Global weak solutions to the Nordström–Vlasov system

Simone Calogero\textsuperscript{a} and Gerhard Rein\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a}Department of Mathematics, Chalmers University of Technology, S-41296 Göteborg, Sweden
\textsuperscript{b}Department of Mathematics, University of Bayreuth, D-95440 Bayreuth, Germany

Received September 17, 2003

Abstract

The Nordström–Vlasov system is a Lorentz invariant model for a self-gravitating collisionless gas. We establish suitable a priori bounds on the solutions of this system, which together with energy estimates and the smoothing effect of “momentum averaging” yield the existence of global weak solutions to the corresponding initial value problem. In the process we improve the continuation criterion for classical solutions which was derived recently. The weak solutions are shown to preserve mass.

\textcopyright{} 2004 Elsevier Inc. All rights reserved.

Keywords: Nordström scalar theory of gravitation; Vlasov equation; Global weak solutions

1. Introduction

Consider a large ensemble of particles which interact by force fields which they create collectively and not by collisions. Such a collisionless gas is often used as a matter model both in plasma physics and in astrophysics. If the particles in the gas interact by electromagnetic fields the dynamics of the ensemble is described by the Vlasov–Maxwell system. If the particles interact by gravitational forces one obtains the Vlasov–Poisson system in the non-relativistic case or the Einstein–Vlasov system in the relativistic case. We refer to \cite{1,6,11} for background information on these systems. In the present investigation we consider a self-gravitating collisionless gas where gravity is described by Nordström’s scalar theory \cite{10}. We write the
Nordström–Vlasov system in the formulation of [4]:

\[ \partial_t^2 \phi - \Delta_x \phi = -\mu, \quad (1.1) \]

\[ \mu(t, x) = \int f(t, x, p) \frac{dp}{\sqrt{1 + p^2}}, \quad (1.2) \]

\[ \partial_t f + \hat{p} \cdot \nabla_x f - [(S\phi) p + (1 + p^2)^{-1/2} \nabla_x \phi] \cdot \nabla_p f = 4fS\phi. \quad (1.3) \]

Here \( t \in \mathbb{R}, x, p \in \mathbb{R}^3 \) stand for time, position, and momentum, \( f = f(t, x, p), \phi = \phi(t, x) \), and

\[ S = \partial_t + \hat{p} \cdot \nabla_x, \quad \hat{p} = \frac{p}{\sqrt{1 + p^2}}, \quad p^2 = |p|^2, \]

\( S \) is the free-transport operator, and \( \hat{p} \) denotes the relativistic velocity of a particle with momentum \( p \). Units are chosen such that the mass of each particle, the gravitational constant, and the speed of light are equal to unity. A solution \((f, \phi)\) of this system is interpreted as follows: The space-time is a four-dimensional Lorentzian manifold with a conformally flat metric which, in the coordinates \((t, x)\), takes the form

\[ g_{\mu\nu} = e^{2\phi} \text{diag}(-1, 1, 1, 1). \]

The particle density on the mass shell in this metric is \( e^{-4\phi} f(t, x, e^{\phi}p) \), but it is more convenient to work with \( f \) and \( \phi \) as the dynamic variables as long as it is kept in mind that \( f \) itself is not the particle density. More details on the derivation of this system are given in [2] where its steady states are investigated.

Like Nordström’s theory, the system is not a physically correct model. Nevertheless, there are good reasons for studying it: The Nordström–Vlasov system is a Lorentz invariant model for a self-gravitating gas which has the correct Newtonian limit, namely the Vlasov–Poisson system, cf. [3]. The system is much simpler than the physically correct but notoriously difficult Einstein–Vlasov system, and yet it captures some typical relativistic effects, such as the propagation of gravitational waves. From a more mathematical point of view, the hope is that by investigating this system one can learn more about the whole class of non-linear partial differential equations in kinetic theory to which this model belongs. For instance, one of the most celebrated results in kinetic theory is the existence of global weak solutions for the Vlasov–Maxwell system due to DiPerna and [5]. It is the purpose of the present paper to investigate this solution concept for the Nordström–Vlasov system.

We explain how the paper proceeds and how global weak solutions to the Nordström–Vlasov system are obtained. First one needs to establish suitable a priori bounds for solutions. One set of such bounds derives from conservation of energy. In addition, for related systems like Vlasov–Poisson or Vlasov–Maxwell the fact that \( f \)
is constant along characteristics and that the characteristic flow preserves measure yields bounds on the $L^q$-norms of $f(t)$, $1 \leq q \leq \infty$. However, these properties do not hold in the present situation. In the next section we overcome this difficulty and derive a priori bounds for the $L^q$-norms of $f(t)$. This is a necessary prerequisite for global existence of weak solutions, and at the same time improves the continuation criterion for classical solutions which was derived in [4]. It is interesting to note that the argument would not work if the sign in the field equation were reversed. In the third section we turn to the construction of weak solutions. First a suitable regularization of the system is introduced such that the regularized system does have global (classical) solutions which satisfy the same a priori bounds. In [5] the Vlasov–Maxwell system was regularized by making the Maxwell equations parabolic. Here we smooth the right-hand side of the field equation (1.1); an analogous approach was followed in [6,8,9]. The resulting system is “closer” to the original one than in [5], in particular, it remains time reversible which allows us to avoid various technical difficulties in what follows. Along a sequence of solutions to regularized systems the a priori bounds are shown to hold uniformly. In order to pass to the limit in the non-linear term in the Vlasov equation we use the smoothing effect due to “velocity averaging” [7], which we now prefer to call “momentum averaging”. As opposed to [5] we use this tool only to pass to the limit in the Vlasov equation and not for the moments of $f$ such as $\mu$, which are dealt with directly. This yields another simplification of the proof. Notice that the Vlasov equation (1.3) can be rewritten in the form

$$\partial_t f + \mathbf{p} \cdot \nabla_x f = \nabla \cdot [(S\phi)p + (1 + p^2)^{-1/2}\nabla_x \phi)f] + fS\phi$$

(1.4)

which is the form needed to apply “momentum averaging”. Once a global weak solution is obtained its properties are of interest. For the related systems it is not known whether weak solutions are unique or preserve energy, and the same is true in the present situation. But we can show that the weak solutions preserve mass, which is non-trivial because as opposed to the Vlasov–Maxwell system this quantity is not just the integral of $f(t)$, cf. (2.1). For this purpose we need to exploit the relativistic nature of the system and to assume that the initial datum for $f$ is bounded. This assumption is not made in [5], but it allows us in addition to remove further technical difficulties from the proof, such as the use of renormalized solutions.

2. Conservation laws and $L^q$-estimates

Formally, solutions of the Nordström–Vlasov system conserve mass and energy:

$$\int e^{-\phi} \int f \, dp \, dx = \text{const},$$

(2.1)

$$\int \int f \sqrt{1 + p^2} \, dp \, dx + \frac{1}{2} \int [\partial_t \phi^2 + (\nabla_x \phi)^2] \, dx = \text{const}.$$  

(2.2)
Again formally, these conservation laws can be obtained by integrating their local counterparts

\[ \partial_t \rho + \nabla_x \cdot j = 0, \] (2.3)
\[ \partial_t e + \nabla_x \cdot p = 0, \] (2.4)

where

\[ \rho(t, x) = e^{-\phi} \int f(t, x, p) \, dp, \] (2.5)
\[ j(t, x) = e^{-\phi} \int \hat{p} f(t, x, p) \, dp, \] (2.6)
\[ e(t, x) = \int \sqrt{1 + p^2} f(t, x, p) \, dp + \frac{1}{2}(\partial_t \phi(t, x))^2 + \frac{1}{2}(\nabla_x \phi(t, x))^2, \] (2.7)
\[ p(t, x) = \int p f(t, x, p) \, dp - \partial_t \phi(t, x) \nabla_x \phi(t, x). \] (2.8)

Let us now denote by \((f, \phi) \in C^1([0, T] \times \mathbb{R}^6) \times C^2([0, T] \times \mathbb{R}^3)\) a classical solution of the Nordström–Vlasov system on the interval \([0, T]\), \(T > 0\), with initial data \(f(0) = f^{\text{in}} \in C^1_c(\mathbb{R}^6)\), \(\phi(0) = \phi^{\text{in}}_0 \in C^2_b(\mathbb{R}^3)\), \(\partial_t \phi(0) = \phi^{\text{in}}_1 \in C^2_b(\mathbb{R}^3)\), with \(f^{\text{in}} \geq 0\). In the notation above the subscript \(c\) indicates that the functions are compactly supported while the subscript \(b\) indicates that they are bounded together with their derivatives up to the indicated order. For such data a unique, classical solution exists at least locally in time, cf. [4]. Assuming in addition that the initial data have finite energy the conservation laws stated above hold.

Let \(X(s) = X(s, t, x, p)\), \(P(s) = P(s, t, x, p)\) denote the characteristics of the Vlasov equation, i.e., the solutions of the characteristic system

\[ \frac{dx}{ds} = \hat{p}, \]
\[ \frac{dp}{ds} = -(S\phi)p - (1 + p^2)^{-1/2} \nabla_x \phi, \]

satisfying \(X(t) = x\), \(P(t) = p\). By the Vlasov equation (1.3) the function \(e^{-4\phi} f\) is constant along these curves, and \(f\) can be represented as

\[ f(t, x, p) = f^{\text{in}}(X(0), P(0)) \exp[4\phi(t, x) - 4\phi^{\text{in}}_0(X(0))]. \] (2.9)
In particular, $f$ and hence also $\mu$ are non-negative. We write the function $\phi$ as

$$\phi = \phi_{\text{hom}} + \psi,$$

where $\phi_{\text{hom}}$ is the solution of the homogeneous wave equation with initial data $\phi_0^\text{in}, \phi_1^\text{in}$, and $\psi$ is the solution of (1.1) with zero initial data. By Duhamel's principle, $\psi \leq 0$. Hence (2.9) and (2.10) imply that

$$f(t,x,p) \leq f^\text{in}(X(0), P(0)) \exp[4\phi^\text{hom}_1(t,x) - 4\phi^\text{in}_0(X(0))],$$

and we have proved the following a priori bound:

**Proposition 1.** For all $t \in [0, T]$,

$$\|f(t)\|_\infty \leq \|f^\text{in}\|_\infty \exp[4(\|\phi^\text{hom}_1(t)\|_\infty + \|\phi^\text{in}_0\|_\infty)].$$

Notice that under the assumptions on the initial data for $\phi$ made above $\|\phi^\text{hom}_1(t)\|_\infty \leq C(1 + t)$ with $C$ depending on the data.

Combining this result with the one in [4] we obtain the following improvement on the continuation criterion for classical solutions:

**Proposition 2.** Initial data as specified above launch a unique classical solution to the Cauchy problem for the Nordström–Vlasov system on a maximal time interval $[0, T_{\text{max}}]$, and if

$$\sup\{|p| : (x,p) \in \text{supp} f(t), \ 0 \leq t < T_{\text{max}}\} < \infty$$

then $T_{\text{max}} = \infty$, i.e., the solution is global.

**Proof.** By Calogero and Rein [4, Theorem 1] a unique solution exists and is global, provided $\phi$ remains bounded on $[0, T_{\text{max}}]$ and condition (2.12) holds. But together with Proposition 1, (2.12) implies that the source term $\mu$ in the field equation (1.1) is bounded. Hence a bound on $\phi$ follows from the assumption on the momentum support, and the proposition is established. □

It is standard that the estimates above result in a priori bounds for quantities like $\mu$ or $\rho$:

$$\mu(t,x) = \int_{|p| \leq R} f(t,x,p) \frac{dp}{\sqrt{1 + p^2}} + \int_{|p| > R} f(t,x,p) \frac{dp}{\sqrt{1 + p^2}}$$

$$\leq \|f(t)\|_\infty \int_{|p| \leq R} \frac{dp}{\sqrt{1 + p^2}} + R^{-2} \int \sqrt{1 + p^2 f} \ dp$$

$$\leq CR^2\|f(t)\|_\infty + R^{-2} \int \sqrt{1 + p^2 f} \ dp = C\left(\|f(t)\|_\infty \int \sqrt{1 + p^2 f} \ dp\right)^{1/2},$$
where for the last step we choose
\[ R = \left( \int \sqrt{1 + p^2 f} \, dp / ||f(t)||_\infty \right)^{1/4}. \]

Squaring both sides of the estimate for \( \mu \), integrating in \( x \), and using Proposition 1 and conservation of energy implies the estimate
\[
||\mu(t)||_2 \leq C e^2 ||\phi_\text{hom}(t)||_\infty, \quad t \in [0, T].
\] (2.13)

Here \( C \) denotes a positive constant which depends only on \( ||f\text{in}||_\infty, ||\phi_0\text{in}||_\infty \), and the energy of the initial data, and which may change from line to line. Similarly, using (2.9),
\[
\rho(t, x) = e^{-\phi} \int_{|p| < R} f \, dp + e^{-\phi} \int_{|p| > R} f \, dp
\leq ||e^{-4\phi_\text{in}} f\text{in}||_\infty e^3 \phi R^3 + e^{-\phi} R^{-1} \int \sqrt{1 + p^2 f} \, dp = C \left( \int \sqrt{1 + p^2 f} \, dp \right)^{3/4},
\]
where we have chosen
\[
R = e^{-\phi} \left( \int \sqrt{1 + p^2 f} \, dp \right)^{1/4}
\]
in the last step. Elevating this estimate to the power \( 4/3 \), integrating in \( x \), and observing conservation of energy we conclude that
\[
||\rho(t)||_{4/3}, \ ||j(t)||_{4/3} \leq C;
\] (2.14)

note that \( \rho \) dominates \( j \).

Combining (2.9) with Liouville’s Theorem one can prove the following \( L^q \) estimates on the distribution function. Since we will assume \( L^\infty \) initial data for \( f \) in the construction of weak solutions Proposition 1 and the bound on the kinetic energy gives us a bound on any \( L^q \)-norm of \( f(t) \), and hence the following estimates are not used in the rest of the paper. But they may be useful if one does not wish to consider bounded initial data for \( f \).

**Proposition 3.** For all \( q \geq 1, \ \gamma \geq 3/q - 4 \), and \( t \in [0, T] \) we have
\[
||e^{[3/q-4]j} f(t)||_q = ||e^{[3/q-4]j^n} f\text{in}||_q \leq ||f\text{in}||_q \exp(7||\phi_0\text{in}||_\infty), \quad (2.15)
\]
\[
||e^{ij} f(t)||_q \leq ||f\text{in}||_q \exp[7(||\phi_\text{hom}(t)||_\infty + ||\phi_0\text{in}||_\infty) + ||j(t)||_{4/3}]. \quad (2.16)
\]
Proof. For any smooth function \( Q : \mathbb{R} \rightarrow \mathbb{R} \), Liouville’s Theorem and (2.9) imply
\[
\iint Q(fe^{-4\phi}) e^{3\phi} \, dp \, dx = \text{const},
\]
(2.17)
note that the \((x,p)\)-divergence of the right-hand side of the characteristic system equals \(-3 S\phi\) and so
\[
\det \frac{\partial (X, P) }{\partial (x, p)} (0, t, x, p) = \exp[3 \phi(t, x) - 3 \phi^{\text{in}}(X(0, t, x, p))].
\]
With the choice \( Q(z) = z^4 \), (2.17) implies (2.15). Moreover
\[
e^{[3/q-4]\phi} f = e^{\gamma \phi} e^{[3/q-4-\gamma] \phi_{\text{hom}}} e^{(3/q-4-\gamma) \psi} f.
\]
Since \([3/q - 4 - \gamma] \psi \geq 0\) this yields
\[
\|e^{[3/q-4]\phi} f(t)\|_q \geq e^{-[3/q+4+\gamma \|\psi\|_{\infty}} \|e^{\gamma \phi} f\|_q,
\]
and (2.16) follows.  

Although we have formulated the results of this section only for going forward in time they hold equally well towards the past since the system is time reversible.

3. Global weak solutions

The purpose of this section is to prove global existence of weak solutions to the Nordström–Vlasov system. We denote by \( L^1_{\text{kin}}(\mathbb{R}^6) \) the Banach space of the measurable functions \( g : \mathbb{R}^6 \rightarrow \mathbb{R} \) for which the norm
\[
\|g\|_{1, \text{kin}} = \iint \sqrt{1 + p^2} |g| \, dp \, dx
\]
is finite.

Theorem. For any triple \((f^{\text{in}}, \phi_0^{\text{in}}, \phi_1^{\text{in}})\) such that for some \( s > 3/2 \),
\[
0 \leq f^{\text{in}} \in L^1_{\text{kin}} \cap L^\infty(\mathbb{R}^6), \quad \phi_0^{\text{in}} \in H^s(\mathbb{R}^3), \quad \phi_1^{\text{in}} \in H^{s-1}(\mathbb{R}^3),
\]
there exists a global weak solution \((f, \phi)\) of the Nordström–Vlasov system, more precisely,
\[
f \in L^\infty(\mathbb{R}; L^1_{\text{kin}}(\mathbb{R}^6)) \cap L^\infty_{\text{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^6)), \quad \phi \in L^\infty_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}^3)),
\]

\[\text{ARTICLE IN PRESS}\]
with \( \partial_t \phi, \nabla_x \phi \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3)), e^{\phi} \in H^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^3), f \geq 0 \text{ a.e.}, \) and the following holds:

(i) \((f, \phi)\) solves (1.1)–(1.3) in the sense of distributions.

(ii) The mapping

\[
F : \mathbb{R} \to L^2(\mathbb{R}^6) \times H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3), \quad t \mapsto (f(t), \phi(t), \partial_t \phi(t))
\]

is weakly continuous with \( F(0) = (f^\text{in}, \phi_0^\text{in}, \phi_1^\text{in}) \). Moreover, for any \( R > 0, \phi \in C(\mathbb{R}; L^2(B_R)) \).

(iii) The energy at any time \( t \) is bounded by its initial value, the local conservation law (2.3) holds in the sense of distributions, and the mass is conserved:

\[
\int \rho(t) \, dx = \int \rho(0) \, dx \text{ for a.a. } t \in \mathbb{R}.
\]

Here \( B_R = \{ x \in \mathbb{R}^3 : |x| < R \} \), and \( H^s(\mathbb{R}^3) \) denotes the usual Sobolev spaces. The proof proceeds in a number of steps:

**Step 1: The regularized system.** Let \( 0 \leq \delta_n \in C^\infty_c(\mathbb{R}^3) \) be a mollifier satisfying the following conditions:

\[
\delta_n(x) = \delta_n(-x), \quad \int \delta_n(x) \, dx = 1, \quad \text{supp} \, \delta_n \subset B_{1/n}, \quad n \in \mathbb{N}.
\]

The Nordström–Vlasov system is regularized by replacing the right-hand side of (1.1) by \(-\mu \star \delta_n \star \delta_n\), where \( \star \) denotes convolution with respect to \( x \). The regularized system is supplied with regularized initial data

\[
f_n^\text{in}, \phi_0^\text{in}, \phi_1^\text{in} = g_n \star \delta_n, h_n \star \delta_n,
\]

where

\[
0 \leq f_n^\text{in} \in C^\infty_c(\mathbb{R}^6), g_n, h_n \in C^\infty_c(\mathbb{R}^3)
\]

are such that, as \( n \to \infty \),

\[
f_n^\text{in} \to f^\text{in} \text{ in } L^1_{\text{kin}} \cap L^\infty, \quad g_n \to \phi_0^\text{in} \text{ in } H^s, \quad h_n \to \phi_1^\text{in} \text{ in } H^{s-1}.
\]

The reason for regularizing \( \mu \) and the data for \( \phi \) in this particular way will become obvious in the energy estimate below. The regularized initial value problem has global smooth solutions. To see this observe that by (2.16), \( \| \mu(t) \|_1 \) is bounded. Hence, for any \( n \in \mathbb{N}, \mu(t) \star \delta_n \star \delta_n \) is smooth and bounded in \( L^\infty, \) together with all its derivatives, the bounds of course depending on \( n \). Hence \( \phi \) and \( \partial_t \phi \) are bounded on any compact time interval, together with all their spatial derivatives. It is then straightforward to see that a standard iteration scheme like the one employed in [4] converges on any time interval to a smooth solution of the regularized initial value problem. For more details we refer to the proof of the analogous result for the relativistic Vlasov–Klein–Gordon system in [9, Section 3]. Note that the regularized
system remains time reversible so that solutions really exist on all of $\mathbb{R}$ and not only on $[0, \infty]$.

In what follows, $(f_n, \phi_n)$ denotes the global smooth solution of the regularized initial value problem. It satisfies the continuity equation

$$\partial_t \rho_n + \nabla_x \cdot j_n = 0,$$

where $\rho_n = e^{-\phi_n} \int f_n \, dp$, $j_n = e^{-\phi_n} \int \tilde{f} f_n \, dp$.

**Step 2:** Uniform bounds on $(f_n, \phi_n)$. First we observe that the energy of $(f_n, \phi_n)$ is not conserved, but one can prove that it is bounded. In fact, a direct computation shows that

$$\frac{d}{dt} \left\{ \int \int \sqrt{1 + p^2} f_n \, dp \, dx + \frac{1}{2} \int (|\partial_t \phi_n|^2 + |\nabla_x \phi_n|^2) \, dx \right\} = \int \partial_t \phi_n (\mu_n - \mu_n \ast \delta_n \ast \delta_n) \, dx.$$

Denote by $\tilde{\phi}_n$ the solution of the wave equation with the *given* right-hand side $-\mu_n \ast \delta_n$ and initial data $\tilde{\phi}_{0,n} = g_n$, $\tilde{\phi}_{1,n} = h_n$. By uniqueness, $\phi_n = \tilde{\phi}_n \ast \delta_n$. On the other hand, the energy of $(f_n, \tilde{\phi}_n)$ is constant, and we conclude that

$$||f_n(t)||_{1, \text{kin}} + \frac{1}{2} ||\partial_t \phi_n(t)||_2^2 + \frac{1}{2} ||\nabla_x \phi_n(t)||_2^2 = C_1,$$

where $C_1$ does not depend on $n$ or $t$. The bound on $\partial_t \phi_n$ implies that $||\phi_n(t)||_2 \leq C_T$, for all $T > 0$ and $t \in [-T, T]$, where $C_T$ depends on $T$ but not on $n$. Hence for any $T > 0$,

$$||\phi_n(t)||_{H^1} \leq C_T, \; t \in [-T, T]. \quad (3.2)$$

Let $\phi_{\text{hom},n}$ denote the homogeneous part of the field in the modified system. By Sobolev estimates for the solution of the homogeneous wave equation and the Sobolev embedding theorem it follows that

$$||\phi_{\text{hom},n}(t)||_{\infty} \leq C_T, \; t \in [-T, T]. \quad (3.3)$$

Hence, the analogue of (2.10) and the fact that $\mu_n \ast \delta_n \ast \delta_n$ is non-negative imply that

$$0 \leq e^{\phi_n(t)} \leq e^{\phi_{\text{hom},n}(t)} \leq C_T, \; t \in [-T, T].$$
Hence
\[ \| \partial_t e^{\phi_n(t)} \|_2 \leq \| e^{\phi_n(t)} \|_\infty \| \partial_t \phi_n(t) \|_2 \leq C_T, \ t \in [-T, T] \]
and similarly for the \( x \)-derivatives. Hence, for any \( T > 0 \) and \( R > 0 \)
\[ \| e^{\phi_n(t)} \|_{H^1([-T,T] \times B_R)} \leq C_{T,R}, \| e^{\phi_n(t)} \|_{H^1(B_R)} \leq C_{T,R}, \ t \in [-T, T], \]  
(3.4)
clearly the same estimates hold for \( e^{2\phi_n} \). From Proposition 1, (3.3), and the fact that
the sequence \( (\phi_{0,n}^\text{in}) \) is uniformly bounded it follows that
\[ e^{-4\phi_n} f_n \leq C, \quad 0 \leq f_n, \leq C_T, \ t \in [-T, T]. \]  
(3.5)
Hence by Eq. (2.13) we conclude from (3.5) and (3.1) that
\[ \| \mu_n(t) \|_2 \leq C_T, \ t \in [-T, T]. \]  
(3.6)
By (2.14) we conclude that
\[ \| \rho_n(t) \|_{4/3}, \| j_n(t) \|_{4/3} \leq C, \ t \in \mathbb{R}. \]  
(3.7)

**Step 3: The weak limit.** Here, we use the estimates proved in Step 2 to obtain a
weakly convergent subsequence of \((f_n, \phi_n)\) whose limit will be a global weak solution.
The repeated extraction of suitable subsequences is not reflected in our notation. By
(3.1), (3.2), and (3.5)–(3.7), there exist
\[ f \in L^\infty(\mathbb{R}; L^1_{\text{kin}}(\mathbb{R}^6)) \cap L^\infty_{\text{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^6)), \phi \in L^\infty_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}^3)), \]
\[ \tilde{\mu} \in L^\infty_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^3)), \tilde{\rho}, \tilde{j} \in L^\infty(\mathbb{R}; L^{4/3}(\mathbb{R}^3)), \]
such that
\[ f \geq 0 \ \text{a.e.}, \ \partial_t \phi, \ \nabla_x \phi \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3)), \]
and up a subsequence and for all \( T > 0 \),
\[ f_n \rightharpoonup f \ \text{in} \ L^2([-T, T] \times \mathbb{R}^6), \]
\[ \phi_n \rightharpoonup \phi \ \text{in} \ H^1([-T, T] \times \mathbb{R}^3), \]
\[ \partial_t \phi_n \rightharpoonup \partial_t \phi \ \text{in} \ L^2([-T, T] \times \mathbb{R}^3), \]
\[ \mu_n \rightharpoonup \tilde{\mu} \ \text{in} \ L^2([-T, T] \times \mathbb{R}^3), \]
\[ \rho_n, j_n \rightharpoonup \tilde{\rho}, \tilde{j} \ \text{in} \ L^{4/3}([-T, T] \times \mathbb{R}^3). \]
By a standard diagonal sequence argument we can choose the subsequence and its limit independent of $T>0$. Next we observe that by (3.4) we can choose the subsequence such that $e^{\phi_n}$ converges weakly in $H^1(\cdot; T, T[\times B_R])$ for any $T>0$ and $R>0$. Since that space is compactly embedded in $L^4(\cdot; T, T[\times B_R])$ it follows that we can choose $\phi_n$ and $e^{\phi_n}$ to converge strongly in $L^4$, and by the Riesz–Fischer theorem also pointwise a.e. Hence

$$e^{\phi_n} \to e^{\phi} \text{ in } H^1_{loc}(\mathbb{R} \times \mathbb{R}^3)$$

and the same is true for $e^{2\phi}$. We shall prove now that $\tilde{\mu} = \mu$, $\tilde{\rho} = \rho$ and $\tilde{j} = j$ almost everywhere, where $\mu$, $\rho$ and $j$ are related to $(f, \phi)$ by (1.2), (2.5), and (2.6), respectively. As to $\tilde{\mu}$, we have for any $\chi \in C^\infty_c(\mathbb{R} \times \mathbb{R}^3)$ and any $R>0$,

$$\int \int \left( \int f \frac{dp}{\sqrt{1 + p^2}} - \tilde{\mu} \right) \chi \, dx \, dt$$

$$= \int \int \left( \int_{|p| \leq R} f \frac{dp}{\sqrt{1 + p^2}} - \tilde{\mu} \right) \chi \, dx \, dt + \int \int \int_{|p| > R} f \frac{dp}{\sqrt{1 + p^2}} \chi \, dx \, dt$$

$$= \lim_{n \to \infty} \int \int \left( \int_{|p| \leq R} f_n \frac{dp}{\sqrt{1 + p^2}} - \mu_n \right) \chi \, dx \, dt + \int \int \int_{|p| > R} f \frac{dp}{\sqrt{1 + p^2}} \chi \, dx \, dt$$

$$= \lim_{n \to \infty} \int \int \int_{|p| > R} f_n \frac{dp}{\sqrt{1 + p^2}} \chi \, dx \, dt + \int \int \int_{|p| > R} f \frac{dp}{\sqrt{1 + p^2}} \chi \, dx \, dt.$$

The last line can be estimated in modulus by

$$\frac{C}{R^2} \left( \sup_{n \in \mathbb{N}} \|f_n(t)\|_{1, \text{kin}} + \|f(t)\|_{1, \text{kin}} \right) \leq \frac{C}{R^2},$$

where the constant $C$ is independent of $R$. Since $R>0$ is arbitrary we conclude that $\tilde{\mu} = \int f (1 + p^2)^{-1/2} \, dp$ a.e., and

$$\mu = \int f \frac{dp}{\sqrt{1 + p^2}} \in L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^3)) \text{ with } \mu_n \to \mu \text{ in } L^2(\cdot; T, T[\times \mathbb{R}^3])$$

for any $T>0$. Finally, if we define $\sigma_n = \int f_n \, dp$ so that $\rho_n = e^{-\phi_n} \sigma_n$ then the sequence $(\sigma_n)$ is bounded in $L^\infty_{loc}(\mathbb{R}; L^{4/3}(\mathbb{R}^3))$, and hence as for $\mu$ we can show that $\sigma_n \to \int f \, dp$ in $L^{4/3}(\cdot; T, T[\times \mathbb{R}^3])$. On the other hand, we have already seen that $e^{\phi_n} \to e^{\phi}$ strongly in $L^4(\cdot; T, T[\times B_R])$. Hence $\sigma_n = e^{\phi_n} \rho_n \to e^{\phi} \tilde{\rho}$ in the sense of distributions,
and so $e^\phi \tilde{\rho} = \int f \, dp$ a.e. Thus we have proved that

$$\rho = e^{-\phi} \int f \, dp \in L^{\infty}(\mathbb{R}; L^{4/3}(\mathbb{R}^3)) \text{ with } \rho_n \to \rho \text{ in } L^{4/3}([-T, T] \times \mathbb{R}^3)$$

for any $T > 0$. It is obvious that the same assertion holds for $j$.

**Remark 1.** It is at this point that we need the $L^\infty$-bound on $f$ and hence on the initial data: Since the fact that $e^\phi \in L^4_{\text{loc}}(\mathbb{R}^3)$ seems optimal we must have $\rho$ in the dual space $L^{4/3}$ which would not be true if only an $L^q$-bound for $f$ with $q < \infty$, say $q = 2$, were available.

The pair $(\phi, \mu)$ satisfies the inhomogeneous wave equation (1.1) in the sense of distributions. In fact,

$$0 = \int \int (\partial_t^2 \phi_n - \Delta \phi_n + \mu_n \star \delta_n \star \delta_n) \chi \, dx \, dt$$

$$= \int \int (\phi_n \partial_t^2 \chi - \phi_n \Delta \chi + \mu_n (\delta_n \star \delta_n \star \chi)) \, dx \, dt \to \int \int (\phi \partial_t^2 \chi - \phi \Delta \chi + \mu \chi) \, dx \, dt$$

for any test function $\chi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3)$. The functions $\rho$ and $j$ satisfy the continuity equation (2.3) in the sense of distributions. This implies quite easily that there exists some constant to which the mass $\int \rho(t) \, dx$ is equal for almost all $t$, but we want to show that this constant is equal to the initial mass the point being that a-priori we have no continuity of the mass as a function of $t$. Conservation of mass in the sense of the theorem is shown in Step 6. The assertion on the total energy in item (iii) of the theorem is standard.

Passing to the limit in the linear part of the Vlasov equation (1.3) is not a problem.

To complete the proof that $(f, \phi)$ is a weak solution of (1.1)–(1.3), we need to take care of the non-linear terms in the Vlasov equation. This is done in the next step.

**Step 4: Momentum averaging.** Let $R > 0$ and $T > 0$, $\psi \in C_c^\infty(\mathbb{R}^3)$ with supp $\psi \subset B_R$, and fix $\xi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \xi \leq 1$ and $\xi = 1$ on $[-T, T]$. Define

$$\tilde{f}_n = \xi f_n, \quad g_0^n = (\xi' + \xi S\phi_n) f_n, \quad g_1^n = \xi F_n f_n,$$

where $F_n = (S\phi_n) p + (1 + p^2)^{-1/2} \nabla_x \phi_n$. Then we obtain

$$S\tilde{f}_n = g_0^n + \nabla_p \cdot g_1^n,$$

cf. Eq. (1.4). By the estimates of Step 3 the sequences $(g_0^n)$ and $(g_1^n)$ are bounded in $L^2(\mathbb{R} \times \mathbb{R}^3 \times B_R)$. Hence, by [5] the sequence

$$\int \tilde{f}_n(\cdot, \cdot, \cdot) \psi(p) \, dp$$
is bounded in $H^{1/4}(\mathbb{R} \times \mathbb{R}^3)$, see also [6,7]. Since for all $R' > 0$ and $T > 0$, $H^{1/4}([0, T] \times B_R)$ is compactly embedded in $L^2([0, T] \times B_R)$, we conclude that after extracting a subsequence, which, by a diagonal sequence argument, can be chosen independent of $T > 0$ and $R' > 0$,

$$\int f_n(\cdot, \cdot, p) \psi(p) \, dp \to \int f(\cdot, \cdot, p) \psi(p) \, dp,$$

strongly in $L^2([0, T] \times B_R)$.

Using this information we can prove that $(f, \phi)$ satisfies the Vlasov equation in form (1.4). To see this let $\chi \in C_c(\mathbb{R} \times \mathbb{R}^6)$ and $\psi \in C_c(\mathbb{R})$. Then for the non-linear part in (1.4) we have

$$\int \int \left[ \nabla_p \cdot (f_n F_n) + f_n S \phi_n \right] \chi(t, x) \psi(p) \, dp \, dx dt$$

$$= \int \int \left[ \int f_n (-p \cdot \nabla_p \psi + \psi) \, dp \right] \partial_t \phi_n$$

$$- \int f_n (p \cdot \nabla_p \psi \hat{p} + (1 + p^2)^{-1/2} \nabla_p \psi - \psi \hat{p}) \, dp \cdot \nabla_x \phi_n \right] \chi \, dx \, dt$$

$$\to \int \int \left[ \int f (-p \cdot \nabla_p \psi + \psi) \, dp \right] \partial_t \phi$$

$$- \int f (p \cdot \nabla_p \psi \hat{p} + (1 + p^2)^{-1/2} \nabla_p \psi - \psi \hat{p}) \, dp \cdot \nabla_x \phi \right] \chi \, dx \, dt,$$

note that the weight functions in the $p$ integrals of $f_n$ are compactly supported smooth functions as required by the momentum averaging argument. Hence the $p$ integrals converge strongly in $L^2([0, T] \times B_R)$, and since $\partial_t \phi_n$ and $\nabla_x \phi_n$ converge weakly in $L^2$ the assertion follows. By a standard density argument, we conclude that (1.4) is satisfied in $L^2(\mathbb{R} \times \mathbb{R}^6)$. This completes the proof of item (i) of the theorem.

**Remark 2.** In the application of the momentum averaging argument the $L^\infty$-bound on $f$ could have been avoided, using the concept of renormalization, cf. [5,6]. However, since we already needed this bound above, cf. Remark 1, we prefer to avoid this technical complication at this point.

**Step 5: Continuity in $t$.** Using the Vlasov equation in the form (1.4) for the approximating sequence we have, for any test function $\chi \in C_c(\mathbb{R}^6)$,

$$\int \int f_n(t) \chi \, dp \, dx = \int \int f_n^m \chi \, dp \, dx$$

$$+ \int_0^t \int f_n(s) [\hat{p} \cdot \nabla_x \chi + (S \phi_n)(s) \chi$$

$$- ((S \phi_n)(s) p + (1 + p^2)^{-1/2} \nabla_x \phi_n(s)) \cdot \nabla_p \chi] \, dp \, ds.$$
The convergence of \( f_n \) and \( \phi_n \) is strong enough to pass to the limit in the right-hand side of this equation, and dropping \( n \) in the right-hand side we can use the resulting expression to define a time-dependent distribution \( \tilde{f}(t) \in \mathcal{D}'(\mathbb{R}^6) \) which obviously is continuous in \( t \) with respect to the usual topology of \( \mathcal{D}' \), satisfies the initial condition \( \tilde{f}(0) = f^{in} \), and by construction coincides for almost all \( t \) with \( f(t) \). By a density argument the \( \mathcal{D}' \)-continuity extends to continuity with respect to the weak topology of \( L^2(\mathbb{R}^6) \).

A similar argument works for \( \phi \) in which case the continuous representative is defined by considering the first order formulation of the wave equation (1.1).

The stronger continuity assertion for \( f \) follows from the Arzela–Ascoli theorem: Since \( \partial_t \phi_n \) is bounded in \( L^\infty(\mathbb{R}; L^2(\mathbb{R}^3)) \) uniformly in \( n \) the sequence \( (\phi_n) \) is equicontinuous as a sequence of \( L^2(\mathbb{R}^3) \)-valued functions on \( \mathbb{R} \). Moreover, for each \( t \in \mathbb{R} \), \( (\phi_n(t)) \) is bounded in \( H^1(\mathbb{R}^3) \) which is compactly embedded in \( L^2(B_R) \).

Step 6: Conservation of mass. Since \( \partial_t \rho_n + \text{div} j_n = 0 \) and \( |j_n| \leq \rho_n \) we have for every \( R > 0 \) and \( t > 0 \),

\[
\frac{d}{dt} \int_{|x| > R + t} \rho_n(t) \, dx = - \int_{|x| = R + t} \rho_n(t) \, dS_x + \int_{|x| > R + t} \partial_t \rho_n(t) \, dx
\]

\[
= - \int_{|x| = R + t} \rho_n(t) \, dS_x - \int_{|x| > R + t} \text{div} j_n(t) \, dx
\]

\[
= - \int_{|x| = R + t} (\rho_n(t) + v \cdot j_n(t)) \, dS_x \leq 0,
\]

where \( v \) is the outer unit normal of the domain \( \{|x| > R + t\} \). The analogous argument works for \( t < 0 \) and the domain \( \{|x| > R - t\} \). Hence

\[
\int_{|x| > R + |t|} \rho_n(t) \, dx \leq \int_{|x| > R} \rho_n(0) \, dx, \quad t \in \mathbb{R}, \ R > 0, n \in \mathbb{N}. \tag{3.8}
\]

We claim that for almost all \( t \in \mathbb{R} \),

\[
\int \rho(t) \, dx = \int \rho(0) \, dx. \tag{3.9}
\]

Let \( \varepsilon > 0 \) be arbitrary. Since \( \rho(0) \) is integrable, then we can choose \( R > 0 \) such that

\[
\int_{|x| > R} \rho(0) \, dx < \varepsilon.
\]

By the convergence of the initial data and (3.8) we conclude that

\[
\int_{|x| > R + |t|} \rho_n(t) \, dx \leq \int_{|x| > R} \rho_n(0) \, dx < \varepsilon.
\]
for all \( t \in \mathbb{R} \) and all sufficiently large \( n \in \mathbb{N} \). Let \( A \subset \mathbb{R} \) be measurable and bounded and denote by \( \lambda(A) \) its Lebesgue measure. Then

\[
\int_A \int \rho(t) \, dx \, dt \geq \int_A \int_{|x| \leq R+|t|} \rho(t) \, dx \, dt = \lim_{n \to \infty} \int_A \int_{|x| \leq R+|t|} \rho_n(t) \, dx \, dt
\]

\[
= \lim_{n \to \infty} \int_A \left( \int \rho_n(t) \, dx - \int_{|x| > R+|t|} \rho_n(t) \, dx \right) \, dt
\]

\[
> \lambda(A) \left( \int \rho(0) \, dx - \varepsilon \right)
\]

and for sufficiently large \( S > 0 \) we have by monotone convergence,

\[
\int_A \int \rho(t) \, dx \, dt \leq \int_A \int_{|x| \leq S} \rho(t) \, dx \, dt + \lambda(A)\varepsilon
\]

\[
= \lim_{n \to \infty} \int_A \int_{|x| \leq S} \rho_n(t) \, dx \, dt + \lambda(A)\varepsilon
\]

\[
\leq \lambda(A) \left( \int \rho(0) \, dx + \varepsilon \right).
\]

Since \( A \) was an arbitrary, bounded, measurable subset of \( \mathbb{R} \) this implies that there exists a set \( M_\varepsilon \subset \mathbb{R} \) of measure zero such that

\[
\int \rho(0) \, dx - \varepsilon \leq \int \rho(t) \, dx \leq \int \rho(0) \, dx + \varepsilon, \quad t \in \mathbb{R} \setminus M_\varepsilon.
\]

Hence (3.9) holds on \( \mathbb{R} \setminus \bigcup_{k \in \mathbb{N}} M_{1/k} \).

**Remark 3.** The argument above makes use of the relativistic nature of the system, i.e., of the finite propagation speed of particles. We do not know if conservation of mass in the sense of (3.9) holds without this property. In particular the above argument would also establish conservation of charge for the relativistic Vlasov–Maxwell system, but not for its *non-relativistic* version in which velocity and momentum of the particles are equal. Note that the latter system is the one studied in detail in [5].

**Acknowledgments**

S. C. acknowledges support by the European HYKE network (contract HPRN-CT-2002-00282).
References