MATHEMATICS

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# On the differential operators on the quasi-affine variety G/N

Communicated by T. A. Springer at the meeting of November 26, 1977

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Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a triangular decomposition of a semi-simple Lie algebra over an algebraically closed field of characteristic zero. Let $U, = U(\mathfrak{g})$, be the universal enveloping algebra of $\mathfrak{g}$, $(U_n)_n$ its natural filtration, and $T$ the principal anti-automorphism of $U$, defined by $X^T = -X$ for $X \in \mathfrak{g}$. Let $\lambda$ and $\varrho$ be the left and right regular representations of $U$ respectively, defined by $\lambda(u)v = uv$, $\varrho(u)v = vu^T$ for $u, v \in U$. On the quotient $M = U/U\mathfrak{n}_+$, the generic Verma module, $\lambda$ and $\varrho$ induce representations of $U$ and $U(\mathfrak{h} \oplus \mathfrak{n}_+)$ respectively, also denoted by $\lambda$ and $\varrho$.

The linear dual $U^*$ of $U$ is made into a commutative associative algebra with unit element by means of the dual of the homomorphism $\Delta : U \to U \otimes U$, defined by

$$\Delta(X_1 \dots X_{n+1}) = \sum_{\substack{i_1 < \dots < i_j \\ 0 \leqslant j \leqslant n+1}} X_{i_1} \dots X_{i_j} \otimes X_1 \dots \hat{X}_{i_1} \dots \hat{X}_{i_j} \dots X_{n+1},$$

for $X_1, \dots, X_{n+1} \in \mathfrak{g}$.

(In the future we shall omit the indication of the summation set, in similar expressions.) The dual $M^*$ of $M$, naturally identified with a subspace of $U^*$, is a subalgebra of $U^*$.

Generally, for a dominant integral element $\delta$ of the linear dual $\mathfrak{h}^*$ of $\mathfrak{h}$, $E^\delta$ will denote a fixed irreducible $U$-module with highest weight $\delta$, and for any $U$-module $W$ we shall denote by $W^\delta$ the sum of its submodules isomorphic to $E^\delta$.

Now let $\mathscr{E} = \sum_\delta (U^*)^\delta$, where $U^*$ is considered as a $U$-module by means of the contragredient representation $\lambda^c$ of $\lambda$; so $\mathscr{E}$ is the subalgebra of $U^*$ consisting of the $U$-finite elements. The use of $\varrho^c$ instead of $\lambda^c$ leads to the same subalgebra $\mathscr{E}$; the restrictions of $\lambda^c$ and $\varrho^c$ to $\mathscr{E}$ will be denoted by $\tilde{\lambda}$ and $\tilde{\varrho}$ respectively. Let $\mathscr{F} = \mathscr{E} \cap M^*$. Then $\mathscr{F}$ is the subalgebra of $\mathscr{E}$ consisting of the elements annihilated by $\tilde{\varrho}(\mathfrak{n}_+)$, $\mathscr{F}^\delta$ is simple for every dominant integral $\delta$, and $\mathscr{F}^{\delta_1} \cdot \mathscr{F}^{\delta_2} = \mathscr{F}^{\delta_1+\delta_2}$. For these, and related, matters, see e.g. G. Hochschild [6], N. Conze [2], J. Dixmier [3]. Moreover, $\mathscr{E}$ can be considered as the algebra of regular functions on $G$, where $G$ is a simply connected algebraic group with Lie algebra $\mathfrak{g}$, and $\mathscr{F}$ as the algebra of regular functions of the quasi-affine algebraic variety $G/N$, where $N$ is the subgroup of $G$ that corresponds to $\mathfrak{n}_+$. The maximal ideal $\mathfrak{m}$ of $\mathscr{F}$ consisting of the elements vanishing in $m_0$, with $m_0 = 1 + U\mathfrak{n}_+$, corresponds to the point $N$ of $G/N$, and is a simple point of the affine variety with affine algebra $\mathscr{F}$.

Generally, if $\alpha$ and $\beta$ are representations of $U$ in $V$ and $W$ respectively, then ad $(\alpha, \beta)$ will denote the representation of $U$ in Hom $(V, W)$ defined by:

$$\text{ad}\,(\alpha, \beta)(X) = \underset{A \in \text{Hom}(V,W)}{\curlyvee} \beta(X) \circ A - A \circ \alpha(X), \text{ for } X \in \mathfrak{g}.$$

(Here "$\curlyvee$" denotes Freudenthal's function symbol, see e.g. [4], p. xviii.) Moreover, $L(V, W)$ will denote the subspace of Hom $(V, W)$ consisting of the elements $U$-finite under ad $(\alpha, \beta)$. For the definition of $L(M, M)$, $L(U, M)$ and $L(U, U)$, we take for $\alpha$ and $\beta$ the appropriate representation $\lambda$.

Now let $H = U(\mathfrak{h})$. Then the representation $\gamma$ of $U \otimes H$ in $M$ defined by $\gamma(u \otimes v) = \lambda(u)\varrho(v)$ for $u \in U$, $v \in H$, is an intertwining operator from the adjoint representation of $U \otimes H$ in $U \otimes H$ to the representation ad $(\gamma, \gamma)$; hence the image of $\gamma$ is contained in $L(M, M)_0$, the subspace of elements of $L(M, M)$ annihilated by ad $(\varrho, \varrho)$ $(\mathfrak{h})$. The representation $\gamma$ is compatible with the identification of the centre $Z(\mathfrak{g})$ of $U$ with a subalgebra of $1 \otimes H$ via the Harish-Chandra mapping $\zeta$, defined as the projection of $U$ to $H$ along $\mathfrak{n}_- U + U\mathfrak{n}_+$, using $\lambda(z)m_0 = \lambda(\zeta(z))m_0$ for $z \in Z(\mathfrak{g})$: if $m \in M$, and $u \in U$ such that $m = \lambda(u)m_0$, then

$$\lambda(z)m = \lambda(z)\lambda(u)m_0 = \lambda(u)\lambda(z)m_0 = \lambda(u)\lambda(\zeta(z))m_0 = \varrho(\zeta(z))m.$$

Hence $\gamma$ factors to a homomorphism

$$\sigma\colon U \otimes_{Z(\mathfrak{g})} H \to L(M, M)_0,$$

with an obvious definition for $U \otimes_{Z(\mathfrak{g})} H$.

We shall prove:

THEOREM 1. $\sigma$ is an isomorphism.

First we show that this is equivalent to a conjecture of I. M. Gel'fand and A. A. Kirillov, see [5]. This conjecture has been proved by N. N. Shapovalov [9], as we belatedly discovered. Though the underlying ideas are essentially the same, we believe that our method of approach, based on [2], offers more insight.

For $n \in \mathbf{Z}$, $n \geqslant 0$, let $\mathscr{R}^n$ be the subspace of the linear endomorphism space End $\mathscr{F}$ of $\mathscr{F}$ consisting of the regular differential operators of order at most $n$ on $G/N$; that is, the element $D$ of End $\mathscr{F}$ belongs to $\mathscr{R}^n$ if and only if

$$\sum (-1)^j f_{i_1} \dots f_{i_j} D(f_1 \dots \hat{f}_{i_1} \dots \hat{f}_{i_j} \dots f_{n+1}) = 0 \text{ for } f_1, \dots, f_{n+1} \in \mathscr{F}.$$

Put $\mathscr{R} = \bigcup_n \mathscr{R}^n$.

For $X \in \mathfrak{g}$, $Y \in \mathfrak{h}$, the mappings $\tilde{\lambda}(X)$ and $\tilde{\varrho}(Y)$, as acting on $\mathscr{F}$, are commuting derivations of $\mathscr{F}$; hence the representation $\tilde{\gamma}$ of $U \otimes H$ in $\mathscr{F}$ maps $U \otimes H$ into $\mathscr{R}_0$, the subspace of elements of $\mathscr{R}$ annihilated by ad $(\tilde{\varrho}, \tilde{\varrho})(\mathfrak{h})$. The conjecture was that $\tilde{\gamma}$ factors to an isomorphism from $U \otimes_{Z(\mathfrak{g})} H$ to $\mathscr{R}_0$. We shall show that transposition yields a natural anti-isomorphism from $L(M, M)_0$ to $\mathscr{R}_0$, which, together with the principal anti-automorphism of $U \otimes H$, transforms $\gamma$ into $\tilde{\gamma}$. The same transposition will yield an anti-isomorphism from $L(M, M)$ to $\mathscr{R}$.

Make $M$ into a topological vector space (with discrete scalar field) by taking the set of its $U$-submodules with finite codimension as a neighbourhood base of 0. Then $\mathscr{F}$ is the continuous dual of $M$. And an element $A$ of End $M$ is continuous if and only if its transpose $A^t$ preserves $\mathscr{F}$; then $\tilde{A}$ will denote the restriction of $A^t$ to $\mathscr{F}$. The elements of $L(M, M)$ are continuous. We make $\mathscr{F}$ also into a topological vector space by means of the $\mathfrak{m}$-adic topology, for which the powers $\mathfrak{m}^s$ of $\mathfrak{m}$ form a neighbourhood base of 0.

It follows immediately from the definitions, that $\mathfrak{m}^{s+1}$ is contained in $\{f \in \mathscr{F} | \langle f, m \rangle = 0 \text{ for } m \in M_s\}$, where $M_s$ is the natural image of $U_s$ in $M$. However, since the transcendence degree of $\mathscr{F}$ equals dim $(\mathfrak{h} + \mathfrak{n}_-)$ (being the dimension of $G/N$) and since $\mathfrak{m}$ is a simple point of the affine variety with affine algebra $\mathscr{F}$, $\dim M_s = \dim \mathscr{F}/\mathfrak{m}^{s+1}$; moreover, since the topological vector space $M$ is separated (see e.g. [1], Lemme 9.3), $M$ naturally identifies with the continuous dual of $\mathscr{F}$. For $D \in$ End $\mathscr{F}$, $D$ continuous, we write $\tilde{D}$ for its dual in End $M$. Then we obtain an anti-isomorphism between the spaces of $\mathfrak{g}$-finite continuous endomorphisms of $\mathscr{F}$ and $M$, which is also an equivalence between the restrictions of the representations ad $(\tilde{\lambda}, \tilde{\lambda})$ and ad $(\lambda, \lambda)$ to them. Note that $D(\mathfrak{m}^{s+n}) \subseteq \mathfrak{m}^s$ for $D \in \mathscr{R}^n$, so that the elements of $\mathscr{R}$ are continuous. It was shown in [2] (see Proposition 9.9), that the elements of $\mathscr{R}$ are $\mathfrak{g}$-finite, in other terms, that $\tilde{\mathscr{R}} \subseteq L(M, M)$, and remarked (Remarque 9.10) that even equality holds true; we proceed to prove this inverse inclusion.

LEMMA 1. Let $A \in L(M, M)$. Take $n$ such that $B(m_0) \in M_n$ for all elements $B$ of some finite-dimensional $U$-invariant subspace $S$ of $L(M, M)$ containing $A$. Then $\tilde{A} \in \mathscr{R}^n$.

PROOF. For $u \in U_n$ one has $\tilde{\lambda}(u) \in \mathscr{R}^n$; thus

$$\sum (-1)^j f_{i_1} \dots f_{i_j} \tilde{\lambda}(u)(f_1 \dots \hat{f}_{i_1} \dots \hat{f}_{i_j} \dots f_{n+1}) = 0 \text{ for } f_1, \dots, f_{n+1} \in \mathscr{F}.$$

If, moreover, $u$ is taken such that $Bm_0 = \lambda(u^T)m_0$, then

$$\begin{aligned}&\sum (-1)^j \langle f_{i_1} \dots f_{i_j} \tilde{B}(f_1 \dots \hat{f}_{i_1} \dots \hat{f}_{i_j} \dots f_{n+1}), m_0\rangle = \\ &= \sum (-1)^j \langle f_{i_1} \dots f_{i_j}, m_0\rangle\langle f_1 \dots \hat{f}_{i_1} \dots \hat{f}_{i_j} \dots f_{n+1}, Bm_0\rangle \\ &= \sum (-1)^j \langle f_{i_1} \dots f_{i_j}, m_0\rangle\langle \tilde{\lambda}(u)(f_1 \dots \hat{f}_{i_1} \dots \hat{f}_{i_j} \dots f_{n+1}), m_0\rangle = 0.\end{aligned}$$

To prove the lemma, it is sufficient to show the vanishing of this sum with $m_0$ replaced by an arbitrary element $m$ of $M$. Writing $m = \lambda(v)m_0$, with $v \in U$, this follows by induction on the filtration degree of $v$, simultaneously for all elements $B$ of $S$, by means of the identity

$$\tilde{\lambda}(X)(f_1 \dots f_i \tilde{B}(f_{i+1} \dots f_{n+1})) = \textstyle\sum_{j=1}^{i} f_1 \dots \tilde{\lambda}(X)(f_j) \dots f_i \tilde{B}(f_{i+1} \dots f_{n+1}) + \\ + \textstyle\sum_{j=i+1}^{n+1} f_1 \dots f_i \tilde{B}(f_{i+1} \dots \tilde{\lambda}(X)(f_j) \dots f_{n+1}) - f_1 \dots f_i(\text{ad}\,(\lambda, \lambda)(X)(B))^{\sim} \\ (f_{i+1} \dots f_{n+1}), \text{ for } X \in \mathfrak{g},\ f_1, \dots, f_{n+1} \in \mathscr{F}.$$

Hence:

THEOREM 2. $L(M, M) \ni A \mapsto \tilde{A} \in \mathscr{R}$ is an anti-isomorphism.

Obviously, under this isomorphism, $L(M, M)_0$ corresponds to $\mathscr{R}_0$.

In exactly the same way one makes $U$ and $\mathscr{E}$ into topological vector spaces, and transposition provides an anti-isomorphism between $L(U, U)$ and the space $\mathscr{S}$ of regular differential operators on $G$.

Let $\varkappa$ be any representation of $U$ in a linear space $V$, and let $e_\varkappa$: $U \otimes V \to V$ be the corresponding evaluation mapping, defined by $e_\varkappa(u \otimes v) = \varkappa(u)v$.

Let $\xi_\varkappa$ be the self-mapping of the space Hom $(U, V)$ of linear mappings from $U$ to $V$ defined by $\xi_\varkappa(A) = e_\varkappa \circ (\iota \otimes AT) \circ \Delta$, where $\iota$ stands for the identity mapping of $U$. Thus, for $X_1, \dots, X_{n+1} \in \mathfrak{g}$:

$$\xi_\varkappa(A)(X_1 \dots X_{n+1}) = \sum (-1)^{n+1-j} \varkappa(X_{i_1} \dots X_{i_j}) A(X_{n+1} \dots \hat{X}_{i_j} \dots \hat{X}_{i_1} \dots X_1).$$

From this one sees: $\xi_\varkappa(A) = \curlyvee_u (\text{ad}\,(\lambda, \varkappa)(u)(A))(1)$.

Also, by means of the fact that $\mu \circ (\iota \otimes T) \circ \Delta$ is the unit element of $\mathscr{E}$ (where $\mu$ is the multiplication of $U$), i.e.

$$\sum (-1)^{n+1-j} X_{i_1} \dots X_{i_j} X_{n+1} \dots \hat{X}_{i_j} \dots \hat{X}_{i_1} \dots X_1 = 0 \text{ for } n+1 \geqslant 1,$$

one verifies: $\xi_\varkappa(\xi_\varkappa(A)) = A$, for $A \in$ Hom $(U, V)$, that is, $\xi_\varkappa$ is an involutory linear bijection of Hom $(U, V)$ onto itself. The linear subspace of continuous linear mappings from $U$ to $V$, where $V$ is provided with the discrete topology, is naturally identified with $\mathscr{E} \otimes V$; to $f \otimes v$ corresponds $\curlyvee_u f(u)v$, for $f \in \mathscr{E}$, $v \in V$. One also verifies, for $X \in \mathfrak{g}$:

$$\operatorname{ad}(\lambda, \varkappa)(X)\xi_\varkappa(A) = \xi_\varkappa(\operatorname{ad}(\varrho, \tau)(X)A),$$

where $\tau$ is the trivial representation of $U$ in $V$. Hence $\xi_\varkappa$ intertwines $\operatorname{ad}(\lambda, \varkappa)$ and $\operatorname{ad}(\varrho, \tau)$. But $\mathscr{E} \otimes V$ consists of the $\operatorname{ad}(\varrho, \tau)$-finite elements of $\operatorname{Hom}(U, V)$. Hence we have:

LEMMA 2. $\xi_\varkappa$ interchanges $L(U, V)$ and $\mathscr{E} \otimes V$.

Now take $V = M$, $\varkappa = \lambda$. By the natural identification of $\operatorname{Hom}(M, M)$ with a subspace of $\operatorname{Hom}(U, M)$, $L(M, M)$ consists of the elements of $L(U, M)$ annihilated by $\operatorname{ad}(\varrho, \tau)(\mathfrak{n}_+)$; by $\xi_\lambda$ this subspace is mapped onto the subspace $\operatorname{Ex} \mathscr{E}$ of $\mathscr{E} \otimes M$ consisting of the elements annihilated by $\operatorname{ad}(\lambda, \lambda)(\mathfrak{n}_+)$ (in other terms, annihilated by $(\tilde{\lambda} \otimes \lambda)(\mathfrak{n}_+)$). Let $\operatorname{Ex}_0 \mathscr{E} = \xi_\lambda(L(M, M)_0)$; then one readily sees that $\operatorname{Ex}_0 \mathscr{E}$ is the subspace of $\operatorname{Ex} \mathscr{E}$ consisting of the elements annihilated by $(\tilde{\lambda} \otimes \operatorname{ad})(\mathfrak{h})$, where ad is the representation of $H$ in $M$ defined by $\operatorname{ad}(X) = \lambda(X) + \varrho(X)$ for $X \in \mathfrak{h}$. Furthermore, $\xi_\lambda\colon L(M, M)_0 \to \operatorname{Ex}_0 \mathscr{E}$ intertwines the representation $\varrho_H$ of $H$ defined by $\varrho_H(X)(A) = \varrho(X)A$ for $X \in \mathfrak{h}$. Generally, the properties mentioned can be taken as the definition of $\operatorname{Ex} W$, as a subspace of $W \otimes M$, and of $\operatorname{Ex}_0 W$, for any given representation $\mu$ in a linear space $W$ instead of $\tilde{\lambda}$ in $\mathscr{E}$. For more information on such a space $\operatorname{Ex} W$ of extreme vectors, see A. van den Hombergh [7].

We shall now prove Theorem 1 by exhibiting a free $H$-basis of the $H$-module $U \otimes_{Z(\mathfrak{g})} H$ (by means of multiplication in the second tensor factor) which by the $H$-module homomorphism $\xi_\lambda \circ \gamma$ is mapped onto a free $H$-basis of the $H$-module $\operatorname{Ex}_0 \mathscr{E}$ (by means of $\varrho_H$).

Let $K$ be the subspace of $U$ spanned by the powers of the ad-nilpotent elements of $\mathfrak{g}$. Then, as shown by B. Kostant [8], multiplication yields an isomorphism $K \otimes Z(\mathfrak{g}) \to U$, and by considering $K$ as a $U$-module under the adjoint representation, $\dim \operatorname{Hom}_\mathfrak{g}(E^\delta, K)$ equals the dimension of the zero weight space of $E^\delta$, for any dominant integral $\delta$. Fix $\delta$, a basis $(e_1, \ldots, e_n)$ of $E^\delta$, and a basis $(\phi_1, \ldots, \phi_s)$ of $\operatorname{Hom}_\mathfrak{g}(E^\delta, K)$; let $(e_1^*, \ldots, e_n^*)$ be the dual basis of $E^{\delta*}$, and $r_{ki} = \curlyvee_u \langle e_k^*, ue_i\rangle$. One knows that $(r_{ki})_{k,i}$ is a basis of $\mathscr{E}^{\delta*}$. Then $(\phi_j(e_i))_{i,j}$ is a basis of $K^\delta$, and a free $H$-basis of $U^\delta \otimes_{Z(\mathfrak{g})} H$. One easily checks:

$$\xi_\lambda \circ \gamma(\phi_j(e_i)) = \textstyle\sum_k r_{ki} \otimes \gamma(\phi_j(e_k))m_0.$$

Now one needs only to prove:

THEOREM 3. $(\sum_k r_{ki} \otimes \gamma(\phi_j(e_k))m_0)_{i,j}$ is an $H$-basis of $\operatorname{Ex}_0 \mathscr{E}^{\delta*}$.

Because, for each $i$, the span of $\{r_{ki} | k = 1, \ldots, n\}$ is invariant under $\tilde{\lambda}$, and the representation in it is equivalent to the representation in $E^{\delta*}$

(by letting $r_{kt}$ correspond to $e_k^*$ for all $k$), it is sufficient to show the following:

LEMMA 3. $(\sum_k e_k^* \otimes \gamma(\phi_j(e_k))m_0)_j$ is an $H$-basis of $\mathrm{Ex}_0\ E^{\delta *}$.

PROOF. We use the following fact (see [3], Prop. 8.4.2). Let $P$ be projection from $U$ to $U(\mathfrak{n}_-)$ along $U\cdot\mathfrak{h}+U\cdot\mathfrak{n}_+$, and

$$K^{\mathfrak{n}_-}=\{u\in K\,|\,(\mathrm{ad}\ \mathfrak{n}_-)(u)=\{0\}\};$$

then: if $u\in K^{\mathfrak{n}_-}\backslash U_{n-1}$, then $P(u)\notin U_{n-1}$, in other words, $P$ preserves the filtration degree of elements of $K^{\mathfrak{n}_-}$.

We may, and shall, assume that $\mathfrak{n}_- e_1=\{0\}$, and that, if $g_j$ is the filtration degree of $\phi_j(e_1)$ $(j=1, \ldots, s)$, then $g_1\leqslant g_2\leqslant\ldots\leqslant g_s$. Then $\phi_j(e_1)\in K^{\mathfrak{n}_-}$, $j=1, \ldots, s$.

Let $(b_i)_i$ be a Poincaré-Birkhoff-Witt basis of $U(\mathfrak{n}_-)$; it is also a free $H$-basis for the right $H$-module $U(\mathfrak{n}_-+\mathfrak{h})$. We shall use that $M$ is a free cyclic $U(\mathfrak{n}_-+\mathfrak{h})$-module with cyclic vector $m_0$. Let $x\in\mathrm{Ex}_0\ E^{\delta *}$; then $x=\sum_{i,k} e_k^*\otimes\gamma(b_i q_{ki})m_0$, for uniquely determined elements $q_{ki}$ in $H$. Now suppose $p\in H$, $p\neq 0$, and $p_j\in H$, $j=1, \ldots, s$, such that

$$\varrho_H(p)x=\sum_j\sum_k \varrho_H(p_j)(e_k^*\otimes\gamma(\phi_j(e_i))m_0),$$

whence

$$\varrho_H(p)x=\sum_j\sum_k e_k^*\otimes\gamma(\phi_j(e_k)p_j)m_0=\sum_{i,k} e_k^*\otimes\gamma(b_i q_{ki}p)m_0.$$

We want to show: $p|p_j$, $j=1, \ldots, s$. If $\phi_j(e_k)$ is decomposed according to $U=U(\mathfrak{n}_-+\mathfrak{h})\oplus U\cdot\mathfrak{n}_+$, then the second summand annihilates $\varrho(p_j)m_0$; let $\phi_j'(e_k)$ be the first summand. Then $P(\phi_j'(e_k))=P(\phi_j(e_k))$, and

$$\varrho_H(p)x=\sum_j\sum_k e_k^*\otimes\gamma(\phi_j'(e_k)p_j)m_0.$$

Then:

$$\sum_j \phi_j'(e_k)p_j=\sum_i b_i q_{ki}p,\ \ k=1, \ldots, n.$$

Hence, for $k=1$:

$$\sum_j P(\phi_j(e_1))p_j+\sum_j(\phi_j'(e_1)-P(\phi_j(e_1)))p_j=\sum_i b_i q_{ki}p.$$

Let $t$ be the smallest index $j$ such that $g_j=g_s$. Expressing $P(\phi_j(e_1))$ and $\phi_j'(e_1)-P(\phi_j(e_1))$ in the free right $H$-basis $(b_i)_i$ of $U(\mathfrak{n}_-+\mathfrak{h})$, we conclude, by the fact noted above and considering the terms in which $b_i$ has degree $g_s$:

$$p|p_j,\ j=t, \ldots, s.$$

Note that the summands $P(\phi_j(e_1))p_j$, for $j<t$, and all the summands $(\phi_j'(e_1)-P(\phi_j(e_1)))p_j$, do not contribute to these terms. Repeated application of this procedure, first to

$$\varrho_H(p)x-\sum_{j=t}^{s}\varrho_H(p_j)(e_k^*\otimes\gamma(\phi_j(e_k))m_0),$$

yields:

$p | p_j$, all $j$, as desired.

The theorem now follows from the fact that $\dim_{K(\mathfrak{h})} (\mathrm{Ex}_0 E^\delta) \otimes_H K(\mathfrak{h}) = s$, where $K(\mathfrak{h})$ is the quotient field of $H$, see [7], Prop. I.1.9.

REMARK. In [1] a conjecture is put forward, that among other things asserts that Ex $\mathscr{E}$ is a free $H$-module for the representation $\varrho_H$. It may be worthwile to state the full conjecture in our formalism. Starting from Lemma 2, take for $\varkappa$ the representation $\lambda$ of $U$ in $U$. Then we get a linear isomorphism $\xi_\lambda : L(U, U) \to \mathscr{E} \otimes U$, and the natural mappings $L(M, M) \to L(U, M) \leftarrow L(U, U)$ are transformed into the natural mappings Ex $\mathscr{E} \to \mathscr{E} \otimes M \leftarrow \mathscr{E} \otimes U$. By

$$U = U(\mathfrak{h} \oplus \mathfrak{n}_-) \oplus U \cdot \mathfrak{n}_+ = H \oplus U(\mathfrak{h} \oplus \mathfrak{n}_-)\mathfrak{n}_- \oplus U \cdot \mathfrak{n}_+$$

we obtain natural mappings $\mathscr{E} \otimes M \to \mathscr{E} \otimes U(\mathfrak{h} \oplus \mathfrak{n}_-) \to \mathscr{E} \otimes H$. By means of the evaluation of elements of $H$ in half the sum of the negative roots a mapping $\mathscr{E} \otimes H \to \mathscr{E}$ is obtained. Hence a composite mapping

$$\mathrm{Ex}\, \mathscr{E} \to \mathscr{E} \otimes M \to \mathscr{E} \otimes H \to \mathscr{E} \otimes 1 = \mathscr{E}$$

results. The conjecture is: this composition admits a linear lifting $\eta : \mathscr{E} \to \mathrm{Ex}\, \mathscr{E}$ such that:

(i) $\eta(f \circ T) = (f \circ T) \otimes m_0$ for $f \in \mathscr{F}$ (note that generally, for $f \in \mathscr{E}$, $(f \circ T) \otimes 1$ corresponds with the multiplication by $f$, as element of $\mathscr{S}$),

(ii) $\eta$ intertwines the representations $\tilde{\varrho} \otimes \tau$ of $U$ and $\varrho_H$ of $H$,

(iii) the mapping $\mathscr{E} \otimes H \to \mathrm{Ex}\, \mathscr{E}$ defined by $f \otimes v \mapsto \varrho_H(v)\eta(f)$ is an $H$-module isomorphism.

However, a verification of the conjecture appears to be difficult.

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