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Smooth group representations on bornological vector spaces

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Abstract

We develop the basic theory of smooth representations of locally compact groups on bornological vector spaces. In this setup, we are able to formulate better general theorems than in the topological case. Nonetheless, smooth representations of totally disconnected groups on vector spaces and of Lie groups on Fréchet spaces remain special cases of our theory. We identify smooth representations with essential modules over an appropriate convolution algebra. We examine smoothening functors on representations and modules and show that they agree if they are both defined. We establish the basic properties of induction and compact induction functors using adjoint functor techniques. We describe the center of the category of smooth representations.

Résumé

Nous développons la théorie basique des représentations lisses des groupes localement compacts sur les espaces vectorielles bornologiques. Dans ce contexte, nous pouvons établir des meilleurs théorèmes que dans la situation topologique. Néanmoins, les représentations lisses des groupes totalement discontinus sur les espaces vectorielles et les représentations lisses des groupes de Lie sur les espaces de Fréchet restent des cas spécialux de notre théorie. Nous identifions des représentations lisses avec des modules essentielles sur une algèbre de convolution convenable. Nous examinons des foncteurs régularisants sur des représentations et des modules et nous montrons qu'ils sont égales s'ils sont définis. Nous établissons les propriétés basiques des foncteurs d'induction et d'induction compact en employant des techniques des foncteurs adjointes. Nous décrivons le centre de la catégorie des représentations lisses.

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1. Introduction

Smooth representations of totally disconnected groups on vector spaces and of Lie groups on locally convex topological vector spaces have already been studied for a long time. It is also known that one can define smooth representations of arbitrary locally compact groups using the spaces of smooth functions introduced by François Bruhat in [4]. We shall consider, instead, smooth representations of locally compact groups on *bornological* vector spaces (see [12]). While this may appear to be only a minor variation on the usual theory, it turns out that there are several small but significant details that make the bornological theory much more pleasant and more powerful. Smooth representations of totally disconnected groups on vector spaces and of Lie groups on Fréchet spaces are special cases of our theory, so that it allows for a unified treatment of these two kinds of representations.

Bornological vector spaces went out of fashion quite some time ago. This is rather unfortunate because they are the ideal setting for noncommutative geometry. As soon as we move beyond Fréchet spaces, we run into annoying problems when we work with topological vector spaces. For instance, the multiplication on an algebra like $\mathcal{D}(\mathbb{R})$ with convolution is only separately continuous and not jointly continuous. Therefore, one has to give *ad hoc* definitions for the complexes that compute the Hochschild and cyclic homology of such convolution algebras. Problems of this nature are artefacts which disappear if we work bornologically instead. Moreover, bornologies are essential for the purposes of local cyclic cohomology, which is a variant of cyclic cohomology that produces better results for Banach algebras like the algebra of continuous functions on a compact space.

A great advantage of bornological versus topological analysis is the adjoint associativity between the completed bornological tensor product $\hat{\otimes}$ and the internal Hom functor: $\mathrm{Hom}(A \hat{\otimes} B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$. In particular, there is a canonical bornology on the space $\mathrm{Hom}(B, C)$ of bounded linear maps between two bornological vector spaces. Adjoint associativity holds for vector spaces and Banach spaces, but not for topological vector spaces. It provides bornological analysis with a much richer algebraic structure than topological analysis. For representation theory this means that the general theory of smooth representations of locally compact groups on bornological vector spaces is very similar to the purely algebraic theory of smooth representations of totally disconnected groups on vector spaces.

An instance of this is our main theorem, which asserts that the category of smooth representations of G is isomorphic to the category of essential modules over the convolution algebra $\mathcal{D}(G)$ of smooth functions with compact support on G. We also have very nice adjointness relations between restriction, induction and compact induction functors, from which we can deduce many properties of these functors.

We now explain our results in greater detail. Throughout this article, G denotes a locally compact topological group. Bruhat [4] defines spaces $\mathcal{D}(G)$ and $\mathcal{E}(G)$ of smooth functions with compact support and with arbitrary growth at infinity, respectively. In the totally disconnected case a function is smooth if and only if it is locally constant. In the Lie group case smoothness has the usual meaning. General locally compact groups are treated using the deep structure theory of almost connected groups. We recall Bruhat's definitions and adapt them to our bornological setup in Section 2. Besides basic facts about these function spaces, we prove some interesting results about metrizable bornological vector spaces.

A representation $\pi: G \to \mathrm{Aut}(V)$ on a complete convex bornological vector space V is called smooth if the map that sends $v \in V$ to the function $g \mapsto \pi(g, v)$ takes values in $\mathcal{E}(G, V)$ and is a bounded linear map $\pi_*: V \to \mathcal{E}(G, V)$. Equivalently, the formula $Wf(g) := g \cdot f(g)$ defines a bounded linear operator on $\mathcal{D}(G, V)$. For totally disconnected G this amounts to the requirement that any bounded set be stabilized by an open subgroup of G. In particular, if V is a vector space with the fine bornology, we get the usual notion of a smooth representation of a totally disconnected group on a complex vector space.

Now suppose G to be a Lie group. A representation is called differentiable if it is k times continuously differentiable for all $k \in \mathbb{N}$. This notion is weaker than smoothness. For instance, the left regular representation on the space of compactly supported distributions $\mathcal{E}'(G)$ is differentiable but not smooth. Differentiability and smoothness are equivalent if V is bornologically metrizable. In particular, this happens if V is a Fréchet space equipped with a reasonable bornology.

Differentiable representations on bornological vector spaces are closely related to smooth representations on topological vector spaces. We show that a bornological representation π is differentiable if and only if it extends to a bounded algebra homomorphism $\int\pi: \mathcal{E}'(G) \to \mathrm{End}(V)$. Similarly, a topological representation π is smooth if and only if it extends to a bounded homomorphism $\int\pi: \mathcal{E}'(G) \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ carries the equicontinuous bornology. Let V be a bornological topological vector space, equip it with the von Neumann bornology. Then there is no difference between the spaces of continuous and bounded maps $V \to V$, equipped with the equicontinuous and equibounded bornology, respectively. Hence topological smoothness is equivalent to bornological differentiability in this case. If V is a Fréchet space, we know that bornological differentiability and smoothness are equivalent, so that the topological and bornological notions of smooth representation agree for Fréchet spaces. For general V the bornological notion of smoothness is more restrictive than the topological one.

If we restrict $\int\pi$ to the convolution algebra $\mathcal{D}(G)$, we turn V into a module over $\mathcal{D}(G)$. A module V over $\mathcal{D}(G)$ is called *essential* if the module action is a bornological quotient map $\mathcal{D}(G) \hat{\otimes} V \to V$. That is, each bounded subset of V is the image of a bounded subset of $\mathcal{D}(G) \hat{\otimes} V$. The following theorem generalizes a well-known and much used fact for totally disconnected groups:

Theorem 1.1. *Let G be a locally compact group. The categories of essential bornological left $\mathcal{D}(G)$-modules and of smooth representations of G on bornological vector spaces are isomorphic. The isomorphism sends a representation $\pi: G \to \mathrm{Aut}(V)$ to the module $\int\pi: \mathcal{D}(G) \to \mathrm{End}(V)$.*

The theorem makes three assertions. First, if $\pi : G \to \mathrm{Aut}(V)$ is smooth, then $\int\pi : \mathcal{D}(G) \hat{\otimes} V \to V$ is a bornological quotient map. In fact, this map even has a bounded linear section. Secondly, any essential module over $\mathcal{D}(G)$ arises in this fashion from a smooth representation of G. Thirdly, a bounded linear map between two smooth representations is G-equivariant if and only if it is a homomorphism of $\mathcal{D}(G)$-modules. In the topological framework it is still true that π is smooth if and only if $\int\pi : \mathcal{D}(G, V) \to V$ has a continuous linear section (see [2]). However, $\mathcal{D}(G, V)$ is no longer a topological tensor product of $\mathcal{D}(G)$ and V. Therefore, we fail to characterize smooth representations in terms of the algebra $\mathcal{D}(G)$.

We study analogues in the category of modules over $\mathcal{D}(G)$ of several constructions with representations, namely, smoothening, restriction, induction and compact induction. Let $H \subseteq G$ be a closed subgroup. Then we only have $\mathcal{D}(H) \subseteq \mathcal{E}'(G)$, so that the restriction of a $\mathcal{D}(G)$-module to a $\mathcal{D}(H)$-module is not always defined. If V is an arbitrary $\mathcal{D}(G)$-module, then $\mathcal{D}(G) \hat{\otimes}_{\mathcal{D}(G)} V$ and $\mathrm{Hom}_{\mathcal{D}(G)}(\mathcal{D}(G), V)$ carry canonical $\mathcal{D}(H)$-module structures. The resulting functors are called the *smooth and rough restriction* functors, S_G^H and R_G^H. In the converse direction, if V is a module over $\mathcal{D}(H)$, we can produce a module over $\mathcal{D}(G)$ in two ways. We define the *compact induction functor* and the *rough induction functor* by

$$\mathrm{Ic}_H^G(V) := \mathcal{D}(G) \hat{\otimes}_{\mathcal{D}(H)} V,$$
$$\mathrm{I}_H^G(V) := \mathrm{Hom}_{\mathcal{D}(H)}\big(\mathcal{D}(G), V\big).$$

The functors $\mathrm{S} := \mathrm{Ic}_G^G = \mathrm{S}_G^G$ and $\mathrm{R} := \mathrm{I}_G^G = \mathrm{R}_G^G$ are called *smoothening* and *roughening*, respectively. Up to a relative modular factor, $\mathrm{S} \circ \mathrm{I}_H^G$ and Ic_H^G agree with the induction and compact induction functors for representations, respectively.

These functors enjoy many useful algebraic properties. For instance, they are exact for appropriate classes of extensions. The exactness of the smoothening functor implies that the class of essential modules is closed under extensions. The content of the roughening functor is the following: roughly speaking, the roughening of a module V is the largest module W that satisfies $\mathrm{S}V = \mathrm{S}W$. Many important properties of the induction and restriction functors follow easily by playing around with adjoint associativity. We prove the Shapiro Lemma in group homology and cohomology in this fashion and we show how to reduce Tor and Ext for the category of essential $\mathcal{D}(G)$-modules to group homology and cohomology. It is remarkable that such results can be proved easily and purely algebraically. There are no analytical difficulties whatsoever.

The smoothening functors for representations and modules also agree. The module smoothening is the range of the map $\int\pi : \mathcal{D}(G) \hat{\otimes} V \to V$. The image of the uncompleted tensor product is known as the Gårding subspace of V. Jacques Dixmier and Paul Malliavin show in [5] that the Gårding subspace is equal to the smoothening for Lie group representations on Fréchet spaces. The same is true for arbitrary continuous representations of locally compact groups on bornological vector spaces.

Finally, we examine the analogue of the Bernstein center of a totally disconnected group. This is the center of the category of smooth representations of G on complex vector spaces, which was studied first by Joseph Bernstein [1]. It plays a crucial role in the representation theory of reductive groups, which is parallel to the role played by the center of the universal enveloping algebra in the Lie group case.

We prove that the center of the category of smooth representations of G is isomorphic to the center of the multiplier algebra of $\mathcal{D}(G)$. In the totally disconnected case this is the same as the Bernstein center. We describe the multiplier algebra of $\mathcal{D}(G)$ and its center as spaces of distributions on G. For Lie groups the multiplier algebra is just $\mathcal{E}'(G)$. For a connected complex Lie group with trivial center, central multipliers are necessarily supported at the identity element. Thus the center of the category of smooth representations of G is isomorphic to the center of the universal enveloping algebra of G in this case.

2. Spaces of smooth functions on locally compact groups

Many results of this section are adaptations to the bornological setting of results of François Bruhat [4]. There are a few issues regarding tensor products and metrizability that do not arise in the topological setting, however.

Since we are only dealing with complete convex bornologies, we drop these adjectives from our notation: whenever we assert or ask that a space be a bornological vector space, it is understood that it is asserted or asked to be a complete convex bornological vector space. Good references for the basic theory of bornological vector spaces are the publications of Henri Hogbe-Nlend [10–12], whose notation we will follow mostly.

2.1. Preliminaries

The structure theory of locally compact groups is crucial for Bruhat's definitions in order to reduce to the case of Lie groups. Although its results are very difficult to prove, they are extremely simple to apply and state.

Let G be a locally compact group. Let $G_0 \subseteq G$ be the connected component of the identity element. The group G is called *totally disconnected* if $G_0 = \{1\}$, *connected* if $G_0 = G$ and *almost connected* if G/G_0 is compact.

A totally disconnected locally compact group has a base for the neighborhoods of the identity element consisting of compact open subgroups (see [9]). Applying this to the totally disconnected group G/G_0, we find that any locally compact group contains an almost connected open subgroup.

Theorem 2.1 [15]. *Let G be an almost connected locally compact group. Then G is isomorphic to a projective limit of Lie groups. More explicitly, there is a directed set I of compact normal subgroups $k \subseteq G$ such that G/k is a Lie group for all $k \in I$ and $\bigcap I = \{1\}$. We have $G = \varprojlim_{k\in I} G/k$ for any such system.*

Definition 2.2. A subgroup $k \subseteq G$ is called *smooth* if its normalizer $N_G(k) \subseteq G$ is open and $N_G(k)/k$ is a Lie group. Let SC or SC(G) be the set of all smooth compact subgroups. A *fundamental system of smooth compact subgroups* in G is a set I of smooth compact subgroups which is directed by inclusion and satisfies $\bigcap I = \{1\}$.

Lemma 2.3. *Let G be a locally compact group. If $k \subseteq G$ is a smooth subgroup, then G/k, $k\backslash G$ and $G//k := k\backslash G/k$ are smooth manifolds in a canonical way. If $k_1 \subseteq k_2$, then the induced maps $G/k_1 \to G/k_2$, etc., are smooth.*

The set $\mathrm{SC}(G)$ is a fundamental system of smooth compact subgroups and in particular directed. We have

$$G \cong \varprojlim G/k \cong \varprojlim k\backslash G \cong \varprojlim G//k,$$

where the limits are taken for $k \in \mathrm{SC}(G)$.

A set of subgroups is a fundamental system of smooth compact subgroups if and only if it is a cofinal subset of $\mathrm{SC}(G)$. The set I can be taken countable and even a decreasing sequence if and only if G is metrizable.

Proof. Let $k \subseteq G$ be a smooth subgroup and let U be its normalizer. Thus U is an open subgroup of G, k is a normal subgroup of U and U/k is a Lie group. The homogeneous space G/k is just a disjoint union of copies gU/k of the Lie group U/k for $g \in G/U$ and hence a smooth manifold. The same applies to $k\backslash G$. The proof of the corresponding assertion for $G//k$ is more complicated. We view this as the orbit space of the action of k on G/k by left multiplication. For any $g \in G$, let $k' := k \cap gUg^{-1}$. Then $k\backslash kgU/k \cong k'\backslash gU/k \cong g^{-1}k'g\backslash U/k$ because G/U is open. The latter double coset space is really a left coset space because k is normal in U. Thus $k\backslash G/k$ is a disjoint union of smooth manifolds as well.

Let $U \subseteq G$ be an open almost connected subgroup. For U instead of G, our assertions follow from Theorem 2.1. Since $\mathrm{SC}(U) \subseteq \mathrm{SC}(G)$ is cofinal, the latter is a fundamental system of smooth compact subgroups in G. We also get the isomorphisms $G \cong \varprojlim G/k$, etc., from the corresponding statement for U. It is clear that any cofinal subset of $\mathrm{SC}(G)$ is still a fundamental system of smooth compact subgroups. Conversely, if I is such a set, then $I \subseteq \mathrm{SC}(G)$. Let $k \in \mathrm{SC}(G)$. Since $\bigcap I = \{1\}$, the set of $k' \in I$ with $k' \subseteq N_G(k)$ is cofinal. Since the Lie group $N_G(k)/k$ does not contain arbitrarily small subgroups, the quotient group k'/k must eventually be trivial, that is, $k' \subseteq k$. This means that I is cofinal in $\mathrm{SC}(G)$. It is clear from $G \cong \varprojlim G/k$ that G is metrizable if and only if we can choose I countable. □

Before we can define smooth functions on locally compact groups, we need some generalities about spaces of smooth functions on manifolds (see [14] for more details). Let M be a smooth manifold and let B be a Banach space. Then we equip the space $\mathcal{D}(M, B)$ of smooth functions with compact support from M to B with the following bornology. A set S of smooth functions is bounded if all $f \in S$ are supported in a fixed compact subset of M and the set of functions $D(S)$ is uniformly bounded for any differential operator D on M. This is the von Neumann bornology for the usual LF-topology on $\mathcal{D}(M, B)$. We let $\mathcal{D}(M)$ be $\mathcal{D}(M, \mathbb{R})$ or $\mathcal{D}(M, \mathbb{C})$, depending on whether we work with real or complex bornological vector spaces. In the following, we will assume that we work with complex vector spaces, but everything works for real vector spaces as well.

If V is a bornological vector space, we let $\mathcal{D}(M, V)$ be the space of all functions $M \to V$ that belong to $\mathcal{D}(M, V_T)$ for some bounded complete disk $T \subseteq V$. A subset of

$\mathcal{D}(M, V)$ is bounded if it is bounded in $\mathcal{D}(M, V_T)$ for some T. (Recall that V_T is the linear span of T equipped with the norm whose closed unit ball is T. Hence it is a Banach space.)

Let $\hat{\otimes}$ be the completed projective bornological tensor product. It is defined by the universal property that bounded linear maps $A \hat{\otimes} B \to C$ correspond to bounded bilinear maps $A \times B \to C$. The natural map $\mathcal{D}(M) \hat{\otimes} B \to \mathcal{D}(M, B)$ is a bornological isomorphism for all Banach spaces B. The functor $\mathcal{D}(M) \hat{\otimes} \sqcup$ commutes with direct limits and preserves injectivity of linear maps because $\mathcal{D}(M)$ is nuclear (see [13]). Therefore, we have

$$\mathcal{D}(M, V) \cong \mathcal{D}(M) \hat{\otimes} V \tag{1}$$

for all bornological vector spaces V. Moreover, for two manifolds M_1, M_2 we have

$$\mathcal{D}(M_1) \hat{\otimes} \mathcal{D}(M_2) \cong \mathcal{D}(M_1 \times M_2).$$

We define the spaces $\mathcal{C}^k_{\mathrm{c}}(M, V)$ of k times continuously differentiable functions with compact support similarly for $k \in \mathbb{N}$. If V is a Banach space, we let $\mathcal{C}^k_{\mathrm{c}}(M, V)$ be the usual LF-space and equip it with the von Neumann bornology. For general V we let $\mathcal{C}^k_{\mathrm{c}}(M, V) := \varinjlim \mathcal{C}^k_{\mathrm{c}}(M, V_T)$. We let $\mathcal{C}^\infty_{\mathrm{c}}(M, V) := \varprojlim \mathcal{C}^k_{\mathrm{c}}(M, V)$ and call functions in $\mathcal{C}^\infty_{\mathrm{c}}(M, V)$ *differentiable* (see also [19]). While there evidently is no difference between smooth functions and $\mathcal{C}^\infty$-functions with values in a Banach space, smoothness is more restrictive than differentiability in general. Smooth functions are easier to work with because of (1), which fails for $\mathcal{C}^\infty_{\mathrm{c}}(M, V)$.

Definition 2.4. A bornological vector space is *metrizable* if for any sequence (S_n) of bounded subsets there is a sequence of scalars (ε_n) such that $\sum \varepsilon_n S_n$ is bounded.

The precompact bornology and the von Neumann bornology on a Fréchet space are metrizable in this sense (see [14]).

Lemma 2.5. *If V is metrizable, then $\mathcal{D}(M, V) = \mathcal{C}^\infty_{\mathrm{c}}(M, V)$.*

Proof. Let $S \subseteq \mathcal{C}^\infty_{\mathrm{c}}(M, V)$ be bounded. That is, S is bounded in $\mathcal{C}^k_{\mathrm{c}}(M, V)$ for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, there is a bounded complete disk $T_k \subseteq V$ such that S is bounded in $\mathcal{C}^k_{\mathrm{c}}(M, V_{T_k})$. By metrizability, we can absorb all T_k in some bounded complete disk $T \subseteq V$. Thus S is bounded in $\mathcal{C}^k_{\mathrm{c}}(M, V_T)$ for all $k \in \mathbb{N}$. This means that S is bounded in $\mathcal{D}(M, V)$. □

Lemma 2.6. *A bornological vector space V is metrizable if and only if the functor $V \hat{\otimes} \sqcup$ commutes with countable direct products.*

Proof. It is easy to see that V is metrizable once $V \hat{\otimes} \prod_{\mathbb{N}} \mathbb{C} \cong \prod_{\mathbb{N}} (V \hat{\otimes} \mathbb{C})$. For the converse implication, we clearly have a bounded linear map $V \hat{\otimes} \prod B_n \to \prod V \hat{\otimes} B_n$. We have to show that $\prod V \hat{\otimes} B_n$ satisfies the universal property of $V \hat{\otimes} \prod B_n$. That is, we need that a bounded bilinear map $l : V \times \prod B_n \to X$ induces a bounded linear map $\prod V \hat{\otimes} B_n \to X$. By definition, a bounded subset S of $\prod_{\mathbb{N}} V \hat{\otimes} B_n$ is contained in $\prod S_n \hat{\otimes} T_n$ with bounded complete disks S_n and T_n in V and B_n, respectively. Here $S_n \hat{\otimes} T_n$ denotes the complete disked hull of $S_n \times T_n$ in $V \hat{\otimes} B_n$. By metrizability, all S_n are absorbed by some bounded complete disk $S' \subseteq V$. Moving the absorbing constants into T_n, we obtain $S \subseteq S' \hat{\otimes} \prod T'_n$. This implies the desired universal property. □

2.2. The definitions of the function spaces

Let G be a locally compact group and let V be a bornological vector space. The spaces $\mathcal{D}(G/k, V)$ are defined for all $k \in \mathrm{SC}(G)$. We pull back functions on G/k to G and thus view $\mathcal{D}(G/k, V)$ as a space of functions on G. If $k_1 \subseteq k_2$, then $\mathcal{D}(G/k_2, V)$ is the subspace of right-k_2-invariant functions in $\mathcal{D}(G/k_1, V)$ and thus a retract of $\mathcal{D}(G/k_1, V)$. The set SC is directed by Lemma 2.3. Hence the spaces $\mathcal{D}(G/k, V)$ for $k \in \mathrm{SC}$ form a strict inductive system. Strict means that the structure maps are bornological embeddings. We let $\mathcal{D}(G, V)$ be its inductive limit. This is just the union of the spaces $\mathcal{D}(G/k, V)$ equipped with the direct union bornology and thus a space of V-valued functions on G. We get the same space if we replace SC by any fundamental system of smooth compact subgroups because the latter are cofinal subsets of SC. In particular, if G is metrizable, then we can use a decreasing sequence of subgroups.

Lemma 2.7. *We have*

$$\mathcal{D}(G, V) = \varinjlim \mathcal{D}(G/k, V) = \varinjlim \mathcal{D}(k\backslash G, V) = \varinjlim \mathcal{D}(G//k, V).$$

Proof. For any compact subset $S \subseteq G/k$ there is $k_2 \in \mathrm{SC}$ that stabilizes all points of S. That is, functions in $\mathcal{D}(G/k, V)$ with support in S are automatically left-k_2-invariant and hence belong to $\mathcal{D}(G//k_2, V)$. This yields the assertions. □

Let $H \subseteq G$ be a closed subgroup. We define $\mathcal{D}(G/H, V)$ and $\mathcal{D}(H\backslash G, V)$ as follows. The double coset space $k\backslash G/H$ can be decomposed as a disjoint union of homogeneous spaces for Lie groups as in the proof of Lemma 2.3 and hence is a smooth manifold for all $k \in I$. We view $\mathcal{D}(k\backslash G/H)$ as a space of left-k-invariant functions on G/H. If $k_1 \subseteq k_2$, then $\mathcal{D}(k_2\backslash G/H)$ is the set of left-k_2-invariant functions in $\mathcal{D}(k_1\backslash G/H)$. Thus the spaces $\mathcal{D}(k\backslash G/H)$ for $k \in I$ form a strict inductive system. We let $\mathcal{D}(G/H, V) := \varinjlim \mathcal{D}(k\backslash G/H, V)$. The definition of $\mathcal{D}(H\backslash G, V)$ is analogous. Lemma 2.7 shows that this reproduces the old definition of $\mathcal{D}(G/H, V)$ if H is normal in G. If H is a compact subgroup, then $\mathcal{D}(G/H, V)$ is canonically isomorphic to the space $\mathcal{D}(G, V)^H$ of elements in $\mathcal{D}(G, V)$ that are invariant under right translation by H.

If G is a Lie group, then G/H is a smooth manifold and $\mathcal{D}(G/H, V)$ evidently agrees with the usual space of smooth functions defined in Section 2.1. If G is totally disconnected, then the spaces $k\backslash G/H$ are discrete. Therefore, $\mathcal{D}(G/H, V)$ is the space of locally constant functions with compact support from G/H to V.

Definition 2.8. A function $f : G/H \to V$ is called smooth if $h \cdot f \in \mathcal{D}(G/H, V)$ for all $h \in \mathcal{D}(G/H)$. We let $\mathcal{E}(G/H, V)$ be the space of smooth functions from G/H to V. A subset S of $\mathcal{E}(G/H, V)$ is bounded if $h \cdot S$ is bounded in $\mathcal{D}(G/H, V)$ for all $h \in \mathcal{D}(G/H)$. We let $\mathcal{E}(G/H) := \mathcal{E}(G/H, \mathbb{C})$.

For a closed subset $S \subseteq G/H$, let $\mathcal{E}_0(S, V)$ be the subspace of $\mathcal{E}(G/H, V)$ of functions supported in S and let $\mathcal{E}(S, V)$ be the quotient of $\mathcal{E}(G/H, V)$ by the ideal of functions vanishing in S. (The latter notation is slightly ambiguous because $\mathcal{E}(S, V)$ also depends on G/H.)

Let $S \subseteq G/H$ be compact. Then there is $h \in \mathcal{D}(G/H)$ with $h|_S = 1$. Therefore, we obtain the same spaces $\mathcal{E}_0(S, V)$ and $\mathcal{E}(S, V)$ if we replace $\mathcal{E}(G, V)$ by $\mathcal{D}(G, V)$ in the above definition. It is evident that $\mathcal{D}(G/H, V) = \varinjlim \mathcal{E}_0(S, V)$ where S runs through the directed set of compact subsets of G/H. Thus $\mathcal{D}(G/H, V)$ is the space of compactly supported elements of $\mathcal{E}(G/H, V)$. However, the space $\mathcal{E}(G/H, V)$ tends to be harder to analyze than $\mathcal{D}(G/H, V)$.

2.3. *Nuclearity and exactness properties*

Next we examine some properties of $\mathcal{D}(G/H)$ and of the functor $V \mapsto \mathcal{D}(G/H, V)$. Since the bornological tensor product commutes with inductive limits, (1) implies

$$\mathcal{D}(G/H, V) \cong \varinjlim \mathcal{D}(k\backslash G/H, V) \cong \varinjlim \mathcal{D}(k\backslash G/H) \hat{\otimes} V \cong \mathcal{D}(G/H) \hat{\otimes} V. \tag{2}$$

Proposition 2.9. *The bornological vector space $\mathcal{D}(G/H)$ is nuclear. More generally, if V is nuclear, so is $\mathcal{D}(G/H, V)$.*

Proof. For $k \in \mathrm{SC}$ and $S \subseteq k\backslash G/H$ compact, the subspace $\mathcal{E}_0(S) \subseteq \mathcal{D}(k\backslash G/H)$ is a nuclear Fréchet space because $k\backslash G/H$ is a smooth manifold. Hence it is nuclear as a bornological vector space as well (see [13]). As an inductive limit of these spaces, the space $\mathcal{D}(G/H)$ is nuclear as well. Since nuclearity is hereditary for tensor products, (2) implies that $\mathcal{D}(G/H, V)$ is nuclear if V is. □

To state the exactness properties of the functor $\mathcal{D}(G/H, \sqcup)$, we recall some natural classes of extensions. A *bornological extension* is a diagram $K \xrightarrow{i} E \xrightarrow{p} Q$ with $i = \operatorname{Ker} p$ and $p = \operatorname{Coker} i$. It is called *linearly split* if it has a bounded linear section. Then it follows that $E \cong K \oplus Q$. It is called *locally linearly split* if for any bounded complete disk $T \subseteq Q$ there is a local bounded linear section $Q_T \to E$ defined on the Banach space Q_T. Equivalently, the sequence

$$0 \to \operatorname{Hom}(B, K) \to \operatorname{Hom}(B, E) \to \operatorname{Hom}(B, Q) \to 0$$

is exact for any Banach space B. Locally linearly split extensions are important for local cyclic cohomology.

Proposition 2.10. *The functor $V \mapsto \mathcal{D}(G/H, V)$ commutes with direct limits. It preserves bornological extensions and injectivity of morphisms. It also preserves locally linearly split and linearly split extensions.*

Proof. For any bornological vector space W, the functor $V \mapsto W \hat{\otimes} V$ commutes with direct limits and preserves linearly split and locally linearly split bornological extensions. Nuclearity of W implies that it also preserves injectivity of morphisms and bornological extensions. This yields the assertions because of (2). □

Now we turn from $\mathcal{D}(G/H,V)$ to $\mathcal{E}(G/H,V)$. For any open covering of G/H there is a subordinate partition of unity consisting of functions in $\mathcal{D}(G/H)$. In order to avoid taking square roots, our convention for partitions of unity is that $\sum \phi_j^2(x) = 1$. We choose such a partition of unity $(\phi_j)_{j\in J}$ on G/H with $\phi_j \in \mathcal{D}(G/H)$ for all $j \in J$ and use it to define maps

$$\begin{aligned} &\iota : \mathcal{E}(G/H,V) \to \prod_{j\in J} \mathcal{D}(G/H,V), \quad \iota(f)_j := f\cdot\phi_j, \\ &\pi : \prod_{j\in J} \mathcal{D}(G/H,V) \to \mathcal{E}(G/H,V), \quad \pi\big((f_j)\big) := \sum_{j\in J} f_j\cdot\phi_j. \end{aligned} \tag{3}$$

It is clear that ι is a well-defined bounded linear map. The map π is a well-defined bounded linear map as well because all but finitely many of the products $f_j\phi_j h$ vanish for $h \in \mathcal{D}(G/H)$. Thus $\mathcal{E}(G/H,V)$ is naturally isomorphic to a retract (that is, direct summand) of $\prod_{j\in J} \mathcal{D}(G/H,V)$.

Proposition 2.11. *The functor $\mathcal{E}(G/H,\sqcup)$ preserves bornological extensions and injectivity of morphisms. It also preserves locally linearly split and linearly split bornological extensions. The space $\mathcal{E}(G/H,V)$ is nuclear if (and only if) V is nuclear and G/H is countable at infinity.*

Proof. The classes of extensions that occur in the proposition are closed under direct products. Hence a retract of a direct product of exact functors is again exact. Using the maps in (3), the assertions about $\mathcal{E}(G/H,\sqcup)$ therefore follow from the corresponding assertions about $\mathcal{D}(G/H,\sqcup)$ in Proposition 2.10. Suppose G/H to be countable at infinity. Then the partition of unity above is countable, so that $\mathcal{E}(G/H,V)$ is a retract of a countable direct product of spaces $\mathcal{D}(G/H,V)$. Since nuclearity is hereditary for *countable* direct products, $\mathcal{E}(G/H,V)$ is nuclear. □

Definition 2.12. Let $l : \mathcal{D}(G/H,V) \to W$ be a bounded linear map. Its *support* $\operatorname{supp} l$ is the smallest closed subset $S \subseteq G/H$ such that $l(f) = 0$ for all $f \in \mathcal{D}(G/H,V)$ that vanish in a neighborhood of S. (An argument using partitions of unity shows that this is well defined.)

Let $\mathcal{D}'(G/H,V)$ be the dual space of $\mathcal{D}(G/H,V)$, equipped with the equibounded bornology. For $S \subseteq G/H$, let $\mathcal{D}'(S,V) \subseteq \mathcal{D}'(G/H,V)$ be the set of linear functionals supported in S. Let $\mathcal{E}'(G/H,V) := \varinjlim \mathcal{D}'(S,V)$, where S runs through the compact subsets of G/H. In particular, for $V = \mathbb{C}$, we obtain the spaces $\mathcal{D}'(G/H)$ and $\mathcal{E}'(G/H)$ of *distributions* and *distributions with compact support* on G/H.

Lemma 2.13. *The natural map from the dual of $\mathcal{E}(G/H,V)$ to $\mathcal{D}'(G/H,V)$ is a bornological isomorphism onto $\mathcal{E}'(G/H,V)$. In particular, $\mathcal{E}'(G/H)$ is the dual space of $\mathcal{E}(G/H)$.*

Proof. It is not hard to see that for any set of bornological vector spaces (V_x), the dual space of $\prod V_x$ is bornologically isomorphic to the direct sum $\bigoplus V_x'$. This together with (3) yields the assertion. □

Lemma 2.14. *If G/H is countable at infinity and V is metrizable, then*

$$\mathcal{E}(G/H, V) \cong \mathcal{E}(G/H) \hat{\otimes} V.$$

Proof. We have already shown that $\mathcal{D}(G/H, V) \cong \mathcal{D}(G/H) \hat{\otimes} V$. Using the maps in (3) and Lemma 2.6, we obtain $\mathcal{E}(G/H, V) \cong \mathcal{E}(G/H) \hat{\otimes} V$ as well. □

However, $\mathcal{E}(G/H, V)$ is not isomorphic to $\mathcal{E}(G/H) \hat{\otimes} V$ in general. All three spaces $\mathcal{E}(G/H \times G/H)$, $\mathcal{E}(G/H, \mathcal{E}(G/H))$ and $\mathcal{E}(G/H) \hat{\otimes} \mathcal{E}(G/H)$ are different unless G/H is a smooth manifold or compact. This is the reason why the regular representation on G/H usually fails to be smooth.

2.4. Functoriality with respect to the group

Definition 2.15. A continuous linear map $f: G_1/H_1 \to G_2/H_2$ between two homogeneous spaces is called *smooth* if for any $x \in G_1/H_1$ and any $k_2 \in \mathrm{SC}(G_2)$, there is $k_1 \in \mathrm{SC}(G_1)$ and an open k_1-invariant neighborhood $V \subseteq G_1/H_1$ of x such that the restriction of f to V descends to a smooth map $k_1 \backslash V \to k_2 \backslash G_2/H_2$.

Lemma 2.16. *A smooth map $f: G_1/H_1 \to G_2/H_2$ induces a bounded linear map*

$$f^*: \mathcal{E}(G_2/H_2, V) \to \mathcal{E}(G_1/H_1, V), \quad f^*(h) := h \circ f.$$

If f is proper as well, f^* restricts to a bounded linear map

$$f^*: \mathcal{D}(G_2/H_2, V) \to \mathcal{D}(G_1/H_1, V), \quad f^*(h) := h \circ f.$$

Proof. Use smooth partitions of unity. □

The following are examples of smooth maps. They induce maps on spaces of smooth functions by Lemma 2.16.

(1) The group multiplication is a smooth map $G \times G \to G$. So are the multiplication maps $G \times G/H \cong G \times G/1 \times H \to G/H$ and $H\backslash G \times G \cong H \times 1\backslash G \times G \to H\backslash G$. The map $G \times G \to G \times G$, $(x, y) \mapsto (x, xy)$, is smooth and so are similar maps involving homogeneous spaces.
(2) The inversion is a smooth map $G \to G$ and $G/H \leftrightarrow H\backslash G$.
(3) Any continuous group homomorphism is smooth.
(4) If $g^{-1}Hg \subseteq H'$, then the map $G/H \to G/H'$ that sends xH to $xHgH' = xgH'$ is smooth.

Thus we can define the left and right *regular representations* λ and ρ of G on $\mathcal{D}(G, V)$ and $\mathcal{E}(G, V)$ by

$$\lambda_g f(x) := f\bigl(g^{-1} \cdot x\bigr), \quad \rho_g f(x) := f(x \cdot g). \tag{4}$$

Lemma 2.17. *The space $\mathcal{E}(G/H, V)$ is naturally isomorphic to the subspace of $\mathcal{E}(G, V)$ of functions f that satisfy $\rho_h f = f$ for all $h \in H$.*

Proof. The projection $G \to G/H$ is smooth and therefore induces a bounded injective map $\mathcal{E}(G/H, V) \to \mathcal{E}(G, V)$, whose range clearly consists of right-H-invariant functions. Let $k \in \mathrm{SC}$ and let U be its normalizer. In order to prove that $k\backslash G/H$ is a smooth manifold, we decomposed $k\backslash G/H$ into a disjoint union of the double coset spaces $k\backslash UgH/H$ for $g \in U\backslash G/H$ and identified the contribution of each double coset with a homogeneous space for a Lie group action. This reduces the assertion to the special case where G is a Lie group. The projection $G \to G/H$ is a submersion in this case and hence has local smooth sections. They together with smooth partitions of unity yield the assertion. □

The modular function $\mu_G : G \to \mathbb{R}_+^\times$ is a continuous group homomorphism. We define it by the convention $\mu_G(x)\,dg = d(gx)$. We have $\mu_G \in \mathcal{E}(G)$ because group homomorphisms are smooth maps and the identical function $\mathbb{R}_+^\times \to \mathbb{R}$ is a smooth function on $\mathbb{R}_+^\times$. Hence multiplication by μ_G is a bornological isomorphism on $\mathcal{D}(G, V)$ and $\mathcal{E}(G, V)$.

If $H \subseteq G$ is an *open* subgroup, then there are bornological embeddings

$$\mathcal{D}(H, V) \to \mathcal{D}(G, V), \qquad \mathcal{E}(H, V) \to \mathcal{E}(G, V),$$

which extend a function on H by 0 outside H. Its range is the space of functions supported in H and thus a retract. Let $(G_i)_{i\in I}$ be a directed family of open subgroups of G with $G = \bigcup G_i$. Then $\mathcal{D}(G, V)$ is the strict inductive limit of the subspaces $\mathcal{D}(G_i, V)$.

We have

$$\mathcal{D}(G_1 \times G_2) \cong \mathcal{D}(G_1) \hat{\otimes} \mathcal{D}(G_2) \cong \mathcal{D}\big(G_1, \mathcal{D}(G_2)\big) \tag{5}$$

for all locally compact groups G_1 and G_2 because the corresponding result holds for manifolds and the bornological tensor product commutes with direct limits. The spaces $\mathcal{E}(G \times G)$, $\mathcal{E}(G) \hat{\otimes} \mathcal{E}(G)$ and $\mathcal{E}(G, \mathcal{E}(G))$ agree if G is a Lie group, but not for arbitrary G.

Let $(G_i)_{i\in I}$ be a set of locally compact groups and let $K_i \subseteq G_i$ be compact open subgroups for all $i \in I \setminus F_0$ with some *finite* set of exceptions F_0. For each finite subset $F \subseteq I$ containing F_0 the direct product

$$G(F) := \prod_{i\in I\setminus F} K_i \times \prod_{i\in F} G_i$$

is a locally compact group. For $F_1 \subseteq F_2$ the group $G(F_1)$ is an open subgroup of $G(F_2)$. The *restricted direct product* $\prod'_{i\in I}(G_i, K_i)$ is the direct union of these groups. The characteristic function of $K_i \subseteq G_i$ is a distinguished element of $\mathcal{D}(G_i)$. The (*restricted*) *tensor product* of the spaces $\mathcal{D}(G_i)$ with respect to these distinguished vectors is defined as follows. For each finite subset $F \subseteq I$ containing F_0, consider the completed tensor product $\bigotimes_{i\in F} \mathcal{D}(G_i)$. We have a map between the associated tensor products for $F_1 \subseteq F_2$ that inserts the factor 1_{K_i} for $i \in F_2 \setminus F_1$. The tensor product is the direct limit of the resulting (strict) inductive system. It is straightforward to show that

$$\mathcal{D}\Bigg(\prod_{i\in I}{}'(G_i, K_i)\Bigg) \cong \bigotimes_{i\in I}\big(\mathcal{D}(G_i), 1_{K_i}\big). \tag{6}$$

2.5. Multiplication and convolution

The pointwise product of smooth functions and of smooth functions with distributions is defined in the usual way. All resulting bilinear maps are clearly bounded.

The group law of G gives rise to a comultiplication

$$\Delta : \mathcal{E}(G) \to \mathcal{E}(G \times G), \quad \Delta f(g,h) := f(gh).$$

We do not have $\mathcal{E}(G \times G) = \mathcal{E}(G) \hat{\otimes} \mathcal{E}(G)$ in general. The resulting problem with the convolution of distributions is fixed by the following lemma:

Lemma 2.18. *There is a unique bounded bilinear map*

$$\mathcal{E}'(G) \times \mathcal{E}'(G) \to \mathcal{E}'(G \times G), \quad (D_1, D_2) \mapsto D_1 \otimes D_2,$$

such that

$$\langle D_1 \otimes D_2, f_1 \otimes f_2 \rangle = \langle D_1, f_1 \rangle \cdot \langle D_2, f_2 \rangle,$$
$$(f_1 \otimes f_2) \cdot (D_1 \otimes D_2) = f_1 \cdot D_1 \otimes f_2 \cdot D_2,$$

for all $D_1, D_2 \in \mathcal{E}'(G)$, $f_1, f_2 \in \mathcal{E}(G)$.

There is a unique bounded linear map

$$\mathcal{E}'(G/H) \to \operatorname{Hom}\bigl(\mathcal{E}(G/H, V), V\bigr), \quad D \mapsto D_V,$$

such that

$$\langle D_V, f \otimes v \rangle = \langle D, f \rangle \cdot v, \quad f \cdot D_V = (f \cdot D)_V$$

for all $D \in \mathcal{E}'(G/H)$, $f \in \mathcal{E}(G/H)$, $v \in V$.

Proof. Fix D_1, D_2 with support contained in some compact subset $S \subseteq G$. There exists $\phi \in \mathcal{D}(G)$ with $\phi = 1$ in a neighborhood of S. Hence $\phi \cdot D_j = D_j$ for $j = 1, 2$. Therefore, we must put $\langle D_1 \otimes D_2, f \rangle := \langle D_1 \hat{\otimes} D_2, (\phi \otimes \phi) \cdot f \rangle$. The right-hand side is well defined because $(\phi \otimes \phi) \cdot f$ has compact support and $\mathcal{D}(G \times G) \cong \mathcal{D}(G) \hat{\otimes} \mathcal{D}(G) \subseteq \mathcal{E}(G) \hat{\otimes} \mathcal{E}(G)$. It is straightforward to see that this definition does not depend on ϕ and has the required properties.

The map D_V is defined similarly. There is $\phi \in \mathcal{D}(G/H)$ with $\phi \cdot D = D$. We must have $\langle D_V, f \rangle := D \hat{\otimes} \mathrm{id}_V(\phi \cdot f)$ for all $f \in \mathcal{E}(G/H, V)$. The right-hand side is defined because $\phi \cdot f \in \mathcal{D}(G/H, V) \cong \mathcal{D}(G/H) \hat{\otimes} V$. $\square$

We define the convolution of two compactly supported distributions by

$$\langle D_1 * D_2, f \rangle := \langle D_1 \otimes D_2, \Delta f \rangle$$

for all $f \in \mathcal{E}(G)$. This turns $\mathcal{E}'(G)$ into a bornological algebra. A similar trick allows to define the convolution of a compactly supported distribution with an arbitrary distribution. All these bilinear maps are evidently bounded.

Fix a left Haar measure dg on G. Then we embed $\mathcal{E}(G) \subseteq \mathcal{D}'(G)$ by the usual map $f \mapsto f\,dg$. We define convolutions involving smooth functions in such a way that

$f_1\,dg * f_2\,dg = (f_1 * f_2)\,dg$, $D * (f\,dg) = (D * f)\,dg$ and $(f\,dg * D) = (f * D)\,dg$. It is straightforward to verify that this defines bounded bilinear maps taking values in $\mathcal{E}(G)$ provided one factor has compact support, and taking values in $\mathcal{D}(G)$ if both factors have compact support. In particular, $\mathcal{D}(G)$ becomes a bornological algebra and a bimodule over $\mathcal{E}'(G)$.

The antipode $\tilde{f}(g) := f(g^{-1})$ on $\mathcal{E}(G)$ gives rise by transposition to an antipode on $\mathcal{E}'(G)$, which is a bounded anti-homomorphism with respect to convolution. Its restriction to the ideal $\mathcal{D}(G) \subseteq \mathcal{E}'(G)$ is given by

$$\big(\tilde{f}^{(1)}\big)(g) := f\big(g^{-1}\big)\mu_G(g)^{-1} \tag{7}$$

because $d(g^{-1}) = \mu_G(g^{-1})\,dg$. This is a bounded anti-homomorphism on $\mathcal{D}(G)$, which we use to turn right $\mathcal{D}(G)$-modules into left modules and vice versa.

3. Smooth representations of locally compact groups

We shall use the following notation and conventions. Let G be a locally compact group and let V be a (complete convex) bornological vector space. The space $\mathrm{End}(V) := \mathrm{Hom}(V, V)$ of bounded linear operators on V is a (complete convex) bornological algebra. Let $\mathrm{Aut}(V)$ be the multiplicative group of invertible elements in $\mathrm{End}(V)$. A *group representation* of G on V is a group homomorphism $\pi : G \to \mathrm{Aut}(V)$. Thus we always assume G to act by bounded linear operators. We write $\pi(g) = \pi_g$ and $\pi_g(v) = \pi(g, v) = g \cdot v$. Let $\mathrm{Map}(G, V) := \prod_{g \in G} V$ be the space of all functions from G to V. The *adjoint* of π is the bounded linear map $\pi_* : V \to \mathrm{Map}(G, V)$ defined by $\pi_*(v)(g) := \pi(g, v)$. We let G act on $\mathrm{Map}(G, V)$ by the right regular representation ρ defined in (4). Then π_* is G-equivariant.

Definition 3.1. The representation π is called *smooth* if π_* is a bounded map into $\mathcal{E}(G, V)$.

3.1. First properties of smooth representations

Lemma 3.2. *The representation π is smooth if and only if $Wf(x) := x \cdot f(x)$ defines an element of* $\mathrm{Aut}(\mathcal{D}(G, V))$. *Even more, π is already smooth if*

$$W_\phi : V \xrightarrow{\phi_*} \mathcal{D}(G, V) \xrightarrow{W} \mathrm{Map}(G, V), \quad v \mapsto \big[g \mapsto \phi(g)\pi(g, v)\big],$$

is a bounded linear map into $\mathcal{D}(G, V)$ for some non-zero $\phi \in \mathcal{D}(G)$.

Proof. We have $W_\phi(v) = W(\phi \otimes v) = M_\phi \pi_*(v)$, where M_ϕ denotes the operator of pointwise multiplication by ϕ on $\mathcal{D}(G, V)$. It follows from the definition of $\mathcal{E}(G, V)$ that π is smooth if and only if W_ϕ is a bounded linear map into $\mathcal{D}(G, V)$ for all ϕ. This is equivalent to W being a bounded linear map. If W is bounded, so is its inverse $W^{-1}f(x) := x^{-1}f(x)$. Hence W belongs to $\mathrm{Aut}(\mathcal{D}(G, V))$ if and only if π is smooth.

It remains to prove that W_ϕ is a bounded map into $\mathcal{D}(G, V)$ for all $\phi \in \mathcal{D}(G)$ once this happens for a single $\phi \neq 0$. Let $X \subseteq \mathcal{D}(G)$ be the subspace of all ϕ for which W_ϕ is a bounded map into $\mathcal{D}(G, V)$. Clearly, X is an ideal for the pointwise product. Since $\pi(g)$ is bounded for all $g \in G$, the operator W_ϕ is bounded if and only if $W_{\rho_g \phi}$ is bounded. Hence for all $g \in G$ there is $\phi \in X$ with $\phi(g) \neq 0$. Since X is an ideal, we get $X = \mathcal{D}(G)$. □

Corollary 3.3. *Let $H \subseteq G$ be an open subgroup. Then a representation of G is smooth if and only if its restriction to H is smooth. Any representation of a discrete group is smooth.*

Lemma 3.4. *Let $H \subseteq G$ be a closed subgroup. The left and right regular representations of G on $\mathcal{D}(G/H, V)$ and $\mathcal{D}(H\backslash G, V)$ are smooth.*

Proof. We observed after Lemma 2.16 that the map $G \times G/H \to G \times G/H$ that sends (x, yH) to (x, xyH) is smooth. Since it is also proper, it induces a bounded linear operator on $\mathcal{D}(G, \mathcal{D}(G/H, V)) \cong \mathcal{D}(G \times G/H, V)$. This is the operator W of Lemma 3.2 for the left regular representation λ on $\mathcal{D}(G/H, V)$. Hence λ is smooth. Similarly, the right regular representation on $\mathcal{D}(H\backslash G, V)$ is smooth. □

The regular representations on $\mathcal{E}(G, V)$ usually fail to be smooth. See Section 3.5 for some positive results on $\mathcal{E}(G, V)$.

The *integrated form* of a smooth representation π is the bounded homomorphism

$$\textstyle\int\pi : \mathcal{E}'(G) \to \operatorname{End}(V), \qquad \int\pi(D)(v) := D_V\big(\pi_*(v)\big).$$

The operator $D_V : \mathcal{E}(G, V) \to V$ is defined in Lemma 2.18. We evidently have $\int\pi(\delta_g) = \pi_g$, so that $\int\pi$ extends π. We omit the straightforward proof that $\int\pi$ is an algebra homomorphism. Let $\mathcal{U}(G) \subseteq \mathcal{E}'(G)$ be the subalgebra of distributions supported at 1_G. If G is a Lie group with Lie algebra $\mathfrak{g}$, then $\mathcal{U}(G)$ is the universal enveloping algebra of $\mathfrak{g}$. Restricting $\int\pi$ to $\mathfrak{g} \subseteq \mathcal{U}(G)$, we obtain a Lie algebra representation $D\pi : \mathfrak{g} \to \operatorname{End}(V)$. We call $D\pi$ the *differential of* π.

3.2. Permanence properties of smooth representations

Lemma 3.5. *Smoothness is hereditary for subrepresentations and quotients, direct limits and finite inverse limits* (*that is, inverse limits of finite diagrams*).

Proof. Let $K \rightarrowtail E \twoheadrightarrow Q$ be a bornological extension of representations of G. Consider the diagram

$$\begin{array}{ccccc} K & \rightarrowtail & E & \twoheadrightarrow & Q \\ \vdots & & \downarrow & & \vdots \\ \mathcal{E}(G, K) & \rightarrowtail & \mathcal{E}(G, E) & \twoheadrightarrow & \mathcal{E}(G, Q). \end{array}$$

The middle vertical map is the adjoint of the representation on E. The bottom row is a bornological extension as well by Proposition 2.11. Since the composition $K \to E \to \mathcal{E}(G, E) \to \mathcal{E}(G, Q)$ vanishes, the dotted arrows exist. They are the adjoints of the induced representations on K and Q. Hence K and Q are smooth representations as well. It is trivial to verify that direct sums of smooth representations are again smooth. Since direct limits are quotients of direct sums and inverse limits are subspaces of direct products, we obtain the asserted smoothness for direct limits and finite inverse limits. □

Remark 3.6. Infinite direct products of smooth representations may fail to be smooth. The class of smooth representations is *not* closed under extensions. A simple counterexample is the representation of $\mathbb{R}$ on $\mathbb{C}^2$ by

$$t \mapsto \begin{pmatrix} 1 & \phi(t) \\ 0 & 1 \end{pmatrix}$$

for some discontinuous group homomorphism $\phi : \mathbb{R} \to \mathbb{R}$.

Lemma 3.7. *Let $\phi : H \to G$ be a continuous group homomorphism and let $\pi : G \to \operatorname{End}(V)$ be a group representation. If π is a smooth representation of G, then $\pi \circ \phi$ is a smooth representation of H. In particular, restrictions of smooth representations to closed subgroups remain smooth. If ϕ is an open surjection, then the converse holds. That is, a representation of a quotient group H/N is smooth if and only if it is smooth as a representation of H.*

Proof. The smoothness of $\pi \circ \phi$ follows from the functoriality of $\mathcal{E}(G, V)$ for continuous group homomorphisms. If ϕ is an open surjection, it is isomorphic to a quotient map $\phi : H \to H/N$. The map $\phi^* : \mathcal{E}(H/N, V) \to \mathcal{E}(H, V)$ is a bornological isomorphism onto its range by Lemma 2.17. Hence $\pi \circ \phi$ is smooth if and only if π is. □

The *external tensor product* $\pi_1 \boxtimes \pi_2$ of two representations $\pi_j : G_j \to \operatorname{Aut}(V_j)$, $j = 1, 2$, is the tensor product representation of $G_1 \times G_2$ on $V_1 \hat{\otimes} V_2$. If $G_1 = G_2$, the *internal tensor product* $\pi_1 \hat{\otimes} \pi_2$ is the restriction of $\pi_1 \boxtimes \pi_2$ to the diagonal $G \subseteq G \times G$. Let $(G_i)_{i \in I}$ and $(K_i)_{i \in I \setminus F_0}$ be the data for a restricted direct product of groups. Let $\pi_i : G_i \to \operatorname{Aut}(V_i)$ be representations of G_i and let $\xi_i \in V_i$ be K_i-invariant for all but finitely many $i \in I$. Then we can form the restricted tensor product $\bigotimes_{i \in I} (V_i, \xi_i)$ and let $\prod'_{i \in I} (G_i, K_i)$ act on it in the evident fashion. We call this the *restricted* (*external*) *tensor product representation*. This recipe is frequently used to construct representations of adelic groups.

Lemma 3.8. *A representation of a direct product group is smooth if and only if its restrictions to the factors are smooth. Restricted external tensor products and external and internal tensor products of smooth representations remain smooth.*

Proof. The straightforward proof of the first assertion is left to the reader. Consider a restricted direct product $G = \prod' (G_i, K_i)$ and a restricted tensor product representation $\bigotimes_{i \in I} (V_i, \xi_i)$ as above. We have

$$\mathcal{D}\Big(G, \bigotimes (V_i, \xi_i)\Big) \cong \bigotimes \big(\mathcal{D}(G_i), 1_{K_i}\big) \hat{\otimes} \bigotimes (V_i, \xi_i) \cong \bigotimes \big(\mathcal{D}(G_i, V_i), 1_{K_i} \otimes \xi_i\big).$$

The restricted tensor product is functorial for families of maps $V_i \to V_i$ preserving the distinguished vectors. Since the operator W of Lemma 3.2 is induced from the analogous operators for the factors, we get the assertion for restricted direct products. This implies the smoothness of finite external tensor products and hence also of internal tensor products by Lemma 3.7. □

3.3. Some constructions with representations

Definition 3.9. The *smoothening* of a representation $\pi : G \to \mathrm{Aut}(V)$ is

$$\mathrm{S}_G V := \{ f \in \mathcal{E}(G, V) \mid f(g) = g \cdot f(1) \text{ for all } g \in G \},$$

equipped with the subspace bornology, the right regular representation and the map $\iota_V : \mathrm{S}_G V \to V$ defined by $\iota_V(f) = f(1)$.

We frequently drop G and just write $\mathrm{S}(V)$ for the smoothening. We write $\mathrm{S}_G(V, \pi)$ if it is important to remember the representation π. A function $f \in \mathcal{E}(G, V)$ belongs to $\mathrm{S}(V)$ if and only if $f = \pi_*(f(1))$. Therefore, the map ι_V is injective and $\mathrm{S}(V)$ is invariant under the right regular representation. The map ι_V is bounded and G-equivariant.

Let $L \subseteq G$ be a compact neighborhood of the identity. Recall that $\mathcal{E}(L, V)$ is defined as a quotient of $\mathcal{E}(G, V)$ in Definition 2.8. However, since L is compact, it is also a quotient of $\mathcal{D}(G, V)$. Therefore, $\mathcal{E}(L, V) \cong \mathcal{E}(L) \hat{\otimes} V$.

Lemma 3.10. *The projection $(v, f) \mapsto f|_L$ is a bornological isomorphism from $\mathrm{S}(V)$ onto the space*

$$\mathrm{S}_L V := \{ f \in \mathcal{E}(L, V) \mid f(g) = g \cdot f(1) \textit{ for all } g \in L \}.$$

In particular, $\mathrm{S}_H V \cong \mathrm{S}_G V$ if $H \subseteq G$ is an open subgroup.

Proof. Restriction to L is a bounded linear map $p : \mathrm{S}(V) \to \mathrm{S}_L V$. Define $jf(g) := g \cdot f(1)$ for all $g \in G$, $f \in \mathrm{S}_L V$. This is a bounded linear map from $\mathrm{S}_L V$ to $\mathrm{S}(V)$ because $j(f)|_{gL} = \pi_g(f)$ and the interiors of the sets gL with $g \in G$ cover G. Clearly, the maps j and p are inverse to each other. □

Proposition 3.11. *The smoothening of V is a smooth representation of G. If W is any smooth representation of G, then there is a natural isomorphism*

$$(\iota_V)_* : \mathrm{Hom}_G(W, V) \cong \mathrm{Hom}_G(W, \mathrm{S}(V)).$$

Proof. The map $(\iota_V)_*$ is injective because ι_V is. A map $T : W \to V$ induces a map $\mathcal{E}(G, T) : \mathcal{E}(G, W) \to \mathcal{E}(G, V)$. We have $\iota_V \circ \mathcal{E}(G, T) \circ \pi_*^W = T$ and $\mathcal{E}(G, T) \circ \pi_*^W$ maps W into $\mathrm{S}(V)$ if T is equivariant. Hence $(\iota_V)_*$ is also surjective.

It remains to prove the smoothness of $\mathrm{S}(V)$. This requires work because the regular representation on $\mathcal{E}(G, V)$ may fail to be smooth. Let $L \subseteq G$ be a compact symmetric neighborhood of 1 and let $L^2 := L \cdot L$. There is a bounded linear map

$$\rho^* : \mathcal{E}(G, V) \to \mathcal{E}(G \times G, V), \quad \rho^* f(g, h) := f(gh).$$

It descends to a bounded map $\mathcal{E}(L^2, V) \to \mathcal{E}(L \times L, V) \cong \mathcal{E}(L, \mathcal{E}(L, V))$, which maps $\mathrm{S}_{L^2}(V)$ into $\mathcal{E}(L, \mathrm{S}_L V)$. The isomorphism $\mathcal{E}(L \times L, V) \cong \mathcal{E}(L, \mathcal{E}(L, V))$ follows immediately from $\mathcal{E}(L, V) \cong \mathcal{E}(L) \hat{\otimes} V$, but it holds only if L is compact. Using Lemma 3.10, we get a bounded map

$$\rho^* : \mathrm{S}(V) \to \mathcal{E}(L, \mathrm{S}(V)), \quad \rho^*(f)(g) := \rho_g(f).$$

Since L is a neighborhood of the identity, the smoothness of $\mathrm{S}(V)$ now follows from Lemma 3.2. □

Let $\widehat{\mathbf{R}}_G$ be the category of representations of G on bornological vector spaces with G-equivariant bounded linear maps as morphisms. Let $\mathbf{R}_G$ be the full subcategory of smooth representations. Proposition 3.11 asserts that $\mathrm{S}: \widehat{\mathbf{R}}_G \to \mathbf{R}_G$ is right adjoint to the embedding $\mathbf{R}_G \subseteq \widehat{\mathbf{R}}_G$.

Let $H \subseteq G$ be a closed subgroup. We have an evident restriction functor $\mathrm{Res}_G^H: \widehat{\mathbf{R}}_G \to \widehat{\mathbf{R}}_H$, which maps $\mathbf{R}_G$ into $\mathbf{R}_H$. The *smooth induction functor* $\mathrm{Ind}_H^G: \mathbf{R}_H \to \mathbf{R}_G$ is defined as the right adjoint of the restriction functor. The following construction shows that it exists.

First we construct a right adjoint to $\mathrm{Res}_G^H: \widehat{\mathbf{R}}_G \to \widehat{\mathbf{R}}_H$. Let

$$I(V) := \{v \in \mathrm{Map}(G, V) \mid f(hg) = h \cdot f(g) \text{ for all } h \in H,\ g \in G\},$$

equipped with the subspace bornology from $\mathrm{Map}(G, V)$ and the right regular representation. A morphism $f: \mathrm{Res}_G^H(W) \to V$ in $\widehat{\mathbf{R}}_H$ induces a morphism $f_*: W \to I(V)$ in $\widehat{\mathbf{R}}_G$ by $f_*(w)(g) := f(gw)$. Any morphism $W \to I(V)$ is of this form for a unique morphism f. That is, I is right adjoint to the restriction functor $\widehat{\mathbf{R}}_G \to \widehat{\mathbf{R}}_H$. It follows easily that the functor

$$\mathrm{Ind}_H^G: \mathbf{R}_H \to \mathbf{R}_G, \quad V \mapsto \mathrm{S}_G I(V),$$

is right adjoint to the restriction functor $\mathbf{R}_G \to \mathbf{R}_H$. Any G-equivariant map $W \to \mathrm{Map}(G, V)$ for a smooth representation W already takes values in $\mathcal{E}(G, V)$. Hence we can use $\mathcal{E}(G, V)$ instead of $\mathrm{Map}(G, V)$ to define of $\mathrm{Ind}_H^G(V)$. However, we still have to smoothen afterwards because $\mathcal{E}(G, V)$ may fail to be smooth.

The support of a function in $I(V)$ is left-H-invariant and can be viewed as a subset of $H\backslash G$. We let $I_c(V)$ be the subspace of compactly supported functions in $I(V)$, equipped with the inductive limit bornology over the compact subsets of $H\backslash G$. We define the *compact induction* functor as

$$\text{c-}\mathrm{Ind}_H^G: \mathbf{R}_H \to \mathbf{R}_G, \quad V \mapsto \mathrm{S}_G I_c(V).$$

Proposition 3.12. *The representation* $\text{c-}\mathrm{Ind}_H^G(V)$ *is isomorphic to the right regular representation of G on*

$$W := \varinjlim \{f \in \mathcal{E}_0(H \cdot S, V) \mid f(hg) = h \cdot f(g) \text{ for all } h \in H,\ g \in G\},$$

where S runs through the compact subsets of $H\backslash G$.

The functor $\text{c-}\mathrm{Ind}_H^G$ *preserves direct limits, injectivity of morphisms, bornological extensions, linearly split extensions and locally linearly split extensions.*

Proof. It is clear that W is a subrepresentation of $I_c(V)$. Furthermore, any map $X \to I_c(V)$ from a smooth representation to $I_c(V)$ must factor through W. We must prove that W is a smooth representation of G. We do this by realizing it naturally as a linearly split quotient of the left regular representation on $\mathcal{D}(G, V)$. Thus the functor $\text{c-}\mathrm{Ind}_H^G$ is a retract of the functor $\mathcal{D}(G, \sqcup)$ if we forget the group representation. Hence it inherits its functorial properties listed in Proposition 2.10.

Consider the maps

$$P : \mathcal{D}(G,V) \to W, \quad Pf(g) := \int\limits_H h \cdot f\bigl(g^{-1}h\bigr)\,d_H h,$$
$$J : W \to \mathcal{D}(G,V), \quad Jf(g) := f\bigl(g^{-1}\bigr) \cdot \phi(g). \tag{8}$$

The map P is bounded and G-equivariant. The map J is a bounded linear left section for P provided $\operatorname{supp}\phi \cap S \cdot H$ is compact for all $S \subseteq G/H$ compact and $\int_H \phi(gh)\,d_H h = 1$ for all $g \in G$. Such a function ϕ clearly exists. As a quotient of the left regular representation on $\mathcal{D}(G,V)$, the representation W is smooth. □

Proposition 3.12 easily implies that

$$\text{c-Ind}_H^G\bigl(\mathcal{D}(H,V)\bigr) \cong \mathcal{D}(G,V), \tag{9}$$
$$\text{c-Ind}_H^G\bigl(\mathbb{C}(1)\bigr) \cong \mathcal{D}(G/H), \tag{10}$$

where $\mathbb{C}(1)$ denotes the trivial representation of H on $\mathbb{C}$ and all function spaces carry the left regular representation.

It is customary to twist the functors Ind_H^G and c-Ind_H^G by a modular factor. Let μ_G and μ_H be the modular functions of G and H, respectively. We call the quasi-character $\mu_{G:H} := \mu_G \mu_H^{-1} : H \to \mathbb{R}_+^\times$ the *relative modular function* of $H \subseteq G$. For a representation $\pi : H \to \text{Aut}(W)$ of H and $\alpha \in \mathbb{R}$, we form the representation $\mu_{G:H}^\alpha \cdot \pi$ on W and plug it into Ind_H^G and c-Ind_H^G instead of W itself. We call the resulting functors the *twisted induction and compact induction functors*. The case $\alpha = 1/2$ is important because it preserves unitary representations.

3.4. *Explicit criteria for smoothness*

Let $U \subseteq G$ be an open subgroup which is a projective limit of Lie groups. Let I be a fundamental system of smooth compact subgroups in U. For a subgroup $L \subseteq G$ we let

$$V^L := \bigl\{v \in V \mid gv = v \text{ for all } g \in L\bigr\}.$$

This is a closed linear subspace of V. The subspaces V^k for $k \in I$ form a strict inductive system. We have $V = \varinjlim V^k$ if and only if any bounded subset of V is contained in V^k for some $k \in I$.

Theorem 3.13. *A representation $\pi : G \to \text{Aut}(V)$ is smooth if and only if $V = \varinjlim_{k \in I} V^k$ and the representation of U/k on V^k is smooth for all $k \in I$.*

Proof. Since π is smooth if and only if its restriction to U is smooth we may assume without loss of generality that $G = U$. We may also assume that there be $k_0 \in I$ with $k \subseteq k_0$ for all $k \in I$. Fix $\phi \in \mathcal{D}(G/k_0)$ with $\phi(1) \neq 0$. The representation π is smooth if and only if the operator W_ϕ in Lemma 3.2 is a bounded map from V to $\mathcal{D}(G,V) \cong \varinjlim \mathcal{D}(G/k,V)$. Evidently, $W_\phi(v)$ is k-invariant if and only if $v \in V^k$. As a result, we must have $V = \varinjlim V^k$

if π is smooth. Suppose now that $V = \varinjlim V^k$. Since smoothness is hereditary for inductive limits and subrepresentations, V is smooth if and only if V^k is smooth for all $k \in I$. Moreover, the representation of G on V^k is smooth if and only if the induced representation of G/k is smooth. This yields the assertion. □

If G is totally disconnected, the quotients U/k are discrete, so that any representation of U/k is smooth. Therefore, π is smooth if and only if $V = \varinjlim V^k$. If V carries the fine bornology, then the latter holds if and only if each $v \in V$ is stabilized by some open subgroup. For arbitrary G the quotients U/k are Lie groups. Hence it remains to describe smooth Lie group representations.

Theorem 3.14. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. A representation $\pi : G \to \mathrm{Aut}(V)$ is smooth if and only if it satisfies the following conditions*:

(i) *the representation is locally equibounded, that is, $\pi(K) \subseteq \mathrm{End}(V)$ is equibounded for any compact subset $K \subseteq G$;*
(ii) *the limits $D\pi(X)(v) := \lim_{t\to 0} t^{-1}(\exp(tX) \cdot v - v)$ exist for all $v \in V$ and the convergence is uniform on bounded subsets of V;*
(iii) *for any bounded subset $S \subseteq V$ there is a bounded disk $T \subseteq V$ such that $D\pi(X_1) \circ \cdots \circ D\pi(X_n)(S)$ is bounded in V_T for all $n \in \mathbb{N}$, $X_1, \ldots, X_n \in \mathfrak{g}$.*

Proof. First we show that smooth representations satisfy (i)–(iii). Conditions (i) and (ii) are obvious with $D\pi(X) = \int\pi(X)$ for all $X \in \mathfrak{g} \subseteq \mathcal{E}'(G)$. Let $S \subseteq V$ be bounded and let $\phi \in \mathcal{D}(G)$ be such that $\phi = 1$ in a neighborhood of the identity. Define $W_\phi(v)(g) := \phi(g)\pi(g, v)$ as in Lemma 3.2. The set $W_\phi(S)$ is bounded in $\mathcal{D}(G, V)$ and hence in $\mathcal{D}(G, V_T)$ for some bounded disk $T \subseteq V$. This yields (iii).

Conversely, suppose (i)–(iii) to hold. We claim that π is smooth. Let $S \subseteq V$ be a bounded complete disk and $K \subseteq G$ compact. Condition (i) allows us to choose a bounded complete disk $S' \subseteq V$ containing $\pi(K)(S)$. Let $S'' \subseteq V$ be a bounded complete disk such that the convergence in (ii) is uniform in $V_{S''}$ for all $v \in S'$. Such a set exists by the definition of uniform convergence. Condition (iii) asserts that there is a bounded complete disk T such that $D\pi(X_1) \circ \cdots \circ D\pi(X_n)(S'')$ is bounded in V_T for all $n \in \mathbb{N}$, $X_1, \ldots, X_n \in \mathfrak{g}$.

We claim that the map $v \mapsto \pi_*(v)|_K$ is a bounded linear map from V_S to $\mathcal{E}(K, V_T)$. This claim implies that π is smooth. Since V_S and V_T are Banach spaces, the claim is equivalent to the smoothness of the Banach space valued map $\pi : K \to \mathrm{Hom}(V_S, V_T)$. This is what we are going to show. The construction of the sets S', S'', T yields the following. The family of operators $\pi(g) : V_S \to V_{S'}$ is uniformly bounded for $g \in K$. Let $X_1, \ldots, X_n, X \in \mathfrak{g}$. The operators $(\pi(\exp(hX)) - \mathrm{id})/h : V_{S'} \to V_{S''}$ converge towards $D\pi$ in operator norm for $h \to 0$. The operator $A := D\pi(X_1) \circ \cdots \circ D\pi(X_n) : V_{S''} \to V_T$ is bounded. Hence

$$\lim_{h\to 0} A \circ \big(\pi\big(\exp(hX)g\big) - \pi(g)\big)/h = A \circ D\pi(X) \circ \pi(g)$$

converges in $\mathrm{Hom}(V_S, V_T)$ and is of the same form as the operator $A \circ \pi(g)$. This means that we can differentiate π with respect to right invariant differential operators. Therefore, π is a $\mathcal{C}^\infty$-map from K to $\mathrm{Hom}(V_S, V_T)$ as claimed. □

3.5. Smooth versus differentiable representations

Let G be a Lie group. Using the spaces $\mathcal{C}^k_{\mathrm{c}}(G,V)$ defined in Section 2.1 instead of $\mathcal{D}(G,V)$, we define the space $\mathcal{C}^k(G,V)$ of $\mathcal{C}^k$-functions $G\to V$ for $k\in\mathbb{N}\cup\{\infty\}$ as in Definition 2.8. We call π a *$\mathcal{C}^k$-representation* if π_* is a bounded map from V to $\mathcal{C}^k(G,V)$. For $k=0$ and $k=\infty$ we get *continuous* and *differentiable* representations, respectively.

Theorem 3.15. *Let $\pi: G\to \operatorname{Aut}(V)$ be a representation of a Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G. The following statements are equivalent*:

(1) *the representation π is differentiable*;
(2) *the representation π is $\mathcal{C}^1$*;
(3) *there is a bounded homomorphism $\int\pi:\mathcal{E}'(G)\to \operatorname{End}(V)$ extending π*;
(4) *the following two conditions hold*:
  (i) *the representation is locally equibounded, that is, for all compact subsets $K\subseteq G$ the set $\pi(K)\subseteq \operatorname{End}(V)$ is equibounded*;
  (ii) *the limits $D\pi(X)(v):=\lim_{t\to 0} t^{-1}(\exp(tX)\cdot v - v)$ exist for all $v\in V$ and the convergence is uniform on bounded subsets of V.*

Proof. It is clear that (1) implies (2). The dual of $\mathcal{C}^1(G)$ is a subspace of $\mathcal{E}'(G)$. It generates $\mathcal{E}'(G)$ as a bornological algebra in the sense that any bounded subset of $\mathcal{E}'(G)$ is contained in S^n for a bounded subset $S\subseteq \mathcal{C}^1(G)'$. A $\mathcal{C}^1$-representation gives rise to a bounded linear map $\mathcal{C}^1(G)'\to \operatorname{End}(V)$, which we can then extend to an algebra homomorphism on all of $\mathcal{E}'(G)$. Hence (2) implies (3). The set of δ_g, $g\in K$, is bounded in $\mathcal{E}'(G)$ and we have convergence $t^{-1}(\delta_{\exp(tX)}-\delta_1)\to X$ in $\mathcal{E}'(G)$ for all $X\in\mathfrak{g}$. Hence (3) implies (4). The proof of the implication (4) $\Rightarrow$ (1) is similar to the proof of Theorem 3.14 and therefore omitted. □

Conditions (i) and (ii) above are the same as in Theorem 3.14. Thus the only difference between smoothness and differentiability is condition (iii) of Theorem 3.14.

Remark 3.16. It follows immediately from Theorem 3.15 that the regular representations on $\mathcal{E}'(G)$ and $\mathcal{D}'(G)$ are differentiable. However, these representations are not smooth. One can verify directly that the third condition of Theorem 3.14 fails. It is also clear that they are not essential as modules over $\mathcal{D}(G)$ because the convolution of a smooth function with a distribution is already a smooth function.

Proposition 3.17. *Let G be a locally compact group that is countable at infinity and let V be a metrizable bornological vector space. Let I be a fundamental system of smooth compact subgroups in G. Then*

$$\mathrm{S}_G\big(\mathcal{E}(G,V),\lambda\big)\cong \varinjlim_{k\in I}\mathcal{E}(k\backslash G,V)\cong \varinjlim_{k\in I}\mathcal{E}(k\backslash G)\mathbin{\hat\otimes} V;$$
$$\mathrm{S}_G\big(\mathcal{E}(G,V),\rho\big)\cong \varinjlim_{k\in I}\mathcal{E}(G/k,V)\cong \varinjlim_{k\in I}\mathcal{E}(G/k)\mathbin{\hat\otimes} V;$$

$$\mathrm{S}_{G\times G}\big(\mathcal{E}(G,V),\lambda\boxtimes\rho\big)\cong\varinjlim_{k\in I}\mathcal{E}(G//k,V)\cong\varinjlim_{k\in I}\mathcal{E}(G//k)\mathbin{\hat{\otimes}}V.$$

Proof. We only compute the smoothening of the left regular representation, the other cases are similar. Let $U\subseteq G$ be an open almost connected subgroup. We can assume all $k\in I$ to be normal subgroups of U. Let $k\in I$. Since V is metrizable and $k\backslash G$ is countable at infinity, Lemma 2.14 yields $\mathcal{E}(k\backslash G,V)\cong\mathcal{E}(k\backslash G)\mathbin{\hat{\otimes}}V$ and hence the last isomorphism. The space $\mathcal{E}(k\backslash G)\mathbin{\hat{\otimes}}V$ is metrizable as well. Hence there is no difference between smooth and differentiable Lie group representations on this space by Proposition 3.18. Since $\mathcal{E}'(U/k)$ evidently acts on $\mathcal{E}(k\backslash G)\mathbin{\hat{\otimes}}V$ by convolution, we conclude that U/k acts smoothly on $\mathcal{E}(k\backslash G)\mathbin{\hat{\otimes}}V$ for all $k\in I$. Therefore, $X:=\varinjlim\mathcal{E}(k\backslash G)\mathbin{\hat{\otimes}}V$ is a smooth representation of G by Theorem 3.13. Since $W=\varinjlim W^k$ for any smooth representation, it is clear that any bounded G-equivariant map $W\to\mathcal{E}(G,V)$ factors through X. Hence X is the smoothening of $\mathcal{E}(G,V)$. □

The assertion of the proposition becomes false if G fails to be countable at infinity or if V fails to be metrizable.

Proposition 3.18. *Differentiable Lie group representations on metrizable bornological vector spaces are smooth.*

Proof. This follows immediately from Lemma 2.5. □

3.6. Smooth representations on topological vector spaces

Let G be a Lie group and let V be a complete locally convex topological vector space. Let $\mathrm{End}(V)$ be the algebra of continuous linear operators on V and let $\mathrm{Aut}(V)$ be its multiplicative group. We equip $\mathrm{End}(V)$ with the equicontinuous bornology, so that it becomes a bornological algebra. There is a topological analogue of the space $\mathcal{E}(G,V)$. A representation $\pi:G\to\mathrm{Aut}(V)$ is called *smooth* if its adjoint is a continuous linear map $\pi_*:V\to\mathcal{E}(G,V)$ (see [3]). The following criterion is similar to the criterion for differentiable representations in Theorem 3.15.

Proposition 3.19. *The representation π is smooth if and only if it can be extended to a bounded homomorphism $\textstyle\int\pi:\mathcal{E}'(G)\to\mathrm{End}(V)$.*

Proof. First suppose π to be smooth. We let $D\in\mathcal{E}'(G)$ act on V as usual by $\textstyle\int\pi(D)(v):=\langle D\mathbin{\hat{\otimes}_\pi}\mathrm{id},\pi_*(v)\rangle$. This is defined because $\mathcal{E}(G,V)\cong\mathcal{E}(G)\mathbin{\hat{\otimes}_\pi}V$ is Grothendieck's projective tensor product [8]. Let $S\subseteq\mathcal{E}'(G)$ be bounded. Then S is an equicontinuous set of linear functionals on $\mathcal{E}(G)$ because $\mathcal{E}(G)$ is a Fréchet space. Hence $\textstyle\int\pi(S)$ is equicontinuous as well. Suppose conversely that $\textstyle\int\pi:\mathcal{E}'(G)\to\mathrm{End}(V)$ is a bounded homomorphism extending π. Then the family of operators π_g for g in a compact subset of G is equicontinuous and $t^{-1}(\exp(tX)\cdot v-v)\to\textstyle\int\pi(X)(v)$ in the strong operator topology for $t\to 0$. This implies that π is smooth, see [3]. □

We now equip V with the von Neumann bornology, which consists of the subsets of V that are absorbed by each neighborhood of zero. Any equicontinuous family of operators on V is equibounded. Hence a topologically smooth representation is bornologically differentiable. The converse implication holds if V is "bornological", that is, a subset that absorbs all von Neumann bounded subsets is already a neighborhood of zero. In that case an equibounded set of linear maps is equicontinuous as well. Thus topologically smooth representations on bornological topological vector spaces are the same as bornologically differentiable representations with respect to the von Neumann bornology.

Next we consider the precompact bornology. Let $\mathrm{Pt}(V)$ be V equipped with the precompact bornology. Let π be topologically smooth. Since any bounded subset of $\mathcal{E}'(G)$ is bornologically compact, the set of operators $\int\pi(S)$ for bounded $S \subseteq \mathcal{E}'(G)$ is even bornologically relatively compact for the equicontinuous bornology on $\mathrm{End}(V)$. This implies that $\int\pi(S)(T)$ is again precompact for precompact T, that is, $\int\pi$ is bounded for the equibounded bornology on $\mathrm{End}(\mathrm{Pt}(V))$. The converse implication holds if a subset of V that absorbs all precompact subsets is already a neighborhood of zero. For instance, this is the case if V is a Fréchet space.

As a result, the topological notion of smooth representation is equivalent to the bornological notion of differentiable representation under mild hypotheses on the topology of V. However, condition (iii) of Theorem 3.14 will usually be violated.

Analogous assertions for continuous representations are false unless V is a Fréchet space. For instance, if V is a continuous representation on a Banach space, then the induced representation on the dual space V' is weakly continuous but usually not norm continuous. However, the weak and the norm topology on V' have the same von Neumann bornology.

Theorem 3.20. *Let $\pi : G \to \mathrm{Aut}(V)$ be a group representation of a Lie group on a Fréchet space. Then the following are equivalent*:

(1) π *is smooth as a representation on a topological vector space*;
(2) π *is smooth with respect to the von Neumann bornology*;
(3) π *is smooth with respect to the precompact bornology.*

Proof. A subset of V that absorbs all null sequences is already a neighborhood of zero. Hence the above discussion shows that topological smoothness is equivalent to bornological differentiability for either the von Neumann or the precompact bornology. Since both bornologies on V are metrizable, the assertion now follows from Proposition 3.18. □

Proposition 3.21. *Let V be a Fréchet space equipped with the precompact or von Neumann bornology and let G be a Lie group. Let $\pi : G \to \mathrm{Aut}(V)$ be a representation. Then the smoothening of V is a Fréchet space with the precompact or the von Neumann bornology, respectively. If V is nuclear, so is* $\mathrm{S}(V)$.

Proof. Let W be the Fréchet space of smooth functions $G \to V$ in the usual topological sense, equipped with the precompact or von Neumann bornology, respectively. It is shown in [14] that $\mathcal{E}(G, V) = W$ as bornological vector spaces, for both bornologies. Here we use that the bornologies of locally uniform boundedness and locally uniform continuity on

$\mathcal{E}(G, V)$ coincide. Since $\mathrm{S}(V)$ is a closed subspace of $\mathcal{E}(G, V) \cong W$, it is a Fréchet space as well. Furthermore, if V is nuclear, so is W and hence its subspace $\mathrm{S}(V)$. □

4. Essential modules versus smooth representations

Let G be a locally compact group. We are going to identify the category of smooth representations of G with the category of essential modules over the convolution algebra $\mathcal{D}(G)$. First we introduce the appropriate notion of an approximate identity in a bornological algebra and define the notion of an essential module. Then we compare essential modules over $\mathcal{D}(G)$ with smooth representations of G. Finally, we investigate analogues of the smoothening, restriction, compact induction and induction functors for representations.

4.1. Approximate identities and essential modules

Definition 4.1. Let A be a bornological algebra. We say that A has an *approximate identity* if for each bornologically compact subset $S \subseteq A$ there is a sequence $(u_n)_{n\in\mathbb{N}}$ in A such that $u_n \cdot x$ and $x \cdot u_n$ converge to x uniformly for $x \in S$.

A subset of a bornological vector space V is *bornologically compact* if it is a compact subset of V_T for some bounded complete disk $T \subseteq V$. The uniform convergence in the above definition means that there is a bounded complete disk $T \subseteq A$ such that $u_n x$ and $x u_n$ converge to x uniformly for $x \in S$ in the Banach space V_T.

Since we may take a different sequence (u_n) for each bornologically compact subset, we are really considering a net $(u_{n,S})$ in A, indexed by pairs (S, n) where $S \subseteq A$ is bornologically compact and $n \in \mathbb{N}$. It is more convenient to work with sequences as in Definition 4.1, however. The above definition is related to the usual notion of an approximate identity in a Banach algebra:

Lemma 4.2. *Let A be a Banach algebra with a* (*multiplier*) *bounded approximate identity in the usual sense. Then A equipped with the von Neumann or precompact bornology has an approximate identity in the sense of Definition* 4.1.

Proposition 4.3. *The bornological algebra $\mathcal{D}(G)$ has an approximate identity for any locally compact topological group G.*

Proof. Let $U \subseteq G$ be open and almost connected. Any element of $\mathcal{D}(G)$ can be written as a finite sum of elements of the form $\delta_g * f$ or of elements of the form $f * \delta_g$ with $g \in G$, $f \in \mathcal{D}(U)$. Therefore, it suffices to construct an approximate identity for $\mathcal{D}(U)$. Let I be a fundamental system of smooth compact subgroups of U. Since $\mathcal{D}(U) = \varinjlim \mathcal{D}(U/k)$, it suffices to construct approximate identities in $\mathcal{D}(U/k)$. Consequently, we may assume G to be a Lie group.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(G)$ with

$$\lim_{n\to\infty}\int_G u_n(g)\,dg = 1, \qquad \lim_{n\to\infty}\operatorname{supp} u_n = \{1\}.$$

The latter condition means that the support of u_n is eventually contained in any neighborhood of 1. We claim that (u_n) is an approximate identity for any bounded subset $S \subseteq \mathcal{D}(G)$. We only check the convergence $u_n * f \to f$. The convergence $f * u_n \to f$ is proved similarly, using that $\lim \int_G u_n(g^{-1})\,dg = 1$ as well.

There is a compact subset $K \subseteq G$ such that f and $f * u_n$ are supported in K for all $f \in S$, $n \in \mathbb{N}$. Hence we are working in the nuclear Fréchet space $\mathcal{E}_0(K)$. It is straightforward to see that $u_n * f$ converges to f with respect to the topology of $\mathcal{E}_0(K)$, even uniformly for $f \in S$. Since $\mathcal{E}_0(K)$ is a Fréchet space equipped with the von Neumann bornology, the topological and bornological notions of uniform convergence of a sequence of operators on precompact subsets in $\mathcal{E}_0(K)$ are equivalent (see [14]). Hence (u_n) is a left approximate identity in the sense of Definition 4.1. □

Let V be a right and W a left bornological A-module. Then we define $V \hat{\otimes}_A W$ as the cokernel of the map

$$b_1' : V \hat{\otimes} A \hat{\otimes} W \to V \hat{\otimes} W, \quad v \otimes a \otimes w \mapsto va \otimes w - v \otimes aw.$$

That is, we divide $V \hat{\otimes} W$ by the *closure* of the range of b_1'. For $V = A$ we also consider the map $b_0' : A \hat{\otimes} W \to W$, $a \otimes w \mapsto aw$. Since $b_0' \circ b_1' = 0$, the map b_0' descends to a map $A \hat{\otimes}_A W \to W$. If V is a B-A-bimodule and W a left A-module, then $V \hat{\otimes}_A W$ is a left B-module in an obvious fashion. In particular, $A \hat{\otimes}_A W$ is a left A-module and the map $A \hat{\otimes}_A W \to W$ is a module homomorphism.

Lemma 4.4. *Let A be a bornological algebra with an approximate identity and let W be a bornological left A-module. The natural map $A \hat{\otimes}_A W \to W$ is always injective. The map $b_0' : A \hat{\otimes} W \to W$ is a bornological quotient map if and only if the map $A \hat{\otimes}_A W \to W$ induced by b_0' is a bornological isomorphism.*

Proof. Everything follows once we know that the range of $b_1' : A \hat{\otimes} A \hat{\otimes} W \to A \hat{\otimes} W$ is dense in the kernel of $b_0' : A \hat{\otimes} W \to W$. Pick $\omega \in \operatorname{Ker} b_0'$. Then there exist bounded complete disks $S \subseteq A$, $T \subseteq W$ such that $\omega \in A_S \hat{\otimes} W_T$. Since A_S and W_T are Banach spaces, we can find null sequences (a_n) in A_S, (w_n) in W_T and (λ_n) in $\ell^1(\mathbb{N})$ such that $\omega = \sum \lambda_n a_n \otimes w_n$ (see [8]). Since the set $\{a_n\}$ is bornologically compact in A, there is a sequence (u_m) in A such that $u_m a_n \to a_n$ for $m \to \infty$ uniformly for $n \in \mathbb{N}$. Thus $u_m \cdot \omega \to \omega$ for $m \to \infty$. We have

$$b_1'(u_m \otimes \omega) = u_m \cdot \omega - u_m \otimes b_0'(\omega) = u_m \cdot \omega.$$

Thus ω is the limit of a sequence in the range of b_1'. □

Definition 4.5. Let A be a bornological algebra with approximate identity. A bornological left A-module V is called *essential* if the map $b_0' : A \hat{\otimes} V \to V$ is a bornological quotient map or, equivalently, $A \hat{\otimes}_A V \cong V$. Essential right modules and bimodules are defined analogously.

If A is unital, then a left A-module is essential if and only if it is unital, that is, 1_A acts as the identity. The term "essential" is a synonym for "non-degenerate", which is not as widely used for other purposes. Grønbæk [6] calls such modules "A-induced".

Let $\widehat{\mathbf{M}}_G$ be the category of all bornological left modules over $\mathcal{D}(G)$. Let $\mathbf{M}_G$ be its full subcategory of essential left modules. We write $V \in \mathbf{M}_G$ if V is an object of $\mathbf{M}_G$ and write $f * v$ for $f \in \mathcal{D}(G)$, $v \in V$, for the module structure.

Proposition 4.6. *For any $V \in \mathbf{M}_G$ there is a natural smooth representation $\pi : G \to \mathrm{Aut}(V)$ such that*

$$f * v = \textstyle\int\!\pi(f\,dg)(v) = \displaystyle\int_G \pi(g, v) \cdot f(g)\,dg$$

for all $f \in \mathcal{D}(G)$, $v \in V$. Naturality means that bounded module homomorphisms are π-equivariant.

Proof. Since V is essential, it is naturally isomorphic to the cokernel of the operator $b'_1 : \mathcal{D}(G) \hat{\otimes} \mathcal{D}(G) \hat{\otimes} V \to \mathcal{D}(G) \hat{\otimes} V$. We let G act on the source and target of b'_1 by the left regular representation on the first tensor factor. This representation is smooth by Lemma 3.4 and b'_1 is G-equivariant. Therefore, its cokernel V carries a representation $\pi : G \to \mathrm{Aut}(V)$, which is smooth by Lemma 3.5. It is trivial to check $\int\!\pi(f_1\,dg)(f_2 * v) = f_1 * f_2 * v$. Since V is essential, this implies $\int\!\pi(f\,dg)(v) = f * v$ for all $f \in \mathcal{D}(G)$, $v \in V$. The construction of π is evidently natural. □

4.2. Representations as modules over convolution algebras

We have seen how an essential module over $\mathcal{D}(G)$ can be turned into a smooth representation of G. Conversely, we now turn a continuous representation $\pi : G \to \mathrm{Aut}(V)$ into a module over $\mathcal{D}(G)$. Continuity implies that $Wf(g) := \pi_g f(g)$ defines a bounded linear operator from $\mathcal{D}(G, V)$ to $L^1(G, V) := L^1(G) \hat{\otimes} V$, where $L^1(G)$ carries the von Neumann bornology. We remark without proof that the converse implication also holds: if W is a bounded linear map $\mathcal{D}(G, V) \to L^1(G, V)$, then π is already continuous. If π is continuous, then

$$\textstyle\int\!\pi(f \otimes v) := \displaystyle\int_G \pi_g(v) \cdot f(g)\,dg$$

defines a bounded linear map from $\mathcal{D}(G, V) \cong \mathcal{D}(G) \hat{\otimes} V$ to V. By adjoint associativity we obtain a bounded linear map $\int\!\pi : \mathcal{D}(G) \to \mathrm{End}(V)$. It is straightforward to check that this is an algebra homomorphism, so that V becomes a module over $\mathcal{D}(G)$. A morphism in $\widehat{\mathbf{R}}_G$ between continuous representations is a $\mathcal{D}(G)$-module homomorphism as well. That is, we have a functor from the category of continuous representations of G to $\widehat{\mathbf{M}}_G$.

Proposition 4.7. *Let $\pi : G \to \mathrm{Aut}(V)$ be a continuous representation. Then the following assertions are equivalent*:

(i) *π is a smooth representation, that is, the adjoint of π is a bounded linear map $V \to \mathcal{E}(G, V)$*;
(ii) *the map $\int\pi : \mathcal{D}(G, V) \to V$ has a bounded linear right section, that is, there is a bounded linear map $\sigma : V \to \mathcal{D}(G, V)$ such that $\int\pi \circ \sigma = \mathrm{id}_V$*;
(iii) *V is an essential module over $\mathcal{D}(G)$, that is, the map $\int\pi : \mathcal{D}(G, V) \to V$ is a bornological quotient map.*

If π is smooth, then the section σ in (ii) *can be constructed explicitly as follows. Choose $\phi \in \mathcal{D}(G)$ with $\int_G \phi(g)\,dg = 1$ and define*

$$\sigma_\phi : V \to \mathcal{D}(G, V), \quad \sigma_\phi(v)(g) := \phi(g)\pi\left(g^{-1}, v\right).$$

If $H \subseteq G$ is compact, the section σ in (ii) *can be chosen H-equivariant.*

Proof. If π is smooth, then the formula for σ_ϕ defines a bounded linear map into $\mathcal{D}(G, V)$ by Lemma 3.2. A trivial computation shows that σ_ϕ is a section for $\int\pi$. Thus (i) implies (ii). If $H \subseteq G$ is compact, we can choose ϕ left-H-invariant. Then the operator σ_ϕ is H-equivariant. The implication (ii) $\Rightarrow$ (iii) is trivial. Suppose (iii). The map $\int\pi : \mathcal{D}(G, V) \to V$ is equivariant with respect to the left regular representation of G on $\mathcal{D}(G, V)$. The latter is smooth by Lemma 3.4. Thus π is a quotient of a smooth representation. Lemma 3.5 shows that π is smooth. □

Theorem 4.8. *Let G be a locally compact group. Then the categories of smooth representations and of essential modules are isomorphic. The isomorphism sends a representation $\pi : G \to \mathrm{Aut}(V)$ to its integrated form $\int\pi : \mathcal{D}(G) \to \mathrm{End}(V)$. In particular, π is smooth if and only if $\int\pi$ is essential.*

Proof. The two constructions in Propositions 4.6 and 4.7 are clearly inverse to each other. They provide the desired isomorphism of categories. □

4.3. Constructions with modules and homological algebra

Most functors between module categories are special cases of two constructions: the balanced tensor product and the Hom functor. Let W be a B-A-bimodule. Then we have a functor $W \hat{\otimes}_A \sqcup$ from left A-modules to left B-modules and a functor $\mathrm{Hom}_B(W, \sqcup)$ from left B-modules to left A-modules. The left A-module structure on $\mathrm{Hom}_B(W, V)$ is given by $a \cdot L(w) := L(w \cdot a)$. These two functors are linked by the adjoint associativity relation

$$\mathrm{Hom}_B(W \hat{\otimes}_A V, X) \cong \mathrm{Hom}_A\left(V, \mathrm{Hom}_B(W, X)\right). \tag{11}$$

Of course, there are similar constructions for right modules.

Let $H \subseteq G$ be a closed subgroup. The embedding $H \subseteq G$ induces an algebra homomorphism $\mathcal{E}'(H) \to \mathcal{E}'(G)$. Embedding $\mathcal{D}(H) \subseteq \mathcal{E}'(H)$ as usual, using a left Haar measure $d_H h$ on H, we obtain an algebra homomorphism $\mathcal{D}(H) \to \mathcal{E}'(G)$. This does not suffice to define a restriction functor $\widehat{\mathbf{M}}_G \to \widehat{\mathbf{M}}_H$. However, we can view $\mathcal{D}(G)$ as a bimodule over $\mathcal{D}(H)$ on the left and $\mathcal{D}(G)$ on the right by $f_0 * f_1 * f_2 := (f_0\, d_H h) * f_1 * f_2$ for $f_0 \in \mathcal{D}(H)$, $f_1, f_2 \in \mathcal{D}(G)$. This yields two functors

$$\mathrm{S}_G^H : \widehat{\mathbf{M}}_G \to \widehat{\mathbf{M}}_H, \quad \mathrm{S}_G^H(V) := \mathcal{D}(G) \hat{\otimes}_{\mathcal{D}(G)} V,$$
$$\mathrm{I}_H^G : \widehat{\mathbf{M}}_H \to \widehat{\mathbf{M}}_G, \quad \mathrm{I}_H^G(V) := \mathrm{Hom}_{\mathcal{D}(H)}\big(\mathcal{D}(G), V\big),$$

called (*smooth*) *restriction functor* and (*rough*) *induction functor*, respectively. An analogous formula allows us to view $\mathcal{D}(G)$ as a bimodule over $\mathcal{D}(G)$ on the left and $\mathcal{D}(H)$ on the right. This yields two functors

$$\mathrm{Ic}_H^G : \widehat{\mathbf{M}}_H \to \widehat{\mathbf{M}}_G, \quad \mathrm{Ic}_H^G(V) := \mathcal{D}(G) \hat{\otimes}_{\mathcal{D}(H)} V,$$
$$\mathrm{R}_G^H : \widehat{\mathbf{M}}_G \to \widehat{\mathbf{M}}_H, \quad \mathrm{R}_G^H(V) := \mathrm{Hom}_{\mathcal{D}(G)}\big(\mathcal{D}(G), V\big),$$

called (*smooth*) *compact induction functor* and *rough restriction functor*, respectively. Finally, we define

$$\mathrm{S} := \mathrm{S}_G^G = \mathrm{Ic}_G^G : \widehat{\mathbf{M}}_G \to \widehat{\mathbf{M}}_G, \quad \mathrm{S}(V) := \mathcal{D}(G) \hat{\otimes}_{\mathcal{D}(G)} V,$$
$$\mathrm{R} := \mathrm{R}_G^G = \mathrm{I}_G^G : \widehat{\mathbf{M}}_G \to \widehat{\mathbf{M}}_G, \quad \mathrm{R}(V) := \mathrm{Hom}\big(\mathcal{D}(G), V\big),$$

the *smoothening* and *roughening* functors.

Our treatment of the compact induction functor as a tensor product is analogous to Marc Rieffel's approach to induced representations [18]. The Banach algebra variant of Rieffel's theory by Niels Grønbæk is even closer to our setup [6,7]. The only difference is that Grønbæk works with $L^1(G)$ instead of $\mathcal{D}(G)$.

The following theorem shows that the smoothening deserves its name. We use the natural map $\mathrm{S}(V) \to V$ induced by $b_0'(f \otimes v) := f * v$.

Theorem 4.9. *The natural map $\mathrm{S}(V) \to V$ is always injective and an isomorphism if and only if $V \in \mathbf{M}_G$. The smoothening is an idempotent functor on $\widehat{\mathbf{M}}_G$ whose range is $\mathbf{M}_G$. As a functor $\widehat{\mathbf{M}}_G \to \mathbf{M}_G$ it is left adjoint to the embedding $\mathbf{M}_G \to \widehat{\mathbf{M}}_G$. Let $\pi : G \to \mathrm{Aut}(V)$ be a continuous representation of G. Then the smoothenings of G as a module and as a representation agree.*

Proof. We know from Lemma 4.4 that the map $\mathrm{S}(V) \to V$ is always injective and an isomorphism if and only if V is essential. Since the left regular representation on $\mathcal{D}(G)$ is smooth, $\mathcal{D}(G)$ is an essential left module over itself by Theorem 4.8. That is, $\mathcal{D}(G) \hat{\otimes}_{\mathcal{D}(G)} \mathcal{D}(G) \cong \mathcal{D}(G)$. Since the balanced tensor product is associative, we obtain $\mathrm{S}^2 = \mathrm{S}$. Since $S(V) \cong V$ if and only if $V \in \mathbf{M}_G$, the range of S is $\mathbf{M}_G$.

Let W be an essential module. Since the map $\mathrm{S}(V) \to V$ is always injective, the induced map $\mathrm{Hom}(W, \mathrm{S}(V)) \to \mathrm{Hom}(W, V)$ is injective. Any bounded module homomorphism $W \to V$ restricts to a bounded module homomorphism $W = \mathrm{S}(W) \to \mathrm{S}(V)$, so that the map $\mathrm{Hom}(W, \mathrm{S}(V)) \to \mathrm{Hom}(W, V)$ is also surjective. This means that the embedding and smoothening functors are adjoint.

Let π be a continuous representation. Let V_0 and V_1 be the smoothenings of V as a representation and as a module, respectively. The natural maps $V_0 \to V$ and $V_1 \to V$

are both injective. Since V_1 is an essential module, it is a smooth representation of G as well. Hence the map $V_0 \to V$ factors through $V_0 \to V_1$ by the universal property of the smoothening. Similarly, since V_0 is an essential module, the map $V_1 \to V$ factors through $V_1 \to V_0$. Both maps $V_0 \to V_1$ and $V_1 \to V_0$ are injective and bounded, hence bornological isomorphisms. □

Eq. (11) specializes to natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}(G)}\bigl(\mathrm{Ic}_H^G(V), W\bigr) \cong \mathrm{Hom}_{\mathcal{D}(H)}\bigl(V, \mathrm{R}_G^H(W)\bigr), \tag{12}$$

$$\mathrm{Hom}_{\mathcal{D}(H)}\bigl(\mathrm{S}_G^H(V), W\bigr) \cong \mathrm{Hom}_{\mathcal{D}(G)}\bigl(V, \mathrm{I}_H^G(W)\bigr). \tag{13}$$

That is, compact induction is left adjoint to rough restriction and rough induction is right adjoint to smooth restriction.

Especially, S is left adjoint to R. Being adjoint to an idempotent functor, R is idempotent as well. Thus R is a projection onto a subcategory of $\widehat{\mathbf{M}}_G$. We may call these modules *rough*. They are usually not smooth, but if G is a Lie group they are differentiable by Theorem 3.15 because they are evidently modules over $\mathcal{E}'(G)$. We have $\mathrm{R} \circ \mathrm{S} \cong \mathrm{R}$ because

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(G)}\bigl(V, \mathrm{R} \circ \mathrm{S}(W)\bigr) &\cong \mathrm{Hom}_{\mathcal{D}(G)}\bigl(\mathrm{S}(V), \mathrm{S}(W)\bigr)\\ &\cong \mathrm{Hom}_{\mathcal{D}(G)}\bigl(\mathrm{S}(V), W\bigr) \cong \mathrm{Hom}_{\mathcal{D}(G)}\bigl(V, \mathrm{R}(W)\bigr)\end{aligned}$$

for all $V, W \in \widehat{\mathbf{M}}_G$. We will prove shortly that $\mathrm{S} \circ \mathrm{R} \cong \mathrm{S}$. Summarizing, we have

$$\mathrm{S} \circ \mathrm{S} \cong \mathrm{S}, \quad \mathrm{S} \circ \mathrm{R} \cong \mathrm{S}, \quad \mathrm{R} \circ \mathrm{S} \cong \mathrm{R}, \quad \mathrm{R} \circ \mathrm{R} \cong \mathrm{R}. \tag{14}$$

The natural map $V \to \mathrm{R}(V)$ is injective if and only if no non-zero vector $v \in V$ satisfies $f * v = 0$ for all $f \in \mathcal{D}(G)$. Let us restrict attention to this class of modules. Then the natural maps $\mathrm{S}(V) \to V \to \mathrm{R}(V)$ are injective. If we have injective maps $\mathrm{S}(V) \to W \to \mathrm{R}(V)$, then $\mathrm{S}(V) = \mathrm{S}(W)$ because already $\mathrm{SR}(V) = \mathrm{S}(V)$ and the smoothening preserves monomorphisms. Conversely, if $\mathrm{S}(W) \cong \mathrm{S}(V)$, then $\mathrm{R}(W) \cong \mathrm{RS}(W) \cong \mathrm{RS}(V) \cong \mathrm{R}(V)$ as well, so that we have injective maps $\mathrm{S}(V) \to W \to \mathrm{R}(V)$. This means that a module W satisfies $\mathrm{S}(W) = \mathrm{S}(V)$ if and only if it lies between $\mathrm{S}(V)$ and $\mathrm{R}(V)$.

In the following we tacitly identify $\mathbf{M}_G$ with $\mathbf{R}_G$ using Theorem 4.8. If we have to view a smooth representation as a right module, we always use the antipode $\tilde{f}^{(1)}$ defined in (7) to turn a left into a right module.

Since $\mathrm{S}(V) = V$ for $V \in \mathbf{M}_G$, we have $\mathrm{S}_G^H|_{\mathbf{M}_G} \cong \mathrm{Res}_G^H$. The universal property of the smoothening and (13) imply that $\mathrm{S} \circ \mathrm{I}_H^G(W) : \mathbf{M}_H \to \mathbf{M}_G$ is right adjoint to Res_G^H. This means that

$$\mathrm{S} \circ \mathrm{I}_H^G \cong \mathrm{Ind}_H^G. \tag{15}$$

Since Ind_G^G is the identical functor, we get the relation $\mathrm{S} \circ \mathrm{R} = \mathrm{S}$ claimed in (14). The relationship between Ic_H^G and $\mathrm{c\text{-}Ind}_H^G$ is more complicated. Before we discuss it we need some other useful results.

Let X and Y be a right and left module over $\mathcal{D}(G)$ and let W be a bornological vector space. Then $\mathrm{Hom}(X, W)$ is a left module over $\mathcal{D}(G)$ in a canonical way and (11) yields

$$\mathrm{Hom}(X \hat{\otimes}_{\mathcal{D}(G)} Y, W) \cong \mathrm{Hom}_{\mathcal{D}(G)}\bigl(Y, \mathrm{Hom}(X, W)\bigr). \tag{16}$$

Let $\mathbb{C}(1)$ be the trivial representation of G on $\mathbb{C}$ viewed as a right module over $\mathcal{D}(G)$. The space $\mathbb{C}(1) \hat{\otimes}_{\mathcal{D}(G)} Y$ is called the *coinvariant space* of Y. If Y is a smooth representation viewed as a left module over $\mathcal{D}(G)$ and $W = \mathbb{C}$, then (16) asserts that the dual space of the coinvariant space of Y is the space of G-invariant linear functionals on Y.

Let X, Y, Z be smooth representations of G. We let G act on $\operatorname{Hom}(Y, Z)$ by the conjugation action $(g \cdot l)(y) := g \cdot l(g^{-1}y)$ and on $X \hat{\otimes} Y$ by the diagonal action $g \cdot (x \otimes y) := gx \otimes gy$. These two constructions are adjoint in the sense that

$$\operatorname{Hom}_G\bigl(X, \operatorname{S}\operatorname{Hom}(Y, Z)\bigr) \cong \operatorname{Hom}_G\bigl(X, \operatorname{Hom}(Y, Z)\bigr) \cong \operatorname{Hom}_G(X \hat{\otimes} Y, Z). \tag{17}$$

The first isomorphism is the universal property of the smoothening. The second is proved by identifying both sides with the space of bilinear maps $l : X \times Y \to Z$ that satisfy the equivariance condition $l(gx, gy) = gl(x, y)$. If we let $X := \mathbb{C}(1)$ be the trivial representation of G on $\mathbb{C}$, we have $\mathbb{C}(1) \hat{\otimes} Y \cong Y$ and

$$\operatorname{Hom}_G\bigl(\mathbb{C}(1), \operatorname{S}\operatorname{Hom}(Y, Z)\bigr) \cong \operatorname{Hom}_G(Y, Z). \tag{18}$$

Next we claim that

$$\mathbb{C}(1) \hat{\otimes}_{\mathcal{D}(G)} (Y \hat{\otimes} Z) \cong Y \hat{\otimes}_{\mathcal{D}(G)} Z, \tag{19}$$

where we view $\mathbb{C}(1)$ and Y as right modules over $\mathcal{D}(G)$. Eq. (19) can easily be verified directly. For the fun of it we use adjointness relations to prove the equivalent assertion that $\operatorname{Hom}(\mathbb{C}(1) \hat{\otimes}_{\mathcal{D}(G)} (Y \hat{\otimes} Z), W) \cong \operatorname{Hom}(Y \hat{\otimes}_{\mathcal{D}(G)} Z, W)$ for all bornological vector spaces W. Eq. (16) implies

$$\begin{aligned}
\operatorname{Hom}(Y \hat{\otimes}_{\mathcal{D}(G)} Z, W) &\cong \operatorname{Hom}_G\bigl(Z, \operatorname{Hom}(Y, W)\bigr),\\
\operatorname{Hom}\bigl(\mathbb{C}(1) \hat{\otimes}_{\mathcal{D}(G)} (Y \hat{\otimes} Z), W\bigr) &\cong \operatorname{Hom}\bigl(Y \hat{\otimes} Z, \operatorname{Hom}(\mathbb{C}(1), W)\bigr)\\
&\cong \operatorname{Hom}(Y \hat{\otimes} Z, W),
\end{aligned}$$

where G acts on $\operatorname{Hom}(Y, W)$ by $g \cdot l(y) := l(g^{-1}y)$ and trivially on W. Since the action on $\operatorname{Hom}(Y, W)$ is the conjugation action for the trivial representation on W, both spaces are isomorphic by (17). This finishes the proof of (19).

Now we are ready to relate the functors Ic_H^G and $\operatorname{c-Ind}_H^G$. Recall that $\mu_{G:H}$ denotes the quasi-character $\mu_G/\mu_H : H \to \mathbb{R}_+^\times$. For a representation $\pi : H \to \operatorname{Aut}(V)$ we write $\mu_{G:H} \cdot V$ for the representation $\mu_{G:H} \cdot \pi$ on V.

Theorem 4.10. *There is a natural isomorphism* $\operatorname{Ic}_H^G(V) \cong \operatorname{c-Ind}_H^G(\mu_{G:H} \cdot V)$ *for all* $V \in \mathbf{M}_H$.

Proof. First we explain the source of the relative modular function in $\operatorname{Ic}_H^G(V)$. The right $\mathcal{D}(G)$-module structure on $\mathcal{D}(G)$ is the integrated form of the twisted right regular representation $\rho \cdot \mu_G$ because $f(g)\,d_G g * \delta_{x^{-1}} = f(gx)\mu_G(x)\,d_G g$. We equip $\mathcal{D}(G)$ and $\mathcal{D}(H)$ with the canonical $\mathcal{D}(H)$-bimodule structure. The restriction map $\mathcal{D}(G) \to \mathcal{D}(H)$ is a left module homomorphism, but we pick up a factor $\mu_{G:H}$ for the right module structure. Therefore, it induces an H-equivariant map $\operatorname{Ic}_H^G(V) \to \mu_{G:H} \cdot V$ and hence a G-equivariant map into $\mathrm{Ind}_H^G(\mu_{G:H}\cdot V)$. This is the desired isomorphism onto $\text{c-Ind}_H^G(\mu_{G:H}\cdot V)$. We now construct it more explicitly. Define

$$\Phi:\mathcal{D}(G,V)\to\mathcal{E}(G,V),\quad \Phi f(g):=\int_H h\cdot f\bigl(g^{-1}\cdot h\bigr)\mu_{G:H}(h)\,d_H h,$$

where $d_H h$ is a left invariant Haar measure on H. Clearly, $\operatorname{supp}\Phi f\subseteq H\cdot(\operatorname{supp} f)^{-1}$ is uniformly compact in $H\backslash G$ for f in a bounded subset of $\mathcal{D}(G,V)$. Moreover, $\Phi f(hg)=\mu_{G:H}(h)h\cdot\Phi f(g)$ for all $h\in H$, $g\in G$. This means that the range of Φ is contained in $\text{c-Ind}_H^G(\mu_{G:H}V)$. Moreover, one computes easily that $\Phi(f_h)=\Phi(f)$ if $f_h(g):=\mu_G(h)h\cdot f(gh)$ for $h\in H$. This means that Φ is H-invariant for the diagonal action of H on $\mathcal{D}(G)\mathbin{\hat\otimes}V$ that occurs in (19). Therefore, Φ descends to a bounded linear map on $\mathcal{D}(G)\mathbin{\hat\otimes}_{\mathcal{D}(H)}V=\mathrm{Ic}_H^G(V)$. Finally, Φ is G-equivariant, that is, $\Phi\lambda_g=\rho_g\Phi$. Summing up, we have constructed a natural transformation

$$\Phi:\mathrm{Ic}_H^G(V)\to\text{c-Ind}_H^G(\mu_{G:H}\cdot V).$$

It remains to verify that Φ is an isomorphism for all V. This is easy for the left regular representations on $\mathcal{D}(H,V)$, where we can compute both sides explicitly. Any essential module over $\mathcal{D}(H)$ is the cokernel of a map $b_1':\mathcal{D}(H\times H,V)\to\mathcal{D}(H,V)$ between left regular modules. The functor Ic_H^G preserves cokernels because it has a right adjoint. The functor c-Ind_H^G also preserves cokernels by Proposition 3.12. Hence Φ is an isomorphism for all V. □

Corollary 4.11. *If $H\subseteq G$ is cocompact, then there is a natural isomorphism*

$$\mathrm{S}\circ\mathrm{I}_H^G(\mu_{G:H}\cdot V)\cong\mathrm{Ic}_H^G(V).$$

Proof. It is clear from the definition that $\text{c-Ind}_H^G=\mathrm{Ind}_H^G$ in this case. Hence the assertion follows from Theorem 4.10 and (15). □

We continue with some further properties of our functors. Let $L\subseteq H\subseteq G$. Since the right $\mathcal{D}(H)$-module structure on $\mathcal{D}(G)$ comes from a smooth representation, we have $\mathcal{D}(G)\mathbin{\hat\otimes}_{\mathcal{D}(H)}\mathcal{D}(H)\cong\mathcal{D}(G)$ and hence

$$\mathrm{Ic}_H^G\circ\mathrm{Ic}_L^H(V)=\mathcal{D}(G)\mathbin{\hat\otimes}_{\mathcal{D}(H)}\mathcal{D}(H)\mathbin{\hat\otimes}_{\mathcal{D}(L)}V\cong\mathcal{D}(G)\mathbin{\hat\otimes}_{\mathcal{D}(L)}V=\mathrm{Ic}_L^G(V).$$

The assertion $\mathrm{S}_H^L\circ\mathrm{S}_G^H=\mathrm{S}_G^L$ is proved similarly. By adjointness we also obtain $\mathrm{I}_H^G\circ\mathrm{I}_L^H=\mathrm{I}_L^G$ and $\mathrm{R}_H^L\circ\mathrm{R}_G^H=\mathrm{R}_G^L$. We evidently have $\mathrm{Res}_H^L\,\mathrm{Res}_G^H=\mathrm{Res}_G^L$ and hence $\mathrm{Ind}_H^G\circ\mathrm{Ind}_L^H=\mathrm{Ind}_L^G$ by adjointness. As special cases we note that

$$\mathrm{R}\circ\mathrm{I}_H^G=\mathrm{I}_H^G=\mathrm{I}_H^G\circ\mathrm{R},\qquad \mathrm{S}\circ\mathrm{Ic}_H^G=\mathrm{Ic}_H^G\circ\mathrm{S}=\mathrm{Ic}_H^G.\tag{20}$$

Together with (14), we obtain further relations like $\mathrm{I}_H^G\circ\mathrm{S}=\mathrm{I}_H^G$ and $\mathrm{Ic}_H^G\circ\mathrm{R}=\mathrm{Ic}_H^G$.

Let V and W be a right and a left module over $\mathcal{D}(G)$ and $\mathcal{D}(H)$, respectively. Then we trivially have

$$V\mathbin{\hat\otimes}_{\mathcal{D}(G)}\mathrm{Ic}_H^G(W)\cong V\mathbin{\hat\otimes}_{\mathcal{D}(G)}\mathcal{D}(G)\mathbin{\hat\otimes}_{\mathcal{D}(H)}W\cong\mathrm{S}_G^H V\mathbin{\hat\otimes}_{\mathcal{D}(H)}W.\tag{21}$$

Let X be a bornological vector space, equip $\mathrm{Hom}(W, X)$ with the canonical right module structure. Then we have canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(G)}\big(V, \mathrm{I}_H^G \mathrm{Hom}(W, X)\big) &\cong \mathrm{Hom}_{\mathcal{D}(H)}\big(\mathrm{S}_G^H V, \mathrm{Hom}(W, X)\big)\\ &\cong \mathrm{Hom}\big(\mathrm{S}_G^H V \hat{\otimes}_{\mathcal{D}(H)} W, X\big)\\ &\cong \mathrm{Hom}\big(V \hat{\otimes}_{\mathcal{D}(G)} \mathrm{Ic}_H^G W, X\big)\\ &\cong \mathrm{Hom}_{\mathcal{D}(G)}\big(V, \mathrm{Hom}(\mathrm{Ic}_H^G W, X)\big).\end{aligned}$$

Since V is arbitrary, we conclude that

$$\mathrm{I}_H^G \mathrm{Hom}(W, X) \cong \mathrm{Hom}\big(\mathrm{Ic}_H^G W, X\big) \tag{22}$$

as left modules over $\mathcal{D}(G)$. Here W is a right module over $\mathcal{D}(H)$ and X is a bornological vector space. For $X = \mathbb{C}$ this is an assertion about induction of dual spaces. The smoothening of the dual is the *contragradient* representation $\widetilde{W}$. Eq. (22) implies

$$\mathrm{Ind}_H^G \widetilde{W} \cong \big(\text{c-Ind}_H^G(\mu_{G:H} \cdot W)\big)^{\sim}. \tag{23}$$

The analogous statements

$$\mathrm{R}_G^H \mathrm{Hom}(W, X) \cong \mathrm{Hom}\big(\mathrm{S}_G^H W, X\big), \qquad \mathrm{R}\,\mathrm{Hom}(W, X) \cong \mathrm{Hom}(\mathrm{S}W, X), \tag{24}$$

about restriction follow easily from (11).

Finally, we do some homological algebra and begin by recalling a few standard notions. Let A^+ be the augmented unital algebra obtained by adjoining a unit element to a bornological algebra A. The category of left modules over A is isomorphic to the category of unital left modules over A^+. Hence the correct definition of a *free left module* over A is $A^+ \hat{\otimes} V$ with the evident left module structure over A. Similar remarks apply to right modules and bimodules. The free module has the universal property that bounded module homomorphisms $A^+ \hat{\otimes} V \to W$ correspond bijectively to bounded linear maps $V \to W$. As a consequence, free modules are projective for linearly split extensions. In the following we say that a module is *relatively projective* if it is projective for this class of extensions. In general, the modules $A \hat{\otimes} V$ need not be relatively projective.

Proposition 4.12. *Let $H \subseteq G$. Then $\mathcal{D}(G)$ is relatively projective as a left or right module over $\mathcal{D}(H)$.*

Proof. It suffices to prove that $\mathcal{D}(G)$ is projective as a left module over $\mathcal{D}(H)$. We are going to construct a bounded $\mathcal{D}(H)$-linear section σ for the convolution map

$$\mu : \mathcal{D}(H \times G) \cong \mathcal{D}(H) \hat{\otimes} \mathcal{D}(G) \to \mathcal{D}(G), \quad \mu f(g) := \int_H f\big(h, h^{-1}g\big)\, dh.$$

Let μ^+ be the extension of μ to $\mathcal{D}(H)^+ \hat{\otimes} \mathcal{D}(G)$, then $\mu^+ \circ \sigma = \mathrm{id}$ as well. Thus $\mathcal{D}(G)$ is relatively projective as a retract of the free module $\mathcal{D}(H)^+ \hat{\otimes} \mathcal{D}(G)$. The map σ is defined by $\sigma f(h, g) := f(hg) \cdot \phi(g)$ for some function $\phi \in \mathcal{E}(G)$. This defines a map to $\mathcal{D}(H \times G)$ if $\operatorname{supp}\phi \cap H \cdot L$ is compact for all compact $L \subseteq G$. It is a section for μ if and only if $\int_H \phi(h^{-1}g)\, d_H h = 1$ for all $g \in G$. Functions ϕ with these properties clearly exist. □

Theorem 4.13. *Let $H \subseteq G$ be a closed subgroup. The functors Ic_H^G and S_H^G preserve bornological extensions, locally linearly split extensions, linearly split extensions and injectivity of morphisms. They commute with arbitrary direct limits. They map relatively projective objects to relatively projective objects. In particular, all this applies to the smoothening functor.*

The functors I_H^G and R_H^G preserve linearly split extensions and injectivity of morphisms. They commute with arbitrary inverse limits. They map relatively injective objects to relatively injective objects. In particular, all this applies to the roughening functor.

Proof. For the exactness assertions we can forget the module structure on $\mathrm{Ic}_H^G(V)$ and $\mathrm{S}_G^H(V)$ and view these spaces just as bornological vector spaces. Thus the exactness assertions about S_G^H follow from the corresponding statements about Ic_G^G. Proposition 4.12 implies that the functor Ic_H^G is a retract of the functor $V \mapsto \mathcal{D}(G) \hat{\otimes} V \cong \mathcal{D}(G, V)$. Hence it inherits the properties of the latter functor listed in Proposition 2.10. Since Ic_H^G and S_G^H have right adjoints, they commute with direct limits. Furthermore, the assertion that Ic_H^G preserves relative projectivity is equivalent to the statement that its right adjoint functor R_H^G is exact for linearly split extensions. This follows from Proposition 4.12. It is evident that I_H^G and R_H^G preserve injectivity of morphisms. Since they have left adjoint functors, they commute with inverse limits. Since their left adjoints are exact for linearly split extensions, they preserve relatively injective objects. □

Theorem 4.14. *Let $K \rightarrowtail E \twoheadrightarrow Q$ be a bornological extension in $\widehat{\mathbf{M}}_G$. Then $E \in \mathbf{M}_G$ if and only if both $K \in \mathbf{M}_G$ and $Q \in \mathbf{M}_G$.*

Proof. Let K', E', Q' be the smoothenings of K, E, Q. Consider the diagram

$$\begin{array}{ccccc} K' & \rightarrowtail & E' & \twoheadrightarrow & Q' \\ \downarrow & & \downarrow & & \downarrow \\ K & \rightarrowtail & E & \twoheadrightarrow & Q. \end{array}$$

Both rows are bornological extensions by Theorem 4.13. If K and Q are essential, then the vertical arrows $K' \to K$ and $Q' \to Q$ are bornological isomorphisms. This implies that the middle arrow is a bornological isomorphism by the Five Lemma. The validity of the Five Lemma for bornological vector spaces can be proved directly. It also follows easily from the observation that the category of bornological vector spaces with the class of bornological extensions is an exact category in the sense of Daniel Quillen (see [16, 17]). Hence E is essential if both K and Q are essential. Conversely, if E is essential, then the module action $\mathcal{D}(G) \hat{\otimes} Q \to Q$ is a bornological quotient map, so that Q is essential. Another application of the Five Lemma shows that K is essential as well. □

We have seen in Section 3.2 that the class of smooth representations of G is hereditary for subrepresentations and quotient representations, but not for extensions in general. We have to assume the representation on E to be continuous. Then we can use Theorem 4.14 to obtain the smoothness of E.

Theorem 4.15. *The category $\mathbf{M}_G \cong \mathbf{R}_G$ has enough relatively projective and injective objects.*

The functor $\mathrm{Ind}_H^G : \mathbf{R}_H \to \mathbf{R}_G$ is exact for linearly split extensions. It preserves monomorphisms and relatively injective objects. It commutes with inverse limits in these subcategories (they differ from those in the larger categories $\widehat{\mathbf{R}}_G$ or $\widehat{\mathbf{M}}_G$!).

The functors c-Ind_H^G and Res_G^H are exact for any class of extensions and preserve monomorphisms and relatively projective objects. They commute with direct limits.

Proof. The exactness assertions about Res_G^H are trivial. The exactness properties of $\mathrm{Ind}_H^G \cong \mathrm{S} \circ \mathrm{I}_H^G$ follow immediately from those of S and I_H^G. Since Res_H^G and Ind_H^G are adjoint, the first preserves direct and the latter preserves inverse limits. The exactness properties imply that Ind_H^G and Res_G^H preserve relatively injective and projective objects, respectively. The assertions about c-Ind_H^G follow immediately from the corresponding properties of Ic_H^G and Theorem 4.10. For the trivial group E, linearly split extensions are already direct sum extensions. Thus any object is relatively injective and projective. By Theorem 4.13 we obtain that $\mathrm{Ic}_E^G(V) = \mathcal{D}(G, V)$ is relatively projective and $\mathrm{Ind}_E^G(V) = \mathrm{S}\mathcal{E}(G, V)$ is relatively injective. If V is an arbitrary smooth representation, then we have a linearly split surjection $\mathcal{D}(G, V) \to V$ by Proposition 4.7 and a linearly split injection $V \to \mathrm{S}\mathcal{E}(G, V)$. □

Thus we can derive functors on the category of smooth representations using relatively projective and injective resolutions. Let us write $\mathbb{L}_* F$ and $\mathbb{R}^* F$, $* \in \mathbb{N}$, for the left and right derived functors of a functor F from $\mathbf{R}_G$ to some additive category. The left derived functors of $V \hat{\otimes}_{\mathcal{D}(G)} \sqcup$ are denoted $\mathrm{Tor}_*^G(V, W)$, the right derived functors of $\mathrm{Hom}_G(V, \sqcup)$ are denoted $\mathrm{Ext}_G^*(V, W)$. If we take V to be the trivial representation on $\mathbb{C}$, we obtain group homology and cohomology, denoted $\mathrm{H}_*(G, V)$ and $\mathrm{H}^*(G, V)$, respectively.

The general machinery of derived functors yields the following results. Since the compact induction functor is exact and preserves relatively projective objects, we have $\mathbb{L}_*(F \circ \mathrm{Ic}_H^G) = (\mathbb{L}_* F) \circ \mathrm{Ic}_H^G$. Since the induction functor Ind_H^G is exact and preserves relatively injective objects, we have $\mathbb{R}^*(F \circ \mathrm{Ind}_H^G) = (\mathbb{R}^* F) \circ \mathrm{Ind}_H^G$. Therefore, the adjointness of restriction and induction and (21) imply

$$\mathrm{Ext}_G^*\big(V, \mathrm{Ind}_H^G(W)\big) \cong \mathrm{Ext}_H^*\big(\mathrm{Res}_G^H V, W\big), \tag{25}$$

$$\mathrm{Tor}_*^G\big(V, \text{c-Ind}_H^G(\mu_{G:H} \cdot W)\big) \cong \mathrm{Tor}_*^H\big(\mathrm{Res}_G^H V, W\big), \tag{26}$$

$$\mathrm{H}^*\big(G, \mathrm{Ind}_H^G(W)\big) \cong \mathrm{H}^*(H, W), \tag{27}$$

$$\mathrm{H}_*\big(G, \text{c-Ind}_H^G(\mu_{G:H} \cdot W)\big) \cong \mathrm{H}_*(H, W). \tag{28}$$

The functors $W \mapsto V \hat{\otimes} W$ with diagonal action and $W \mapsto \mathrm{Hom}(V, W)$ with conjugation action are evidently exact for linearly split extensions. Since they are adjoint by (17), the first preserves relative projectivity and the second preserves relative injectivity. Reasoning as above (18) and (19) imply

$$\mathrm{Ext}_G^*(V, W) \cong \mathrm{Ext}_G^*\big(\mathbb{C}(1), \mathrm{S}\,\mathrm{Hom}(V, W)\big) = \mathrm{H}^*\big(G, \mathrm{S}\,\mathrm{Hom}(V, W)\big), \tag{29}$$

$$\mathrm{Tor}_*^G(V, W) \cong \mathrm{Tor}_*^G\big(\mathbb{C}(1), V \hat{\otimes} W\big) = \mathrm{H}_*(G, V \hat{\otimes} W). \tag{30}$$

That is, group homology and cohomology already determine the bivariant homology and cohomology theories.

4.4. *The Gårding subspace*

The smoothening for modules is closely related to the Gårding subspace. Let V be a continuous representation of a locally compact group on a bornological vector space. The *Gårding subspace* of V is defined as the linear subspace spanned by $\int \pi(f)(v)$ with $f \in \mathcal{D}(G)$, $v \in V$. This is the image of the uncompleted tensor product $\mathcal{D}(G) \otimes V$ in V. In contrast, $\mathrm{S}(V)$ is the image of the completed tensor product $\mathcal{D}(G) \hat{\otimes} V$. It seems that everything that can be done with the Gårding subspace can also be done with $\mathcal{D}(G) \hat{\otimes}_{\mathcal{D}(G)} V$. However, it is actually true that the Gårding subspace is always equal to $\mathrm{S}(V)$. This is proved by Jacques Dixmier and Paul Malliavin in [5] for Lie group representations on Fréchet spaces. The same argument actually works in much greater generality:

Theorem 4.16. *Let $\pi : G \to \mathrm{Aut}(V)$ be a continuous representation of a locally compact group G on a bornological vector space V. The Gårding subspace of V is equal to $\mathrm{S}(V)$. Especially, any element of $\mathcal{D}(G)$ is a finite linear combination of products $f_1 * f_2$ with $f_1, f_2 \in \mathcal{D}(G)$.*

Proof. We may assume that the representation V is already smooth because we only make the problem more difficult if we shrink V to $\mathrm{S}(V)$. Any $v \in V$ already belongs to V^k for some smooth compact subgroup $k \subseteq G$. We can replace the representation of G on V by the smooth representation of the Lie group $N_G(k)/k$ on V^k. Thus we may assume G to be a Lie group without loss of generality. The class of smooth representations for which the theorem holds is evidently closed under inductive limits and under quotients. If V is a smooth representation, then it is a quotient of the left regular representation on $\mathcal{D}(G, V)$. The latter is the inductive limit of the left regular representations on $\mathcal{D}(G, V_T)$ for the small complete disks $T \subseteq V$. Hence it suffices to prove the assertion for the left regular representation on $\mathcal{D}(G, V_T)$ for a Banach space V_T. This case can be dealt with by literally the same argument that Jacques Dixmier and Paul Malliavin use in [5] to prove that the Gårding subspace of $\mathcal{D}(G)$ is $\mathcal{D}(G)$. □

5. The center of the category of smooth representations

Definition 5.1. Let A be a bornological algebra with the property that $A \cdot A$ spans a dense subspace of A.

Let $\mathcal{M}_\mathrm{l}(A)$ and $\mathcal{M}_\mathrm{r}(A)^{\mathrm{op}}$ be the algebras of bounded right and left module homomorphisms $A \to A$, equipped with the equibounded bornology. These are the *left and right multiplier algebras* of A. By convention, the multiplication in $\mathcal{M}_\mathrm{r}(A)$ is the opposite of the composition of operators. The (*two-sided*) *multiplier algebra* $\mathcal{M}(A)$ of A is the algebra of pairs (l, r) of a left and a right multiplier such that $a \cdot (l \cdot b) = (a \cdot r) \cdot b$ for all $a, b \in A$.

All three multiplier algebras are unital bornological algebras and there are obvious bounded algebra homomorphisms from A into them. We claim that A is a bornological unital $\mathcal{M}_{\mathrm{l}}(A)$-$\mathcal{M}_{\mathrm{r}}(A)$-bimodule. The only point that is not obvious is that $(l \cdot a) \cdot r = l \cdot (a \cdot r)$ for all $a \in A$, $l \in \mathcal{M}_{\mathrm{l}}(A)$, $r \in \mathcal{M}_{\mathrm{r}}(A)$. If $a = bc$ with $b, c \in A$, then $(l \cdot bc) \cdot r = (lb) \cdot (cr) = l \cdot (bc \cdot r)$. The claim follows because the linear span of elements of the form bc is dense in A.

We denote the center of an algebra A by $\mathrm{Z}(A)$. A left multiplier l of A is called *central* if $a \cdot l \cdot b = l \cdot a \cdot b$ for all $a, b \in A$. That is, the pair (l, l) is a two-sided multiplier of A. Since we know that left and right multipliers commute with each other, it follows that l commutes with any left or right multiplier on A. Thus l belongs to the centers of all three multiplier algebras. Conversely, if l is central, say, in $\mathcal{M}_{\mathrm{l}}(A)$, then it is a central multiplier in the above sense because $A \subseteq \mathcal{M}_{\mathrm{l}}(A)$. As a result, the multiplier algebras all have the same center, which consists exactly of the central multipliers.

Definition 5.2. The *center* $\mathrm{Z}(\mathcal{C})$ of an additive category $\mathcal{C}$ is the ring of natural transformations from the identity functor $\mathrm{id}: \mathcal{C} \to \mathcal{C}$ to itself.

Equivalently, an element of $\mathrm{Z}(\mathcal{C})$ is a family of morphisms $\gamma_X : X \to X$ for each object X of $\mathcal{C}$ such that $f \circ \gamma_X = \gamma_Y \circ f$ for any morphism $f : X \to Y$ in $\mathcal{C}$. The center of the category of smooth representations of a totally disconnected group on vector spaces is studied by Joseph Bernstein in [1] and plays a crucial role in the representation theory of reductive groups over non-Archimedean local fields.

Lemma 5.3. *Let A be a bornological algebra with an approximate identity. Suppose that $A \hat{\otimes}_A A \cong A$. Then the center of the category of essential A-modules is naturally isomorphic to the algebra of central multipliers of A.*

Proof. Let $\mathcal{C}$ be the category of essential bornological left A-modules. The center of $\mathcal{C}$ maps into the center of the endomorphism ring of A because $A \in \mathcal{C}$. By definition, this endomorphism ring is $\mathcal{M}_{\mathrm{r}}(A)^{\mathrm{op}}$. Hence its center is the algebra of central multipliers. Thus we obtain a homomorphism $\alpha : \mathrm{Z}(\mathcal{C}) \to \mathrm{Z}\mathcal{M}(A)$. We have to check that this map is bijective.

For injectivity suppose that $\Phi \in \mathrm{Z}(\mathcal{C})$ vanishes on A. Let $V \in \mathcal{C}$ and $v \in V$. Then the map $a \mapsto av$ is a morphism $A \to V$ in $\mathcal{C}$. Hence $\Phi_V(av) = \Phi_A(a)v = 0$. Since elements of the form av generate V, we get $\Phi_V = 0$. Thus α is injective. For surjectivity let l be a central multiplier. Since A is a bimodule over $\mathcal{M}_{\mathrm{l}}(A)$ and A, there is a canonical $\mathcal{M}_{\mathrm{l}}(A)$-module structure on $A \hat{\otimes}_A V$, that is, on any essential module. Thus l acts in a canonical way on any $V \in \mathcal{C}$. Centrality implies that l acts by left module homomorphisms. Thus we obtain an element of $\mathrm{Z}(\mathcal{C})$. □

The center of the category of all modules over A is equal to the center of A^+ because modules over A are the same as essential modules over A^+. Hence we may get a much smaller center than for essential modules.

Theorem 5.4. *Let G be a locally compact group. Then the center of the category of smooth representations of G is naturally isomorphic to $\mathrm{Z}\mathcal{M}(\mathcal{D}(G))$, the algebra of central multipliers of $\mathcal{D}(G)$.*

Proof. Theorem 4.8 asserts that $\mathbf{R}_G$ is isomorphic to $\mathbf{M}_G$ and hence has an isomorphic center. We know that $\mathcal{D}(G)$ satisfies the hypotheses of Lemma 5.3. Hence $\mathrm{Z}(\mathbf{R}_G) \cong \mathrm{Z}\mathcal{M}(\mathcal{D}(G))$. □

Lemma 5.5. *A left multiplier L of $\mathcal{D}(G)$ is of the form $f \mapsto D * f$ for a uniquely determined distribution $D \in \mathcal{D}'(G)$. A right multiplier is of the form $f \mapsto f * D$ for a uniquely determined distribution $D \in \mathcal{D}'(G)$. If a pair (D_1, D_2) of distributions gives an element of $\mathcal{M}(A)$, then $D_1 = D_2$. Thus $\mathcal{M}(A)$ is the intersection of $\mathcal{M}_{\mathrm{l}}(A)$ and $\mathcal{M}_{\mathrm{r}}(A)$ inside $\mathcal{D}'(G)$.*

Proof. Let $L \in \mathcal{M}_{\mathrm{l}}(\mathcal{D}(G))$. Then we define a distribution $D_L \in \mathcal{D}'(G)$ by $D_L(f) := L(f)(1_G)$. We view $\mathcal{D}(G)$ as an essential right module over $\mathcal{D}(G)$ and L as a bounded module homomorphism. The right module structure on $\mathcal{D}(G)$ is the integrated form of the representation $\mu_G \cdot \rho$. Theorem 4.8 yields that L is equivariant with respect to this representation of G. A straightforward computation now shows that $Lf = D_L * f$ for all $f \in \mathcal{D}(G)$. If $D * f = 0$ for all $f \in \mathcal{D}(G)$, then $D * f(1) = 0$ for all f and hence $D = 0$. Thus the distribution and the left multiplier $D * \sqcup$ determine each other uniquely. The antipode on $\mathcal{D}(G)$ extends to an algebra isomorphism between $\mathcal{M}_{\mathrm{l}}(\mathcal{D}(G))$ and $\mathcal{M}_{\mathrm{r}}(\mathcal{D}(G))$. Hence the description of left multipliers above yields a description of right multipliers. If the pair (D_1, D_2) determines a two-sided multiplier, then $(a * D_2) * b = a * (D_1 * b)$ for all $a, b \in \mathcal{D}(G)$. Thus the right multiplier associated to the distribution $(D_2 - D_1) * b$ vanishes for all b. This implies $(D_2 - D_1) * b = 0$. Since b is arbitrary, we obtain $D_2 = D_1$. □

It remains to identify the distributions on G that give rise to left, right and two-sided multipliers. Let I be a fundamental system of smooth compact subgroups of G. For $k \in I$ let μ_k be the normalized Haar measure on k, viewed as a distribution on G. Thus the convolution with μ_k on the left and right averages a function over left or right k-cosets.

Proposition 5.6. *A distribution $D \in \mathcal{D}'(G)$ is a left multiplier of $\mathcal{D}(G)$ if and only if $D * \mu_k \in \mathcal{E}'(G)$ for all $k \in I$ and a right multiplier if and only if $\mu_k * D \in \mathcal{E}'(G)$ for all $k \in I$. There are bornological isomorphisms*

$$\mathcal{M}_{\mathrm{l}}\big(\mathcal{D}(G)\big) \cong \varprojlim_{k \in I} \mathcal{E}'(G/k) \cong \Big(\varinjlim_{k \in I} \mathcal{E}(G/k)\Big)';$$

$$\mathcal{M}_{\mathrm{r}}\big(\mathcal{D}(G)\big) \cong \varprojlim_{k \in I} \mathcal{E}'(k\backslash G) \cong \Big(\varinjlim_{k \in I} \mathcal{E}(k\backslash G)\Big)'.$$

Proof. We only prove the isomorphisms for $\mathcal{M}_{\mathrm{l}}(\mathcal{D}(G))$. The structure maps in the projective system $\mathcal{E}'(G/k)$ are right convolution with μ_k. Recall that $\mathcal{D}(G) = \varinjlim \mathcal{D}(k\backslash G)$ and that left convolution with μ_k is a projection onto $\mathcal{D}(k\backslash G)$. Thus $D \in \mathcal{M}_{\mathrm{l}}(\mathcal{D}(G))$ if and only if left convolution with $D * \mu_k$ is a bounded map from $\mathcal{D}(k\backslash G)$ to $\mathcal{D}(G)$. Clearly, this is the case if $D * \mu_k$ has compact support. Conversely, if $D * \mu_k$ does not have compact support, then there exist functions $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(k\backslash G)$ whose support is contained in a fixed compact subset $L \subseteq G$ for which $D * \mu_k * \phi_n$ does not have a common compact support. Multiplying the functions ϕ_n by appropriate scalars we can achieve that $\{\phi_n\}$ is a bounded subset of $\mathcal{D}(k\backslash G)$. By construction, $D * \{\phi_n\}$ is not a bounded subset of $\mathcal{D}(G)$, so that D is not a left multiplier. Thus $D \in \mathcal{M}_{\mathrm{l}}(\mathcal{D}(G))$ if and only if $D * \mu_k$ has compact support for all $k \in I$. An analogous computation for a set $S \subseteq \mathcal{D}'(G)$ of distributions shows that S is bounded in $\mathcal{M}_{\mathrm{l}}(\mathcal{D}(G))$ if and only if $S * \mu_k$ is bounded in $\mathcal{E}'(G/k)$ for all $k \in I$. This proves the first isomorphism. The second one follows from the universal property of direct limits. □

Corollary 5.7. *If G is a projective limit of Lie groups, then*

$$\mathcal{M}_{\mathrm{l}}\big(\mathcal{D}(G)\big) = \mathcal{M}_{\mathrm{r}}\big(\mathcal{D}(G)\big) = \mathcal{M}\big(\mathcal{D}(G)\big).$$

If G is a Lie group then all three multiplier algebras are equal to $\mathcal{E}'(G)$.

The spaces $\mathcal{E}(G/k)$ for $k \in I$ are nuclear Fréchet spaces and hence reflexive. We can rewrite the inductive limit $\varinjlim_{k\in I} \mathcal{E}(G/k)$ as a direct sum. If G is metrizable, this is quite easy: choose I to be a sequence and notice that $\mathcal{E}(G/k_n)$ is a retract of $\mathcal{E}(G/k_{n+1})$ for any $n \in \mathbb{N}$. If G is not metrizable, the assertion is still correct, but the proof is more complicated. Therefore, $\varinjlim \mathcal{E}(G/k)$ is reflexive, so that $\mathcal{M}_{\mathrm{l}}(\mathcal{D}(G))' \cong \varinjlim \mathcal{E}(G/k)$. Furthermore, if G is countable at infinity, then Proposition 3.17 shows that $\varinjlim_{k\in I} \mathcal{E}(G/k)$ is the smoothening of the right regular representation on $\mathcal{E}(G)$.

Proposition 5.8. *Let $D \in \mathcal{D}'(G)$. Then D is a central multiplier of $\mathcal{D}(G)$ if and only if $\mu_k * D * \mu_k \in \mathrm{Z}\mathcal{E}'(G//k)$ for all $k \in I$. There is a natural isomorphism of bornological algebras*

$$\mathrm{Z}\mathcal{M}\big(\mathcal{D}(G)\big) \cong \varprojlim \mathrm{Z}\mathcal{E}'(G//k).$$

Proof. If D is a central multiplier of $\mathcal{D}(G)$, then $\mu_k * D * \mu_k$ belongs to the center of $\mu_k \mathcal{M}(\mathcal{E}(G))\mu_k$. Proposition 5.6 yields an isomorphism of bornological algebras $\mu_k \mathcal{M}(\mathcal{E}(G))\mu_k = \mathcal{E}'(G//k)$. Hence we have a bounded homomorphism $\mathrm{Z}\mathcal{M}(\mathcal{D}(G)) \to \varprojlim \mathrm{Z}\mathcal{E}'(G//k)$.

Suppose conversely that $\mu_k D \mu_k$ be a central element of $\mathcal{E}'(G//k)$ for all $k \in I$. For any $j \in I$, $j \subseteq k$, $f \in \mathcal{D}(G//k)$, we have

$$\mu_j * D * f = \mu_j * D * \mu_j * f * \mu_k = f * \mu_j * D * \mu_j * \mu_k = f * \mu_k * D * \mu_k.$$

Since this is independent of j, we obtain $D * f = f * \mu_k * D * \mu_k$. In particular, D is a left multiplier. A similar computation for $f * D$ shows $f * D = D * f$ because $\mu_k D \mu_k$ commutes with f. Hence D is central, so that we obtain an isomorphism $\mathrm{Z}\mathcal{M}(\mathcal{D}(G)) \cong \varprojlim \mathrm{Z}\mathcal{E}'(G//k)$. It is easy to check that it is bornological. □

If G is totally disconnected, then the spaces $G//k$ are all discrete, so that $\mathcal{E}'(G//k) = \mathcal{D}(G//k)$. This special case is covered in [1]. Now let G be a connected Lie group. If

$[D, X] = 0$ for all $X \in \mathfrak{g}$, then $[D, \delta_g] = 0$ for all $g \in G$ and hence D is central. Thus a distribution is central if and only if it commutes with $\mathfrak{g}$. In particular, the center of the universal enveloping algebra of G is contained in the center of $\mathcal{E}'(G)$. The latter can be bigger than $Z\mathcal{U}(G)$. This happens, for instance, if G has non-trivial center or if G is compact. However, there are also many Lie groups for which we have $Z\mathcal{U}(G) = Z\mathcal{E}'(G)$, that is, any central distribution is supported at 1. The following proposition only gives one class of examples.

Proposition 5.9. *Let G be a connected complex Lie group with trivial center. Then $\mathrm{Z}\mathcal{M}(\mathcal{D}(G))$ is equal to the center of the universal enveloping algebra.*

Proof. Since G has trivial center, the adjoint representation of G on its Lie algebra $\mathfrak{g}$ is faithful, so that $G \subseteq \mathrm{Gl}(\mathfrak{g})$. Let $D \in \mathrm{Z}\mathcal{E}'(G)$ and $y \in \operatorname{supp} D$. Since $\operatorname{supp} D$ is compact and conjugation invariant, the holomorphic function

$$\mathbb{C} \ni s \mapsto \exp(sX)\, y \exp(-sX) \in \mathrm{Gl}(\mathfrak{g})$$

is bounded for any $X \in \mathfrak{g}$. Liouville's Theorem yields that it is constant, that is, $[X, y] = 0$. This implies $\operatorname{supp} D = \{1\}$ because G has trivial center. Now use the identification of distributions supported at 1 with the universal enveloping algebra. Since G is connected, a distribution is central if and only if it commutes with $\mathfrak{g}$. □

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