Contributions to multivariate analysis by Professor Yasunori Fujikoshi

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Abstract

The purpose of this article is to review the findings of Professor Fujikoshi which are primarily in multivariate analysis. He derived many asymptotic expansions for multivariate statistics which include MANOVA tests, dimensionality tests and latent roots under normality and nonnormality. He has made a large contribution in the study on theoretical accuracy for asymptotic expansions by deriving explicit error bounds. A large contribution has been also made in an important problem involving the selection of variables with introducing “no additional information hypotheses” in some multivariate models and the application of model selection criteria. Recently he is challenging to a high-dimensional statistical problem. He has been involved in other topics in multivariate analysis, such as power comparison of a class of tests, monotone transformations with improved approximations, etc.

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1. Introduction

Professor Fujikoshi received his B.S., M.S. and D.S. degrees in science from Hiroshima University. He spent a total 38 years at Hiroshima University, as a student from 1960 to 1966, as a research associate from 1966 to 1971, and as a professor from 1978 until his retirement in 2005. He was at Kobe University from 1971 to 1978. At present, he is a visiting professor at Chuo University and the University of Tokyo. He has made several excellent contributions in the area of...
statistics, especially multivariate analysis. He has served on the editorial board of many scientific journals. In particular, he served on the editorial board of the Journal of Multivariate Analysis from 1982 to 2002. He has supervised the doctoral work of 25 graduate students and researchers. Finally, he was President of the Japan Statistical Society from 2003 to 2005.

The purpose of this article is to review the findings of Professor Fujikoshi, which are primarily in multivariate analysis. For multivariate tests, there are three well-known test criteria: the likelihood ratio criterion $T_{LR}$, the Lawley–Hotelling trace criterion $T_{LH}$ and the Bartlett–Nanda–Pillai trace criterion $T_{BNP}$. Fujikoshi [9] obtained asymptotic expansions of the null and nonnull distributions of $T_{BNP}$ for the multivariate linear hypothesis and independence. The nonnull distribution of $T_{LR}$ was solved by Sugiura and Fujikoshi [103]. Furthermore, Fujikoshi [27] theoretically established asymptotic power comparison among invariant tests, as well as among the three criteria. He extended the results for the three types of statistics to dimensionality in MANOVA and canonical correlation models. The development of asymptotic theory under nonnormality was important. Related to this problem, Fujikoshi [19] derived an asymptotic expansion of the distribution of the latent roots of the sample covariance matrix. Furthermore, Fujikoshi [35] derived an asymptotic expansion for Hotelling’s test statistic, which was a breakthrough in the derivation of asymptotic expansions for several test statistics related to mean vectors under nonnormality.

Fujikoshi [24] succeeded in deriving explicit error bounds for an asymptotic expansion of the distribution of the MLE in the growth curve model. In addition, in cooperation with his colleagues, he has made a large contribution in deriving explicit error bounds for asymptotic expansions of a number of statistical distributions. The statistics they considered include the linear discriminant functions, $T_{LR}$, $T_{LH}$, as described by, for example, Fujikoshi [34] and Fujikoshi et al. [62]. Professor Fujikoshi is currently investigating an important problem involving the selection of variables with introducing “no additional information hypotheses” in some multivariate models and the application of model selection criteria. Fujikoshi and Satoh [56] proposed modified AIC and $C_p$-statistics for the multivariate regression model that are more faithful estimators of the risks of the selection of variables. One of his recent interests is the development of a high-dimensional statistical problem, as described by Fujikoshi et al. [42].

In the following, we present a detailed review of the above-mentioned contributions as well as a number of other contributions.

2. Asymptotic expansions for major multivariate test statistics

There are three major test statistics for the multivariate linear hypothesis on the mean parameter matrix:

- Likelihood ratio test: $T_{LR} = -n \log \{|S_e|/|S_e + S_h|\}$,
- Lawley–Hotelling’s statistic: $T_{LH} = n \text{tr} S_h S_e^{-1}$,
- Bartlett–Nanda–Pillai’s statistic: $T_{BNP} = n \text{tr} S_h (S_e + S_h)^{-1}$,

where $S_e$ and $S_h$ are independently distributed as a Wishart distribution $W_p(n, \Sigma)$ and a noncentral Wishart distribution $W_p(q, \Sigma; \Omega)$, respectively. In the following, we assume that $n \geq p$. We can assume that $\Sigma = I_p$ and $\Omega = \text{diag}(\omega_1, \ldots, \omega_p)$ without loss of generality. The above statistics are used for testing the hypothesis “$\Omega = O$”. Since the exact distributions of the test statistics are generally complicated, their asymptotic expansions have been investigated for the case in which $n$ is large. Box [5] derived an asymptotic expansion of the null distribution of $T_{LR}$. Asymptotic expansions of the null and nonnull distributions of $T_{LH}$ were given by Siotani [97,98] and Ito [72,73]. The moment of $T_{LR}$ was obtained by Constantine [6] in terms of the hypergeometric
function $\text{I}_1 F_1$ of the matrix argument, which is a power series of zonal polynomials reported by James [74,75]. Sugiura and Fujikoshi [103] derived asymptotic expansions of the nonnull distribution of $T_{LR}$ by expanding the characteristic function, which is expressed in terms of the hypergeometric function $\text{I}_1 F_1$ of the matrix argument. Moreover, they obtained formulas of weighted sums of zonal polynomials. Fujikoshi [9] derived asymptotic expansions of the null and nonnull distributions of $T_{BNP}$ by proving formulas of new types of weighted sums of zonal polynomials.

3. A class of test statistics and power comparison

The asymptotic expansions of the null and nonnull distributions of the three statistics allows us to compare the powers of the tests. Rothenberg [92] showed that if the value of

$$\gamma = \frac{\sigma_{\bar{\omega}}^2}{\bar{\omega}^2} - \frac{(p - 1)(p + 2)}{pq + 2} \left( \bar{\omega} = \sum_{i=1}^{p} \omega_i, \sigma_{\bar{\omega}}^2 = \frac{1}{p} \sum_{i=1}^{p} (\omega_i - \bar{\omega})^2 \right)$$

(3.1)

is positive, then the power of $T_{LH}$ is greater than that of $T_{LR}$, which in turn is greater than that of $T_{BNP}$, neglecting the terms of order $n^{-2}$. If the value of $\gamma$ is negative, then the order of the powers is reversed. Fujikoshi [27] extended Rothenberg’s result for a class of tests:

$$T = \sum_{i=1}^{p} h(\ell_i),$$

where $\ell_1, \ldots, \ell_p$ are the latent roots of $S_h S_e^{-1}$ and the function $h$ is an increasing function with $h(0) = 0, h(1)(0) = 1$.

Wakaki et al. [106] considered a class of test statistics for testing a general covariance structure, based on the sample covariance matrix. They hypothesized that the covariance matrix has a functional form $\Sigma(\xi)$ with an unknown parameter vector $\xi$. Let $h$ be a $C^4$–class function on $(0, \infty)$ that satisfies $h(1) = 0, h^{(1)}(0) = 1$. The test statistic associated with $h$ is defined by

$$T_h = n \inf_{\xi} \sum_{i=1}^{p} h(\ell_i),$$

where $\ell_1, \ldots, \ell_p$ are the latent roots of $S^{-1} \Sigma(\xi)$ and $S$ is the sample covariance matrix. They derived an asymptotic expansion of the null distribution of $T_h$ and found the condition for $T_h$ to be Bartlett-correctable.

4. Testing the hypothesis on dimensionality

The dimension in canonical discriminant analysis is defined by the number of significant discriminant functions, which is equal to the dimension of the linear space spanned by the contrasts of the mean vectors. Similarly, the dimension in the canonical correlation analysis is defined by the number of significant canonical variables and is equal to the rank of the covariance matrix between two random vectors. Let $S_h$ and $S_e$ be the between-classes matrix and the within-classes matrix, respectively, of sums of squares and products based on samples from $q + 1$ normal populations $N_p(\mu_i, \Sigma), (i = 1, \ldots, q + 1)$. In addition, let $\ell_1, \ldots, \ell_m$ $(m = \min\{p, q\})$ be the latent roots of $S_h S_e^{-1}$. The hypothesis that the dimension is $k$ in canonical discriminant analysis can be
tested using

\[ T_{LR; k} = -n \log \prod_{j=k+1}^{m} (1 + \ell_j), \]

Lawley–Hotelling’s statistic: \[ T_{LH; k} = n \sum_{j=k+1}^{m} \ell_j, \]

Bartlett–Nanda–Pillai’s statistic: \[ T_{BNP; k} = n \sum_{j=1}^{m} \ell_j / (1 + \ell_j). \]

Anderson [2] proved that \( T_{LR; k} \) is the likelihood ratio test by Lagrange’s method of undetermined multipliers. Fujikoshi [11] gave another algebraic proof and derived the likelihood ratio criteria for testing the dimension in canonical correlation analysis and the growth curve model. In addition, asymptotic expansions of the null and nonnull distributions were derived by Fujikoshi [17,20]. Moreover, Fujikoshi [28] found a general result of Rothenberg [92] regarding power comparison. The criterion corresponding to (3.1) is given by

\[ \gamma_k = \frac{\sigma^2_{\bar{\omega}}}{\bar{\omega}^2} - \frac{(p - k - 1)(p - k + 2)}{(p - k)(q - k) + 2} \left( \bar{\omega} = \sum_{i=k+1}^{p} \omega_i, \sigma^2_{\bar{\omega}} = \frac{1}{p} \sum_{i=k+1}^{p} (\omega_i - \bar{\omega})^2 \right). \]

Fujikoshi [17] also derived an asymptotic expansion for the likelihood ratio test statistic for the equality of the smallest latent roots of a covariance matrix in principal component analysis.

5. Asymptotic expansion of the distribution of latent roots

In discriminant analysis, it is important to derive the joint distribution of the latent roots of \( S_h S_v^{-1} \), where \( S_c \) and \( S_h \) are independently distributed as a Wishart distribution \( W_p(n, \Sigma) \) and a noncentral Wishart distribution \( W_p(q, \Sigma; \Omega) \), respectively. The exact null and nonnull distributions have been investigated extensively. However, the exact nonnull distributions are very complicated. The limiting distribution of the latent roots was derived by Hsu [70,71], while Fujikoshi [15,16] derived asymptotic expansions for the case in which all of the population latent roots are simple roots and for the case in which that some of the population latent roots are multiple roots. In these derivations, Fujikoshi proposed a method of deriving asymptotic expansions of the distributions of the sample latent roots when the population latent roots have multiplicity.

Professor Fujikoshi [15,16] also derived the asymptotic expansions of the joint distribution of the latent roots in principal component analysis and canonical correlations. The result in the principle component analysis is a generalization of the result by Anderson [3], who derived the limiting distribution, and the result by Sugiura [101], who derived an asymptotic expansion for the case in which all of the population roots are simple roots.

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6. Growth curve model

The growth curve model of Potthoff and Roy [87] is represented as

\[ Y = AX + E, \]

where \( Y \) is the \( N \times p \) matrix of observations, \( A \) and \( X \) are a between-individuals design matrix of size \( N \times k \) and a within-individuals design matrix of size \( q \times p \), respectively, \( \Xi \) is an unknown matrix of parameters, and \( E \) is an error matrix having rows that are independently distributed, a mean of zero, and common unknown covariance matrix \( \Sigma \). The matrices \( S_h \) and \( S_e \) of sums of squares and products for the hypothesis and the error, respectively, are defined as in MANOVA for testing linear hypothesis \( H_0 : CD = O \) with respect to \( H_1 : CD \neq O \) with known matrices \( C : c \times k \) and \( D : q \times p \). The test statistics \( T_{LR}, T_{LH} \) and \( T_{BNP} \) were proposed by Khatri [81] and Gleser and Olkin [67].

With respect to the joint distribution of \( S_h \) and \( S_e \), Fujikoshi [10] showed that

1. \( S_h | B \sim W_d(c, \Psi; \Omega^{1/2} B \Omega^{1/2}), \quad B \sim \beta_{p-q} \left[ \frac{1}{2} (n + p - q), \frac{1}{2} c \right] \)
2. \( S_e \) is independent with \( \{S_h, B\} \), and \( S_e \sim W_d(n, \Psi) \).

Using these results, he found that if the power of a test based on \( \tilde{S}_h \sim W_d(c, \Psi; \Omega) \) and \( S_e \) in MANOVA is an increasing function of the latent roots of \( \Omega \), then the power of the corresponding test based on \( S_h \) and \( S_e \) is also increasing. Fujikoshi [12] derived an asymptotic expansion of the nonnull distribution of \( T_{LR} \) using the fact that the characteristic function is represented in the term of the hypergeometric function \( _2F_2 \) of a matrix argument and satisfies a certain differential equation given by Fujikoshi [14]. Fujikoshi [13] also derived asymptotic expansions of the nonnull distributions of the three test statistics.

Fujikoshi [11] considered the following testing problem:

\[ H_{0k} : \text{rank}(CD) = k \quad \text{against} \quad H_{1k} : \text{rank}(CD) \neq k \]

and showed that the likelihood ratio test is given by \( T_{LR;k} \) (see Section 4). He also proposed other tests \( T_{LH;k}, T_{BNP;k} \), and derived asymptotic expansions of the null and nonnull distributions of these three test statistics.

Other studies on the extended growth curve model include the following: Fujikoshi et al. [45], Yokoyama and Fujikoshi [112], Yokoyama and Fujikoshi [113], Fujikoshi and Satoh [55], Fujikoshi [36], Fujikoshi et al. [44], Kanda et al. [78], Fujikoshi et al. [52], and Fujikoshi and von Rosen [65].

7. Selection of variables in multivariate models

There are several types of variables in the variable selection problem in multivariate analysis. For example, in multivariate regression analysis we want to select the best subset of explanatory variables, whereas in discriminant analysis or inverse regression, we want to select the best subset of main variables or response variables. In the growth curve model, the best subset of covariates, as well as the explanatory variables, is explored. In canonical analysis, it is important to decide the number of significant canonical variables.

Professor Fujikoshi developed the following approach. First, a class of parametric models is formulated such that a subset of variables is useful, but the remainder variables are redundant.
or have no additional information. A model selection criterion is then applied to the class of parametric models. Fujikoshi extended the concept proposed by Rao [88,89] in two-group discriminant analysis to various models, including multiple discriminant analysis, canonical correlation analysis, the multivariate inverse model, and the growth curve model. Fujikoshi [30] gave a good review of the redundancy of variables in multivariate analysis.

Along the above lines, Fujikoshi posed selection criteria for several problems of selecting variables based on the AIC ([1]) and \( C_p \) ([84]). Fujikoshi and Veitch [63] derived criteria for selecting canonical variables in canonical correlation analysis and canonical discriminant analysis. Fujikoshi [23,25] derived criteria for selecting the main variables and gave the asymptotic properties thereof in discriminant analysis, canonical correlation analysis, and multivariate linear models. Fujikoshi [26] showed that the AIC-type criterion is asymptotically equivalent to minimizing misclassification probabilities in discriminant analysis for two groups. Fujikoshi and Nishii [51] treated a problem of selecting the main variables in multivariate inverse regression analysis, and Fujikoshi and Rao [54] treated a problem of selecting covariates in growth curve models.

The AIC is a first-order unbiased estimator of the risk based on Kullback–Leibler divergence. Sugiura [102] presented corrected AICs in some normal models. The CAIC is unbiased for an over-specified candidate model, where a candidate model is referred to as over-specified or under-specified according to whether or not the model involves the true model. Fujikoshi and Satoh [56] proposed the modified \( C_p \) as well as the modified AIC in multivariate linear model taking into account the fact that the true model may not be included in a subset of candidate models but is included in the full model. For example, consider a \( p \)-variate linear regression model \( M_F \) with \( k \) explanatory variables, and a candidate model \( M_J \) with \( j \) explanatory variables. Then the modified AIC and \( C_p \) for \( M_J \) were proposed as

\[
MAIC = n \log |n^{-1}S_J| + np(\log 2\pi + 1) + \frac{2(pj + p(p + 1)/2)}{1 - (j + p + 1)/n} + 2j \text{tr} R_{FJ} - (\text{tr} R_{FJ})^2 - \text{tr} R_{FJ}^2,
\]

\[
MC_p = (n - k - p - 1) \text{tr} S_F^{-1} S_J + 2pj + p(p + 1),
\]

where \( n \) is the sample size, \( S_F \) and \( S_J \) are the matrices of residual sums of squares and products under \( M_F \) and \( M_J \), respectively, and \( R_{FJ} = [(n - j)/(n - k)]S_F S_J^{-1} \). Similar criteria were also proposed by Satoh et al. [93] in the growth curve model and by Yanagihara et al. [111] in logistic regression models.

Fujikoshi has made contributions to tests of redundancy, or the no-additional-information hypothesis. Fujikoshi [22] derived the likelihood ratio criterion for the redundancy hypothesis in the canonical correlation model. Fujikoshi and Khatri [46] treated a similar problem in discriminant analysis with covariates. Fujikoshi et al. [47] reported an asymptotic distribution of additional information in some multivariate models.

8. Error bounds for asymptotic expansions

Fujikoshi, along with his colleagues, has made a large contribution to the derivation of explicit and computable error bounds for asymptotic expansions of the distributions of some multivariate statistics. First, we explain in detail the meaning of explicit error bounds for asymptotic expansions. Let \( F_n(x) \) be a distribution function depending on some parameter \( n \), typically a sample size.
Suppose that \( F_n(x) \) can be approximated by an asymptotic expansion with \( k \) terms. Generally,

\[
G_{k,n}(x) = G(x) + \sum_{j=1}^{k-1} n^{-j/2} p_j(x) g(x),
\]

where \( G(x) \) is the limit distribution function of \( F_n(x) \) as \( n \to \infty \), \( g(x) \) is the pdf of \( G(x) \), and \( p_j(x) \) are suitable polynomials. For a wide class of statistics the error \( R_{k,n}(x) = F_n(x) - G_{k,n}(x) \) has been proved to satisfy

\[
R_{k,n}(x) = O(n^{-k/2}), \tag{8.1}
\]

uniformly in \( x \). Note that the order estimate of \( R_{k,n} \) is qualitative rather than quantitative. It does not give any explicit error bounds for a given \( n \). In fact, according to definition, (8.1) is equivalent to stating that there exists a positive constant \( C_k \) and a positive number \( N_k \) such that

\[
|R_{k,n}(x)| \leq n^{-k/2} C_k \quad \text{for all} \quad n \geq N_k. \tag{8.2}
\]

The values of \( C_k \) and \( N_k \) are usually not specified. In some cases, \( F_n \) depends on the population parameter \( \theta \in \Theta \). Then, \( C_k \) and \( N_k \) may depend on \( \theta \) as well, i.e., \( C_k = C_k(\theta) \), \( N_k = N_k(\theta) \). However, the functions \( C_k(\theta) \) and \( N_k(\theta) \) remain unknown. The information about the error bound becomes more definite when an explicit constant \( C_k \) can be found for a given \( N_0 \) such that (8.2) holds for all \( n \geq N_0 \). Here, \( C_k \) may depend on \( \theta \), but it is a function of \( \theta \) in explicit form without other unknown constants. Such a bound is referred to as an explicit or computable error bound, or simply as an error bound.

Fujikoshi [24] succeeded in deriving an explicit error bound for an asymptotic expansion of the distribution of the MLE in the growth curve model, and, together with his colleagues, he has attempted to extend his results to a wide class of statistical distributions. The error bounds have been obtained for asymptotic expansions of the distributions of the maximums of correlated \( t \)- and \( F \)-distributions and the linear discriminant function, \( T_{LR} \), \( T_{LH} \), as reported by, for example, Fujikoshi [29,32], Fujikoshi and Mukaihata [49], Fujikoshi et al. [62], Fujikoshi and Ulyanov [60], and Ulyanov et al. [105]. These distributions are expressed as distributions of univariate or multivariate scale mixtures and the functions thereof. An error bound is given for an asymptotic expansion of the distribution of a scale mixture \( X = Sz \), where \( S \) and \( Z \) are independent, \( S \) is a positive random variable that takes values near one with high probability, and \( Z \) has a smooth distribution, such as a normal distribution or a chi–square distribution. This error bound was extended to the multivariate case by Fujikoshi and Shimizu [58,59] and Fujikoshi et al. [62]).

The error bound for \( T_0^2 = n \text{tr } S_n S_e^{-1} \) or Lawley–Hotelling’s statistic (see Section 2) was given as follows:

\[
|P(T_0^2 \leq x) - P(x_{pq}^2 \leq x)| \leq \frac{1}{2n} p(p + 1) 
\times \left[ \left( \xi_2(q) + (\xi_1(q) + 2)^{1/2} \right)^2 + \frac{1}{2} \xi_1(q)(\xi_1(q) + 2) \right],
\]

where \( \xi_1(q) = (1/2) \int_0^\infty |x - q| g_q(x) \, dx \), \( \xi_2(q) = (1/8) \int_0^\infty |x^2 - 2qx + q(q - 2)| g_q(x) \, dx \), and \( g_q \) is the density function of \( X_q^2 \)-variate. Furthermore, the result was extended to an asymptotic expansion of the distribution of \( T_0^2 \) up to the order \( O(n^{-1}) \).
Fujikoshi [33] reviewed error bounds for asymptotic expansions. The results including the above-mentioned results will be published as a monograph by Fujikoshi et al. [61].

9. Asymptotic expansions under nonnormality

Several procedures in multivariate analysis require normality of the underlying distributions. It is important to investigate the affect of the violation of the assumption of normality. Fujikoshi has revealed these affects by deriving asymptotic expansions of the statistics used in multivariate analysis under non-normality.

Fujikoshi [19] derived an asymptotic expansion for the latent roots of the sample covariance matrix. Seo et al. [94,95] derived asymptotic expansions for the latent roots in canonical correlation analysis and MANOVA when the underlying distributions are elliptical. Fujikoshi [35] and Kano [79] independently derived an asymptotic expansion for Hotelling’s $T^2$ statistic, where the assumptions for the underlying distribution were only the smoothness and existence of the fourth-order moments. Fujikoshi, together with his colleagues, derived asymptotic expansions under nonnormality for MANOVA tests, tests on multivariate linear hypothesis, and a test on the hypothesis on additional information, among others. These expansions are described in a number of studies, including Fujikoshi [39,41], Wakaki et al. [108], Gupta et al. [69].

10. Monotone transformations

Let $T$ be a statistic distributed asymptotically as a $\chi^2_f$ distribution as $n$ tends to infinity. Bartlett adjustment improves the $\chi^2$-approximation in the sense that $E[\bar{T}] = f + o(n^{-1})$. The Bartlett adjustment for the likelihood ratio test improves the approximation in the sense of distribution, that is $E(\tilde{T}) = P(\bar{T}^2 \leq f) + o(n^{-1})$. However, with respect to the Bartlett adjustment for Lawley–Hotelling’s $T_{\text{LH}}$ and Bartlett–Nanda–Pillai’s $T_{\text{BNP}}$, the improvements are only for the expectations. Fujikoshi [34] considered monotone transformations to improve $\chi^2$-approximations with respect to the distribution for statistics of which the distribution functions have asymptotic expansions of the form:

$$P(T \leq x) = G_f(x) + \frac{1}{n} \sum_{j=0}^{k} a_j G_{f+2j} + o(n^{-1}),$$

where $G_r$ is the distribution function of $\chi^2_r$. He found desired monotone transformations for the case in which $k = 2, 3$. For the case of general $k$, Fujisawa [66] and Kakizawa [76] gave different types of monotone transformations independently. In some cases, the coefficients $a_j$ depend on unknown parameters, and the monotone transformations with estimated coefficients do not perform well. Fujikoshi [38] considered the problem of adjusting the first- and second-order moments and gave such monotone transformations.

11. High-dimensional statistical problems

Classical methods in multivariate analysis assume the sample size $n$ to be greater than the dimension $p$ of the vector of variables. Approximation formulas based on large-samples asymptotic theory perform worse as $p$ increases. Fujikoshi and Seo [57] gave approximation formulas of the distribution functions in a class of discriminant functions that involve the linear and the quadratic
discriminant functions in a framework of $p/n \rightarrow c \in (0, 1)$ and found that the approximations are very accurate, even in case of small $p$. Fujikoshi [37] derived error bounds for high-dimensional approximation formulas of the misclassification probabilities of the linear discriminant function. Tonda and Fujikoshi [104] derived a high-dimensional asymptotic expansion for the likelihood ratio test in MANOVA, and Wakaki et al. [107] derived a high-dimensional asymptotic expansion for the null and nonnull distributions of $T_{LR}$, $T_{LH}$ and $T_{BNP}$. Fujikoshi [40] treated a problem of correcting the AIC for selecting variables in discriminant analysis for a high-dimensional case.

Recently, problems of analyzing data such that $n << p$ are frequently encountered. Fujikoshi et al. [42] derived an asymptotic expansions of the test statistic $T_D = (\text{tr} S_h)/(\text{tr} S_e)$ proposed by Dempster [7,8] when both $n$ and $p$ are large and found that $T_D$ performs better than the classical three test statistics when the variation of the latent roots of the covariance matrix is small. Srivastava and Fujikoshi [100] proposed high-dimensional tests in the MANOVA model using the Moore–Penrose Inverse $S_e^+$, rather than $S_e^{-1}$, because the usual inverse of $S_e$ does not exist.

12. Concluding remarks

Although we herein examined only studies related to multivariate analysis, Professor Fujikoshi has been involved in several studies in other areas of statistics, such as multivariate inference, economics, time series analysis, experimental design, inference of directional data, corresponding analysis, and applications to actual data. For some of these studies, see Fujikoshi and Isogai [43], Fujikoshi and Ochi [53], Siotani and Fujikoshi [99], Fujikoshi and Nishii [50], Fujikoshi et al. [48], Kariya et al. [80], Fujikoshi and Watamori [64], Otake et al. [86], Fujikoshi [31], Seo et al. [96], Naito et al. [85], Kanda and Fujikoshi [77], Gupta et al. [68], etc. At the time of his retirement of Hiroshima University, the number of published papers written by Professor Fujikoshi was 124, and after retirement, he has written another eight papers. We hope to introduce his future achievements on his 70th birthday.

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