A zero-free interval for chromatic polynomials

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Abstract


It is proved that, for a wide class of near-triangulations of the plane, the chromatic polynomial has no zeros between 2 and 2.5. Together with a previously known result, this shows that the zero of the chromatic polynomial of the octahedron at 2.546602· · · is the smallest non-integer real zero of any chromatic polynomial of a plane triangulation.

1. Introduction and motivation

A near-triangulation of the plane is a loopless planar multigraph G drawn in the plane in such a way that one face is bounded by a circuit (connected 2-regular subgraph C_k) with k ⩾ 3 edges, and every other face is bounded by a triangle C_3. In diagrams we shall always draw the exceptional face as the outside face, and its bounding circuit will be called the bounding circuit of G. If k = 3 then G is a triangulation, and it can be thought of as a near-triangulation with any one of its faces as the 'exceptional face'. All near-triangulations are 2-connected. A triangulation is 3-connected if and only if it is a simple graph (see Fig. 1).

Let P(G, t) denote the chromatic polynomial of G. It follows from a result of Birkhoff and Lewis [1] that for every plane triangulation G, and hence for every planar graph G, P(G, t) > 0 whenever t ⩾ 5, and they conjectured (inter alia) that this is true whenever t ⩾ 4. The case t = 4 is the four-colour theorem, but the conjecture remains open for 4 < t < 5.

It follows from another result in [1] (see [3], Theorems 2 and 3) that, for every plane near-triangulation G, P(G, t) ≠ 0 whenever t < 2 except for simple zeros at 0 and 1 and a zero at 2, the last being simple if and only if G is a 3-connected triangulation. Now, the chromatic polynomial of the octahedron C_4 + K_2 (which is a 3-connected triangulation, called H_4,4,4 in Fig. 2) is

\[ t(t - 1)(t - 2)(t^3 - 9t^2 + 29t - 32), \]

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Fig. 1. A plane triangulation that is not 3-connected.

Fig. 2.
which has a zero at 2.546602 · · · . By adapting the method used by Birkhoff and Lewis for the case \( t \geq 5 \), I proved in [2] that if \( G \) is a 3-connected triangulation with \( n \) vertices, and one defines a quotient polynomial \( q(G, t) \) by

\[
P(G, t) = (-1)^{n-1}t(t-1)(t-2)q(G, t),
\]

then \( q(G, t) > 0 \) whenever \( 2.5 < t < 2.546602 \ · · · ; \) and I conjectured that this is true whenever \( t < 2.546602 \ · · · \). It follows from Birkhoff and Lewis’s result when \( t \leq 2 \), but it has remained open until now for \( 2 < t < 2.5 \).

The proof in [2] fails when \( t < 2.5 \) for the same reason that Birkhoff and Lewis failed to prove that \( P(G, t) > 0 \) when \( 4 < t < 5 \): it is the same equation that ‘goes wrong’ in each case—the equation for reducing a vertex of degree 5—although it is a different term in the equation that gives trouble. Looked at from this point of view, the conjecture for \( 2 < t < 2.5 \) seems to be of comparable difficulty to that for \( 4 < t < 5 \), which is presumably at least as hard as the four-colour theorem. However, in the present paper we by-pass the difficulty when \( 2 < t < 2.5 \) by approaching the conjecture from the other end, extending the method of near-triangulations that Birkhoff and Lewis used for \( t \leq 2 \). The problem with this is that the result does not hold for all near-triangulations. For example, a near-triangulation with all its \( n \) vertices in the bounding circuit has chromatic polynomial \( t(t-1)(t-2)^{n-2} \), which is non-zero throughout the interval \( (2, 2.5) \) but has the ‘wrong’ sign when \( n \) is even. And the graph of the octahedron minus an edge (\( H_{4,4} \) in Fig. 2) has chromatic polynomial

\[
t(t-1)(t-2)(t^3 - 8t^2 + 23t - 23),
\]

with a zero at about 2.43. So we need to find a way of excluding all near-triangulations for which the result is false.

A separating circuit in a plane graph is a circuit that has at least one vertex inside it and at least one vertex outside it. If \( G \) is a plane near-triangulation and we remove from \( G \) every vertex that lies inside a separating triangle of \( G \), then we obtain a new near-triangulation \( s(G) \), which we shall call the shell of \( G \), that has no separating triangles and has the same bounding circuit as \( G \). \( N(v) \) will denote the set of vertices neighbouring the vertex \( v \). We shall prove the following results.

**Theorem.** Let \( G \) be a 3-connected plane near-triangulation with bounding circuit \( C \). Suppose that:

(i) for each vertex \( v \) in \( s(G) - C \), \( N(v) \cap C \) is a (possibly empty) set of consecutive vertices of \( C \);

(ii) if \( |V(s(G) - C)| = 2 \) and one vertex of \( s(G) - C \) has degree 4 in \( s(G) \), then the other has odd degree in \( s(G) \).

Let \( q(G, t) \) be defined by (1). Then \( q(G, t) > 0 \) whenever \( t \leq 2.5 \).
Note that the hypotheses of the Theorem hold for every 4-connected near-triangulation, and for every 3-connected near-triangulation in which every separating set of three vertices induces a triangle.

**Corollary.** (a) If $G$ is a 3-connected plane triangulation then $q(G, t) > 0$ whenever $t < 2.546602 \cdots$.

(b) If $G$ is a plane triangulation, then $P(G, t)$ has no non-integer real zeros in the interval $(-\infty, 2.546602 \cdots)$.

I still cannot prove the following conjecture from [2], in which $2.677814 \cdots$ denotes a zero of the chromatic polynomial

$$t(t - 1)(t - 2)(t - 3)(t^3 - 6t^2 + 30t - 35)$$

of the graph $C_5 + K_2$.

**Conjecture.** If $G$ is a plane triangulation, then $P(G, t)$ is nonzero throughout the interval $(2.677814 \cdots, 3)$. If $G$ is 4-connected and non-Eulerian, then $P(G, t)$ has the sign of $(-1)^n$ in $(2.677814 \cdots, 3)$, and has a unique zero in the interval $(2.546602 \cdots, 2.677814 \cdots)$.

The remainder of this paper is devoted to proofs of the above Theorem and Corollary, starting with the Corollary because it is easier.

### 2. Proof of the Corollary

A 3-connected plane triangulation certainly satisfies the hypotheses of the Theorem. (Since $|V(C)| = 3$ and $s(G)$ has no separating triangle by definition, it is not possible that $|V(s(G) - C)| = 2$.) So $q(G, t) > 0$ whenever $t \leq 2.5$, by the Theorem. The same holds whenever $2.5 < t < 2.546602 \cdots$ by Theorem 7 of [2]. Thus (a) is proved.

We prove (b) by induction on $|V(G)|$, noting that it is true by (a) when $G$ is 3-connected, and true also when $G = C_3 (= K_3)$ since $P(C_3, t) = t(t - 1)(t - 2)$. If $G$ is not 3-connected and not $C_3$, then $G$ has a separating digon ($C_2$: see Fig. 1). Let $G_1$ consist of the digon and everything inside it, and $G_2$ consist of the digon and everything outside it, but remove one edge of the digon in each case so that $G_1$ and $G_2$ become plane triangulations with fewer vertices than $G$. We may suppose inductively that the result holds for $G_1$ and $G_2$. But

$$P(G, t) = \frac{P(G_1, t)P(G_2, t)}{t(t - 1)}$$

by a well-known formula, and so the result holds also for $G$. This completes the proof of the Corollary. □
3. Proof of the Theorem

If \( t < 2 \), then the result follows from the result of Birkhoff and Lewis already referred to ([1], page 397; [2], Theorem 5; [3], Theorem 3); so we shall suppose from now on that 2 \( \leq t \leq 2.5 \). We shall prove the result by induction on |\( V(G) \)|, considering four cases, of which Case 2 is by far the longest.

Case 1: \( G \) has a separating triangle.

Let \( G_1 \) consist of this triangle and everything inside it, and \( G_2 \) consist of the triangle and everything outside it. Then \( G_1 \) is a 3-connected triangulation, and \( G_2 \) is a 3-connected near-triangulation with the same shell and bounding circuit \( C \) as \( G \). We may suppose inductively that \( q(G_1, t) > 0 \) and \( q(G_2, t) > 0 \). But |\( V(G_1) \)| + |\( V(G_2) \)| = |\( V(G) \)| + 3, and

\[
P(G, t) = \frac{P(G_1, t)P(G_2, t)}{t(t-1)(t-2)},
\]

and so \( q(G, t) = q(G_1, t)q(G_2, t) > 0 \) as required.

Thus we shall suppose from now on that \( G \) has no separating triangle, whence \( s(G) = G \).

Case 2: |\( V(G - C) \)| \( \leq 3 \).

Clearly |\( V(G - C) \)| \( \geq 1 \) since \( G \) is 3-connected, so \( G - C \) comprises either a single vertex with degree \( a \geq 3 \) in \( G \), or two mutually adjacent vertices with degrees \( a \geq 4 \) and \( b \geq 4 \) in \( G \), or three mutually adjacent vertices (inducing a triangle, by condition (i)) with degrees \( a \geq 4 \), \( b \geq 4 \) and \( c \geq 4 \) in \( G \). Denote these three types of near-triangulation by \( H_a \) (a wheel), \( H_{a,b} \) and \( H_{a,b,c} \) respectively (see Fig. 2).

Recall the addition-identification formula for chromatic polynomials, which says that

\[
P(G, t) = P(G_1, t) + P(G_2, t),
\]

where \( G_2 \) is obtained from \( G \) by identifying two non-adjacent vertices \( u \) and \( w \), and \( G_1 = G + uw \). If \( u, v \) and \( w \) are three consecutive vertices of \( C \) that are all adjacent to the same vertex of \( G - C \), so that \( v \) has degree 3 in \( G \), then we can apply this formula and remove \( v \) from \( G_1 \) and \( G_2 \) to deduce that, for example,

\[
P(H_a, t) = (t - 3)P(H_{a-1}, t) + (t - 2)P(H_{a-2}, t) \quad (a \geq 4),
\]

whence

\[
q(H_a, t) = (3 - t)q(H_{a-1}, t) + (t - 2)q(H_{a-2}, t) \quad (a \geq 4),
\]

which is positive whenever 2 \( \leq t \leq 2.5 \) if \( q(H_{a-1}, t) \) and \( q(H_{a-2}, t) \) both are.

Similarly,

\[
q(H_{a,b}, t) = (3 - t)q(H_{a-1,b}, t) + (t - 2)q(H_{a-2,b}, t) \quad (a \geq 4, b \geq 4)
\]

and

\[
q(H_{a,b,c}, t) = (3 - t)q(H_{a-1,b,c}, t) + (t - 2)q(H_{a-2,b,c}, t) \quad (a \geq 5, b \geq 4, c \geq 4).
\]
For the purposes of calculation we shall consider smaller values of \( a \) than we are ultimately interested in. We find by direct calculation (see Fig. 2) that
\[
P(H_2, t) = t(t - 1)(t - 2), \quad q(H_2, t) = 1 > 0,
\]
\[
P(H_3, t) = t(t - 1)(t - 2)(t - 3), \quad q(H_3, t) = -(t - 3) > 0,
\]
and it follows inductively from these and (3) that \( q(H_n, t) > 0 \) whenever \( 2 \leq t \leq 2.5 \), for all \( a \geq 3 \), as required.

Similarly, we find by direct calculation that
\[
P(H_{2,4}, t) = t(t - 1)(t - 2)^2, \quad q(H_{2,4}, t) = -(t - 2),
\]
\[
P(H_{3,4}, t) = t(t - 1)(t - 2)(t - 3), \quad q(H_{3,4}, t) = -(t - 3),
\]
\[
P(H_{3,5}, t) = t(t - 1)(t - 2)(t - 3)^2, \quad q(H_{3,5}, t) = -(t - 3)^2,
\]
and it follows by double induction from these and (4) that
\[
q(H_{a,b}, t) = \frac{(3 - t)(t - 1)^2 - [1 - (2 - t)^{a-1}][1 - (2 - t)^{b-1}]}{(t - 2)(t - 1)^2}, \quad \text{(6)}
\]
using the fact that
\[
(3 - t)[1 - (2 - t)^{a-1}] + (t - 2)[1 - (2 - t)^{a-1}] = 1 - (2 - t)^{a-1}.
\]
If \( b \) (say) is odd and \( a \geq 4 \), then (in the range \( 2 \leq t \leq 2.5 \)) the numerator of (6) is strictly greater than
\[
(3 - t)(t - 1)^2 - [1 + (t - 2)^3] = (5 - 2t)(t - 1)(t - 2) \geq 0.
\]
And if \( a \geq 6 \) and \( b \geq 6 \), then (6) is at least as large as
\[
\frac{(3 - t)(t - 1)^2 - [1 + (t - 2)^3]^2}{(t - 2)(t - 1)^2} \geq \frac{1 - (t - 2) - (t - 2)^3 - 2(t - 2)^4 - (t - 2)^9}{(t - 1)^2},
\]
which is positive since \( 0 \leq t - 2 \leq \frac{1}{2} \). Thus the conclusion of the Theorem holds for all near-triangulations \( H_{a,b} \) that are not explicitly excluded in the statement.

Turning to \( H_{a,b,c} \), we find by direct calculation that
\[
q(H_{3,4,4}, t) = (t - 3)^2, \quad q(H_{3,4,5}, t) = -(t - 3)^3,
\]
\[
q(H_{3,5,5}, t) = (t - 3)q(H_{4,4}, t) = (t - 3)(t^3 - 8t^2 + 23t - 23).
\]

Recall the deletion-contraction formula for chromatic polynomials,
\[
P(G, t) = P(G - e, t) - P(G/e, t),
\]
where \( G - e \) and \( G/e \) are obtained from \( G \) by, respectively, deleting and contracting the edge \( e \). It follows that
\[
q(G, t) = q(G - e, t) + q(G/e, t).
\]
Thus
\[
q(H_{4,4,4}, t) = q(H_{4,4}, t) + q(H_{3,4,4}, t)
\]
\[
= -(t^3 - 9t^2 + 29t - 32) > 0 \quad \text{if} \ 2 \leq t < 2.546602 \cdots.
\]
Using (5) we find that

\[ q(H_{5,4,4}, t) = (3 - t)q(H_{4,4,4}, t) + (t - 2)q(H_{3,4,4}, t) \]

\[ = (t - 3)(t^3 - 8t^2 + 24t - 26) > 0 \quad \text{if } 2 < t < 2.6388 \ldots, \]

\[ q(H_{5,4,5}, t) = (3 - t)q(H_{4,4,5}, t) + (t - 2)q(H_{3,4,5}, t) \]

\[ = -(t - 3)^2(t^3 - 7t^2 + 19t - 20) > 0 \quad \text{if } 2 < t < 2.715 \ldots \]

and

\[ q(H_{5,5,5}, t) = (3 - t)q(H_{4,5,5}, t) + (t - 2)q(H_{3,5,5}, t) \]

\[ = (t - 3)(t^3 - 12t^4 + 60t^3 - 158t^2 + 222t - 134) > 0 \quad \text{if } 2 < t < 2.54176 \ldots. \]

It follows inductively from these and (5) that \(q(H_{a,b,c}, t) > 0\) whenever \(2 < t < 2.5\) and \(a, b, c \geq 4\). This completes the discussion of Case 2.

We shall suppose from now on that \(|V(G - C)| \geq 4\).

Case 3: Some vertex \(v\) of \(C\) has degree 3 in \(G\).

Let \(u, w\) be the neighbours of \(v\) on \(C\). The result will follow from the addition-identification formula (2), applied exactly as in Case 2, provided that the graphs \(G - v\) and \(G_2 - v\) satisfy the hypotheses of the Theorem (after merging a pair of parallel edges, in the latter case). The only problem arises if \(G\) has the form in Fig. 3 (in which there are additional vertices and edges triangulating the polygons labelled ?), in which case \(G_2\) has a separating triangle and \(s(G_2 - v)\) will violate hypothesis (ii) if \(x\) has degree 4 and \(y\) has even degree in \(s(G_2 - v)\). But in this case \(y\) is not adjacent to \(w\) in \(G\), nor \(x\) to \(u\), by hypothesis (i). Also, \(z\) is not adjacent to both \(x\) and \(y\) in \(G\), since there is at least one vertex inside \(C\) apart from \(x, y, z\) and \(G\) has no separating triangles. And \(v_1\) and \(v_2\) have degree 3 in \(G\). Thus we can deduce the result from the addition-identification formula (2) exactly as in Case 2, by applying it to \(v_1\) or \(v_2\) and its two neighbours on \(C\) rather than to \(u, v\) and \(w\): if \(y\) is not adjacent to \(z\) then we apply this reduction to \(v_2\), and if \(y\) is adjacent to \(z\) then \(x\) is not adjacent to \(z\) and we apply the reduction to \(v_1\). In either case, the result is proved.
Case 4: None of the previous three cases applies.

Let $e$ be an edge of $C$ and apply the deletion-contraction formula (7). The result will follow if both $G - e$ and $G/e$ satisfy the hypotheses of the Theorem. $G/e$ satisfies the hypotheses unless $G$ has the form in Fig. 4, which is not possible in Case 4 since the case when $C$ contains a vertex of degree 3 was disposed of in Case 3. So it suffices to choose $e$ so that $G - e$ satisfies the hypotheses. To do this, note that (i) forces $G$ to have the general shape depicted in Fig. 5, with \{v \in G - C: \text{N}(v) \cap C \neq \emptyset\} forming a circuit $C_1$ inside $C$; and, for each edge $e_i$ of $C$, there is a vertex $v_i$ in $C_1$ that is adjacent to both end vertices of $e_i$. $G - e_i$ satisfies the hypotheses unless $v_i$ is adjacent to some vertex $u$ of $C_1$ that is not adjacent to either end vertex of $e_i$. If this happens, choose $v_i$ and $u$ so that the shorter segment of $C_1$ connecting $v_i$ to $u$ is as short as possible. Then $v_{i-1}$ or $v_{i+1}$—whichever lies in this segment—is not adjacent to any vertex of $C_1$ that is not adjacent to an end vertex of $e_{i-1}$ or $e_{i+1}$ respectively. Thus either $G - e_{i-1}$ or $G - e_{i+1}$ satisfies the hypotheses, and the proof of the Theorem is complete. $\square$

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**Fig. 5.**
References