation further complicates our attempt to characterize higher separation properties in topological fundamental groups. To be able to make any general statement for when $\pi_1^{top}(X)$ is Hausdorff, it will be necessary to find a basis of open neighborhoods at the identity of $\pi_1^{top}(X)$. While computationally challenging, there is a method of constructing a basis of open neighborhoods for any quotient space. We take this approach so that if $q : Y \to Z$ is a quotient map, a basis for Z may be described in terms of open coverings of Y.

Definition 2.16. For any space Y, a *pointwise open cover* of Y is an open cover $\mathscr{U} = \{U^y\}_{y \in Y}$ where each point $y \in Y$ has a distinguished open neighborhood U^y containing it. Let $Cov(Y)$, be the directed set of pointed open covers of Y where the direction is given by pointwise refinement: If $\mathscr{U} = \{U^y\}_{y \in Y}, \mathscr{V} = \{V^y\}_{y \in Y} \in Cov(Y)$, then we say $\mathscr{U} \preccurlyeq \mathscr{V}$ when $V_y \subseteq U_y$ for each $y \in Y$.

We also make use of the following notation: If $\mathscr{U} = \{U^y\}_{y \in Y} \in Cov(Y)$ is a pointwise open covering of Y and $A \subseteq Y$, let $\mathscr{U}(A) = \bigcup_{a \in A} U^a$.

Construction of basis 2.17. Suppose $q : Y \to Z$ is quotient map, $z \in Z$, and $\mathscr{U} \in Cov(Y)$ is a fixed point-wise open cover. We construct open neighborhoods of z in Z in the most unabashed way, that is, by recursively "collecting" the elements of Y so that our collection is both open and saturated. We begin by letting $\mathcal{O}_q^0(z, \mathscr{U}) = \{z\}$. For integer $n \geqslant 1$, we define $\mathcal{O}_q^n(z, \mathscr{U}) \subseteq Z$ as

$$\mathcal{O}_q^n(z, \mathscr{U}) = q\big(\mathscr{U}\big(q^{-1}\big(\mathcal{O}_q^{n-1}(z, \mathscr{U})\big)\big)\big).$$

It is clear that $\mathcal{O}_q^{n-1}(z,\mathscr{U}) \subseteq \mathcal{O}_q^n(z,\mathscr{U})$ for all $n \geqslant 1$. We then may take the union

$$\mathcal{O}_q(z,\mathscr{U}) = \bigcup_n \mathcal{O}_q^n(z,\mathscr{U}).$$

Note that if $y \in q^{-1}(\mathcal{O}_q(z,\mathscr{U}))$, then $U^y \subset q^{-1}(\mathcal{O}_q(z,\mathscr{U}))$ so that $\mathcal{O}_q(z,\mathscr{U})$ is open in Z. Also, if $\mathscr{W} = \{W^y\}_{y\in Y}$ is another point-wise open cover of Y such that $q(W^y) \subseteq q(U^y)$ for each $y \in Y$, then $\mathcal{O}_q(z,\mathscr{W}) \subseteq \mathcal{O}_q(z,\mathscr{U})$. It is easy to see that for each open neighborhood V of z in Z, there is a pointwise open covering $\mathscr{V} \in Cov(Y)$ such that $z \in \mathcal{O}_q(z,\mathscr{V}) \subseteq V$. In particular, let $V^y = q^{-1}(V)$ when $q(y) \in V$ and $V^y = Y$ otherwise.

In the case of the quotient map $\pi : \Omega(X) \to \pi_1^{top}(X)$, the neighborhoods $\mathcal{O}_\pi([c_x],\mathscr{U})$ for $\mathscr{U} = \{U^\beta\}_{\beta\in\Omega(X)} \in Cov(\Omega(X))$ give a basis for the topology of $\pi_1^{top}(X)$ at the identity $[c_x]$. The loops in $\pi^{-1}(\mathcal{O}_\pi([c_x],\mathscr{U}))$ can be described as follows: For each $\alpha \in \pi^{-1}(\mathcal{O}_\pi([c_x],\mathscr{U}))$, there is an integer $n \geqslant 1$ and a sequence of loops $\gamma_0, \gamma_1, \dots, \gamma_{2n+1}$ where $\gamma_0 = c_x$, $\gamma_{2i} \simeq \gamma_{2i+1}$ for $i = 0, 1, \dots, n$, $\gamma_{2i+1} \in U^{\gamma_{2i+2}}$ for $i = 0, 1, \dots, n-1$, and $\gamma_{2n+1} = \alpha$. In this sense, the neighborhood $\mathcal{O}_\pi([c_x],\mathscr{U})$ is an alternating "collection" of homotopy classes and nearby loops (the nearby being determined by the elements of $\mathscr{U}$). We can state what it means for $\pi_1^{top}(X)$ to be Hausdorff in these terms.

Proposition 2.18. *$\pi_1^{top}(X)$ is Hausdorff if and only if for each class $[\beta] \in \pi_1^{top}(X) - \{[c_x]\}$, there is a pointwise open covering $\mathscr{U} \in Cov(\Omega(X))$ such that $\mathscr{O}_\pi([c_x],\mathscr{U}) \cap \mathscr{O}_\pi([\beta],\mathscr{U}) = \emptyset$.*

Proof. If $\pi_1^{top}(X)$ is Hausdorff and $[\beta] \in \pi_1^{top}(X) - \{[c_x]\}$, we can find disjoint open neighborhoods W of $[c_x]$ and V of $[\beta]$. Now we may find pointwise open coverings $\mathscr{W} = \{W^\alpha\}_{\alpha\in\Omega(X)}, \mathscr{V} = \{V^\alpha\}_{\alpha\in\Omega(X)} \in Cov(\Omega(X))$ such that $\mathscr{O}_\pi([c_x],\mathscr{W}) \subset W$ and $\mathscr{O}_\pi([\beta],\mathscr{V}) \subset V$. We let $\mathscr{W} \cap \mathscr{V} = \{W^\alpha \cap V^\alpha\}_{\alpha\in\Omega(X)} \in Cov(\Omega(X))$ be the intersection of the two. Since $\mathscr{W}, \mathscr{V} \preccurlyeq \mathscr{W} \cap \mathscr{V}$, we have $\mathscr{O}_\pi([c_x],\mathscr{W}\cap\mathscr{V}) \subseteq \mathscr{O}_\pi([c_x],\mathscr{W}) \subset W$ and $\mathscr{O}_\pi([\beta],\mathscr{W}\cap\mathscr{V}) \subseteq \mathscr{O}_\pi([\beta],\mathscr{V}) \subset V$.

If the second statement holds, we suppose that $[\beta_1]$ and $[\beta_2]$ are distinct classes in $\pi_1^{top}(X)$. Therefore $[\beta_1 * \beta_2^{-1}] \neq [c_x]$ and by assumption there is a $\mathscr{U} \in Cov(\Omega(X))$ such that $\mathscr{O}_\pi([c_x],\mathscr{U}) \cap \mathscr{O}_\pi([\beta_1 * \beta_2^{-1}],\mathscr{U}) = \emptyset$. Since right multiplication by $[\beta_2]$ is a homeomorphism, we have that $\mathscr{O}_\pi([\beta_1 * \beta_2^{-1}],\mathscr{U})[\beta_2]$ is open containing $[\beta_1]$ and $\mathscr{O}_\pi([c_x],\mathscr{U})[\beta_2]$ is open containing $[\beta_2]$. But $(\mathscr{O}_\pi([\beta_1 * \beta_2^{-1}],\mathscr{U})[\beta_2]) \cap (\mathscr{O}_\pi([c_x],\mathscr{U})[\beta_2]) = \emptyset$ and so $\pi_1^{top}(X)$ is Hausdorff. □

Though this approach is quite general and offers intuition for the topological structure of topological fundamental groups, it is difficult to use in practice. The author thanks Paul Fabel for pointing out the following use of shape homotopy groups. We refer to [18] for all preliminaries of shape theory.

Remark 2.19. Since the homotopy category of Polyhedra $\mathbf{hPol}_*$ is dense in $\mathbf{Top}_*$, every based space X has an $\mathbf{hPol}_*$-expansion $X \to (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ consisting of maps $p_\lambda : X \to X_\lambda$ such that $p_\lambda = p_{\lambda\lambda'} p_{\lambda'}$ whenever $\lambda' \geqslant \lambda$. Since each X_λ has the homotopy type of a polyhedron, each $\pi_1^{top}(X_\lambda)$ is a discrete group by Theorem 2.12. The maps p_λ induce continuous homomorphisms $(p_\lambda)_* : \pi_1^{top}(X) \to \pi_1^{top}(X_\lambda)$ which, in turn, induce a continuous homomorphism $\phi : \pi_1^{top}(X) \to \varprojlim_\lambda \pi_1^{top}(X_\lambda)$ to the inverse limit of discrete groups. The underlying group $\varprojlim_\lambda \pi_1(X_\lambda)$ is the first shape homotopy group of X. If ϕ is injective, $\pi_1^{top}(X)$ continuously injects into a functionally Hausdorff[3] topological group and must be functionally Hausdorff. Some recent results on the injectivity of ϕ include [19–22].

3. A computation of $\pi_1^{top}(\Sigma(X_+))$

3.1. The spaces $\Sigma(X_+)$

Let X be an arbitrary topological space and $X_+ = X \sqcup \{*\}$ be the based space with added isolated basepoint. Let

$$\big(\Sigma(X_+), x_0\big) = \left(\frac{X_+ \times I}{X \times \{0,1\} \cup \{*\} \times I}, x_0\right)$$

be the reduced suspension of X_+ with canonical choice of basepoint and $x \wedge s$ denote the image of $(x,s) \in X \times I$ under the quotient map $X_+ \times I \to \Sigma(X_+)$. For subsets $A \subseteq X$ and $S \subseteq I$, let $A \wedge S = \{a \wedge s \mid a \in A,\ s \in S\}$. A subspace $P \wedge I$ where $P \in \pi_0(X)$ is a path component of X is called a *hoop* of $\Sigma(X_+)$.

Suppose $\mathscr{B}_X$ is a basis for the topology of X which is closed under finite intersections. For a point $x \wedge t \in X \wedge (0,1) = \Sigma(X_+) - \{x_0\}$, a subset $U \wedge (c,d)$ where $x \in U$, $U \in \mathscr{B}_X$ and $t \in (c,d) \subseteq (0,1)$ is an open neighborhood of $x \wedge t$. Open neighborhoods of x_0 may be given in terms of open coverings of $X \times \{0,1\}$ in $X \times I$. If $U^x \in \mathscr{B}_X$ is an open neighborhood of x in X and $t_x \in (0, \frac{1}{8})$, the set

[3] A space is functionally Hausdorff if distinct points may be separated by continuous real valued functions.

$$\bigcup_{x\in X}\big(U^x \wedge [0,t_x)\cup(1-t_x,1]\big)$$

is an open neighborhood of x_0 in $\Sigma(X_+)$. The collection $\mathscr{B}_{\Sigma(X_+)}$ of neighborhoods of the form $U\wedge(c,d)$ and $\bigcup_{x\in X}(U^x\wedge[0,t_x)\cup(1-t_x,1])$ is a basis for the topology $\Sigma(X_+)$ which is closed under finite intersection.

Remark 3.1. For an arbitrary space X:

1. $\Sigma(X_+)$ is path-connected.
2. $\Sigma(X_+)-\{x_0\}=X\wedge(0,1)\cong X\times(0,1)$.
3. Every basic neighborhood $V\in\mathscr{B}_{\Sigma(X_+)}$ containing x_0 is arc connected and simply connected.
4. For each $t\in(0,1)$, the closed subspaces $X\wedge[0,t]$ and $X\wedge[t,1]$ are homeomorphic to CX the cone of X, and are contractible to the basepoint point x_0.
5. $\Sigma(X_+)$ is Hausdorff if and only if X is Hausdorff, but the following holds for arbitrary X: For each point $x\wedge t\in X\wedge(0,1)$, there are disjoint open neighborhoods separating $x\wedge t$ and the basepoint x_0.

Remark 3.2. It is a well-known fact that the reduced suspension functor $\Sigma:\mathbf{Top}_*\to\mathbf{Top}_*$ is left adjoint to the loop space functor $\Omega:\mathbf{Top}_*\to\mathbf{Top}_*$. Additionally, adding isolated basepoint to an unbased space $(-)_+:\mathbf{Top}\to\mathbf{Top}_*$ is left adjoint to the functor $U:\mathbf{Top}_*\to\mathbf{Top}$ forgetting basepoint. Taking composites, we see the construction $\Sigma((-)_+):\mathbf{Top}\to\mathbf{Top}_*$ is a functor left adjoint to $U\Omega$. For a map $f:X\to Y$, the map $\Sigma(f_+):\Sigma(X_+)\to\Sigma(Y_+)$ is defined by $\Sigma(f_+)(x\wedge s)=f(x)\wedge s$. The adjunction is illustrated by natural homeomorphisms

$$M_*\big(\Sigma(X_+),Y\big)\cong M_*\big(X_+,\Omega(Y)\big)\cong M\big(X,U\Omega(Y)\big).$$

This adjunction immediately gives motivation for our proposed computation of $\pi_1^{top}(\Sigma(X_+))$. We say a quotient group G/N of a quasitopological group G is a *topological quotient group* when G/N has the quotient topology with respect to the canonical projection $G\to G/N$.

Proposition 3.3. *Every topological fundamental group $\pi_1^{top}(Y)$ is the topological quotient group of $\pi_1^{top}(\Sigma(X_+))$ for some space X.*

Proof. Let $cu:\Sigma(\Omega(Y)_+)\to Y$ be the adjoint of the unbased identity of $\Omega(Y)$. The basic property of counits gives that the unbased map $U\Omega(cu):\Omega(\Sigma(\Omega(Y)_+))\to\Omega(Y)$ is a topological retraction. Applying the path component functor, we obtain a group epimorphism $\pi_1^{top}(\Sigma(\Omega(Y)_+))\to\pi_1^{top}(Y)$ which is, by Remark 2.3, a quotient map of spaces. Take $X=\Omega(Y)$. □

Since topological quotient groups of topological groups are topological groups the spaces $\Sigma(X_+)$ are prime candidates for counterexamples to the claim that π_1^{top} is a functor to the category of topological groups. It is convenient to view the spaces $\Sigma(X_+)$ as natural generalizations of wedges of circles. Intuitively, one might think of $\Sigma(X_+)$ as a "wedge of circles parameterized by the space X." Let $\bigvee_X S^1$ be the wedge of circles indexed by the underlying set of X. Suppose $\epsilon:I\to S^1$ is the exponential map and a point in the x-th summand of the wedge is denoted as $\epsilon(t)_x$ for $t\in I$. The pushout property implies that every map $f:X\to Y$ induces a map $\bigvee_X S^1\to\bigvee_Y S^1$ given by $\epsilon(t)_x\mapsto\epsilon(t)_{f(x)}$ for all $t\in I$, $x\in X$. It is easy to see that $\bigvee_{(-)}S^1:\mathbf{Top}\to\mathbf{Top}_*$ is a functor which we may relate to $\Sigma((-)_+)$ in the following way.

Remark 3.4. There is a natural transformation $\gamma:\bigvee_{(-)}S^1\to\Sigma((-)_+)$ where each component $\gamma_X:\bigvee_X S^1\to\Sigma(X_+)$ given by $\gamma_X(\epsilon(t)_x)=x\wedge t$ is a continuous bijection. Moreover, γ_X is a homeomorphism if and only X has the discrete topology.

According to this remark, if X has the discrete topology, then $\Sigma(X_+)$ is homeomorphic to a wedge of circles. By the Van Kampen Theorem and Theorem 2.12, $\pi_1^{top}(\Sigma(X_+))$ must be isomorphic to the discrete free group $F(X)$. We will see later on that $\pi_1^{top}(\Sigma(X_+))$ is discrete if and only if X is semilocally 0-connected.

3.2. Free topological monoids and the James map

Prior to computing $\pi_1^{top}(\Sigma(X_+))$, we recall some common constructions in algebraic topology and topological algebra. Let Y^{-1} denote a homeomorphic copy of an unbased space Y. The *free topological monoid* on the disjoint union $Y\sqcup Y^{-1}$ is

$$M_T^*(Y)=\coprod_{n\geqslant 0}\big(Y\sqcup Y^{-1}\big)^n\quad\text{where }\big(Y\sqcup Y^{-1}\big)^0=\{e\}.$$

A non-empty element of $M_T^*(Y)$ is a word $y_1^{\epsilon_1}y_2^{\epsilon_2}\dots y_n^{\epsilon_n}$ where $\epsilon_i\in\{\pm1\}$ and $y_i^{\epsilon_i}\in Y^{\epsilon_i}$. Multiplication is concatenation of words and the identity is the disjoint empty word e. The underlying monoid of $M_T^*(Y)$ is $M^*(Y)$. The length of a word $w=y_1^{\epsilon_1}y_2^{\epsilon_2}\dots y_n^{\epsilon_n}$ is $|w|=n$ and we set $|e|=0$. For each finite (possibly empty) sequence $\zeta=\{\epsilon_i\}_{i=1}^n$ where $\epsilon_i\in\{\pm1\}$, let $Y^\zeta = \{y_1^{\epsilon_1} y_2^{\epsilon_2} \ldots y_n^{\epsilon_n} \mid y_i \in Y\}$ (so $Y^\emptyset = \{e\}$ in the empty case). Since $M_T^*(Y) = \coprod_\zeta Y^\zeta$ each word $w = y_1^{\epsilon_1} y_2^{\epsilon_2} \ldots y_n^{\epsilon_n}$ has a neighborhood base consisting of products $U_1^{\epsilon_1} U_2^{\epsilon_2} \ldots U_n^{\epsilon_n}$ where U_i is an open neighborhood of y_i in Y. A word w is *reduced* if, for each $i = 1, 2, \ldots, n-1$, $y_i = y_{i+1}$ implies $\epsilon_i = \epsilon_{i+1}$. The empty word is vacuously reduced. The collection of reduced words forms the free group $F(Y)$ on the underlying set of Y and the monoid epimorphism $R : M^*(Y) \to F(Y)$ denotes the usual reduction of words. We can describe the functorial nature of M_T^* using the next definition.

Definition 3.5. A *topological monoid with continuous involution* is a pair (M, s) where M is a topological monoid with identity e and $s : M \to M$ is a continuous involution (i.e. $s^2 = id$, $s(mn) = s(n)s(m)$, and $s(e) = e$). A morphism $f : (M_1, s_1) \to (M_2, s_2)$ of two such pairs is a continuous homomorphism $f : M_1 \to M_2$ such that f preserves involution, i.e. $f \circ s_1 = s_2 \circ f$. Let **TopMon*** be the category of topological monoids with continuous involution and continuous, involution-preserving homomorphisms.

A continuous involution for $M_T^*(Y)$ is given by $(y_1^{\epsilon_1} y_2^{\epsilon_2} \ldots y_n^{\epsilon_n})^{-1} = y_n^{-\epsilon_n} y_{n-1}^{-\epsilon_{n-1}} \ldots y_1^{-\epsilon_1}$ making the pair $(M_T^*(Y), ^{-1})$ an object of **TopMon***. One simply uses the universal properties of coproducts and free topological monoids to prove the next proposition.

Proposition 3.6. *The functor* $M_T^* :$ **Top** $\to$ **TopMon*** *is left adjoint to the forgetful functor* **TopMon*** $\to$ **Top**.

We relate free topological monoids to $\pi_1^{top}(\Sigma(X_+))$, via the unbased "James map" $u : X \to \Omega(\Sigma(X_+))$, $u(x)(t) = u_x(t) = x \wedge t$. This allows us to define a natural embedding $\mathscr{J} : M_T^*(X) \to \Omega(\Sigma(X_+))$ taking the empty word to the constant map and $\mathscr{J}(x_1^{\epsilon_1} x_2^{\epsilon_2} \ldots x_n^{\epsilon_n}) = *_{i=1}^n (u_{x_i}^{\epsilon_i})$. These constructions follow the well-known James construction [23, 5.3] used originally by I.M. James to study the geometry of $\Omega(\Sigma Z, *)$ for a connected CW-complex Z.

3.3. The fundamental group $\pi_1(\Sigma(X_+))$

Throughout the rest of this section let $\pi_X : X \to \pi_0^{top}(X)$ and $\pi_\Omega : \Omega(\Sigma(X_+)) \to \pi_1^{top}(\Sigma(X_+))$ denote the canonical quotient maps.

To study $\pi_1^{top}(\Sigma(X_+))$, we must first understand the algebraic structure of $\pi_1(\Sigma(X_+))$. We begin by observing that the James map $u : X \to \Omega(\Sigma(X_+))$ induces a continuous map $u_* : \pi_0^{top}(X) \to \pi_0^{top}(\Omega(\Sigma(X_+))) = \pi_1^{top}(\Sigma(X_+))$ on path component spaces. The underlying function $u_* : \pi_0(X) \to \pi_1(\Sigma(X_+))$ induces a group homomorphism $h_X : F(\pi_0(X)) \to \pi_1(\Sigma(X_+))$ on the free group generated by the path components of X. In particular, h_X takes the reduced word $P_1^{\epsilon_1} P_2^{\epsilon_2} \ldots P_k^{\epsilon_k}$ (where $P_i \in \pi_0(X)$ and $\epsilon_i \in \{\pm 1\}$) to the homotopy class $[u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \cdots * u_{x_k}^{\epsilon_k}]$ where $x_i \in P_i$ for each i. We show that h_X is a group isomorphism. To do this, we require the next definition which will also be used in the proof of Theorem 1.1.

Definition 3.7. A loop $\alpha \in M(I, \{0,1\}; Y, \{y\})$ is *simple* if $\alpha^{-1}(y) = \{0, 1\}$. The subspace of $M(I, \{0,1\}; Y, \{y\})$ consisting of simple loops is denoted $\Omega_s(Y)$.

Remark 3.8. Ω_s is not a functor since it is not well defined on morphisms. It is easy to see, however, that $\Omega_s(\Sigma((-)_+)) :$ **Top** $\to$ **Top** is a functor.

The map $X \to \{*\}$ collapsing X to a point induces a retraction $r : \Sigma(X_+) \to \Sigma S^0 \cong S^1$. This, in turn, induces a retraction $r_* : \pi_1^{top}(\Sigma(X_+)) \to \pi_1^{top}(S^1) \cong \mathbb{Z}$ onto the discrete group of integers. By the previous remark, if $\alpha \in \Omega_s(\Sigma(X_+))$, then $r \circ \alpha : I \to S^1$ is a simple loop in S^1. But the homotopy class of a simple loop in S^1 is either the identity or a generator of $\pi_1^{top}(S^1)$. Therefore $r_*([\alpha])$ must take on the value $1, 0$ or -1.

Definition 3.9. A simple loop $\alpha \in \Omega_s(\Sigma(X_+))$ has *positive (resp. negative) orientation* if $[\alpha] \in r_*^{-1}(1)$ (resp. $[\alpha] \in r_*^{-1}(-1)$). If $[\alpha] \in r_*^{-1}(0)$, then we say α has no orientation and is *trivial*. The subspaces of $\Omega_s(\Sigma(X_+))$ consisting of simple loops with positive, negative, and no orientation are denoted $\Omega_{+s}(\Sigma(X_+))$, $\Omega_{-s}(\Sigma(X_+))$, and $\Omega_{0s}(\Sigma(X_+))$ respectively.

The fact that $\mathbb{Z}$ is discrete, allows us to write the loop space $\Omega(\Sigma(X_+))$ as the disjoint union of the subspaces $\pi_\Omega^{-1}(r_*^{-1}(n))$, $n \in \mathbb{Z}$. Consequently, we may write $\Omega_s(\Sigma(X_+))$ as disjoint union

$$\Omega_s\big(\Sigma(X_+)\big) = \Omega_{+s}\big(\Sigma(X_+)\big) \sqcup \Omega_{0s}\big(\Sigma(X_+)\big) \sqcup \Omega_{-s}\big(\Sigma(X_+)\big).$$

We also note that $\Omega_{-s}(\Sigma(X_+)) = \Omega_{+s}(\Sigma(X_+))^{-1}$. Thus loop inversion give a homeomorphism $\Omega_{+s}(\Sigma(X_+)) \cong \Omega_{-s}(\Sigma(X_+))$. The next two lemmas are required to prove the surjectivity of h_X.

Lemma 3.10. *A simple loop $\alpha \in \Omega_s(\Sigma(X_+))$ is null-homotopic if and only if it is trivial.*

Proof. By definition, a simple loop which has orientation is not null-homotopic. Therefore, it suffices to show that any trivial loop is null-homotopic. If α is trivial, then α does not traverse any hoop of $\Sigma(X_+)$, i.e. there is a $t \in (0,1)$ such that α has image in either $X \wedge [0,t]$ or $X \wedge [t,1]$. By Remark 3.1.4, α is null-homotopic. □

The subspaces $P \wedge (0,1)$, $P \in \pi_0(X)$ are precisely the path components of $X \wedge (0,1)$. Therefore, if $p : I \to \Sigma(X_+)$ is a path such that $p(0) \in P_1 \wedge (0,1)$ and $p(1) \in P_2 \wedge (0,1)$ for distinct $P_1, P_2 \in \pi_0(X)$ (i.e. the endpoints of p lie in distinct hoops and are not the basepoint x_0), then there is a $t \in (0,1)$ such that $p(t) = x_0$. This implies that the image of each simple loop lies entirely within a single hoop.

Lemma 3.11. *If simple loops α and β have the same orientation and have image in the same hoop $P \wedge I$, then they are homotopic.*

Proof. Suppose α and β have positive orientation and image in $P \wedge I$. Since $P \wedge (1,0)$ is a path component of $X \wedge (0,1)$, we may find a $t \in (0,1)$ and a path $p : I \to X \wedge (0,1)$ such that $p(0) = \alpha(t)$ and $p(1) = \beta(t)$. Now

$$\alpha_{[0,t]} * p * \beta^{-1}_{[0,t]} \quad \text{and} \quad \beta^{-1}_{[t,1]} * p^{-1} * \alpha_{[t,1]}$$

are trivial simple loops which by the previous lemma must be null homotopic. This gives fixed endpoint homotopies of paths

$$\alpha_{[0,t]} \simeq \beta_{[0,t]} * p^{-1} \quad \text{and} \quad \alpha_{[t,1]} \simeq p * \beta_{[t,1]}.$$

The concatenation of these two gives

$$\alpha \simeq \alpha_{[0,t]} * \alpha_{[t,1]} \simeq \beta_{[0,t]} * p^{-1} * p * \beta_{[t,1]} \simeq \beta_{[0,t]} * \beta_{[t,1]} \simeq \beta.$$

One may simply invert loops to prove the case of negative orientation. □

We require the next lemma and remark to prove the injectivity of h_X.

Lemma 3.12. *If $w = P_1^{\epsilon_1} \dots P_n^{\epsilon_n} \in F(\pi_0(X))$ is a non-empty reduced word such that $\sum_{i=1}^n \epsilon_i \neq 0$, then $h_X(w)$ is not the identity of $\pi_1(\Sigma(X_+))$.*

Proof. The retraction $r : \Sigma(X_+) \to S^1$ induces an epimorphism $r_* : \pi_1(\Sigma(X_+)) \to \mathbb{Z}$ on fundamental groups, where $r_*([u_x]^\epsilon) = \epsilon$ for each $x \in X$ and $\epsilon \in \{\pm 1\}$. Therefore, if $\sum_{i=1}^n \epsilon_i \neq 0$, then $r_*(h_X(w)) = r_*([u_{x_1}^{\epsilon_1} * \dots * u_{x_n}^{\epsilon_n}]) = \sum_{i=1}^n \epsilon_i \neq 0$ (where $x_i \in P_i$) and $h_X(w)$ cannot be the identity of $\pi_1(\Sigma(X_+))$. □

Remark 3.13. Let $P_1^{\epsilon_1} \dots P_n^{\epsilon_n} \in F(\pi_0(X))$ be a reduced word.

1. If $1 \leqslant k \leqslant m \leqslant n$, the subword $P_k^{\epsilon_k} \dots P_m^{\epsilon_m}$ is also reduced.
2. If $n \geqslant 2$ and $\sum_{i=1}^n \epsilon_i = 0$, then there are $i_0, i_1 \in \{1, 2, \dots, n\}$ such that $P_{i_0} \neq P_{i_1}$.

Theorem 3.14. *$h_X : F(\pi_0(X)) \to \pi_1(\Sigma(X_+))$ is an isomorphism of groups.*

Proof. To show that h_X is surjective, we suppose $\alpha \in \Omega(\Sigma(X_+))$ is an arbitrary loop. The pullback $\alpha^{-1}(\Sigma(X_+) - \{x_0\}) = \coprod_{m \in M}(c_m, d_m)$ is an open subset of $(0,1)$. Each restriction $\alpha_m = \alpha_{[c_m,d_m]}$ is a simple loop, and by Remark 3.1.5, all but finitely many of the α_m have image in the simply connected neighborhood $X \wedge [0, \frac{1}{8}) \sqcup (\frac{7}{8}, 1]$. Therefore α is homotopic to a finite concatenation of simple loops $\alpha_{m_1} * \alpha_{m_2} * \dots * \alpha_{m_n}$. By Lemma 3.10, we may suppose that each α_{m_i} has orientation $\epsilon_i \in \{\pm 1\}$ and image in hoop $P_i \wedge I$. Lemma 3.11 then gives that $\alpha_{m_i} \simeq u_{x_i}^{\epsilon_i}$ for any $x_i \in P_i$. But then

$$h_X\big(P_1^{\epsilon_1} P_2^{\epsilon_2} \dots P_n^{\epsilon_n}\big) = \big[u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \dots * u_{x_n}^{\epsilon_n}\big] = [\alpha_{m_1} * \alpha_{m_2} * \dots * \alpha_{m_n}] = [\alpha].$$

For injectivity, we suppose $w = P_1^{\epsilon_1} P_2^{\epsilon_2} \dots P_n^{\epsilon_n}$ is a non-empty reduced word in $F(\pi_0(X))$. It suffices to show that $h_X(w) = [u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \dots * u_{x_n}^{\epsilon_n}]$ is non-trivial when $x_i \in P_i$ for each i. We proceed by induction on n and note that Lemma 3.12 gives the first step of induction $n = 1$. Suppose $n \geqslant 2$ and $h_X(v)$ is non-trivial for all reduced words $v = Q_1^{\delta_1} Q_2^{\delta_2} \dots Q_j^{\delta_j}$ of length $j < n$. By Lemma 3.12, it suffices to show that $h_X(w) = [u_{x_1}^{\epsilon_1} * u_{x_1}^{\epsilon_1} * \dots * u_{x_n}^{\epsilon_n}]$ is non-trivial when $\sum_{i=1}^n \epsilon_i = 0$. We suppose otherwise, i.e. that there is a homotopy of based loops $H : I^2 \to \Sigma(X_+)$ such that $H(t,0) = x_0$ and $H(t,1) = (u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \dots * u_{x_n}^{\epsilon_n})(t)$ for all $t \in I$. For $j = 0, 1, \dots, 2n$, we let $b_j = (\frac{j}{2n}, 1) \in I^2$. Remark 3.1.5 indicates that the singleton $\{x_0\}$ is closed in $\Sigma(X_+)$ so that $H^{-1}(x_0)$ is a compact subset of I^2. Since each $u_{x_i}^{\epsilon_i}$ is simple we have that

$$H^{-1}(x_0) \cap \partial\big(I^2\big) = \{0,1\} \times I \cup I \times \{0\} \sqcup \coprod_{i=1}^{n-1} \{b_{2i}\}$$

where ∂ denotes boundary in $\mathbb{R}^2$. We also have that $H(b_{2i-1}) = u_{x_i}^{\epsilon_i}(\frac{1}{2}) = x_i \wedge \frac{1}{2} \neq x_0$ for each $i = 1, \ldots, n$. This allows us to find an $r_0 > 0$ so that when $U_i = B(b_{2i-1}, r_0) \cap I^2$ is the ball of radius r_0 about b_{2i-1} in I^2, we have $H^{-1}(x_0) \cap \bigcup_{i=1}^n U_j = \emptyset$. Now we find an $r_1 \in (0, r_0)$ and cover $H^{-1}(x_0)$ with finitely many open balls $V_l = B(z_l, r_1) \cap I^2$ so that

$$\left(\bigcup_l \overline{V_l}\right) \cap \left(\bigcup_{i=1}^n \overline{U_i}\right) = \emptyset \quad \text{and} \quad H\left(\bigcup_l V_l\right) \subseteq \left[0, \frac{1}{8}\right) \sqcup \left(\frac{7}{8}, 1\right] \wedge X$$

(which is possible since H is continuous). Note that if $q : I \to \bigcup_l V_l$ is a path with endpoints $q(0), q(1) \in H^{-1}(x_0)$, then the loop $H \circ q : I \to \Sigma(X_+)$ is based at x_0 and has image in the simply connected neighborhood $X \wedge [0, \frac{1}{8}) \sqcup (\frac{7}{8}, 1]$, and therefore must be null-homotopic. We note that there is no path $q : I \to \bigcup_l V_l$ such that $q(0) = b_{2k}$, $q(1) = b_{2m}$ for $1 \leqslant k < m \leqslant n$. If $q : I \to \bigcup_l V_l$ is such a path, the concatenation $u_{x_{k+1}}^{\epsilon_{k+1}} * u_{x_{k+2}}^{\epsilon_{k+2}} * \cdots * u_{x_m}^{\epsilon_m}$ is null-homotopic since $H \circ q$ is null-homotopic and $(H \circ q) \simeq u_{x_{k+1}}^{\epsilon_{k+1}} * u_{x_{k+2}}^{\epsilon_{k+2}} * \cdots * u_{x_m}^{\epsilon_m}$. This means that $h_X(P_{k+1}^{\epsilon_{k+1}} P_{k+2}^{\epsilon_{k+2}} \ldots P_m^{\epsilon_m}) = [u_{x_{k+1}}^{\epsilon_{k+1}} * u_{x_{k+2}}^{\epsilon_{k+2}} * \cdots * u_{x_m}^{\epsilon_m}]$ is the identity of $\pi_1(\Sigma(X_+))$. But by Remark 3.13.1 $P_{k+1}^{\epsilon_{k+1}} P_{k+2}^{\epsilon_{k+2}} \ldots P_m^{\epsilon_m}$ is a reduced word of length $< n$ and so by our induction hypothesis $h_X(P_{k+1}^{\epsilon_{k+1}} P_{k+2}^{\epsilon_{k+2}} \ldots P_m^{\epsilon_m})$ cannot be the identity.

Since such paths q do not exist, each b_{2i} lies in a distinct path component (and consequently connected component) of $\bigcup_l V_l$ for each $i = 1, \ldots, n$. Let $C_i = \bigcup_{m=1}^{M_i} V_{l_m}^i$ be the path component of $\bigcup_l V_l$ containing b_{2i}. But this means the b_{2i-1}, $i = 1, \ldots, n$, all lie in the same path component of $I^2 - \bigcup_l V_l$. Specifically, the subspace

$$\left(\partial\left(I^2\right) - \bigcup_l V_l\right) \cup \left(\partial\left(\bigcup_{i=1}^n C_i\right) - \partial\left(I^2\right)\right)$$

is path connected and contains each of the b_{2i-1}. Since we were able to assume that $\sum_i \epsilon_i = 0$, we know by Remark 3.13.2 that there are $i_0, i_1 \in \{1, \ldots, n\}$ such that $P_{i_0} \neq P_{i_1}$. We have shown that there is a path $p : I \to I^2 - \bigcup_l V_l$ with $p(0) = b_{2i_0-1}$ and $p(1) = b_{2i_1-1}$. But then $H \circ p : I \to \Sigma(X_+)$ is a path with $x_0 \notin H \circ p(I)$, $H(p(0)) = \frac{1}{2} \wedge x_{i_0}$, and $H(p(1)) = \frac{1}{2} \wedge x_{i_1}$. But this is impossible as $H(p(0))$ and $H(p(1))$ lie in different hoops of $\Sigma(X_+)$. Therefore $u_{x_1}^{\epsilon_1} * u_{x_2}^{\epsilon_2} * \cdots * u_{x_n}^{\epsilon_n}$ cannot be null-homotopic. □

Corollary 3.15. *The fibers of the map* $\pi_\Omega \circ \mathscr{J} : M_T^*(X) \to \pi_1^{top}(\Sigma(X_+))$ *are equal to those of* $R \circ M_T^*(\pi_X) : M_T^*(X) \to M_T^*(\pi_0^{top}(X)) \to F(\pi_0(X))$.

Since $u_x \simeq u_y$ if and only if x and y lie in the same path component of X, we denote the homotopy class of u_x by $[u_P]$ where P is the path component of x in X. Thus $\{[u_P] | P \in \pi_0(X)\}$ freely generates $\pi_1(\Sigma(X_+))$. This computation also indicates that the map $u_* : \pi_0^{top}(X) \to \pi_1^{top}(\Sigma(X_+))$ is injective.

3.4. Topologies on free groups

In the effort to recognize the topological structure of $\pi_1^{top}(\Sigma(X_+))$, we observe three comparable but potentially distinct topologies on $F(\pi_0(X))$. We proceed from the coarsest to the finest topology.

Free Topological Groups 3.16. The *free (Markov) topological group* on an unbased space Y is the unique topological group $F_M(Y)$ with a continuous map $\sigma : Y \to F_M(Y)$ universal in the sense that for any map $f : Y \to G$ to a topological group G, there is a unique continuous homomorphism $\tilde{f} : F_M(Y) \to G$ such that $f = \tilde{f} \circ \sigma$. Using Taut liftings or the General Adjoint Functor Theorem [24], it can be shown that $F_M(Y)$ exists for every space Y and that $F_M : \mathbf{Top} \to \mathbf{TopGrp}$ is a functor left adjoint to the forgetful functor $\mathbf{TopGrp} \to \mathbf{Top}$. Moreover, the underlying group of $F_M(Y)$ is the free group $F(Y)$ on the underlying set of Y and $\sigma : Y \to F_M(Y)$ is the canonical injection of generators. The reader is referred to [25] for proofs of the following basic facts. Thomas actually proves that the free abelian topological group functor preserves quotients, however, the non-abelian case follows in the same manner.

Remark 3.17. Let Y be an arbitrary unbased space.

1. $F_M(Y)$ is Hausdorff if and only if Y is functionally Hausdorff.
2. $\sigma : Y \to F_M(Y)$ is an embedding if and only if Y is completely regular.
3. If $q : X \to Y$ is a quotient map, then so is $F_M(q) : F_M(X) \to F_M(Y)$.

Reduction Topology 3.18. Since we expect the canonical map $\Omega(\Sigma(X_+)) \to \pi_1(\Sigma(X_+))$ to make identifications similar to those made by reduction of words $R : M^*(\pi_0(X)) \to F(\pi_0(X))$, a natural choice of topology on the free group $F(\pi_0(X))$ is the quotient topology with respect to $R : M_T^*(\pi_0^{top}(X)) \to F(\pi_0(X))$. More generally, give the free group $F(Y)$ the quotient topology with respect to $R : M_T^*(Y) \to F(Y)$ for each space Y. We refer to this topology as the *reduction topology* and denote the resulting group with topology as $F_R(Y)$. Using arguments similar to those in the proof of Lemma 2.8 one can easily prove:

Proposition 3.19. *F_R : **Top** $\to$ **qTopGrp** is a functor and $R : M_T^* \to F_R$ is a natural transformation each component of which is a monoid epimorphism and topological quotient map.*

Proposition 3.20. *The identity homomorphism $F_R(Y) \to F_M(Y)$ is continuous and is a homeomorphism if and only if $F_R(Y)$ is a topological group. If Y is Tychonoff, then $F_R(Y)$ is a topological group if and only if Y satisfies the equivalent conditions in Fact* 1.3.

Proof. Let $\sigma' : Y \hookrightarrow M_T^*(Y)$ and $\sigma = R \circ \sigma' : Y \to F_R(Y)$ be the canonical continuous injections. Since $F_M(Y)$ is a topological monoid with continuous involution and $\sigma : Y \to F_M(Y)$ is continuous, there is a unique, involution-preserving, continuous monoid homomorphism $\tilde{\sigma} : M_T^*(Y) \to F_M(Y)$ such that $\sigma = \tilde{\sigma} \circ \sigma'$. But the fibers of $\tilde{\sigma}$ are equal to those of the quotient map $R : M_T^*(Y) \to F_R(Y)$. Therefore the identity $F_R(Y) \to F_M(Y)$ is continuous. If $F_R(Y)$ is a topological group, $\sigma : Y \to F_R(Y)$ induces the continuous identity $F_M(Y) \to F_R(Y)$. Clearly then $id : F_R(Y) \cong F_M(Y)$ if and only if $F_R(Y)$ is a topological group. The last statement follows from the obvious fact that $M_T^*(Y) \to F_M(Y)$ is quotient if and only if $id : F_R(Y) \cong F_M(Y)$. □

***q*-Reduction Topology 3.21.** Unfortunately, the groups $\pi_1^{top}(\Sigma(X_+))$ and $F_R(\pi_0^{top}(X))$ are not always homeomorphic. For this reason, we give the following more general construction: Fix a quotient map $q : X \to Y$. The continuous monoid epimorphism $M_T^*(q) = \coprod_{n \geqslant 0} (q \sqcup q)^n : M_T^*(X) \to M_T^*(Y)$ is quotient if and only if $q^n : X^n \to Y^n$ is quotient for each $n \geqslant 1$. Give the monoid $M^*(Y)$ the quotient topology with respect to $M^*(q) : M_T^*(X) \to M^*(Y)$. We denote the resulting monoid with topology as $M_q^*(Y)$. Now give $F(Y)$ the quotient topology (which we refer to as the *q-reduction topology*) with respect to $R : M_q^*(Y) \to F(Y)$ and denote the resulting group with topology as $F_R^q(Y)$. The composite $R \circ M^*(q) : M_T^*(X) \to F_R^q(Y)$ is a monoid epimorphism which is also quotient. The identities $M_q^*(Y) \to M_T^*(Y)$ and $F_R^q(Y) \to F_R(Y)$ are continuous by the universal property of quotient spaces and if $q = id_Y$, then $M_q^*(Y) = M_T^*(Y)$ and $F_R^q(Y) = F_R(Y)$. The following are immediate consequences of these constructions.

Lemma 3.22. *Let $q : X \to Y$ be a quotient map.*

1. *$\sigma' : Y \to M_q^*(Y)$ is an embedding.*
2. *The following are equivalent*:
   (a) *$M_q^*(Y)$ is a topological monoid.*
   (b) *The identity $id : M_q^*(Y) \to M_T^*(Y)$ is a homeomorphism.*
   (c) *$q^n : X^n \to Y^n$ is quotient for each $n \geqslant 1$.*
3. *$\sigma : Y \to F_R^q(Y)$ is continuous.*
4. *The following are equivalent*:
   (a) *The identity $id : F_R^q(Y) \to F_R(Y)$ is a homeomorphism.*
   (b) *$F_R(q) : F_R(X) \to F_R(Y)$ is a topological quotient map.*
5. *If Y satisfies the equivalent conditions in* 2., *then Y satisfies the equivalent conditions in* 4.
6. *The following are equivalent*:
   (a) *$F_R^q(Y)$ is a topological group.*
   (b) *The identities $id : F_R^q(Y) \to F_R(Y)$ and $id : F_R(Y) \to F_M(Y)$ are homeomorphisms.*
   (c) *$F(q) : F_R(X) \to F_M(Y)$ is a quotient map.*

Proof. 1. This follows from the fact that the quotient map $q : X \to Y$ occurs as a summand of the quotient map $M^*(q) : M_T^*(X) \to M_q^*(Y)$.

2. Clearly $id : M_q^*(Y) \cong M_T^*(Y)$ if and only if $M_T^*(q) = \coprod_{n \geqslant 0} (q \sqcup q)^n : M_T^*(X) \to M_T^*(Y)$ is quotient if and only if q^n is quotient for each $n \geqslant 1$. It then suffices to show that $id : M_T^*(Y) \to M_q^*(Y)$ is continuous whenever $M_q^*(Y)$ is a topological monoid. But this follows from 1. and the universal property of $M_T^*(Y)$.

3. This follows from 1. and the equation $\sigma = R \circ \sigma'$.

4. This follows from the fact that $R \circ M^*(q)$ and R in the diagram

$$\begin{array}{ccccc}
M_T^*(X) & \xrightarrow{R} & F_R(X) & \xrightarrow{id} & F_M(X) \\
\downarrow{\scriptstyle R \circ M^*(q)} & & \downarrow{\scriptstyle F_R(q)} & & \downarrow{\scriptstyle F_M(q)} \\
F_R^q(Y) & \xrightarrow[id]{} & F_R(Y) & \xrightarrow[id]{} & F_M(Y)
\end{array}$$

are quotient.

5. If $id : M_q^*(Y) \cong M_T^*(Y)$, then clearly the quotients $F_R^q(Y)$ and $F_R(Y)$ are homeomorphic.

6. (a) ⇔ (b) If $F_R^q(Y)$ is a topological group, then $\sigma : Y \to F_R^q(Y)$ induces the continuous identity $F_M(Y) \to F_R^q(Y)$. Since the identities $F_R^q(Y) \to F_R(Y) \to F_M(Y)$ are always continuous, the topologies of $F_R^q(Y)$, $F_R(Y)$ and $F_M(Y)$ must agree. The converse is obvious. Similar to the proof of 4., (b) ⇔ (c) follows from the fact that $R \circ M^*(q)$ and R in the above diagram are quotient. □

Corollary 3.23. *If $q : X \to Y$ is a quotient map and $F_R(X)$ is a topological group, then $F_R^q(Y)$ is a topological group.*

Proof. Since F_M preserves quotients, $F_M(q) : F_M(X) \to F_M(Y)$ is quotient. If $F_R(X)$ is a topological group, then $id : F_R(X) \cong F_M(Y)$ by Proposition 3.20 and the composite $F(q) : F_R(X) \cong F_M(X) \to F_M(Y)$ is quotient. By 6. of the previous lemma, $F_R^q(Y)$ is a topological group. □

Definition 3.24. A *semitopological monoid* is a monoid M with topology such that multiplication $M \times M \to M$ is separately continuous. A *semitopological monoid with continuous involution* is a pair (M, s) where M is a semitopological monoid and $s : M \to M$ a continuous involution. A morphism of semitopological monoids with continuous involution is a continuous, involution-preserving, monoid homomorphism. The category consisting of such objects and morphisms is denoted **sTopMon***.

The same arguments used to prove the functorality of π_1^{top} and F_R may be used to prove the next proposition. Let **Quo**(**Top**) be the category of quotient maps. A morphism of quotient maps is a commuting square in **Top**.

Proposition 3.25. *$(q : X \to Y) \mapsto M_q^*(Y)$ is a functor* **Quo**(**Top**) → **sTopMon*** *and $(q : X \to Y) \mapsto F_R^q(Y)$ is a functor* **Quo**(**Top**) → **qTopGrp**. *Additionally, $R : M_q^* \to F_R^q$ is a natural transformation each component of which is a monoid epimorphism and topological quotient map.*

Example 3.26. We are most interested in applying these constructions to the quotient maps $\pi_X : X \to \pi_0^{top}(X)$. Together the quotient maps π_X form a natural transformation $\pi : id_{\mathbf{Top}} \to \pi_0^{top}$ and so the constructions $X \mapsto M_{\pi_X}^*(\pi_0^{top}(X))$ and $X \mapsto F_R^{\pi_X}(\pi_0^{top}(X))$ give functors **Top** → **sTopMon*** and **Top** → **qTopGrp** respectively. Moreover, the quotient maps $\pi_{M_T^*(X)} : M_T^*(X) \to \pi_0^{top}(M_T^*(X))$ and $M^*(\pi_X) : M_T^*(X) \to M_{\pi_X}^*(\pi_0^{top}(X))$ make the same identifications and so there is a natural homeomorphism $\psi_X : \pi_0^{top}(M_T^*(X)) \to M_{\pi_X}^*(\pi_0^{top}(X))$ such that $\psi_X \circ \pi_{M_T^*(X)} = M^*(\pi_X)$. We let $\pi_0^{top}(M_T^*(X))$ inherit the monoid structure of $M^*(\pi_0(X))$ so that ψ is a natural isomorphism of semitopological monoids and $R \circ \psi_X : \pi_0^{top}(M_T^*(X)) \to F_R^{\pi_X}(\pi_0^{top}(X))$ is both a monoid epimorphism and a topological quotient map.

The James map $u : X \to \Omega(\Sigma(X_+))$ has image in $\Omega_{+s}(\Sigma(X_+))$ and the map $u : X \to \Omega_{+s}(\Sigma(X_+))$ with restricted codomain induces a continuous bijection $u_* : \pi_0^{top}(X) \to \pi_0^{top}(\Omega_{+s}(\Sigma(X_+)))$ on path component spaces. The fact that u_* is also a homeomorphism follows from the next lemma which will be used in the proof of Theorem 1.1. For a map $f : X \to Y$, let $f_{**} = \pi_0^{top}(M_T^*(f))$ be the induced, continuous, involution-preserving monoid homomorphism.

Lemma 3.27. *The James map $u : X \to \Omega_{+s}(\Sigma(X_+))$ induces a natural isomorphism of semitopological monoids with continuous involution $u_{**} : \pi_0^{top}(M_T^*(X)) \to \pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+))))$.*

Proof. We note that on generators u_{**} is given by $u_{**}(P) = [u_P]$. The naturality of $\psi : \pi_0(M^*(-)) \to M^*(\pi_0(-))$ applied to the James map makes the following diagram commute in the category of monoids (without topology)

$$\begin{array}{ccc} \pi_0(M_T^*(X)) & \xrightarrow[\cong]{\psi_X} & M^*(\pi_0(X)) \\ \downarrow u_{**} & & \cong \downarrow M^*(u_*) \\ \pi_0(M_T^*(\Omega_{+s}(\Sigma(X_+)))) & \xleftarrow[\psi^{-1}_{\Omega_{+s}(\Sigma(X_+))}]{\cong} & M^*(\pi_0(\Omega_{+s}(\Sigma(X_+)))) \end{array}$$

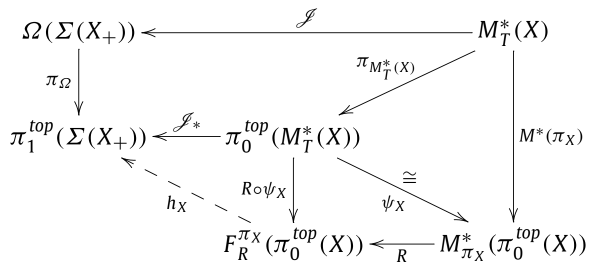
Since u_* is a bijection, $M^*(u_*)$ is a monoid isomorphism. Therefore $u_{**} : \pi_0^{top}(M_T^*(X)) \to \pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+))))$ is a continuous, involution-preserving monoid isomorphism and it suffices to show the inverse is continuous. Let $r : \Omega_{+s}(\Sigma(X_+)) \to M((0,1),(0,1) \times X)$ be the map taking each positively oriented simple loop $\alpha : I \to \Sigma(X_+)$ to the restricted map $\alpha|_{(0,1)} : (0,1) \to X \wedge (0,1) \cong X \times (0,1)$ and $p : M((0,1), X \times (0,1)) \to M((0,1), X)$ be post-composition with the projection $X \times (0,1) \to X$. For any $t \in (0,1)$, consider the composite map

$$v : \Omega_{+s}\big(\Sigma(X_+)\big) \xrightarrow{j_t} (0,1) \times \Omega_{+s}\big(\Sigma(X_+)\big) \xrightarrow{id \times (p \circ r)} (0,1) \times M\big((0,1), X\big) \xrightarrow{ev} X$$

where $j_t(\alpha) = (t, \alpha)$ and ev is the evaluation map $ev(t, f) = f(t)$. If $\alpha \in \Omega_{+s}(\Sigma(X_+))$ such that $\alpha(t) = x \wedge s$, then $\nu(\alpha) = x$. It is easy to check that the continuous homomorphism $\nu_{**} : \pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+)))) \to \pi_0^{top}(M_T^*(X))$ is the inverse of u_{**} since on the generator $[u_P]$ of $\pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+))))$, we have $\nu_{**}([u_P]) = \nu_{**}([u_x]) = P$ (where $x \in P$). $\square$

3.5. Proof of Theorem 1.1

To prove Theorem 1.1, we prove that $F_R^{\pi_X}(\pi_0^{top}(X))$ and $\pi_1^{top}(\Sigma(X_+))$ are isomorphic quasitopological groups. We use the following commutative diagram to relate these groups.



Here $\mathscr{J}_* = \pi_0^{top}(\mathscr{J})$ is the map induced by $\mathscr{J}$ on path component spaces. Recall from Corollary 3.15, that the fibers of the composites $\pi_\Omega \circ \mathscr{J} : M_T^*(X) \to \pi_1^{top}(\Sigma(X_+))$ and $R \circ M^*(\pi_X) = R \circ (\psi_X \circ \pi_{M_T^*(X)}) : M_T^*(X) \to F_R^{\pi_X}(\pi_0^{top}(X))$ are equal. Since

$$R \circ M^*(\pi_X) : \coprod_{n \geqslant 0} \big(X \sqcup X^{-1}\big)^n \to F_R^{\pi_X}\big(\pi_0^{top}(X)\big)$$

is quotient, the group isomorphism $h_X : F_R^{\pi_X}(\pi_0^{top}(X)) \to \pi_1^{top}(\Sigma(X_+))$ is always continuous and Theorem 1.1 is equivalent to the following theorem.

Theorem 3.28. *For an arbitrary space X, $h_X : F_R^{\pi_X}(\pi_0^{top}(X)) \to \pi_1^{top}(\Sigma(X_+))$ is a homeomorphism.*

Note that Corollary 1.2 follows directly from Theorem 3.28 and Lemma 3.22.6.

Remark 3.29. This description of $\pi_1^{top}(\Sigma(X_+))$ becomes remarkably simple when X is totally path disconnected (i.e. $\pi_X : X \cong \pi_0^{top}(X)$). In this case

$$\pi_1^{top}\big(\Sigma(X_+)\big) \cong F_R^{\pi_X}\big(\pi_0^{top}(X)\big) \cong F_R\big(\pi_0^{top}(X)\big) \cong F_R(X)$$

where the middle isomorphism comes from Lemma 3.22.5.

Outline of Proof. We have already proven that h_X is a continuous, group isomorphism. To prove that h_X is open, we take the following approach: It is shown in the proof of Theorem 3.14 that for $\alpha \in \Omega(\Sigma(X_+))$, all but finitely many of the restrictions $\alpha_{[c_m,d_m]}$ which are simple loops have image in the contractible neighborhood $X \wedge [0, \frac{1}{8}) \sqcup (\frac{7}{8}, 1]$. We assign to α, the word $\mathscr{D}(\alpha) = \alpha_{[c_{m_1},d_{m_1}]} \ldots \alpha_{[c_{m_n},d_{m_n}]}$ where the letters $\alpha_{[c_{m_i},d_{m_i}]}$ are the non-trivial simple loops. This gives a "decomposition" function $\mathscr{D} : \Omega(\Sigma(X_+)) \to M_T^*(\Omega_{+s}(\Sigma(X_+)))$ to the free topological monoid on the space $\Omega_{+s}(\Sigma(X_+)) \sqcup \Omega_{+s}(\Sigma(X_+))^{-1} = \Omega_{+s}(\Sigma(X_+)) \sqcup \Omega_{-s}(\Sigma(X_+))$ of non-trivial simple loops. The use of this free topological monoid provides a convenient setting for forming neighborhoods of arbitrary loops from strings of neighborhoods of simple loops and is the key to proving Theorem 1.1 for arbitrary X.

Identifying path components in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$, gives the semitopological monoid $\pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+)))) \cong \pi_0^{top}(M_T^*(X)) \cong M_{\pi_X}^*(\pi_0^{top}(X))$ which consists of words with letters P^ϵ where $P \in \pi_0(X)$, $\epsilon \in \{\pm 1\}$. The map $R \circ \psi_X : \pi_0^{top}(M_T^*(X)) \to F_R^{\pi_X}(\pi_0^{top}(X))$ is quotient by definition. The composite

$$K : \Omega\big(\Sigma(X_+)\big) \to M_T^*\big(\Omega_{+s}\big(\Sigma(X_+)\big)\big) \to \pi_0^{top}\big(M_T^*\big(\Omega_{+s}\big(\Sigma(X_+)\big)\big)\big) \cong \pi_0^{top}\big(M_T^*(X)\big) \to F_R^{\pi_X}\big(\pi_0^{top}(X)\big)$$

of these four maps (illustrated in Step 3) satisfies $h_X \circ K = \pi_\Omega$. It then suffices to show that K is continuous. Since the last three maps in the above composition are continuous, we show that for open $U \subset M_T^*(\Omega_{+s}(\Sigma(X_+)))$ saturated with respect to $M_T^*(\Omega_{+s}(\Sigma(X_+))) \to F_R^{\pi_X}(\pi_0^{top}(X))$, $\mathscr{D}^{-1}(U)$ is open in $\Omega(\Sigma(X_+))$. This is achieved by "piecing together" convenient neighborhoods of simple loops studied in Step 1 and using the fact that $\Sigma(X_+)$ is locally contractible and arc connected at x_0.

Step 1. *The topology of simple loops.*

Throughout the rest of this section let $U = X \wedge [0, \frac{1}{8}) \sqcup (\frac{7}{8}, 1]$. This is an arc-connected, contractible neighborhood and by the definition of $\mathscr{B}_{\Sigma(X_+)}$ contains all basic open neighborhoods of the basepoint x_0. We now prove a basic property of open neighborhoods of simple loops in the free path space $M(I, \Sigma(X_+))$. Recall that basic open neighborhoods in $M(I, \Sigma(X_+))$ are those described in Lemma 2.6 with respect to the basis $\mathscr{B}_{\Sigma(X_+)}$.

Lemma 3.30. *Suppose $0 < \epsilon < \frac{1}{8}$ and $W = \bigcap_{i=1}^{m} \langle K_m^i, W_i \rangle$ is a basic open neighborhood of simple loop $\alpha : I \to \Sigma(X_+)$ in the free path space $M(I, \Sigma(X_+))$. There is a basic open neighborhood V_0 of x_0 in $\Sigma(X_+)$ contained in $X \wedge [0, \epsilon) \sqcup (1-\epsilon, 1]$ and a basic open neighborhood $V = \bigcap_{j=1}^{n} \langle K_n^j, V_j \rangle$ of α in $M(I, \Sigma(X_+))$ contained in W such that*:

1. *$V_0 = V_1 = V_2 = \cdots = V_l = V_k = V_{k+1} = \cdots = V_n$ for integers $1 \leqslant l < k \leqslant n$.*
2. *The open neighborhoods $V_{l+1}, \ldots, V_{k-1}$ are of the form $A \wedge (a, b)$ where $A \in \mathscr{B}_X$ and $b - a < \epsilon$.*

Proof. Let $V_0 = (W_1 \cap W_m) \cap (X \wedge [0, \epsilon) \sqcup (1-\epsilon, 1]) \subset U$. Since $\mathscr{B}_{\Sigma(X_+)}$ is closed under finite intersection $V_0 \in \mathscr{B}_{\Sigma(X_+)}$. There is an integer $M > 3$ such that m divides M and $\alpha(K_M^1 \sqcup K_M^M) \subseteq V_0$. Since α is simple we have $\alpha([\frac{1}{M}, \frac{M-1}{M}]) \subset X \wedge (0, 1)$. When $p = 2, \ldots, M-1$ and $K_M^p \subseteq K_m^i$ we may cover $\alpha(K_M^p)$ with finitely many open neighborhoods contained in $W_i \cap (X \wedge (0, 1))$ of the form $A \wedge (a, b)$ where $A \in \mathscr{B}_X$ and $b - a < \epsilon$. We then apply the Lebesgue lemma to take even subdivisions of I to find open neighborhoods $Y_i = \bigcap_{q=1}^{N_p} \langle K_{N_p}^q, Y_{p,q}^i \rangle \subseteq \langle I, W_i \rangle$ of the restricted path $\alpha_{K_M^p}$. Here each $Y_{p,q}^i$ is one of the open neighborhoods $A \wedge (a, b) \subseteq W_i$. We now use the induced neighborhoods of Section 1.2 to define

$$V = \langle K_M^1 \sqcup K_M^M, V_0 \rangle \cap \bigcap_{p=2}^{M-1} \left((Y_i)^{K_M^p} \right).$$

This is an open neighborhood of α by definition, and it suffices to show that $V \subseteq W$. We suppose $\beta \in V$ and show that $\beta(K_m^i) \subseteq W_i$ for each i. Clearly, $\beta(K_M^1 \sqcup K_M^M) \subseteq W_1 \cap W_m$. If $p = 2, \ldots, M-1$ and $K_M^p \subseteq K_m^i$, then $\beta_{K_M^p} \in V_{K_M^p} \subseteq Y_i$ and $\beta(K_M^p) = \beta_{K_M^p}(I) \subseteq \bigcup_{q=1}^{N_p} Y_{p,q}^i \subseteq W_i$. We may write V as $V = \bigcap_{j=1}^{n} \langle K_n^j, V_j \rangle$ simply by finding an integer n which is divisible by M and every N_p and reindexing the open neighborhoods V_0 and $Y_{p,q}^i$. In particular, we can set $V_j = V_0$ when $K_n^j \subseteq K_M^1 \cup K_M^M$. Additionally, if $H_{K_M^p}^{-1} : I \to K_M^p$ is the unique linear homeomorphism (as in Section 1.2), then we let $V_j = Y_{p,q}^i$ whenever

$$K_n^j \subseteq H_{K_M^p}^{-1}\left(K_{N_p}^q\right) \subseteq K_M^p \subseteq K_m^i.$$

It is easy to see that both 1. and 2. in the statement are satisfied by V. $\square$

We note some additional properties of the neighborhood V constructed in the previous lemma:

Remark 3.31. For each path $\beta \in V$ we have,

$$\beta\left(\left(\bigcup_{j=1}^{l} K_n^j\right) \cup \left(\bigcup_{j=k}^{n} K_n^j\right)\right) \subseteq U \quad \text{and} \quad x_0 \notin \beta\left(\bigcup_{j=l+1}^{k-1} K_n^j\right).$$

This follows directly from the conditions 1. and 2. in the lemma.

Remark 3.32. It is previously noted that there are disjoint open neighborhoods W_+, W_0, and W_- in $M(I, \Sigma(X_+))$ containing $\Omega_{+s}(\Sigma(X_+))$, $\Omega_{0s}(\Sigma(X_+))$, and $\Omega_{-s}(\Sigma(X_+))$ respectively. Consequently, if α has positive orientation, then we may take $V \subseteq W_+$ such that $V \cap \Omega_s(\Sigma(X_+)) \subseteq \Omega_{+s}(\Sigma(X_+))$, i.e. all simple loops in V also have positive orientation. The same holds for the negative and trivial case. In some sense, this means that V, when thought of as an instruction set, is "good enough" to distinguish orientations of simple loops.

Remark 3.33. We now give a construction necessary for Step 4 which produces a simple loop $\mathscr{S}_V(\beta) \in V$ for each path $\beta \in V$. For brevity, we let $[0, r] = \bigcup_{j=1}^{l} K_n^j$, $[r, s] = \bigcup_{j=l+1}^{k-1} K_n^j$, and $[s, 1] = \bigcup_{j=k}^{n} K_n^j$ and define $\mathscr{S}_V(\beta)$ piecewise by letting $\mathscr{S}_V(\beta)$ be equal to β on the middle interval $[r, s]$ (i.e. $\mathscr{S}_V(\beta)_{[r,s]} = \beta_{[r,s]}$). We then demand that $\mathscr{S}_V(\beta)$ restricted to $[0, r]$ is an arc in V_0 connecting x_0 to $\beta(r)$ and similarly $\mathscr{S}_V(\beta)$ restricted to $[s, 1]$ is an arc in V_0 connecting $\beta(s)$ to x_0. Since the image of $\mathscr{S}_V(\beta)$ on $[0, r] \cup [s, 1]$ remains in V_0, it follows that $\mathscr{S}_V(\beta) \in V$. Additionally, Remark 3.31 and the use of arcs to define $\mathscr{S}_V(\beta)$ means that $\mathscr{S}_V(\beta)$ is a simple loop.

Step 2. *Decomposition of arbitrary loops.*

Here we assign to each loop in $\Sigma(X_+)$, a (possibly empty) word of simple loops with orientation. We again use the observation, that $\Omega_{+s}(\Sigma(X_+))$ and $\Omega_{-s}(\Sigma(X_+)) = \Omega_{+s}(\Sigma(X_+))^{-1}$ are disjoint homeomorphic subspaces of $M(I, \Sigma(X_+))$. The free topological monoid on $\Omega_{+s}(\Sigma(X_+)) \sqcup \Omega_{-s}(\Sigma(X_+)) = \Omega_{+s}(\Sigma(X_+))^{-1}$ is just the free topological monoid with continuous involution $M_T^*(\Omega_{+s}(\Sigma(X_+)))$. We make no distinction between the one letter word α^{-1} in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$ and the reverse loop $\beta = \alpha^{-1} \in \Omega_{+s}(\Sigma(X_+))^{-1}$. Similarly, a basic open neighborhood of α^{-1} in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$ corresponds to an open neighborhood of β in $\Omega_{+s}(\Sigma(X_+))^{-1}$. We now define the "decomposition" function $\mathscr{D} : \Omega(\Sigma(X_+)) \to M_T^*(\Omega_{+s}(\Sigma(X_+)))$. In Step 4 we refer to the details described here.

Decomposition 3.34. Suppose $\beta \in M(I, \{0,1\}; \Sigma(X_+), \{x_0\})$ is an arbitrary loop. First, if β has image contained in U (i.e. $\beta \in \langle I, U\rangle$), then we let $\mathscr{D}(\beta) = e$ be the empty word. Suppose then that $\beta(I) \nsubseteq U$. The pullback $\beta^{-1}(X \wedge (0,1)) = \coprod_{m\in M}(c_m, d_m)$ is open in I where M is a countable indexing set with ordering induced by the ordering of I. Each restricted loop $\beta_m = \beta_{[c_m,d_m]} : I \to \Sigma(X_+)$ is a simple loop. Remark 3.1.5 implies that all but finitely many of these simple loops have image in U and so we may take $m_1 < \cdots < m_k$ to be the indices of M corresponding to those $\beta_{m_1}, \ldots, \beta_{m_k}$ with image not contained in U. Note that if $C = I - \bigcup_{i=1}^{k}(c_{m_i}, d_{m_i})$, then $\beta \in \langle C, U\rangle$. If none of the β_{m_i} have orientation, we again let $\mathscr{D}(\beta) = e$. On the other hand, if one of the β_{m_i} has orientation, we let $m_{i_1} < \cdots < m_{i_n}$ be the indices corresponding to the simple loops $\beta_j = \beta_{m_{i_j}}$ which have either positive or negative orientation. We then let $\mathscr{D}(\beta)$ be the word $\beta_1\beta_2\ldots\beta_n$ in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$.

Remark 3.35. Informally, $\mathscr{D}(\beta)$ denotes the word composed of the simple loops of β which contribute a letter in the unreduced word of the homotopy class $[\beta]$. We may suppose that β_j has image in $P_j \wedge I$ and orientation $\epsilon_j \in \{\pm 1\}$, or equivalently that $[\beta_j] = [u_{P_j}]^{\epsilon_j}$. Clearly $\beta \simeq *_{j=1}^{n}\beta_j$ and $h_X^{-1}([\beta]) = R(P_1^{\epsilon_1} P_n^{\epsilon_n} \ldots P_n^{\epsilon_n}) \in F(\pi_0(X))$.

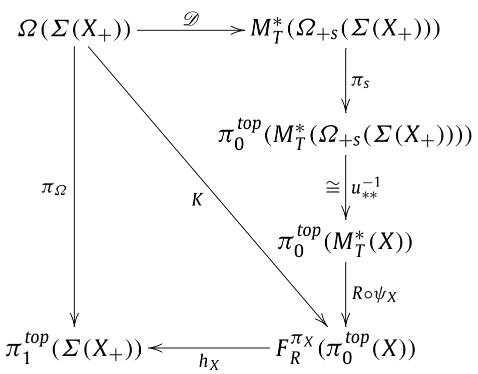
Step 3. *Factoring* π_Ω.

We factor the quotient map $\pi_\Omega : \Omega(\Sigma(X_+)) \to \pi_1^{top}(\Sigma(X_+))$ into a composite using the following functions:

1. The decomposition function $\mathscr{D} : \Omega(\Sigma(X_+)) \to M_T^*(\Omega_{+s}(\Sigma(X_+)))$.
2. The quotient map $\pi_s : M_T^*(\Omega_{+s}(\Sigma(X_+))) \to \pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+))))$ identifying path components (homotopy classes of positively oriented simple loops).
3. The natural homeomorphism $u_{**}^{-1} : \pi_0^{top}(M_T^*(\Omega_{+s}(\Sigma(X_+)))) \to \pi_0^{top}(M_T^*(X))$ of Lemma 3.27.
4. The quotient map $R \circ \psi_X : \pi_0^{top}(M_T^*(X)) \to F_R^{\pi_X}(\pi_0^{top}(X))$.
5. The continuous, group isomorphism $h_X : F_R^{\pi_X}(\pi_0^{top}(X)) \to \pi_1^{top}(\Sigma(X_+))$.

We let $K = R \circ \psi_X \circ u_{**}^{-1} \circ \pi_s \circ \mathscr{D} : \Omega(\Sigma(X_+)) \to F_R^{\pi_X}(\pi_0^{top}(X))$ be the composite of 1.–4. and $K' = R \circ \psi_X \circ u_{**}^{-1} \circ \pi_s : M_T^*(\Omega_{+s}(\Sigma(X_+))) \to F_R^{\pi_X}(\pi_0^{top}(X))$ be the continuous (and even quotient) composite of 2.–4.

Lemma 3.36. *The following diagram commutes*:



The function $\mathscr{D}$ will not be continuous even when X contains only a single point (i.e. $\Sigma(X_+) \cong S^1$). This is illustrated by the fact that any open neighborhood of a concatenation $\alpha * \alpha^{-1}$ for simple loop α contains a trivial simple loop β which may be found by "pulling" the middle of $\alpha * \alpha^{-1}$ off of x_0 within a sufficiently small neighborhood of x_0.

Step 4. *Continuity of* K.

Lemma 3.37. $K : \Omega(\Sigma(X_+)) \to F_R^{\pi_X}(\pi_0^{top}(X))$ *is continuous.*

Proof. Suppose W is open in $F_R^{\pi_X}(\pi_0^{top}(X))$ and $\beta \in K^{-1}(W)$. We now refer to the details of the decomposition of β in Step 2. If β has image in U, then clearly $\beta \in \langle I, U\rangle \subseteq \mathscr{D}^{-1}(e) \subseteq K^{-1}(W)$. Suppose, on the other hand, that some simple loop restriction β_{m_i} has image intersecting $\Sigma(X_+) - U$ and $\mathscr{D}(\beta) = \beta_1\beta_2 \ldots \beta_n$ is the (possibly empty) decomposition of β. Recall from our the notation in Step 2, that $\beta_j = \beta_{m_{i_j}}$, $j = 1, \ldots, n$ are the β_{m_i} with orientation. Since $K' = R \circ \psi_X \circ u_{**}^{-1} \circ \pi_s$ is continuous, $(K')^{-1}(W)$ is an open neighborhood of $\mathscr{D}(\beta)$ in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$.

Recall that $\beta \in \langle C, U\rangle$, where $C = I - \bigcup_{i=1}^{k}(c_{m_i}, d_{m_i})$. We construct the rest of desired open neighborhood of β by defining an open neighborhood of each β_{m_i} and taking the intersection of the induced neighborhoods.

If $i \neq i_j$ for any $j = 1, \ldots, n$, then β_{m_i} does not appear as a letter in the decomposition of β and must be trivial. We apply Lemma 3.30, to find an open neighborhood $V_i = \bigcap_{l=1}^{N_i}\langle K_{N_i}^l, V_l^i\rangle$ of β_{m_i} in $M(I, \Sigma(X_+))$ which satisfies both 1. and 2. in the statement. By Remark 3.32, we may also assume that $V_i \cap \Omega_s(\Sigma(X_+)) \subseteq \Omega_{0s}(\Sigma(X_+))$, i.e. all simple loops in V_i are trivial.

If $i = i_j$ for some $j = 1, \ldots, n$, then $\beta_j = \beta_{m_{i_j}}$ has orientation ϵ_j. Since $(K')^{-1}(W)$ is an open neighborhood of $\mathscr{D}(\beta) = \beta_1\beta_2 \ldots \beta_n$ in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$ and basic open neighborhoods in $M_T^*(\Omega_{+s}(\Sigma(X_+)))$ are products of open neighborhoods in $\Omega_{+s}(\Sigma(X_+))$ and $\Omega_{+s}(\Sigma(X_+))^{-1}$, we can find basic open neighborhoods $V_{i_j} = \bigcap_{l=1}^{N_{i_j}}\langle K_{N_{i_j}}^l, U_l^{i_j}\rangle$ of β_j in $M(I, \Sigma(X_+))$ such that

$$W_j = V_{i_j} \cap \Omega_{+s}\big(\Sigma(X_+)\big)^{\epsilon_j} \quad \text{and} \quad \beta_1\beta_2 \ldots \beta_n \in W_1W_2 \ldots W_n \subseteq \big(K'\big)^{-1}(W).$$

We assume each V_{i_j} satisfies 1. and 2. of Lemma 3.30 and by Remark 3.32 that $V_{i_j} \cap \Omega_s(\Sigma(X_+)) \subseteq \Omega_{+s}(\Sigma(X_+))^{\epsilon_j}$.

Let

$$\mathscr{U} = \langle C, U\rangle \cap \left(\bigcap_{i=1}^{k}\big(V_i^{[c_{m_i}, d_{m_i}]}\big)\right)$$

so that $\mathscr{V} = \mathscr{U} \cap \Omega(\Sigma(X_+))$ is an open neighborhood of β in the loop space. We claim that each loop $\gamma \in \mathscr{V}$ is homotopic to a loop γ' such that $\mathscr{D}(\gamma') \in (K')^{-1}(W)$. If this is done, we have

$$h_X\big(K(\gamma)\big) = \pi_\Omega(\gamma) = \pi_\Omega\big(\gamma'\big) = h_X\big(K\big(\gamma'\big)\big)$$

and since h_X is a bijection,

$$K(\gamma) = K\big(\gamma'\big) = K'\big(\mathscr{D}\big(\gamma'\big)\big) \in W.$$

This gives $K(\mathscr{V}) \subseteq W$, proving the continuity of K.

We define γ' piecewise and begin by setting $\gamma'(C) = x_0$. The restricted path $\gamma_i = \gamma_{[c_{m_i}, d_{m_i}]} : I \to \Sigma(X_+)$ lies in the open neighborhood $\mathscr{U}_{[c_{m_i}, d_{m_i}]} \subseteq V_i$. We now define γ' on $[c_{m_i}, d_{m_i}]$ by using the construction of Remark 3.33. We set

$$\gamma_i{}' = \big(\gamma'\big)_{[c_{m_i}, d_{m_i}]} = \mathscr{S}_{V_i}(\gamma_i)$$

which by construction is a simple loop in V_i. Intuitively, we have replaced the portions of γ which are close to x_0 ("close" meaning with respect to $\mathscr{V}$) with arcs and constant paths. Since $\gamma_i{}' = \gamma'(C) = x_0 \in U$ and $(\gamma')_{[c_{m_i}, d_{m_i}]} \in V_i$ for each i, it follows that $\gamma' \in \mathscr{V}$. Moreover, since $\gamma(t) \neq \gamma'(t)$ only when $\gamma(t)$ and $\gamma'(t)$ both lie in the path connected, contractible neighborhood U, it is obvious that $\gamma \simeq \gamma'$. It now suffices to show that $\mathscr{D}(\gamma') \in (K')^{-1}(W)$. We begin by checking which of the simple loops $\gamma_i{}' \in V_i \cap \Omega(\Sigma(X_+))$ have orientation and will appear in the word $\mathscr{D}(\gamma')$. If $i \neq i_j$ for any j, all simple loops in V_i, including $\gamma_i{}'$ are trivial. Therefore $\gamma_i{}'$ has no orientation and will not appear as a letter in $\mathscr{D}(\gamma')$. If this is the case for all i so that $\mathscr{D}(\beta)$ is the empty word, then $\mathscr{D}(\gamma')$ must also be the empty word $e \in (K')^{-1}(W)$. Suppose on the other hand that $\mathscr{D}(\beta) = \beta_1\beta_2 \ldots \beta_n \neq e$ and $i = i_j$ for some j. The neighborhood $V_i = V_{i_j}$ was chosen so that all simple loops in V_{i_j} have orientation ϵ_j. Since $\gamma_i{}' \in V_{i_j}$ is simple, it has orientation ϵ_j and we have $\gamma_i{}' \in V_{i_j} \cap \Omega_{+s}(\Sigma(X_+))^{\epsilon_j} = W_j$. Therefore $\mathscr{D}(\gamma') = \gamma_{i_1}{}'\gamma_{i_2}{}' \ldots \gamma_{i_n}{}' \in W_1W_2 \ldots W_n \subseteq (K')^{-1}(W)$. □

Since K is continuous, $\pi_\Omega = h_X \circ K$ is quotient, and h_X is bijective, h_X is a homeomorphism.

3.6. The weak suspension spaces $w\Sigma(X_+)$ and $\pi_1^{top}(w\Sigma(X_+))$

We pause here to note a deficiency of the suspension spaces $\Sigma(X_+)$: $\Sigma(X_+)$ is not always first countable at x_0. It is easy to see that if X is compact, then there is a countable neighborhood base at the basepoint x_0 consisting of neighborhoods of the form $B_n = X \wedge [0, \frac{1}{n}) \sqcup (\frac{n-1}{n}, 1]$. Unfortunately, if X is a non-compact, first-countable (resp. metric) space, then $\Sigma(X_+)$ may not be first countable (resp. a metric space). For this reason, we consider a slightly weaker topology on the underlying set of $\Sigma(X_+)$, and denote the resulting space as $w\Sigma(X_+)$. A basis for the topology of $w\Sigma(X_+)$ is given by subsets of the form $V \wedge (a, b)$ and B_n, where $V \in \mathscr{B}_X$ and $n \geqslant 2$. The identity function $id : \Sigma(X_+) \to w\Sigma(X_+)$ is continuous and, in fact, is a homotopy equivalence. A homotopy inverse $w\Sigma(X_+) \to \Sigma(X_+)$ is given by the quotient map $w\Sigma(X_+) \to w\Sigma(X_+)/B_3 \cong \Sigma(X_+)$. One could also repeat the arguments of the previous section to prove the following theorem.

Theorem 3.38. *The identity map $id : \Sigma(X_+) \to w\Sigma(X_+)$, induces a natural isomorphism of quasitopological groups $id : \pi_1^{top}(\Sigma(X_+)) \to \pi_1^{top}(w\Sigma(X_+))$.*

The "weak suspension" $w\Sigma(X_+)$ has a few advantages over $\Sigma(X_+)$ including the fact that if X is a subspace of $\mathbb{R}^n$, then $w\Sigma(X_+)$ may be embedded as a subspace of $\mathbb{R}^{n+1}$.

Example 3.39. For $a \in [0, \infty)$, let

$$C_a = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; (x-a)^2 + y^2 = (1+a)^2\right\}.$$

If $X \subset [0, \infty)$ is any subspace of the non-negative real line, then $w\Sigma(X_+)$ is homeomorphic to $\bigcup_{a\in X} C_a \subset \mathbb{R}^2$ with basepoint $(-1, 0)$. It is not necessarily true that $\Sigma(X_+) \cong \bigcup_{a\in X} C_a$ if X is non-compact.

4. The topology of $\pi_1^{top}(\Sigma(X_+))$

Here we study the topological structure of $\pi_1^{top}(\Sigma(X_+))$ by studying the isomorphic quasitopological group $F_R^q(Y)$ for $q = \pi_X$. First, we give a nice characterization of discreteness.

Corollary 4.1. *$\pi_1^{top}(\Sigma(X_+))$ is a discrete group if and only if $\pi_0^{top}(X)$ is discrete.*

Proof. If $\pi_0^{top}(X)$ is discrete, then so is $M_T^*(\pi_0^{top}(X))$ and its quotient $F_R(\pi_0^{top}(X))$. Since the topology of $\pi_1^{top}(\Sigma(X_+)) \cong F_R^{\pi_X}(\pi_0^{top}(X))$ is finer than that of $F_R(\pi_0^{top}(X))$, $\pi_1^{top}(\Sigma(X_+))$ must also be discrete. Conversely, if $\pi_1^{top}(\Sigma(X_+))$ is discrete, the continuity of the injection $u_* : \pi_0^{top}(X) \to \pi_1^{top}(\Sigma(X_+))$ gives the discreteness of $\pi_0^{top}(X)$. □

4.1. Separation in $\pi_1^{top}(\Sigma(X_+))$

Fix a quotient map $q : X \to Y$. The next two definitions are important for characterizing the topological properties of $F_R^q(Y)$.

Definition 4.2. We say $q : X \to Y$ is *separating* if whenever $q(x_1) = y_1 \neq y_2 = q(x_2)$ there are open neighborhoods U_i of x_i in X such that $q(U_1) \cap q(U_2) = \emptyset$.

Of course

$$Y \text{ is Hausdorff} \quad \Rightarrow \quad q \text{ is separating} \quad \Rightarrow \quad Y \text{ is } T_1$$

and the identity $id : Y \to Y$ is separating if and only if Y is Hausdorff.

Definition 4.3. A neighborhood $U = U_1^{\epsilon_1} \dots U_n^{\epsilon_n}$ of $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ in $M_T^*(X)$ is *q-separating* if $q(U_i) \cap q(U_j) = \emptyset$ whenever $q(x_i) \neq q(x_j)$. We say U is *separating* when $q = id_Y$.

Remark 4.4. If $U_1^{\epsilon_1} \dots U_n^{\epsilon_n}$ is a q-separating neighborhood of $x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$, then

$$U_1^{\epsilon_1} \dots U_{i-1}^{\epsilon_{i-1}} U_{i+1}^{\epsilon_{i+1}} \dots U_n^{\epsilon_n}$$

is a q-separating neighborhood of $x_1^{\epsilon_1} \dots x_{i-1}^{\epsilon_{i-1}} x_{i+1}^{\epsilon_{i+1}} \dots x_n^{\epsilon_n}$. This will be particularly useful when we remove letters by word reduction.

Let $Q = M^*(q) : M_T^*(X) \to M_q^*(Y)$ be the induced monoid homomorphism which takes word $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ to $Q(w) = q(x_1)^{\epsilon_1} \dots q(x_n)^{\epsilon_n}$ and is quotient by definition. Additionally, the composite $RQ : M_T^*(X) \to M_q^*(Y) \to F_R^q(Y)$ is quotient.

Lemma 4.5. *Let $q : X \to Y$ be a separating quotient map, $w = x_1^{\epsilon_1} \dots x_n^{\epsilon_n}$ be a non-empty word in $M_T^*(X)$, $y_i = q(x_i)$, and W be an open neighborhood of w.*

1. *There is a q-separating neighborhood $U = U_1^{\epsilon_1} \dots U_n^{\epsilon_n}$ of w contained in W.*
2. *$Q(w)$ is reduced if and only if $Q(v)$ is reduced for each $v \in U$.*
3. *If $v \in U$, then $|RQ(w)| \leqslant |RQ(v)| \leqslant |Q(w)|$.*

Proof. 1. If all of the y_i are the same, there is an open $V = U_i$ of x_i such that $U = U_1^{\epsilon_1} \dots U_n^{\epsilon_n} \subseteq W$. On the other hand, if $y_i \neq y_j$, there are open neighborhoods $V_{i,j}$ of x_i and $V_{j,i}$ of x_j such that $q(V_{i,j}) \cap q(V_{j,i}) = \emptyset$. For each i, take open neighborhood $U_i \subseteq \bigcap_{q(x_i) \neq q(x_j)} V_{i,j}$ of x_i such that $U = U_1^{\epsilon_1} \dots U_n^{\epsilon_n} \subseteq W$. Clearly U is a q-separating neighborhood of w contained in W.

2. Suppose first that $Q(w)$ is a reduced word. Then for each $i \in \{1, \dots, n-1\}$ either $y_i \neq y_{i+1}$ or $\epsilon_i = \epsilon_{i+1}$. Suppose $v = z_1^{\epsilon_1} \dots z_n^{\epsilon_n}$ lies in the q-separating neighborhood U and $i \in \{1, \dots, n-1\}$ such that $\epsilon_i = -\epsilon_{i+1}$. If $q(z_i) = q(z_{i+1})$, then we must have $y_i = q(x_i) = q(x_{i+1}) = y_{i+1}$. But this cannot be since $Q(w)$ is reduced. Therefore $Q(v)$ is reduced. The converse is obvious since if $Q(w)$ is not reduced then U already contains $Q(w)$.

3. Suppose $v = z_1^{\epsilon_1} \dots z_n^{\epsilon_n} \in U$. The second inequality is obvious since $|RQ(v)| \leqslant |Q(v)| = |Q(w)|$. To prove the first inequality, Remark 4.4 indicates that it suffices prove that for every reduction in $Q(v)$, there is a corresponding reduction in $Q(w)$. This follows directly from 2. □

Let $F_R^q(Y)_n$ denote $F(Y)_n = \{w \in F(Y) \mid |w| \leqslant n\}$ with the subspace topology of $F_R^q(Y)$.

Corollary 4.6. *If $q : X \to Y$ is separating and $n \geqslant 0$, then $F_R^q(Y)_n$ is closed in $F_R^q(Y)$.*

Proof. Suppose $w \in M_T^*(X)$ such that $|RQ(w)| > n$. Now take any q-separating neighborhood U of w in $M_T^*(X)$. Lemma 4.5 asserts that if $v \in U$, then $|RQ(v)| \geqslant |RQ(w)| > n$. Consequently, $w \in U \subset M_T^*(X) - RQ^{-1}(F_R^q(Y)_n)$ proving that $RQ^{-1}(F_R^q(Y)_n)$ is closed in $M_T^*(X)$. Since RQ is quotient $F_R^q(Y)_n$ is closed in $F_R^q(Y)$. □

We now observe some properties of $F_R^q(Y)$ which are often desirable in free topological groups. Let Z denote the set of all finite sequences $\zeta = \epsilon_1, \dots, \epsilon_n$ with $\epsilon_i \in \{\pm 1\}$, including the empty sequence. For each $\zeta = \epsilon_1, \dots, \epsilon_n \in Z$, let $X^\zeta = \{x_1^{\epsilon_1} \dots x_n^{\epsilon_n} \mid x_i \in X\}$ and recall that $M_T^*(X) = \coprod_{\zeta \in Z} X^\zeta$. Let $|\zeta|$ denote the length of each sequence and $(RQ)_\zeta : X^\zeta \to F_R^q(Y)_{|\zeta|}$ be the restriction of the quotient map $RQ : M_T^*(X) \to F_R^q(Y)$. The proof of the next proposition is based on that of Statement 5.1 in [10].

Proposition 4.7. *Let $q : X \to Y$ be separating.*

1. *$F_R^q(Y)$ has the inductive limit topology of the sequence of closed subspaces $\{F_R^q(Y)_n\}_{n \geqslant 0}$.*
2. *For each $n \geqslant 0$, the restriction $(RQ)_n : \coprod_{i=0}^n (X \sqcup X^{-1})^i \to F_R^q(Y)_n$ of RQ is a quotient map.*

Proof. 1. Suppose $C \subseteq F_R^q(Y)$ such that $C \cap F_R^q(Y)_n$ is closed in $F_R^q(Y)_n$ for each $n \geqslant 0$. Since $(RQ)_\zeta$ is continuous $RQ^{-1}(C) \cap X^\zeta = (RQ)_\zeta^{-1}(C \cap F_R^q(Y)_{|\zeta|})$ is closed in X^ζ for each ζ. But $M_T^*(X)$ is the disjoint union of the X^ζ and so $(RQ)^{-1}(C)$ is closed in $M_T^*(X)$. Since RQ is quotient, C is closed in $F_R^q(Y)$.

2. Suppose $A \subseteq F_R^q(Y)_n$ such that $(RQ)_n^{-1}(A)$ is closed in $\coprod_{i=0}^n (X \sqcup X^{-1})^i = \coprod_{|\zeta| \leqslant n} X^\zeta$. Since $F_R^q(Y)_n$ is closed in $F_R^q(Y)$, RQ is a quotient map, and $M_T^*(X)$ is the disjoint union of the X^ζ, it suffices to show that

$$(RQ)_\zeta^{-1}(A) = \left\{a = a_1^{\epsilon_1} \dots a_k^{\epsilon_k} \in M_T^*(X) \;\middle|\; RQ(a) \in A\right\} = RQ^{-1}(A) \cap X^\zeta$$

is closed in X^ζ for each $\zeta = \epsilon_1, \dots, \epsilon_k \in Z$. If $|\zeta| \leqslant n$, then $(RQ)_\zeta^{-1}(A) \cap X^\zeta = (RQ)_n^{-1}(A) \cap X^\zeta$ is closed by assumption. For $|\zeta| > n$, a simple induction (on $|\zeta|$) argument and application of Lemma 4.5 quite similar to the one below in the proof of Theorem 4.25 gives that $(RQ)_\zeta^{-1}(A)$ is closed in X^ζ for all $\zeta \in Z$. □

For each $n \geqslant 1$, let Y_q^n denote the product Y^n with the quotient topology from the product function $q^n : X^n \to Y^n$. Of course, since q is quotient, $Y_q^1 = Y$ and if $q = \pi_X : X \to \pi_0^{top}(X)$, then $Y_q^n \cong \pi_0^{top}(X^n)$. Similarly, denote Y_q^ζ and $(Y \sqcup Y^{-1})_q^n$ as the quotients of X^ζ and $(X \sqcup X^{-1})^n$ with respect to q and its powers and sums. In these terms, we have

$$M_q^*(Y) = \coprod_{n \geqslant 0} \left(Y \sqcup Y^{-1}\right)_q^n = \coprod_{\zeta \in Z} Y_q^\zeta.$$

Let $Q_n : \coprod_{i=0}^{n}(X \sqcup X^{-1})^i \to \coprod_{i=0}^{n}(Y \sqcup Y^{-1})_q^i$ and $R_n : \coprod_{i=0}^{n}(Y \sqcup Y^{-1})_q^i \to F_R^q(Y)_n$ be the respective restrictions of R and Q. Since $R_n \circ Q_n = (RQ)_n$, the previous proposition implies:

Corollary 4.8. *If $q : X \to Y$ is separating and $n \geqslant 0$, the restriction $R_n : \coprod_{i=0}^{n}(Y \sqcup Y^{-1})_q^i \to F_R^q(Y)_n$ of $R : M_q^*(Y) \to F_R^q(Y)$ is quotient.*

Theorem 4.9. *The following are equivalent*:

1. *$q : X \to Y$ is separating.*
2. *$F_R^q(Y)$ is T_1.*
3. *For each $n \geqslant 1$, the canonical map $\sigma_n : Y_q^n \to F_R^q(Y)$ taking $(y_1, \ldots, y_n)$ to the word $y_1 \ldots y_n$ is a closed embedding.*

Proof. 1. ⇒ 2. If $q : X \to Y$ is separating, the singleton $F_R^q(Y)_0 = \{e\}$ containing the identity is closed by Corollary 4.6. Since $F_R^q(Y)$ is a quasitopological group, it is T_1.

2. ⇒ 1. Suppose $q : X \to Y$ is not separating. There are distinct $y_1, y_2 \in Y$ such that whenever $q(x_i) = y_i$ and U_i is an open neighborhood of x_i, then $q(U_1) \cap q(U_2) \neq \emptyset$. Suppose W is any open neighborhood of reduced word $y_1 y_2^{-1}$ in $F_R^q(Y)$ and $x_i \in q^{-1}(y_i)$. Since RQ is continuous, there are open neighborhoods U_i of x_i such that $x_1 x_2^{-1} \in U_1 U_2^{-1} \subset RQ^{-1}(W)$. But there is a $y_3 \in q(U_1) \cap q(U_2)$ by assumption and so $Q(U_1 U_2^{-1}) \subset R^{-1}(W)$ contains the word $y_3 y_3^{-1}$. Therefore $e = R(y_3 y_3^{-1}) \in W$. But if every neighborhood of $y_1 y_2^{-1}$ in $F_R^q(Y)$ contains the identity, then $F_R^q(Y)$ is not T_1.

1. ⇒ 3. Suppose A is a closed subspace of Y_q^n and q is separating. Let $j : Y_q^n \hookrightarrow \coprod_{i=0}^{n}(Y \sqcup Y^{-1})_q^i$ be given by $j(y_1, \ldots, y_n) = y_1 \ldots y_n$ so that $R_n \circ j = \sigma_n$. Since j is a closed embedding, $R_n^{-1}(\sigma_n(A)) = j(A)$ is closed in $\coprod_{i=0}^{n}(Y \sqcup Y^{-1})_q^i$. But R_n is quotient by Corollary 4.8 and $F_R^q(Y)_n$ is closed in $F_R^q(Y)$. Therefore $\sigma_n(A)$ is closed in $F_R^q(Y)$.

3. ⇒ 1. If q is not separating, the argument for 2. ⇒ 1. implies that there are distinct $y_1, y_2 \in Y$ such that any open neighborhood of the three letter word $y_1 y_2 y_1^{-1}$ in $F_R^q(Y)$ contains the one letter word y_1 which lies in the image of σ_1. Therefore, if q is not separating, the image of σ_1 cannot be closed. □

Corollary 4.10. *$\pi_1^{top}(\Sigma(X_+))$ is T_1 if and only if for each $x_1, x_2 \in X$ lying in distinct path components, there are open neighborhoods U_i of x_i such that $\pi_X(U_1) \cap \pi_X(U_2) = \emptyset$.*

Corollary 4.11. *Suppose (P) is a topological property hereditary to closed subspaces. If $\pi_1^{top}(\Sigma(X_+))$ has property (P), then so does $\pi_0^{top}(X^n)$ for each $n \geqslant 1$.*

Determining when $F_R^q(Y)$ is Hausdorff requires a much greater effort that we avoid here. We can see quite easily, however, that $\pi_1^{top}(\Sigma(X_+))$ may fail to be Hausdorff even when $\pi_0^{top}(X)$ is Hausdorff.

Remark 4.12. Using an argument similar to the one used to prove 2. ⇒ 1. in Theorem 4.9 one can show that $X \times X/\Delta$ (where $\Delta \subset X \times X$ is the diagonal) is Hausdorff whenever $F_R(X)$ is Hausdorff. The author does not know if the converse is true.

Example 4.13. We begin by defining the underlying set of a space X. Let $K = \{\frac{1}{n} \mid n \geqslant 1\}$, $-K = \{-k \mid k \in K\}$, and $X = (K \times (-K \cup \{0\} \cup K)) \sqcup \{a, b\}$. We define a basis for the topology of X as follows. For each $(r, s) \in K \times (-K \cup K)$, the singleton $\{(r, s)\}$ is open in X. Let K_m be the set $K_m = \{\frac{1}{k} \mid k \geqslant m\}$ for each integer $m \geqslant 1$. A basic open neighborhood of $(r, 0)$ is of the form $\{r\} \times (-K_m \cup \{0\} \cup K_m)$, of a is of the form $\{a\} \cup (K_m \times K)$, of b is of the form $\{b\} \cup (K_m \times (-K))$. Since X is Hausdorff and totally path disconnected, the group $\pi_1^{top}(\Sigma(X_+)) \cong F_R(X)$ is T_1. But any two neighborhoods of a and b in X are both intersected by any neighborhood of $(r, 0)$ for some $r \in K$. Clearly $X \times X/\Delta$ is not Hausdorff and $\pi_1^{top}(\Sigma(X_+))$ cannot be Hausdorff.

While determining when $\pi_1^{top}(\Sigma(X_+))$ is Hausdorff is difficult, the next result is a direct consequence of Theorem 1.1.

Corollary 4.14. *$\pi_1^{top}(\Sigma(X_+))$ is functionally Hausdorff if and only if $\pi_0^{top}(X)$ is functionally Hausdorff.*

Proof. Since the injection $u_* : \pi_0^{top}(X) \to \pi_1^{top}(\Sigma(X_+))$ is continuous, $\pi_0^{top}(X)$ is functionally Hausdorff, whenever $\pi_1^{top}(\Sigma(X_+))$ is. Conversely, if $\pi_0^{top}(X)$ is functionally Hausdorff, then $F_M(\pi_0^{top}(X))$ is a Hausdorff topological group. Since Hausdorff topological groups are functionally Hausdorff and there is a continuous injection $h_X^{-1} : \pi_1^{top}(\Sigma(X_+)) \to F_M(\pi_0^{top}(X))$, $\pi_1^{top}(\Sigma(X_+))$ must also be functionally Hausdorff. □

4.2. First countability

The arguments used to prove the next statements are based on the arguments used by Fabel [26] to show that the Hawaiian earring group $\pi_1^{top}(\mathbb{HE})$ is not first countable. Given a sequence of integers N_m, we write $\lim_{m\to\infty} N_m = \infty$ when for each $M \geqslant 1$, there is an m_0 such that $N_m \geqslant M$ for all $m \geqslant m_0$.

Lemma 4.15. *If $q : X \to Y$ is separating and w_m is a sequence of reduced words in $F_R^q(Y)$ such that $\lim_{m\to\infty} |w_m| = \infty$, then the set $\{w_m\}_{m\geqslant 1}$ is closed in $F_R^q(Y)$.*

Proof. Let $C = \{w_m\}_{m\geqslant 1} \subset F_R^q(Y)$. Since $RQ : M_T^*(X) \to F_R^q(Y)$ is quotient, it suffices to show that $RQ^{-1}(C)$ is closed in $M_T^*(X)$. Let $z_k \in RQ^{-1}(C)$, $k \in K$ be a net ($(K, \geqslant)$ is a directed set) in $M_T^*(X)$ converging to $z \in X^{\zeta_0} \subset M_T^*(X)$ such that $RQ(z) \notin C$. For each $k \in K$, we write $RQ(z_k) = w_{m_k}$, which implies $|z_k| \geqslant |w_{m_k}|$. Since X^{ζ_0} is open in $M_T^*(X)$, there is a $k_0 \in K$ such that $z_k \in X^{\zeta_0}$ (and consequently $|z_k| = |z|$) for every $k \geqslant k_0$. If the net of integers m_k is bounded by integer M, then $RQ(z_k) \in \{w_1, w_2, \ldots, w_M\}$ for each $k \in K$. But $F_R^q(Y)$ is T_1 by Theorem 4.9 and so the finite set $\{w_1, w_2, \ldots, w_M\}$ is closed in $F_R^q(Y)$. Since $RQ(z_k) \to RQ(z)$, we must have $RQ(z) \in \{w_1, w_2, \ldots, w_M\} \subseteq C$ but this is a contradiction. Suppose, on the other hand, that m_k is unbounded and $k_0 \in K$. Since $\lim_{m\to\infty} |w_m| = \infty$, there is an m_0 such that $|w_m| > |z|$ for all $m \geqslant m_0$. Since m_k is unbounded, there is a $k_1 \geqslant k_0$ such that $m_{k_1} > m_0$. But this means

$$|z_{k_1}| \geqslant |w_{m_{k_1}}| > |z|.$$

This contradicts that $|z_k|$ is eventually $|z|$. Therefore we must have that $RQ(z) \in C$ which again is a contradiction. Since any convergent net in $RQ^{-1}(C)$ has limit in $RQ^{-1}(C)$, this set must be closed in $M_T^*(X)$. □

Corollary 4.16. *Let $q : X \to Y$ be separating and w_m be a sequence in $F_R^q(Y)$ such that $\lim_{m\to\infty} |w_m| = \infty$. Then w_m does not have a subsequence which converges in $F_R^q(Y)$.*

Proof. If $\lim_{m\to\infty} |w_m| = \infty$, then $\lim_{m\to\infty} |w_{m_j}| = \infty$ for any subsequence w_{m_j}. Therefore, it suffices to show that w_m does not converge in $F_R^q(Y)$ whenever $\lim_{m\to\infty} |w_m| = \infty$. Suppose $w_m \to v$ for some $v \in F_R^q(Y)$. There is a subsequence w_{m_j} of w_m such that $|w_{m_j}| > |v|$ for each $j \geqslant 1$. But $\lim_{m\to\infty} |w_{m_j}| = \infty$ and so $C = \{w_{m_j}\}_{j\geqslant 1}$ is closed in $F_R^q(Y)$ by Lemma 4.15. This implies $v \in C$ which is impossible. □

Corollary 4.17. *If $q : X \to Y$ is separating and K is a compact subset of $F_R^q(Y)$, then $K \subseteq F_R^q(Y)_n$ for some $n \geqslant 1$.*

Proof. Suppose $K \nsubseteq F_R^q(Y)_n$ for any $n \geqslant 1$. Take $w_1 \in K$ such that $|w_1| = n_1$. Inductively, if we have $w_m \in K \cap F_R^q(Y)_{n_m}$, there is an $n_{m+1} > n_m$ and a word $w_{m+1} \in K \cap (F_R^q(Y)_{n_{m+1}} - F_R^q(Y)_{n_m})$. Now we have a sequence $w_m \in K$ such that $|w_1| < |w_2| < \cdots$ which clearly gives $\lim_{m\to\infty} |w_m| = \infty$. Corollary 4.16 then asserts that w_m has no converging subsequence in $F_R^q(Y)$, however, this contradicts the fact that K is compact. □

Theorem 4.18. *Let $q : X \to Y$ be a separating quotient map. The following are equivalent*:

1. *Y is a discrete space.*
2. *$F_R^q(Y)$ is a discrete group.*
3. *$F_R^q(Y)$ is first countable.*

Proof. 1. ⇒ 2. ⇒ 3. is clear. To prove 3. ⇒ 1. we suppose Y is non-discrete and $F_R^q(Y)$ is first countable. Since q is quotient and separating, Y must be T_1. Let $y_0 \in Y$ such that the singleton $\{y_0\}$ is not open. Since q is quotient $q^{-1}(y_0)$ is not open in X. There is an $x_0 \in q^{-1}(y_0)$ such that every open neighborhood U of x_0 in X satisfies $q(U) \neq \{y_0\}$. In fact, $q(U)$ must be infinite, since if $q(U) = \{y_0, y_1, \ldots, y_m\}$, then $U \cap \bigcap_{i=1}^m (X - q^{-1}(y_i))$ is an open neighborhood of x_0 contained in $q^{-1}(y_0)$. Suppose $\{B_1, B_2, \ldots\}$ is a countable basis of open neighborhoods at the identity e in $F_R^q(Y)$ where $B_{i+1} \subseteq B_i$ for each i. Choose any $z \in X$ such that $q(z) \neq q(x_0) = y_0$ and let $w_n = (x_0 z x_0 x_0^{-1} z^{-1} x_0^{-1})^n \in M_T^*(X)$. It is clear that $RQ(w_n) = R((y_0 q(z) y_0 y_0^{-1} q(z)^{-1} y_0^{-1})^n) = e$ and therefore $w_n \in RQ^{-1}(B_i)$ for all $i, n \geqslant 1$. Since q is separating, there is an open neighborhood U_n of x_0 and V_n of z such that $\mathscr{U}_n = (U_n V_n U_n U_n^{-1} V_n^{-1} U_n^{-1})^n$ is a q-separating neighborhood of w_n contained in $RQ^{-1}(B_n)$. Recall that $\mathscr{U}_n$ being a q-separating neighborhood means $q(U_n) \cap q(V_n) = \emptyset$. Since $q(U_n)$ is infinite, we can find $y_n \in q(U_n)$ distinct from y_0 and $x_n \in U_n \cap q^{-1}(y_n)$. Since $q(U_n) \cap q(V_n) = \emptyset$, the three elements $y_0, y_n, q(z)$ of Y are distinct for each $n \geqslant 1$. Now we have

$$v_n = \left(x_0 z x_0 x_n^{-1} z^{-1} x_n^{-1}\right)^n \in \mathscr{U}_n \subseteq RQ^{-1}(B_n)$$

which satisfies

$$RQ(v_n) = R\big(\big(y_0 q(z) y_0 y_n^{-1} q(z)^{-1} y_n^{-1}\big)^n\big) = \big(y_0 q(z) y_0 y_n^{-1} q(z)^{-1} y_n^{-1}\big)^n \in B_n.$$

Note that $|RQ(v_n)| = 6n$ and so $\lim_{n\to\infty} |RQ(v_n)| = \infty$. By Corollary 4.16, the sequence $RQ(v_n)$ cannot converge to the identity of $F_R^q(Y)$. But since $\{B_i\}$ is a countable basis at e and $RQ(v_n) \in B_n$, we must have $RQ(v_n) \to e$. This is a contradiction. □

Corollary 4.19. *Suppose $\pi_1^{top}(\Sigma(X_+))$ is T_1. Then $\pi_1^{top}(\Sigma(X_+))$ is first countable if and only if it is discrete.*

4.3. When is $\pi_1^{top}(\Sigma(X_+))$ a topological group?

One of the most interesting problems involving $\pi_1^{top}(\Sigma(X_+))$ is the characterization of spaces X for which $\pi_1^{top}(\Sigma(X_+))$ is a topological group. Before our proof of Theorem 1.4, we consider a few simple cases. The following corollary is a direct application of Corollary 4.11 and allows us to construct us our first explicit example of a space whose topological fundamental group fails to be a topological group.

Corollary 4.20. *If $\pi_1^{top}(\Sigma(X_+))$ is a Hausdorff topological group (which is also normal), then $\pi_0^{top}(X^n)$ is Tychonoff (normal) for each $n \geqslant 1$.*

Example 4.21. Let $\mathbb{Q}_K$ denote the rational numbers with the subspace topology of the real line with the K-topology [27]. Then $\pi_0^{top}(\mathbb{Q}_K) \cong \mathbb{Q}_K$ is Hausdorff and totally path disconnected but is not regular. Therefore $\pi_1^{top}(\Sigma(\mathbb{Q}_K)_+) \cong F_R(\mathbb{Q}_K)$ is not a topological group.

Example 4.22. Using Remark 2.4, we can produce a large class of spaces each with topological fundamental group failing to be a topological group. For every Hausdorff, non-completely regular space Y, there is a paracompact Hausdorff space $X = \mathcal{H}(Y)$ such that $\pi_0^{top}(X) \cong Y$. Since $\pi_0^{top}(X)$ is Hausdorff, $\pi_1^{top}(\Sigma(X_+))$ is T_1 but by the previous corollary $\pi_1^{top}(\Sigma(X_+))$ cannot be a topological group.

Corollary 4.23. *If X is a Tychonoff k_ω-space, then $\pi_1^{top}(\Sigma(X_+))$ is a topological group. If, in addition, X is totally path disconnected, then $\pi_1^{top}(\Sigma(X_+)) \cong F_M(X)$.*

Proof. As mentioned in the introduction, if X is a Tychonoff, k_ω-space, then $F_R(X)$ is a topological group [12]. By Corollary 3.23, $F_R^{\pi_X}(\pi_0^{top}(X)) \cong \pi_1^{top}(\Sigma(X_+))$ is a topological group. □

Example 4.24. Let $X = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\} \subset \mathbb{R}$. Then $\Sigma(X_+)$ is homeomorphic to the planar continuum

$$\bigcup_{r\in X} \big\{(x, y) \in \mathbb{R}^2 \;\big|\; (x-r)^2 + y^2 = (1+r)^2\big\}$$

and $\pi_1^{top}(\Sigma(X_+))$ is isomorphic to the free topological group $F_M(X)$.

Theorem 4.25. *Let $q : X \to Y$ be a quotient map. If $q^n : X^n \to Y^n$ is a quotient map for all $n \geqslant 1$, then the continuous homomorphism $F_R(q) : F_R(X) \to F_R(Y)$ is a topological quotient map. If X and Y are Hausdorff, the converse holds.*

Proof. If $q^n : X^n \to Y^n$ is a quotient map for each $n \geqslant 1$, then $M_T^*(q) = \coprod_{n\geqslant 0} (q \sqcup q)^n : M_T^*(X) \to M_T^*(Y)$ is quotient. Since the diagram

$$\begin{array}{ccc} M_T^*(X) & \xrightarrow{M_T^*(q)} & M_T^*(Y) \\ {\scriptstyle R_X}\downarrow & & \downarrow{\scriptstyle R_Y} \\ F_R(X) & \xrightarrow[F_R(q)]{} & F_R(Y) \end{array}$$

commutes and the reduction maps (distinguished with subscripts) are quotient, $F_R(q)$ is quotient.

To prove the converse, let $\sigma_n^X : X^n \to F_R(X)$ and $\sigma_n^Y : Y^n \to F_R(Y)$ be the canonical, closed embeddings of Theorem 4.9 and $\tilde{X}^n = \sigma_n^X(X^n)$ and $\tilde{Y}^n = \sigma_n^Y(Y^n)$ be their images. We show the restriction $p = F_R(q)|_{\tilde{X}^n} : \tilde{X}^n \to \tilde{Y}^n$ is a quotient map using the commutative diagram

$$\begin{array}{ccccc} X^n & \xrightarrow{\sigma_X^n} & F_R(X) & \xleftarrow{R_X} & M_T^*(X) \\ \downarrow{\scriptstyle q^n} & & \downarrow{\scriptstyle F_R(q)} & & \downarrow{\scriptstyle M_T^*(q)} \\ Y^n & \xrightarrow[\sigma_Y^n]{} & F_R(Y) & \xleftarrow[R_Y]{} & M_T^*(Y) \end{array}$$

To see that p being quotient implies q^n is quotient, take $C \subseteq Y^n$ such that $(q^n)^{-1}(C)$ is closed in X^n. Then $\sigma_n^X((q^n)^{-1}(C)) = p^{-1}(\sigma_n^Y(C))$ is closed in $\tilde{X}^n$ and consequently $\sigma_n^Y(C)$ is closed in $\tilde{Y}^n$. Since σ_n^Y is a continuous injection, C is closed in Y^n.

Suppose $A \subseteq \tilde{Y}^n$ such that $p^{-1}(A)$ is closed in $\tilde{X}^n$. Since R_X is quotient and $F_R(q)$ is assumed to be quotient and $\tilde{Y}^n$ is closed in $F_R(Y)$, it suffices to show that

$$B^\zeta = R_X^{-1}\big(F_R(q)^{-1}(A)\big) \cap X^\zeta = \big\{x = x_1^{\epsilon_1} \dots x_k^{\epsilon_k} \;\big|\; R_Y\big(M_T^*(q)(x)\big) = R_Y\big(q(x_1)^{\epsilon_1} \dots q(x_k)^{\epsilon_k}\big) \in A\big\}$$

is closed in X^ζ for each $\zeta = \epsilon_1, \dots, \epsilon_k$. We proceed by induction on $|\zeta| = k$. It is clear that if $|\zeta| < n$, then $B^\zeta = \emptyset$. Additionally, if $|\zeta| = n$ and $\zeta \neq 1, 1, \dots, 1$, then $B^\zeta = \emptyset$. On the other hand, if $|\zeta| = n$ and $\zeta = 1, 1, \dots, 1$, then $B^\zeta = \{x_1 \dots x_n \mid q(x_1) \dots q(x_n) \in A\} = R_X^{-1}(p^{-1}(A)) \cap X^\zeta$ is closed by assumption. Now we suppose that $|\zeta| > n$ and B^ζ is closed in X^δ for all δ such that $|\delta| = n, n+1, \dots, |\zeta| - 1$. Let $x = x_1^{\epsilon_1} \dots x_k^{\epsilon_k} \in X^\zeta - B^\zeta$ and $y = M_T^*(q)(x) = q(x_1)^{\epsilon_1} \dots q(x_k)^{\epsilon_k}$. Since $x \notin B^\zeta$, we have $R_Y(y) \notin A$. Let $E = E_1^{\epsilon_1} \dots E_k^{\epsilon_k}$ be a separating neighborhood of y in $M_T^*(Y)$. Since $M_T^*(q)$ is continuous, there is a separating neighborhood $D = D_1^{\epsilon_1} \dots D_k^{\epsilon_k}$ of x, such that $q(D_i) \subseteq E_i$ for each $i \in \{1, \dots, k\}$. Since E is a separating neighborhood, if $q(x_i) \neq q(x_j)$, then $q(D_i) \cap q(D_j) = \emptyset$. Now we consider the cases when y is and is not reduced.

If y is reduced and $v \in D$, then $M_T^*(q)(v) \in E$ must also be reduced by Lemma 4.5. Therefore $n < |\zeta| = |y| = |R_Y(M_T^*(v))|$, i.e. the reduced word of $M_T^*(q)(v)$ has length greater than n and cannot lie in $A \subseteq \tilde{Y}^n$. Therefore $D \cap B^\zeta = \emptyset$.

If y is not reduced, then for each $i \in \{1, \dots, k-1\}$ such that $q(x_i) = q(x_{i+1})$ and $\epsilon_i = -\epsilon_{i+1}$, we let $w_i = q(x_1)^{\epsilon_1} \dots q(x_{i-1})^{\epsilon_{i-1}} q(x_{i+2})^{\epsilon_{i+2}} \dots q(x_k)^{\epsilon_k} \in M_T^*(Y)$ and $u_i = x_1^{\epsilon_1} \dots x_{i-1}^{\epsilon_{i-1}} x_{i+2}^{\epsilon_{i+2}} \dots x_k^{\epsilon_k}$ be the words obtained by removing the i-th and $(i+1)$-th letters from y and x respectively. We also let $\zeta_i = \epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+2}, \dots \epsilon_k$. This gives $F_R(q)(R_X(u_i)) = R_Y(M_T^*(u_i)) = R_Y(w_i) = R_Y(y) \notin A$ and consequently $u_i \in X^{\zeta_i} - B^{\zeta_i}$. We know by our induction hypothesis that B^{ζ_i} is closed in X^{ζ_i} and so we may find a separating neighborhood $V_i = A_1^{\epsilon_1} \dots A_{i-1}^{\epsilon_{i-1}} A_{i+2}^{\epsilon_{i+2}} \dots A_k^{\epsilon_k}$ of u_i contained in $X^{\zeta_i} - B^{\zeta_i}$. Let $A_i = A_{i+1} = X$ so that

$$U_i = A_1^{\epsilon_1} \dots A_{i-1}^{\epsilon_{i-1}} A_i^{\epsilon_i} A_{i+1}^{\epsilon_{i+1}} A_{i+2}^{\epsilon_{i+2}} \dots A_k^{\epsilon_k}$$

is an open neighborhood of x. Now take a separating neighborhood U of x such that $U \subseteq D \cap \bigcap_i U_i$ where the intersection ranges over the $i \in \{1, \dots, k-1\}$ such that $q(x_i) = q(x_{i+1})$ and $\epsilon_i = -\epsilon_{i+1}$. It now suffices to show that $F_R(q)(R_X(v)) = R_Y(q(z_1)^{\epsilon_1} \dots q(z_k)^{\epsilon_k}) \notin A$ whenever $v = z_1^{\epsilon_1} \dots z_k^{\epsilon_k} \in U$. If $M_T^*(q)(v) = q(z_1)^{\epsilon_1} \dots q(z_k)^{\epsilon_k}$ is reduced, then $n < |\zeta| = |x| = |R_Y(M_T^*(q))|$ and $R_Y(M_T^*(q)(v)) \notin A$. On the other hand, suppose $q(z_1)^{\epsilon_1} \dots q(z_k)^{\epsilon_k}$ is not reduced. There is an $i_0 \in \{1, \dots, k-1\}$ such that $q(z_{i_0}) = q(z_{i_0+1})$ and $\epsilon_{i_0} = -\epsilon_{i_0+1}$. But $z_{i_0} \in D_{i_0}$ and $z_{i_0+1} \in D_{i_0+1}$, so we must have $q(x_{i_0}) = q(x_{i_0+1})$. Since $v \in U \subseteq U_{i_0}$, we have

$$v_{i_0} = z_1^{\epsilon_1} \dots z_{i_0-1}^{\epsilon_{i_0-1}} z_{i_0+2}^{\epsilon_{i_0+2}} \dots z_k^{\epsilon_k} \in V_{i_0} \subseteq X^{\zeta_{i_0}} - B^{\zeta_{i_0}}.$$

Therefore

$$F_R(q)\big(R_X(v)\big) = R_Y\big(M_T^*(q)(v)\big) = R_Y\big(M_T^*(q)(v_{i_0})\big) = F_R(q)\big(R_X(v_{i_0})\big) \notin A$$

proving that $U \cap B^\zeta = \emptyset$ and B^ζ is closed in X^ζ. □

Theorem 1.4 is the case $q = \pi_X$ of the following theorem.

Theorem 4.26. *If $q : X \to Y$ is a quotient map where X is Hausdorff, then $F_R^q(Y)$ is a Hausdorff topological group if and only if all four of the following conditions hold*:

1. *Y is Tychonoff.*
2. *$q^n : X^n \to Y^n$ is quotient for each $n \geqslant 1$.*
3. *The free topological group $F_M(Y)$ has the inductive limit topology of subspaces $F_M(Y)_n$ consisting of words of length $\leqslant n$.*
4. *For every $n \geqslant 1$, the canonical multiplication map $\coprod_{i=0}^n (Y \sqcup Y^{-1})^i \to F_M(Y)_n$ is a quotient map.*

Proof. Suppose X is Hausdorff and $F_R^q(Y)$ is a Hausdorff topological group. By Lemma 3.22, we have that $id : F_R^q(Y) \cong F_R(Y) \cong F_M(Y)$ is Hausdorff. By Remark 3.17.1 Y must be functionally Hausdorff and therefore q is separating. By Theorem 4.9, the canonical map $\sigma : Y \to F_R^q(Y) \cong F_M(Y)$ is an embedding. Since Y embeds into a Tychonoff space, it must be Tychonoff. By Lemma 3.22.4, we have that $F_R(q) : F_R(X) \to F_R(Y)$ is quotient. Since both X and Y are Hausdorff, Theorem 4.25 applies and $q^n : X^n \to Y^n$ is quotient for each $n \geqslant 1$. The last statement of Proposition 3.20 gives the last two conditions.

To prove the converse, suppose the four conditions in the statement of the theorem hold. Since 1., 3., and 4. hold, Fact 1.3 applies and $M_T^*(Y) \to F_M(\pi_0^{top}(X))$ is quotient. Therefore $id : F_R(Y) \cong F_M(Y)$. Additionally, condition 2. implies that $F_R(q)$ is quotient. Thus $F_R^q(Y) \cong F_R(Y)$ by Lemma 3.22.4. Moreover, $F_R^q(Y) \cong F_R(Y) \cong F_M(Y)$ is Hausdorff since Y is Tychonoff. □

Example 4.27. We obtain a particularly interesting example when we let $X = \mathbb{Q} \cong \mathbb{Q} \cap (0,1)$. According to Example 3.39, $w\Sigma(X_+)$ may be embedded as a subspace of $\mathbb{R}^2$. Since $\mathbb{Q}$ is functionally Hausdorff and totally path disconnected, $\pi_1^{top}(w\Sigma(X_+)) \cong F_R(\mathbb{Q})$ is a functionally Hausdorff quasitopological group. It is shown in [28] that both conditions 3. and 4. of Theorem 4.26 do not hold for the case $Y = \mathbb{Q}$. Therefore $w\Sigma(X_+)$ is a locally simply connected (but not locally path connected) metric space whose topological fundamental group fails to be a topological group.

5. Conclusions

This computation and analysis of $\pi_1^{top}(\Sigma(X_+))$ offers new insight into the nature of topological fundamental groups and provides a geometric interpretation of many quasitopological and free topological groups. We note here how these ideas may be extended to higher dimensions and abelian groups, i.e. to the higher topological homotopy groups $\pi_n^{top}(X,x) = \pi_0^{top}(\Omega^n(X,x))$ and free abelian topological groups. These quasitopological abelian groups were first studied in [5] and [29], however, these authors assert that $\pi_n^{top}(X,x)$ is a topological group without sufficient proof. This misstep is noted in [30] and the following problem remains open.

Problem 5.1. For $n \geqslant 2$, is π_n^{top} a functor to the category abelian topological groups?

As mentioned in the introduction, Fabel has shown that the topological fundamental group of the Hawaiian earring fails to be a topological group. This particular complication seems to disappear in higher dimensions since, for $n \geqslant 2$, the n-th topological fundamental group of the n-dimensional Hawaiian earring is indeed a topological group [30]. The results in this paper, however, indicate that Problem 5.1 is likely to have a negative answer. Just as in Proposition 3.3, we have

Proposition 5.2. *For every based space Y, $\pi_n^{top}(Y)$ is a topological quotient group of $\pi_n^{top}(\Sigma^n(\Omega^n(Y)_+))$.*

Therefore, if $\pi_n^{top}(\Sigma^n(X_+))$ is a topological group for every X, then $\pi_n^{top}(Y)$ is a topological group for every Y. Consequently, the spaces $\Sigma^n(X_+)$ are prime candidates for producing counterexamples to Problem 5.1. Let $Z_R^q(Y)$ (resp. $Z_R(Y)$) be the free abelian group on the underlying set of Y viewed as the quotient space of $F_R^q(Y)$ (resp. $F_R(Y)$) with respect to the abelianization map. These groups have many of the same topological properties as their non-abelian counterparts. In particular, $Z_R^q(Y)$ (resp. $Z_R(Y)$) either fails to be a topological group or is the free abelian topological group $Z_M(Y)$ on Y. The current paper indicates the likelihood of the following statement.

Conjecture 5.3. *For an arbitrary space X, the canonical map $\pi_0^{top}(X) \to \pi_n^{top}(\Sigma^n(X_+))$ induces an isomorphism $h_X : Z_R^{\pi_X}(\pi_0^{top}(X)) \to \pi_n^{top}(\Sigma^n(X_+))$ of quasitopological groups which are not always topological groups.*

If this is indeed the case, then π_n^{top} will be a functor to the category of quasitopological abelian groups but not to the category of topological abelian groups. A computation of $\pi_n^{top}(\Sigma^n(X_+))$ for $n \geqslant 2$ should then provide an answer to Problem 5.1.

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