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An intermixed algorithm for strict pseudo-contractions in Hilbert spaces

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Abstract

An intermixed algorithm for two strict pseudo-contractions in Hilbert spaces have been presented. It is shown that the suggested algorithms converge strongly to the fixed points of two strict pseudo-contractions, independently. As a special case, we can find the common fixed points of two strict pseudo-contractions in Hilbert spaces.

MSC: 47H09; 47H10

Keywords: intermixed algorithm; strict pseudo-contraction; fixed point; strong convergence

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with its inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

Definition 1.1 A mapping $T: C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T.

Definition 1.2 A mapping $T: C \to C$ is said to be strictly pseudo-contractive if there exists a constant $0 \leq \lambda < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda \bigl\|(I - T)x - (I - T)y\bigr\|^2, \quad \forall x, y \in C.$$

Remark 1.3 It is well known that the class of strictly pseudo-contractive mappings properly includes the class of nonexpansive mappings.

Iterative construction of fixed points of nonlinear mappings has a long history and is still an active field in the nonlinear functional analysis. Let C be a nonempty closed convex subset of a real Hilbert space. Let $T: C \to C$ be a nonlinear mapping. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. For arbitrarily fixed $x_0 \in C$, define a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0. \tag{1.1}$$

Iteration (1.1) is said to be a Mann iteration [1]; it has been studied extensively in the literature. If T is a nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$ and $\{\alpha_n\}$ satisfies the condition $\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm converges weakly to a fixed point of T [2]. Now, it is well known that Mann's algorithm fails, in general, to converge strongly in the setting of infinite-dimensional Hilbert spaces [3]. Iterative methods for nonexpansive mappings have been investigated extensively in the literature; see [2–27] and the references therein. However, iterative methods for strictly pseudo-contractive mappings are far less developed than those for nonexpansive mappings though Browder and Petryshyn [4] initiated their work in 1967. However, strictly pseudo-contractive mappings have more powerful applications than nonexpansive mappings, for example, to solve inverse problems (see Scherzer [21]). Therefore it is interesting to develop the algorithms for finding the fixed points of strictly pseudo-contractive mappings. Now, we know that Mann's algorithm is not good enough for approximating fixed points of (even if Lipschitz continuous) pseudo-contractions. Thus, we have to find other type of iterative algorithms; see [28–35]. The first such an attempt was done by Ishikawa [7] who introduced the following Ishikawa algorithm:

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n, \end{aligned} \quad n \geq 0,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0,1]$, T is a (nonlinear) self-mapping of C, and the initial guess $x_0 \in C$ is selected arbitrarily. (Ishikawa's algorithm can be viewed as a double-step (or two-level) Mann's algorithm.) Ishikawa proved that his algorithm converges in norm to a fixed point of a Lipschitz pseudo-contraction T if $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy certain conditions and if T is compact.

On the other hand, iterative methods for approximating the common fixed points of a finite (or an infinite) family of nonlinear mappings have been considered by many authors. For the related work, we refer the reader to [22–26, 32, 33]. Above discussion suggests the following question.

Question 1.4 Could we construct an iterative algorithm such that it converges strongly to the fixed points of a finite family of strict pseudo-contractions?

It is our purpose in this paper to construct redundant intermixed algorithms for two strict pseudo-contractions. It is shown that the suggested algorithms converge strongly to the fixed points of two strict pseudo-contractions, independently. As a special case, we can find the common fixed points of two strict pseudo-contractions in Hilbert spaces.

2 Preliminaries

Let C be a nonempty closed convex subset of H. The (nearest point or metric) projection from H onto C is defined as follows: for each point $x \in H$, $P_C x$ is the unique point in C

with the property:

$$\|x - P_C x\| \leq \|x - y\|, \quad y \in C.$$

Note that P_C is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, y - P_C x\rangle \leq 0, \quad y \in C.$$

Consequently, P_C is nonexpansive.

In order to prove our main results, we need the following well-known lemmas.

Lemma 2.1 ([28]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strictly pseudo-contractive mapping. Then $I - T$ is demi-closed at* 0, *i.e., if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.2 ([18]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$ *for all* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.3 ([17]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that* $a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n$, $n \geq 0$ *where $\{\gamma_n\}$ is a sequence in* $(0,1)$ *and $\{\delta_n\}$ is a sequence in R such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a λ-strict pseudo-contraction. Let $f : C \to H$ be a ρ_1-contraction and $g : C \to H$ be a ρ_2-contraction. (A mapping $f : C \to H$ is said to be contractive if $\|f(x) - f(y)\| \leq \rho\|x - y\|$ for some $\rho \in [0,1)$ and for all $x, y \in C$.) Let $k \in (0, 1 - \lambda)$ be a constant.

Now we propose the following redundant intermixed algorithm for two strict pseudo-contractions S and T.

Algorithm 3.1 For arbitrarily given $x_0 \in C$, $y_0 \in C$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n], & n \geq 0, \\ y_{n+1} = (1 - \beta_n)y_n + \beta_n P_C[\alpha_n g(x_n) + (1 - k - \alpha_n)y_n + kSy_n], & n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0,1)$.

Remark 3.2 Note that the definition of the sequence $\{x_n\}$ is involved in the sequence $\{y_n\}$ and the definition of the sequence $\{y_n\}$ is also involved in the sequence $\{x_n\}$. So, this algorithm is said to be the redundant intermixed algorithm. We can use this algorithm to find the fixed points of S and T, independently.

Theorem 3.3 *Suppose that* $\mathrm{Fix}(S) \neq \emptyset$ *and* $\mathrm{Fix}(T) \neq \emptyset$. *Assume the following conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $\beta_n \in [\xi_1, \xi_2] \subset (0,1)$ *for all* $n \geq 0$.

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *generated by* (3.1) *converge strongly to the fixed points* $P_{\mathrm{Fix}(T)} f(y^*)$ *and* $P_{\mathrm{Fix}(S)} g(x^*)$ *of* T *and* S, *respectively, where* $x^* \in \mathrm{Fix}(T)$ *and* $y^* \in \mathrm{Fix}(S)$.

Proof First, we give the following propositions.

Proposition 3.4 *The sequences* $\{x_n\}$ *and* $\{y_n\}$ *are bounded.*

In order to prove this proposition, we need the following result.

Proposition 3.5 *The mapping* $P_C[\alpha f + (1-k-\alpha)I + kT]$ *is contractive for small enough* α.

Proof Let $x, y \in C$. Then we have

$$\begin{aligned}
&\bigl\|P_C\bigl[\alpha f(x) + (1-k-\alpha)x + kTx\bigr] - P_C\bigl[\alpha f(y) + (1-k-\alpha)y + kTy\bigr]\bigr\|^2 \\
&\quad\leq \bigl\|\alpha\bigl(f(x) - f(y)\bigr) + (1-k-\alpha)(x-y) + k(Tx - Ty)\bigr\|^2 \\
&\quad= \left\|\alpha\bigl(f(x) - f(y)\bigr) + (1-\alpha)\left[\frac{1-k-\alpha}{1-\alpha}(x-y) + \frac{k}{1-\alpha}(Tx-Ty)\right]\right\|^2 \\
&\quad\leq \alpha\bigl\|f(x) - f(y)\bigr\|^2 + (1-\alpha)\left\|\frac{1-k-\alpha}{1-\alpha}(x-y) + \frac{k}{1-\alpha}(Tx-Ty)\right\|^2 \\
&\quad\leq \alpha\rho_1\|x-y\|^2 + \frac{(1-k-\alpha)^2}{1-\alpha}\|x-y\|^2 + \frac{k^2}{1-\alpha}\|Tx-Ty\|^2 \\
&\qquad+ \frac{2(1-k-\alpha)k}{1-\alpha}\langle Tx - Ty, x-y\rangle \\
&\quad\leq \alpha\rho_1\|x-y\|^2 + \frac{(1-k-\alpha)^2}{1-\alpha}\|x-y\|^2 + \frac{k^2}{1-\alpha}\Bigl[\|x-y\|^2 + \lambda\bigl\|(I-T)x - (I-T)y\bigr\|^2\Bigr] \\
&\qquad+ \frac{2(1-k-\alpha)k}{1-\alpha}\left[\|x-y\|^2 - \frac{1-\lambda}{2}\bigl\|(I-T)x - (I-T)y\bigr\|^2\right] \\
&\quad= \alpha\rho_1\|x-y\|^2 + \frac{1}{1-\alpha}\bigl[\lambda k^2 - (1-\lambda)(1-k-\alpha)k\bigr]\bigl\|(I-T)x - (I-T)y\bigr\|^2 \\
&\qquad+ (1-\alpha)\|x-y\|^2 \\
&\quad= \frac{k}{1-\alpha}\bigl[k - (1-\alpha)(1-\lambda)\bigr]\bigl\|(I-T)x - (I-T)y\bigr\|^2 + \bigl[1 - (1-\rho_1)\alpha\bigr]\|x-y\|^2.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\bigl\|P_C\bigl[\alpha f(x) + (1-k-\alpha)x + kTx\bigr] - P_C\bigl[\alpha f(y) + (1-k-\alpha)y + kTy\bigr]\bigr\| \\
&\quad\leq \left[1 - \frac{(1-\rho_1)\alpha}{2}\right]\|x-y\|
\end{aligned}$$

for all $x, y \in C$ as $k \leq (1-\alpha)(1-\lambda)$ (that is, $\alpha \leq 1 - \frac{k}{1-\lambda}$). □

Next, we prove Proposition 3.4.

Proof Since $\mathrm{Fix}(S) \neq \emptyset$ and $\mathrm{Fix}(T) \neq \emptyset$, we can choose $x^* \in \mathrm{Fix}(T)$ and $y^* \in \mathrm{Fix}(S)$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1-\beta_n)x_n + \beta_n P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n] - x^*\| \\
&\leq \beta_n \|P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n] - x^*\| \\
&\quad + (1-\beta_n)\|x_n - x^*\| \\
&\leq \beta_n\alpha_n \|f(y_n) - x^*\| + \beta_n \|(1-k-\alpha_n)(x_n - x^*) + k(Tx_n - Tx^*)\| \\
&\quad + (1-\beta_n)\|x_n - x^*\| \\
&\leq \beta_n\alpha_n \|f(y_n) - f(y^*)\| + \beta_n\alpha_n \|f(y^*) - x^*\| + (1-\beta_n)\|x_n - x^*\| \\
&\quad + \beta_n(1-\alpha_n)\|x_n - x^*\| \\
&\leq \rho_1\beta_n\alpha_n \|y_n - y^*\| + \beta_n\alpha_n \|f(y^*) - x^*\| + (1-\alpha_n\beta_n)\|x_n - x^*\| \\
&\leq \rho\beta_n\alpha_n \|y_n - y^*\| + \beta_n\alpha_n \|f(y^*) - x^*\| + (1-\alpha_n\beta_n)\|x_n - x^*\|,
\end{aligned}$$

where $\rho = \max\{\rho_1, \rho_2\}$. Similarly, we have

$$\begin{aligned}
\|y_{n+1} - y^*\| &\leq \rho_2\beta_n\alpha_n \|x_n - x^*\| + \beta_n\alpha_n \|g(x^*) - y^*\| + (1-\alpha_n\beta_n)\|y_n - y^*\| \\
&\leq \rho\beta_n\alpha_n \|x_n - x^*\| + \beta_n\alpha_n \|g(x^*) - y^*\| + (1-\alpha_n\beta_n)\|y_n - y^*\|.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\quad \leq [1 - (1-\rho)\alpha_n\beta_n](\|x_n - x^*\| + \|y_n - y^*\|) + \alpha_n\beta_n(\|f(y^*) - x^*\| + \|g(x^*) - y^*\|) \\
&\quad \leq \max\left\{\|x_n - x^*\| + \|y_n - y^*\|, \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1-\rho}\right\}.
\end{aligned}$$

By induction, we have

$$\begin{aligned}
&\|x_n - x^*\| + \|y_n - y^*\| \\
&\quad \leq \max\left\{\|x_0 - x^*\| + \|y_0 - y^*\|, \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1-\alpha}\right\}.
\end{aligned}$$

So, $\{x_n\}$ and $\{y_n\}$ are bounded. □

Proposition 3.6 $\|x_n - Tx_n\| \to 0$ *and* $\|y_n - Sy_n\| \to 0$.

Proof We first estimate $\|x_{n+1} - x_n\|$. Set $u_n = P_C[\alpha_n f(y_n) + (1-k-\alpha_n)x_n + kTx_n]$, $n \geq 0$. It follows that

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|\alpha_{n+1} f(y_{n+1}) + (1-k-\alpha_{n+1})x_{n+1} + kTx_{n+1} \\
&\quad - \alpha_n f(y_n) - (1-k-\alpha_n)x_n + kTx_n\| \\
&\leq \|(1-k-\alpha_{n+1})(x_{n+1} - x_n) + k(Tx_{n+1} - Tx_n)\|
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_{n+1}\big(\|f(y_{n+1})\| + \|x_n\|\big) + \alpha_n\big(\|f(y_n)\| + \|x_n\|\big) \\
&\leq (1-\alpha_{n+1})\|x_{n+1}-x_n\| + \alpha_{n+1}\big(\|f(y_{n+1})\| + \|x_n\|\big) \\
&\quad + \alpha_n\big(\|f(y_n)\| + \|x_n\|\big).
\end{aligned}
$$

Since $\alpha_n \to 0$, we deduce that

$$\limsup_{n\to\infty}\big(\|u_{n+1}-u_n\| - \|x_{n+1}-x_n\|\big) \leq 0.$$

From Lemma 2.2, we get

$$\lim_{n\to\infty}\|u_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|x_{n+1}-x_n\| = 0.$$

From (3.1), we derive

$$
\begin{aligned}
\|x_{n+1}-Tx_n\| &\leq (1-\beta_n)\|x_n - Tx_n\| + \beta_n\alpha_n\|f(y_n) - Tx_n\| \\
&\quad + \beta_n(1-k-\alpha_n)\|x_n - Tx_n\| \\
&= \big[1-(k+\alpha_n)\beta_n\big]\|x_n - Tx_n\| + \beta_n\alpha_n\|f(y_n) - Tx_n\|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\leq \big[1-(k+\alpha_n)\beta_n\big]\|x_n - Tx_n\| + \beta_n\alpha_n\|f(y_n) - Tx_n\| \\
&\quad + \|x_n - x_{n+1}\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_n - Tx_n\| &\leq \frac{1}{(k+\alpha_n)\beta_n}\big(\|x_n - x_{n+1}\| + \beta_n\alpha_n\|f(y_n) - Tx_n\|\big) \\
&\to 0.
\end{aligned}
$$

Similarly, we can obtain

$$\lim_{n\to\infty}\|y_n - Sy_n\| = 0. \qquad \square$$

By Proposition 3.5, we know that the mapping $P_C[\alpha f + (1-k-\alpha)I + kT]$ is contractive for small enough α. Thus, the equation $x = P_C[tf(x) + (1-k-t)x + kTx]$ has a unique fixed point, denoted by x_t, that is,

$$x_t = P_C\big[tf(x_t) + (1-k-t)x_t + kTx_t\big] \tag{3.2}$$

for small enough t. In order to prove Theorem 3.3, we need the following lemma.

Lemma 3.7 *Suppose* Fix$(T) \neq \emptyset$. *Then, as* $t \to 0$, *the net* $\{x_t\}$ *defined by* (3.2) *converges strongly to a fixed point of* T.

Proof Let $x^* \in \mathrm{Fix}(T)$. From (3.2), we have

$$\begin{aligned}\|x_t - x^*\| &= \|P_C[tf(x_t) + (1-k-t)x_t + kTx_t] - x^*\| \\ &\le t\|f(x_t) - x^*\| + \|(1-k-t)(x_t - x^*) + k(Tx_t - x^*)\| \\ &\le t\rho_1\|x_t - x^*\| + t\|f(x^*) - x^*\| + (1-t)\|x_t - x^*\|,\end{aligned}$$

hence,

$$\|x_t - x^*\| \le \frac{1}{1-\rho_1}\|f(x^*) - x^*\|.$$

Thus, $\{x_t\}$ is bounded. Again, from (3.2), we get

$$\|x_t - Tx_t\| \le t\|f(x_t) - Tx_t\| + (1-k-t)\|x_t - Tx_t\|.$$

It follows that

$$\|x_t - Tx_t\| \le \frac{t}{k+t}\|f(x_t) - Tx_t\| \to 0.$$

Let $\{t_n\} \subset (0,1)$. Assume that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. We have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Set $y_t = tf(x_t) + (1-k-t)x_t + kTx_t$, for all t. Then we have $x_t = P_C y_t$, and for any $x^* \in \mathrm{Fix}(T)$,

$$\begin{aligned}x_t - x^* &= x_t - y_t + y_t - x^* \\ &= x_t - y_t + t(f(x_t) - x^*) + (1-k-t)(x_t - x^*) + k(Tx_t - x^*).\end{aligned}$$

From the property of the metric projection, we deduce

$$\langle x_t - y_t, x_t - x^*\rangle \le 0.$$

So,

$$\begin{aligned}\|x_t - x^*\|^2 &= \langle x_t - y_t, x_t - x^*\rangle + \langle (1-k-t)(x_t - x^*) + k(Tx_t - x^*), x_t - x^*\rangle \\ &\quad + t\langle f(x_t) - x^*, x_t - x^*\rangle \\ &\le \|(1-k-t)(x_t - x^*) + k(Tx_t - x^*)\|\|x_t - x^*\| \\ &\quad + t\langle f(x_t) - f(x^*), x_t - x^*\rangle + t\langle f(x^*) - x^*, x_t - x^*\rangle \\ &\le [1 - (1-\rho_1)t]\|x_t - x^*\|^2 + t\langle f(x^*) - x^*, x_t - x^*\rangle.\end{aligned}$$

Hence,

$$\|x_t - x^*\|^2 \le \frac{1}{(1-\rho_1)}\langle f(x^*) - x^*, x_t - x^*\rangle, \quad \forall x^* \in \mathrm{Fix}(T).$$

By similar arguments to [28], we find that the net $\{x_t\}$ converges strongly to $x^* \in \mathrm{Fix}(T)$. This completes the proof. □

Remark 3.8 From Lemma 3.7, we know that the net $\{x_t\}$ defined by $x_t = P_C[tu + (1 - k - t)x_t + kTx_t]$ where $u \in H$, converges to $P_{\mathrm{Fix}(T)}u$. Let $x^* \in \mathrm{Fix}(T)$ and $y^* \in \mathrm{Fix}(S)$. If we take $u = f(y^*)$, then the net $\{x_t\}$ defined by $x_t = P_C[tf(y^*) + (1 - k - t)x_t + kTx_t]$, converges to $P_{\mathrm{Fix}(T)}f(y^*)$.

Finally, we prove that $x_n \to P_{\mathrm{Fix}(T)}f(y^*)$ and $y_n \to P_{\mathrm{Fix}(S)}g(x^*)$, where $x^* \in \mathrm{Fix}(T)$ and $y^* \in \mathrm{Fix}(S)$. We note the following fact. If the sequence $\{w_n\}$ is bounded and $\|w_n - Tw_n\| \to 0$, we easily deduce that

$$\limsup_{n\to\infty}\bigl\langle f\bigl(P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr)\bigr) - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr), w_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\rangle \le 0.$$

Set $v_n = P_C[\alpha_n g(x_n) + (1-k-\alpha_n)y_n + kSy_n]$ for all $n \ge 0$. Thus, we deduce that the sequences $\{u_n\}$ and $\{v_n\}$ satisfy: (1) $\{u_n\}$ and $\{v_n\}$ are bounded; (2) $\|u_n - Tu_n\| \to 0$ and $\|v_n - Sv_n\| \to 0$. Therefore,

$$\limsup_{n\to\infty}\bigl\langle f\bigl(P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr)\bigr) - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr), u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\rangle \le 0$$

and

$$\limsup_{n\to\infty}\bigl\langle g\bigl(P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr) - P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr), v_n - P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr)\bigr\rangle \le 0.$$

Next, we estimate $\|u_n - P_{\mathrm{Fix}(T)}f(y^*)\|$. Set $\tilde{u}_n = \alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n$ and $\tilde{v}_n = \alpha_n g(x_n) + (1 - k - \alpha_n)y_n + kSy_n$ for all n. We have

$$\begin{aligned}
&\bigl\|u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|^2 \\
&\quad = \bigl\|P_C[\tilde{u}_n] - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|^2 \\
&\quad \le \bigl\langle \tilde{u}_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr), u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\rangle \\
&\quad = \bigl\langle \alpha_n f(y_n) + (1 - k - \alpha_n)x_n + kTx_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr), u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\rangle \\
&\quad \le \alpha_n\bigl\langle f(y_n) - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr), u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\rangle \\
&\qquad + (1 - \alpha_n)\bigl\|x_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|\bigl\|u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\| \\
&\quad \le \frac{1 - \alpha_n}{2}\bigl\|x_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|^2 + \frac{1}{2}\bigl\|u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|^2 \\
&\qquad + \alpha_n\bigl\langle f(y_n) - f\bigl(P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr)\bigr), u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\rangle \\
&\qquad + \alpha_n\bigl\langle f\bigl(P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr)\bigr) - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr), u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\rangle \\
&\quad \le \frac{1 - \alpha_n}{2}\bigl\|x_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|^2 + \frac{1}{2}\bigl\|u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|^2 \\
&\qquad + \alpha_n\rho\bigl\|y_n - P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr)\bigr\|\bigl\|u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\| \\
&\qquad + \alpha_n\bigl\langle f\bigl(P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr)\bigr) - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr), u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\rangle \\
&\quad \le \frac{1 - \alpha_n}{2}\bigl\|x_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|^2 + \frac{1}{2}\bigl\|u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|^2 \\
&\qquad + \frac{\alpha_n\rho}{2}\bigl(\bigl\|y_n - P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr)\bigr\|^2 + \bigl\|u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\|^2\bigr) \\
&\qquad + \alpha_n\bigl\langle f\bigl(P_{\mathrm{Fix}(S)}g\bigl(x^*\bigr)\bigr) - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr), u_n - P_{\mathrm{Fix}(T)}f\bigl(y^*\bigr)\bigr\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|u_n - P_{\mathrm{Fix}(T)}f(y^*)\|^2 \\
&\quad\le \frac{1-\alpha_n}{1-\alpha_n\rho}\|x_n - P_{\mathrm{Fix}(T)}f(y^*)\|^2 + \frac{\alpha_n\rho}{1-\alpha_n\rho}\|y_n - P_{\mathrm{Fix}(S)}g(x^*)\|^2 \\
&\qquad + \frac{2\alpha_n}{1-\alpha_n\rho}\langle f(P_{\mathrm{Fix}(S)}g(x^*)) - P_{\mathrm{Fix}(T)}f(y^*), u_n - P_{\mathrm{Fix}(T)}f(y^*)\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|x_{n+1} - P_{\mathrm{Fix}(T)}f(y^*)\|^2 \\
&\quad\le (1-\beta_n)\|x_n - P_{\mathrm{Fix}(T)}f(y^*)\|^2 + \beta_n\|u_n - P_{\mathrm{Fix}(T)}f(y^*)\|^2 \\
&\quad\le \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n\right)\|x_n - P_{\mathrm{Fix}(T)}f(y^*)\|^2 + \frac{\alpha_n\beta_n\rho}{1-\alpha_n\rho}\|y_n - P_{\mathrm{Fix}(S)}g(x^*)\|^2 \\
&\qquad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle f(P_{\mathrm{Fix}(S)}g(x^*)) - P_{\mathrm{Fix}(T)}f(y^*), u_n - P_{\mathrm{Fix}(T)}f(y^*)\rangle.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&\|y_{n+1} - P_{\mathrm{Fix}(S)}g(x^*)\|^2 \\
&\quad\le \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\beta_n\right)\|y_n - P_{\mathrm{Fix}(S)}g(x^*)\|^2 + \frac{\alpha_n\beta_n\rho}{1-\alpha_n\rho}\|x_n - P_{\mathrm{Fix}(T)}f(y^*)\|^2 \\
&\qquad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle g(P_{\mathrm{Fix}(T)}f(y^*)) - P_{\mathrm{Fix}(S)}g(x^*), v_n - P_{\mathrm{Fix}(S)}g(x^*)\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|x_{n+1} - P_{\mathrm{Fix}(T)}f(y^*)\|^2 + \|y_{n+1} - P_{\mathrm{Fix}(S)}g(x^*)\|^2 \\
&\quad\le \left(1 - \frac{1-2\rho}{1-\alpha_n\rho}\alpha_n\beta_n\right)\left(\|x_n - P_{\mathrm{Fix}(T)}f(y^*)\|^2 + \|y_n - P_{\mathrm{Fix}(S)}g(x^*)\|^2\right) \\
&\qquad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle f(P_{\mathrm{Fix}(S)}g(x^*)) - P_{\mathrm{Fix}(T)}f(y^*), u_n - P_{\mathrm{Fix}(T)}f(y^*)\rangle \\
&\qquad + \frac{2\alpha_n\beta_n}{1-\alpha_n\rho}\langle g(P_{\mathrm{Fix}(T)}f(y^*)) - P_{\mathrm{Fix}(S)}g(x^*), v_n - P_{\mathrm{Fix}(S)}g(x^*)\rangle.
\end{aligned}$$

We can check that all assumptions of Lemma 2.3 are satisfied. Therefore, $x_n \to P_{\mathrm{Fix}(T)}f(y^*)$ and $y_n \to P_{\mathrm{Fix}(S)}g(x^*)$. This completes the proof. □

Algorithm 3.9 For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = (1-\beta_n)x_n + \beta_n P_C[(1-k-\alpha_n)x_n + kTx_n], \quad n \ge 0, \tag{3.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0,1)$.

Theorem 3.10 *Suppose* $\mathrm{Fix}(T) \neq \emptyset$. *Assume the following conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $\beta_n \in [\xi_1, \xi_2] \subset (0,1)$ *for all* $n \geq 0$.

Then the sequence $\{x_n\}$ *generated by* (3.3) *converge strongly to the fixed points* $P_{\mathrm{Fix}(T)}(0)$, *which is the minimum norm element in* $\mathrm{Fix}(T)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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