Electroelastic behavior of doubly periodic piezoelectric fiber composites under antiplane shear

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Abstract

The piezoelectric composites with a doubly periodic parallelogrammic array of piezoelectric fibers are dealt with under antiplane shear coupled with inplane electrical load. A rigorous analytical method is developed by using the doubly quasi-periodic Riemann boundary value problem theory integrated with the eigenstrain and eigen-electrical-field concepts. The numerical results are presented and a comparison with finite element calculations, experimental data and micromechanical analysis is made to demonstrate the efficiency and accuracy of the present method. Subsequently, the present solutions are used to study two important topics in piezoelectric fiber composites, i.e., (1) stress and electrical field fluctuations in the microstructure, (2) the macroscopic effective electroelastic moduli. The relation between the macroscopic properties of the composites and their microstructural parameters is discussed and many interesting electroelastic interaction phenomena are revealed, which are useful to estimate and optimize the performance of piezoelectric composites.

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1. Introduction

During the last two decades, the application of piezoelectric composites has been dramatically increasing with the development of smart materials and structures. Recently, much attention has been given to piezocomposites with a fine periodic structure. These composites have many important applications, for example, they allow higher operating frequencies, and thus increase the resolution in medical imaging transducers (Chan and Guy, 1994; Janas and Safari, 1995).

Different methods are available to study materials with periodic microstructures. Aboudi (1991) developed a unified micromechanical theory based on the study of interacting periodic cells. In his work, homogeneous boundary conditions (plane-remains-plane) were imposed on the unit cell boundary. These boundary condi-
tions are approximate, because they are over-constrained and violate the stress/strain periodicity conditions (Xia et al., 2003; Needleman and Tvergaard, 1993). The Fourier series expansion technique (Nemat-Nasser and Hori, 1999) is an important tool in analyzing periodic composites. However, in general, it is a difficult task to determine the exact Fourier coefficient of the corresponding eigenstrain for solids with periodic inclusions (see Nemat-Nasser and Taya, 1981, 1985, and Nunan and Keller, 1984, for a discussion of alternative methods of solution). A good approximation may be obtained if the eigenstrain is replaced by its average value and the result is entered into the consistency condition (Nemat-Nasser and Hori, 1999). The asymptotic homogenization theory (Benssousan et al., 1978; Bakhalov and Panasenko, 1984; Suquet, 1987; Hori and Nemat-Nasser, 1999; Bravo-Castillero et al., 2001; Sabina et al., 2001) can be regarded as a mathematical model. It uses explicit periodic boundary conditions in modeling materials with periodic microstructures and can yield more accurate results. The finite element method (FEM) has been extensively used to determine the mechanical properties of composites by analyzing a periodic unit cell (see, for instance, Adams and Crane, 1984; Bonora et al., 1994; Xia et al., 2003). A difficulty in FEM analysis is how to impose reasonable unit cell boundary conditions, which would satisfy the displacement and traction periodicity. Xia et al. (2003) pointed out that the homogeneous displacement boundary conditions overestimate the effective moduli whereas the homogeneous traction boundary conditions underestimate the effective moduli. At the same time, the application of the homogeneous displacement boundary generally would not guarantee to produce a periodic boundary traction, and the application of the homogeneous traction boundary conditions would not guarantee the displacement periodicity at the boundaries. They presented an explicit unified form of boundary conditions for a periodic representative volume element and obtained results with high accuracy. Their method needs to apply a large number of the constraint equations, which can be inserted in a FEM package by using certain automatic schemes. Berger et al. (2005) dealt with the modeling of 1–3 periodic composites made of piezoceramic (PZT) fibers in a soft non-piezoelectric matrix (polymer). They presented two ways, an analytical method based on the asymptotic homogenization method and a numerical approach based on the finite element method. Special attention is given on the definition of appropriate boundary conditions for the unit cell to ensure periodicity.

The theory of doubly periodic and doubly quasi-periodic Riemann boundary problems provides an elegant analytical method (Lu, 1993; Li, 1999), which only requires prescribing far-field conditions. The inner boundary conditions of a unit cell are shown to depend on the microstructures and can be automatically determined by the method. Unfortunately, the method is only suitable for homogeneous materials, and it cannot provide useful numerical results for general problems of materials with periodic inhomogeneities. This hinders the application of the method in analysis of composites.

In the present work, by using Eshelby’s equivalent inclusion concept, problems of composites with doubly periodic piezoelectric fibers are transformed into ones of homogeneous materials with doubly periodic eigenstrains and eigen-electrical-fields. A rigorous analytical method is developed for piezoelectric composites with doubly periodically distributed piezoelectric fibers under antiplane shear coupled with inplane electrical field. The numerical results are presented and a comparison with finite element calculations, experimental data and micromechanical analysis is made to demonstrate the efficiency and accuracy of the present method. In the FEM analysis, to examine the finite boundary influence, the computational object contains 1, 9, 25, 49 (repeated) cell or cells, respectively, and the variation of the results for the central cell with the cell number is shown. Subsequently two important topics in analysis of such composites are studies: (1) stress and field fluctuations in the microstructure, (2) the macroscopic effective electroelastic moduli.

2. Problem and basic equations

2.1. Statement of the problem

As shown in Fig. 1, the cross-section of a piezoelectric composite with a doubly periodic array of piezoelectric fibers lies on the complex plane, \( z = x + iy \). \( 2\omega_1 \) and \( 2\omega_2 \) denote two fundamental periods with \( \text{Im} \omega_2 / \omega_1 > 0 \), where \( \text{Im} \) denotes the imaginary part. \( P_{90} \) is the fundamental parallelogram (one of the largest parallelograms without congruent points) or fundamental cell with the boundary \( \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \) oriented clockwise and the vertices \( \omega_1 + \omega_2, -\omega_1 + \omega_2, -\omega_1 - \omega_2 \) and \( \omega_1 - \omega_2 \). Without loss of generality, assume
that the fundamental cell and the circular fiber within it share the common center (the origin), as otherwise, a suitable coordinate translation may be introduced. Let \( S^+ \) denote the region occupied by the fiber bounded by the contour \( L_0 \) oriented counter-clockwise, \( S^+/C_0 = \Gamma_0 \) and \( S^\pm \) represents the union of \( S^\pm \) and its periodically congruent regions, and \( L \) represents \( L_0 \) and its periodically congruent contours. Assume that the piezoelectric composite is poled in the fiber direction and subjected to longitudinal shear stresses \( \tau_{xz}^\infty, \tau_{yz}^\infty \) at infinity (which are not depicted in the figure).

2.2. Basic equations

For a piezoelectric composite under antiplane shear coupled with inplane electrical field (Fig. 1), there are only the non-trivial antiplane displacement \( w \), strain components \( \gamma_{xz} \) and \( \gamma_{yz} \), stress components \( \tau_{xz} \) and \( \tau_{yz} \), inplane electrical potential \( \phi \), electrical field components \( E_x \) and \( E_y \), electrical displacement components \( D_x \) and \( D_y \), with all field quantities being only the functions of coordinates \( x \) and \( y \).

The strain–displacement relation can be written as

\[
\gamma_{xz} = 2\varepsilon_{xz} = \frac{\partial w}{\partial x}, \quad \gamma_{yz} = 2\varepsilon_{yz} = \frac{\partial w}{\partial y}.
\]

The electrical field components are related to the electrical potential by

\[
E_x = -\frac{\partial \phi}{\partial x}, \quad E_y = -\frac{\partial \phi}{\partial y}
\]

The mechanically and electrically coupled constitutive equations (Tiersten, 1969) can be expressed as

\[
\tau_{xz} = C_{44} \frac{\partial w}{\partial x} + e_{15} \frac{\partial \phi}{\partial x} = C_{44} \gamma_{xz} - e_{15} E_x, \quad \tau_{yz} = C_{44} \frac{\partial w}{\partial y} + e_{15} \frac{\partial \phi}{\partial y} = C_{44} \gamma_{yz} - e_{15} E_y,
\]

\[
D_x = e_{15} \frac{\partial w}{\partial x} - d_{11} \frac{\partial \phi}{\partial x} = e_{15} \gamma_{xz} + d_{11} E_x, \quad D_y = e_{15} \frac{\partial w}{\partial y} - d_{11} \frac{\partial \phi}{\partial y} = e_{15} \gamma_{yz} + d_{11} E_y.
\]
where \( C_{44} \) is the longitudinal shear modulus at a constant electrical field, \( e_{15} \) and \( d_{11} \) are the piezoelectric modulus and dielectric modulus at a constant strain, respectively. The equilibrium equation and charge equation (Tiersten, 1969) can be reduced to the harmonic equations

\[
\nabla^2 w = 0, \quad \nabla^2 \varphi = 0,
\]

where \( \nabla^2 = (\partial^2 / \partial x^2 + (\partial^2 / \partial y^2) \) is the Laplace operator. Eq. (5) shows that the antiplane displacement \( w \) and electrical potential \( \varphi \) can be expressed by two analytical functions \( F(z) \) and \( \Phi(z) \), respectively,

\[
w = \text{Re} F(z) = \frac{1}{2} [F(z) + \overline{F(z)}], \quad \varphi = \text{Re} \Phi(z) = \frac{1}{2} [\Phi(z) + \overline{\Phi(z)}]
\]

where \( z = x + iy \), \( \text{Re} \) denotes the real part of a complex function and the overbar denotes the conjugate. Substituting Eq. (6) into Eqs. (3) and (4), and casting them into complex form, we obtain

\[
\tau_{xz} - i\tau_{yz} = C_{44} \frac{dF(z)}{dz} + e_{15} \frac{d\Phi(z)}{dz}, \quad D_x - id_x = e_{15} \frac{dF(z)}{dz} - d_{11} \frac{d\Phi(z)}{dz}.
\]

To facilitate the following analysis, we introduce

\[
T_x = - \int_A^B \left( \tau_{xz} \, dx - \tau_{yz} \, dy \right) = \text{Im} \{ e_{15} F(z) \} |_A^B, \quad T_D = - \int_A^B \left( D_x \, dx - D_y \, dy \right) = \text{Im} \{ e_{15} F(z) - d_{11} \Phi(z) \} |_A^B,
\]

where \( A \) is a fixed point, \( B \) is a moving point and the path \( AB \) does not go across the interface between adjacent phases.

3. Analysis and solution

3.1. Eigenstrain, eigen-electric-field and equivalent homogeneous piezoelectric medium

To formulate the problem, the concepts of the eigenstrain, eigen-electrical-field and equivalent piezoelectric medium are introduced (refer to Dunn and Taya, 1993).

Consider two equivalent electroelastic cases:

**Case 1.** An infinite homogeneous piezoelectric solid with electroelastic constants \( C_{44}^M, e_{15}^M, d_{11}^M \) (the same as the matrix) is subjected to uniform far-field antiplane antiplane shear stresses \( \tau_{xz}^\infty, \tau_{yz}^\infty \) coupled with inplane electrical displacements \( D_x^\infty, D_y^\infty \). Apparently, in the entire plane, the stresses and electrical displacements are uniform, i.e., \( \tau_{xz}^0 = \tau_{xz}^\infty, \tau_{yz}^0 = \tau_{yz}^\infty, D_x^0 = D_x^\infty, D_y^0 = D_y^\infty \). From Eqs. (3) and (4), the strain and electrical field components are also uniform

\[
\begin{align*}
\gamma_{xz}^0 &= \frac{d_{11}^M \tau_{xz}^\infty + e_{15}^M D_x^\infty}{d_{11}^M C_{44}^M + (e_{15}^M)^2}, \\
\gamma_{yz}^0 &= \frac{d_{11}^M \tau_{yz}^\infty + e_{15}^M D_y^\infty}{d_{11}^M C_{44}^M + (e_{15}^M)^2}, \\
E_x^0 &= \frac{C_{44}^M D_x^\infty - e_{15}^M \tau_{yz}^\infty}{d_{11}^M C_{44}^M + (e_{15}^M)^2}, \\
E_y^0 &= \frac{C_{44}^M D_y^\infty - e_{15}^M \tau_{xz}^\infty}{d_{11}^M C_{44}^M + (e_{15}^M)^2}.
\end{align*}
\]

For an actual piezoelectric fiber composite with fiber electroelastic constants \( C_{44}^l, e_{15}^l, d_{11}^l \) as shown in Fig. 1, the presence of the doubly periodic region \( S^+ \) with a different electroelasticity, i.e., the existence of a material mismatch, disturbs the uniform strains, stresses, electrical displacements and electrical fields. In this case, the constitutive equations (3) and (4) can be written as

\[
\begin{align*}
\tau_{xz}^0 + \tau'_{xz,im} &= C_{44}^l (\gamma_{xz}^0 + \gamma'_{xz,im}) - e_{15}^l (E_x^0 + E_{x,im}), \\
\tau_{yz}^0 + \tau'_{yz,im} &= C_{44}^l (\gamma_{yz}^0 + \gamma'_{yz,im}) - e_{15}^l (E_y^0 + E_{y,im}), \\
D_x^0 + D'_{x,im} &= e_{15}^l (\gamma_{xz}^0 + \gamma'_{xz,im}) + d_{11}^l (E_y^0 + E_{y,im}), \\
D_y^0 + D'_{y,im} &= e_{15}^l (\gamma_{yz}^0 + \gamma'_{yz,im}) + d_{11}^l (E_x^0 + E_{x,im}),
\end{align*}
\]

in \( S^+ \),
where the superscript prime denotes the corresponding disturbance quantity and the subscripts “in” and “out” denote the quantities corresponding to the fibers and matrix, respectively.

**Case 2.** An infinite homogeneous piezoelectric solid with electroelastic constants \( C_{44}^M \), \( e_{15}^M \), \( d_{11}^M \) (the same as the matrix) is subjected to uniform far-field antiplane shear stresses \( \tau_{xz}^0 \), \( \tau_{yz}^0 \) coupled with inplane electrical displacements \( D_x^0 \), \( D_y^0 \). Instead of dealing with the presence of actual doubly periodic fibers with a different electroelasticity, we introduce a suitable eigenstrain field \( \gamma_{xz}^0 \), \( \gamma_{yz}^0 \) and a suitable eigen-electrical-field \( E_x^0 \), \( E_y^0 \) in \( S^- \), such that the equivalent homogeneous solid has the same electromechanical coupling field as the actual doubly periodic piezoelectric composite under the applied antiplane shear stress and inplane electrical field at infinity, whichever may be the case. The introduction of the eigenstrain and eigen-electrical-field disturbs the uniform electromechanical field, with the disturbance field quantities being denoted by

\[
\begin{align*}
\gamma_{xz,\text{in}}'' &+ \gamma_{yz,\text{in}}'' = E_{x,\text{in}}'' + E_{y,\text{in}}'' \quad \text{in } S^+, \\
\gamma_{xz,\text{in}}'' &+ \gamma_{yz,\text{in}}'' = E_{x,\text{in}}'' + E_{y,\text{in}}'' \quad \text{in } S^- \nonumber
\end{align*}
\]

In this case, the constitutive equations (3) and (4) can be written as

\[
\begin{align*}
\tau_{xz}^0 + \gamma_{xz,\text{in}}'' &+ \gamma_{yz,\text{in}}'' - \tau_{xz,\text{in}}^0 = C_{44}^M (\gamma_{xz}^0 + \gamma_{yz}^0 - \gamma_{xz,\text{in}}^0) - e_{15}^M (E_{x,\text{in}}'' + E_{y,\text{in}}'' - E_{x,\text{in}}^0) \quad \text{in } S^+, \nonumber \\
\tau_{yz}^0 + \gamma_{yz,\text{in}}'' &+ \gamma_{xz,\text{in}}'' = C_{44}^M (\gamma_{yz}^0 + \gamma_{xz}^0 - \gamma_{yz,\text{in}}^0) - e_{15}^M (E_{y,\text{in}}'' + E_{x,\text{in}}'') \quad \text{in } S^+ \nonumber \\
D_x^0 + D_{x,\text{in}}'' &+ d_{11}^M (E_{x,\text{in}}'' + E_{y,\text{in}}'') \quad \text{in } S^- \nonumber \\
D_y^0 + D_{y,\text{in}}'' &+ d_{11}^M (E_{y,\text{in}}'' + E_{x,\text{in}}'') \quad \text{in } S^- \nonumber \\
\end{align*}
\]

The conditions of strain and electrical field equivalence for Cases 1 and 2 require

\[
\begin{align*}
\gamma_{xz,\text{out}}'' &+ \gamma_{yz,\text{out}}'' = \gamma_{xz,\text{out}}'' + \gamma_{yz,\text{out}}'' \quad \text{in } S^+, \nonumber \\
\gamma_{xz,\text{out}}'' &+ \gamma_{yz,\text{out}}'' = \gamma_{xz,\text{out}}'' + \gamma_{yz,\text{out}}'' \quad \text{in } S^- \nonumber
\end{align*}
\]

Noting Eq. (17), a comparison of Eqs. (13) and (15) shows that the stresses and electrical displacements in \( S^- \) are the same for the two cases. Noting Eqs. (12), (14) and (16), the conditions of stress and electrical displacement equivalence in \( S^- \) require

\[
\begin{align*}
C_{44}^M (\gamma_{xz,\text{in}}'' + \gamma_{yz,\text{in}}'') - e_{15}^M (E_{x,\text{in}}'' + E_{y,\text{in}}'') & = C_{44}^M (\gamma_{xz,\text{in}}'' + \gamma_{yz,\text{in}}'') - e_{15}^M (E_{x,\text{in}}'' + E_{y,\text{in}}'') \quad \text{in } S^+, \\
C_{44}^M (\gamma_{yz,\text{in}}'' + \gamma_{xz,\text{in}}'') - e_{15}^M (E_{y,\text{in}}'' + E_{x,\text{in}}'') & = C_{44}^M (\gamma_{yz,\text{in}}'' + \gamma_{xz,\text{in}}'') - e_{15}^M (E_{y,\text{in}}'' + E_{x,\text{in}}'') \quad \text{in } S^+ \nonumber \\
e_{15}^M (\gamma_{xz,\text{in}}'' + \gamma_{yz,\text{in}}'') + d_{11}^M (E_{x,\text{in}}'' + E_{y,\text{in}}'') & = e_{15}^M (\gamma_{xz,\text{in}}'' + \gamma_{yz,\text{in}}'') + d_{11}^M (E_{y,\text{in}}'' + E_{x,\text{in}}'') \quad \text{in } S^- 
onumber \\
e_{15}^M (\gamma_{yz,\text{in}}'' + \gamma_{xz,\text{in}}'') + d_{11}^M (E_{y,\text{in}}'' + E_{x,\text{in}}'') & = e_{15}^M (\gamma_{yz,\text{in}}'' + \gamma_{xz,\text{in}}'') + d_{11}^M (E_{x,\text{in}}'' + E_{y,\text{in}}'') \quad \text{in } S^- \nonumber
\end{align*}
\]
which can be cast into complex form

\[
\begin{align*}
  (C^4_{44} - C^M_{44})(\gamma_{zz}^0 - i\gamma_{z\zeta}^0) + (C^4_{44} - C^M_{44})(\gamma''_{zz,0} - i\gamma''_{z\zeta,0}) + C^4_{44}(\gamma'_{zz}^0 - i\gamma'_{z\zeta}^0) \\
  - (e_1^4 - e^M_1)(E_x^0 - iE_y^0) - (e_1^4 - e^M_1)(E''_{x,0} - iE''_{y,0}) - e_1^4(E_x^0 - iE_y^0) = 0, \\
  (e_1^4 - e^M_1)(\gamma''_{zz,0} - i\gamma''_{z\zeta,0}) + (e_1^4 - e^M_1)(\gamma''_{zz,0} - i\gamma''_{z\zeta,0}) + e_1^4(\gamma'_{zz}^0 - i\gamma'_{z\zeta}^0) \\
  + \sum_{l=1}^{d_{11} - d^M_{11}}(E_x^0 - iE_y^0) + (d_{11}^1 - d^M_{11})(E''_{x,0} - iE''_{y,0}) + d_{11}^1(E_x^0 - iE_y^0) = 0
\end{align*}
\]

Eqs. (19) contain four unknown strain components \(\gamma'_{zz}^0, \gamma''_{zz,0}, \gamma'_{z\zeta}^0, \gamma''_{z\zeta,0}\) and four unknown electrical field components \(E_x^0, E_y^0, E''_{x,0}, E''_{y,0}\). To determine these quantities, we must study the electroelastic field induced by the doubly periodic eigenstrain \(\gamma'_{zz}^0, \gamma''_{zz,0}\) and eigen-electrical-field \(E_x^0, E_y^0\).

### 3.2. Electroelastic field induced by the eigenstrain and eigen-electrical-field

This section addresses the electroelastic field induced by the eigenstrain and eigen-electrical-field, which leads to a doubly quasi-periodic Riemann boundary problem.

The eigenstrain and eigen-electrical-field are doubly periodic, hence the corresponding eigen-displacement and eigen-electrical-potential are doubly quasi-periodic.

Noting Eq. (6), we can define two analytical functions \(F^0(z)\) and \(\Phi^0(z)\) corresponding to the eigenstrain and eigen-electrical-field, respectively, and expand them to Taylor series in \(S^+_0\)

\[
\begin{align*}
  F^0(z) &= \sum_{k=1}^{\infty} A_k z^k, \\
  \Phi^0(z) &= \sum_{k=1}^{\infty} B_k z^k, \\
  z \in S^+_0,
\end{align*}
\]

where the terms of constants have been ignored without loss of generality. The eigenstrain and eigen-electrical-field can be expressed as

\[
\begin{align*}
  \gamma'_{zz}^0 - i\gamma'_{z\zeta}^0 &= \frac{dF^0(z)}{dz} = \sum_{k=1}^{\infty} kA_k z^{k-1}, \\
  E_x^0 - iE_y^0 &= -\frac{d\Phi^0(z)}{dz} = -\sum_{k=1}^{\infty} kB_k z^{k-1},
\end{align*}
\]

From the definition of the eigenstrain and eigen-electrical-field, the jump conditions of the disturbance displacement and electrical potential on \(L = L_0 + \Omega_{mn}\) where \(\Omega_{mn} = 2m\omega + 2n\omega\) \((m, n = 0, \pm 1, \pm 2, \ldots)\), can be written as

\[
\begin{align*}
  w^+(t) - w^-(t) &= -w_0(t), & t \in L, \\
  \varphi^+(t) - \varphi^-(t) &= -\varphi_0(t), & t \in L,
\end{align*}
\]

where \(w_0(t)\) and \(\varphi_0(t)\) are the eigen-displacement and eigen-electrical-potential on \(L\), the superscripts "+" and "−" signify the function value as approached from \(S^+\) and \(S^−\), respectively.

Let \(\Phi(z)\) and \(F(z)\) denote the complex functions corresponding to the electroelastic field induced by the eigenstrain \((\gamma'_{zz}^0, \gamma''_{zz,0})\) and eigen-electrical-field \((E_x^0, E_y^0)\). The substitution of Eq. (6) into Eqs. (22) and (23) yields

\[
\begin{align*}
  \left[ F^+(t) + F_1^+(t) \right] - \left[ F^-(t) + F_1^-(t) \right] &= -2w_0(t), & t \in L, \\
  \left[ \Phi^+(t) + \Phi_1^+(t) \right] - \left[ \Phi^-(t) + \Phi_1^-(t) \right] &= -2\varphi_0(t), & t \in L.
\end{align*}
\]

The continuity conditions of the stress and electrical displacement on \(L\) can be written as

\[
\begin{align*}
  T^+(t) - T^-(t) &= T_1(t), & t \in L, \\
  T_2^+(t) - T_2^-(t) &= T_2(t), & t \in L.
\end{align*}
\]
From Eqs. (24) and (30), it is seen that
\[ F^+(t) - F^+(t) - F^-(t) - F^-(t) = 0, \quad t \in L, \]  
\[ \Phi^+(t) - \Phi^+(t) - \Phi^-(t) - \Phi^-(t) = 0, \quad t \in L. \]  

As the electroelastic constants are non-zero, Eqs. (28) and (29) can be solved, which yields:
\[ \begin{align*}
[F^+(t) - F^+(t)] - [F^-(t) - F^-(t)] &= 0, \quad t \in L, \\
[\Phi^+(t) - \Phi^+(t)] - [\Phi^-(t) - \Phi^-(t)] &= 0, \quad t \in L.
\end{align*} \]

From Eqs. (24) and (30), it is seen that
\[ F^+(t) - F^-(t) = -w_0(t), \quad t \in L. \]

From Eqs. (25) and (31), it is seen that
\[ \Phi^+(t) - \Phi^-(t) = -\varphi_0(t), \quad t \in L. \]

Letting \( R \) be the radius of piezoelectric fiber, then \( t = Re^{i\theta}, \bar{t} = Re^{-i\theta}, \bar{t} = R^2/\bar{t} \) on \( L_0 \). Substituting Eq. (20) into Eq. (6), we can obtain the expansions of the eigen-displacement \( w_0(t) \) and eigen-electrical-potential \( \varphi_0(t) \) on \( L_0 \):
\[ w_0(t) = \frac{1}{2} [F^+(t) + F^-(t)] = \frac{1}{2} \left( \sum_{k=1}^{\infty} A_k t^k + \sum_{k=1}^{\infty} A_k R^{2k} t^k \right), \quad t \in L_0, \]
\[ \varphi_0(t) = \frac{1}{2} [\Phi^+(t) + \Phi^-(t)] = \frac{1}{2} \left( \sum_{k=1}^{\infty} B_k t^k + \sum_{k=1}^{\infty} B_k R^{2k} t^k \right), \quad t \in L_0. \]

According to the new results for the doubly quasi-periodic Riemann boundary problem (Lu, 1993), the general solutions of Eqs. (32) and (33) can be expressed as
\[ F(z) = C_1 + C_2 z - \frac{1}{2\pi i} \int_{L_0} w_0(t) \zeta(t - z) dt, \quad z \in S_0, \]
\[ \Phi(z) = C_2 + C_2 z - \frac{1}{2\pi i} \int_{L_0} \varphi_0(t) \zeta(t - z) dt, \quad z \in S_0, \]
where \( \zeta(t) \) is the Weierstrass Zeta function; \( C_{10} \) and \( C_{20} \) are constants, which do not affect the stress and electrical field and can be ignored; \( C_1 \) and \( C_2 \) are complex constants to be determined. The derivation of the general solutions (36) and (37) refers to Appendix A.

Substituting Eqs. (34) and (35) into (36) and (37) and completing the integrals in these equations, we obtain

\[ F(z) = \begin{cases} 
C_1 z - \frac{1}{2} \left[ \sum_{k=1}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} A_k R^{2k} \left( \zeta(z) - \frac{1}{z} \right)^{(k-1)} \right], & z \in S_0^+, \\
C_1 z - \frac{1}{2} \left[ \sum_{k=1}^{\infty} (-1)^k \frac{A_k R^{2k}}{(k-1)!} \zeta^{(k-1)}(z) \right], & z \in S_0^-, 
\end{cases} \]

\[ \Phi(z) = \begin{cases} 
C_2 z - \frac{1}{2} \left[ \sum_{k=1}^{\infty} B_k z^k + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} B_k R^{2k} \left( \zeta(z) - \frac{1}{z} \right)^{(k-1)} \right], & z \in S_0^+, \\
C_2 z - \frac{1}{2} \left[ \sum_{k=1}^{\infty} (-1)^k \frac{B_k R^{2k}}{(k-1)!} \zeta^{(k-1)}(z) \right], & z \in S_0^-, 
\end{cases} \]

where the superscript \((k-1)\) denotes the \((k-1)\)th order derivative. The derivation of Eq. (38) refers to Appendix B.

Now consider the determination of the complex constants \( C_1 \) and \( C_2 \). For the eigenstrain and eigen-electrical-field problem, no electrical and mechanical loads at infinity are applied. According to the periodicity,
the resultant stress and resultant electrical displacement on each boundary \( \Gamma_k \) \((k = 1, 2, 3, 4; \text{ see Fig. 1}) of the fundamental cell \( P_{00} \) vanishes. Thus, from Eqs. (8) and (9), it follows that

\begin{align}
C_M^M |F(z) - \overline{F(z)}|_{\Gamma_k} + e_{15}^M |\Phi(z) - \overline{\Phi(z)}|_{\Gamma_k} &= 0, \quad (40) \\
e_{15}^M |F(z) - \overline{F(z)}|_{\Gamma_k} - d_{11}^M |\Phi(z) - \overline{\Phi(z)}|_{\Gamma_k} &= 0. \quad (41)
\end{align}

As the electroelastic constants are non-zero, the above two equations can be solved, which yields:

\begin{align}
[F(z) - \overline{F(z)}]_{\Gamma_k} &= 0, \quad (42) \\
[\Phi(z) - \overline{\Phi(z)}]_{\Gamma_k} &= 0. \quad (43)
\end{align}

Substituting Eqs. (38) and (39) into Eqs. (42) and (43), the constants \( C_1 \) and \( C_2 \) can be determined:

\begin{align}
C_1 &= \frac{\pi R^2}{2S} (A_1 - \delta_2 A_1), \quad (44) \\
C_2 &= \frac{\pi R^2}{2S} (B_1 - \delta_2 B_1), \quad (45)
\end{align}

where \( S = 2i(\omega_1 \omega_2 - \omega_2 \omega_1) \) is the area of the fundamental cell \( P_{00} \) and \( \delta_2 = \frac{2}{\pi} (\omega_1 \eta_2 - \omega_2 \eta_1) \) with \( \eta_k = \zeta(\omega_k) \) \((k = 1, 2)\), where \( \omega_1 \) and \( \omega_2 \) are two fundamental half-periods and the overbar denotes the conjugate.

From Eqs. (21), (38) and (39), the disturbance fields induced by the eigenstrain and eigen-electrical-field can be expressed as

\begin{align}
\gamma''_{zz, \text{in}} - i\gamma''_{yz, \text{in}} &= F'(z) = \\
\gamma''_{zz, \text{out}} - i\gamma''_{yz, \text{out}} &= \Phi'(z) = \\
\end{align}

\begin{align}
\frac{1}{2} \left[ C_1 - \frac{1}{2} \sum_{k=1}^{\infty} k A_k z^{k-1} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} A_k R^{2k} \left( \zeta(z) - \frac{1}{z} \right)^{(k)} \right], \\
C_1 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} A_k R^{2k} \zeta(z)^{(k)}, \\
\frac{1}{2} \left[ C_2 + \frac{1}{2} \sum_{k=1}^{\infty} k B_k z^{k-1} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} B_k R^{2k} \left( \zeta(z) - \frac{1}{z} \right)^{(k)} \right], \\
C_2 + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} B_k R^{2k} \zeta(z)^{(k)}. \quad (46, 47)
\end{align}

3.3. The stress and electrical displacement fields in the fibers and matrix

The substitution of Eqs. (21), (46) and (47) into Eq. (19) yields

\begin{align}
(C_{44}^e - C_{44}^M)(\gamma''_{xx} - i\gamma''_{xy}) + (C_{44}^d - C_{44}^M) \left[ C_1 - \frac{1}{2} \sum_{k=1}^{\infty} k A_k z^{k-1} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} A_k R^{2k} \left( \zeta(z) - \frac{1}{z} \right)^{(k)} \right] + C_{44}^e \left( \sum_{k=1}^{\infty} k A_k z^{k-1} \right) \\
= (e_{15}^M)(E''_{x} - iE''_{y}) + (e_{15}^d - e_{15}^M) \left[ C_2 + \frac{1}{2} \sum_{k=1}^{\infty} k B_k z^{k-1} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} B_k R^{2k} \left( \zeta(z) - \frac{1}{z} \right)^{(k)} \right] + e_{15}^d \left( \sum_{k=1}^{\infty} k B_k z^{k-1} \right), \\
\times (e_{15}^M)(\gamma''_{xx} - i\gamma''_{xy}) + (e_{15}^d - e_{15}^M) \left[ C_1 - \frac{1}{2} \sum_{k=1}^{\infty} k A_k z^{k-1} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} A_k R^{2k} \left( \zeta(z) - \frac{1}{z} \right)^{(k)} \right] + e_{15}^d \left( \sum_{k=1}^{\infty} k A_k z^{k-1} \right) \\
+ (d_{11}^d - d_{11}^M)(E''_{x} - iE''_{y}) + (d_{11}^d - d_{11}^M) \left[ C_2 + \frac{1}{2} \sum_{k=1}^{\infty} k B_k z^{k-1} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} B_k R^{2k} \left( \zeta(z) - \frac{1}{z} \right)^{(k)} \right] + d_{11}^d \left( \sum_{k=1}^{\infty} k B_k z^{k-1} \right) \\
= 0. \quad (48)
\end{align}

From Eq. (48), the coefficients \( A_k, B_k \) \((k = 1, 2, 3, \ldots)\) are uniquely determined.
Noting Eqs. (16) and (17), the electroelastic constitutive equations (12) and (13) can be rewritten as

\[
\tau_{xz,\text{in}} - i\tau_{yz,\text{in}} = (\tau_{xz}^0 + \tau_{yz}^0) - i(\tau_{xz}^0 + \tau_{yz}^0)
\]

\[
= C_{44}^1 \left[ (\gamma_{xz}^0 - i\gamma_{yz}^0) + (\gamma_{xz}^0 - i\gamma_{yz}^0) + (\gamma_{xz,\text{in}}^0 - i\gamma_{yz,\text{in}}^0) \right]
\]

\[
- e_{15}^1 \left[ (E_x^0 - iE_y^0) + (E_x^0 - iE_y^0) + (E_{x,\text{in}}^0 - iE_{y,\text{in}}^0) \right]
\]

in the fibers,

\[
D_{x,\text{in}} - iD_{y,\text{in}} = (D_x^0 + D_y^0) - i(D_y^0 + D_x^0)
\]

\[
= e_{15}^1 \left[ (\gamma_{xz}^0 - i\gamma_{yz}^0) + (\gamma_{xz}^0 - i\gamma_{yz}^0) + (\gamma_{xz,\text{in}}^0 - i\gamma_{yz,\text{in}}^0) \right]
\]

\[
+ d_{11}^1 \left[ (E_x^0 - iE_y^0) + (E_x^0 - iE_y^0) + (E_{x,\text{in}}^0 - iE_{y,\text{in}}^0) \right]
\]

in the fibers,

\[
\tau_{xz,\text{out}} - i\tau_{yz,\text{out}} = (\tau_{xz}^0 + \tau_{yz}^0) - i(\tau_{xz}^0 + \tau_{yz}^0)
\]

\[
= C_{44}^M \left[ (\gamma_{xz}^0 - i\gamma_{yz}^0) + (\gamma_{xz,\text{out}}^0 - i\gamma_{yz,\text{out}}^0) \right]
\]

\[
- e_{15}^M \left[ (E_x^0 - iE_y^0) + (E_{x,\text{out}}^0 - iE_{y,\text{out}}^0) \right]
\]

in the matrix,

\[
D_{x,\text{out}} - iD_{y,\text{out}} = (D_x^0 + D_y^0) - i(D_y^0 + D_x^0)
\]

\[
= e_{15}^M \left[ (\gamma_{xz}^0 - i\gamma_{yz}^0) + (\gamma_{xz,\text{out}}^0 - i\gamma_{yz,\text{out}}^0) \right]
\]

\[
+ d_{11}^M \left[ (E_x^0 - iE_y^0) + (E_{x,\text{out}}^0 - iE_{y,\text{out}}^0) \right]
\]

in the matrix.

Let \( F_{\text{tot}}(z) \) and \( \Phi_{\text{tot}}(z) \) denote the total mechanical and electrical complex potentials in the fundamental cell under uniform antiplane shear and inplane electrical displacement at infinity. Then from Eqs. (49)–(52), it follows:

\[
F_{\text{tot}}(z) = \begin{cases} 
(\gamma_{xz}^0 - i\gamma_{yz}^0)z + F^*(z) + F(z), & z \in S_0^+, \\
(\gamma_{xz}^0 - i\gamma_{yz}^0)z + F(z), & z \in S_0^- 
\end{cases}
\]

(53)

\[
\Phi_{\text{tot}}(z) = \begin{cases} 
-(E_x^0 - iE_y^0)z + \Phi^*(z) + \Phi(z), & z \in S_0^+, \\
-(E_x^0 - iE_y^0)z + \Phi(z), & z \in S_0^- 
\end{cases}
\]

(54)

It should be noted that for the equivalent homogeneous medium (Case 2), \( F^*(z) \) and \( \Phi^*(z) \) are not related to the stress and electrical displacement.

Now the problem has been solved.

4. Some special cases

It is interesting to examine some special cases of the present solution.

(1) A piezoelectric fiber in an infinite piezoelectric medium.

Letting \( \tau_{xz}^\infty = 0, D_{yz}^\infty = 0 \), then from Eqs. (10) and (11), \( \gamma_{xz}^0 = 0, E_x^0 = 0 \). Letting \( \omega_1 \to \infty, \omega_2 \to \infty \), then \( \zeta(z) = 1/z, C_1 = 0, C_2 = 0 \). Thus from Eq. (48), we have

\[
A_1 = \left\{ -2[(C_{44}^1 - C_{44}^M)(d_{11}^1 + d_{11}^M) + (e_{15}^1 - e_{15}^M)(e_{15}^1 + e_{15}^M)](-i\gamma_{yz}^0) \\
+ 2[(e_{15}^1 - e_{15}^M)(d_{11}^1 + d_{11}^M) - (e_{15}^1 + e_{15}^M)(d_{11}^1 - d_{11}^M)](-iE_{y}^0) \right\}/A,
\]

(55)

\[
B_1 = \left\{ -2[(C_{44}^1 - C_{44}^M)(e_{15}^1 + e_{15}^M) - (C_{44}^1 + C_{44}^M)(e_{15}^1 - e_{15}^M)](-i\gamma_{yz}^0) \\
+ 2[(e_{15}^1 - e_{15}^M)(e_{15}^1 + e_{15}^M) + (C_{44}^1 + C_{44}^M)(d_{11}^1 - d_{11}^M)](-iE_{x}^0) \right\}/A,
\]

(56)

\[
A = (C_{44}^1 + C_{44}^M)(d_{11}^1 + d_{11}^M) + (e_{15}^1 + e_{15}^M)^2,
\]

(57)

\[
A_k = B_k = 0, \quad k = 2, 3, \ldots
\]

(58)
Letting
\[ -\gamma^0_{zx} = A^1_r, \quad E_y = B^1_r, \]
then the Eqs. (53) and (54) are reduced to
\[
F_{\text{tot}}(z) = \begin{cases} 
  i A^1_1 z, & z \in S^+_0, \\
  i \left( A^1_{-1} \frac{1}{z} + A^1_{+1} z \right), & z \in S^-_0,
\end{cases}
\]
\[
\Phi_{\text{tot}}(z) = \begin{cases} 
  i B^1_1 z, & z \in S^+_0, \\
  i \left( B^1_{-1} \frac{1}{z} + B^1_{+1} z \right), & z \in S^-_0,
\end{cases}
\]
where
\[
A^1 = 2 \left\{ \left[ C^M_{44}(d_{11}^l + d_{11}^M) + e_{15}(e_{15}^l + e_{15}^M) \right] A^1_1 + (e_{15}^l d_{11}^l - e_{15}^M d_{11}^M) B^1_1 \right\} / A, \]
\[
B^1 = 2 \left\{ \left[ C^M_{44}(d_{11}^l + d_{11}^M) A^1_1 + (e_{15}^l e_{15}^M + e_{15}^l) C^M_{44} + C^M_{44} \right] B^1_1 \right\} / A, \]
\[
A^1_{-1} = -R^2 \left\{ \left[ C^M_{44} - C^M_{44} \right] (d_{11}^l + d_{11}^M) + (e_{15}^M)^2 - (e_{15}^l)^2 \right\} A^1_1 + 2(e_{15}^l d_{11}^l - e_{15}^M d_{11}^M) B^1_1 \right\} / A, \]
\[
B^1_{-1} = -R^2 \left\{ 2 \left[ C^M_{44} e_{15}^l - C^M_{44} e_{15}^l \right] A^1_1 + \left[ C^M_{44} + C^M_{44} \right] (d_{11}^l + d_{11}^M) + (e_{15}^M)^2 - (e_{15}^l)^2 \right\} B^1_1 \right\} / A. \]

This solution is in agreement with the result for a single piezoelectric fiber in an infinite piezoelectric matrix (for example, see Pak, 1992; Jiang and Cheung, 2001).

(2) The fibers and matrix are the same piezoelectric material.

In this case, \( C^M_{44} = C^M_{44}, e_{15}^l = e_{15}^l, d_{11}^l = d_{11}^l \). From Eq. (48), we have \( A_k = B_k = 0 (k = 1, 2, 3, \ldots) \). Thus in the entire domain, the stress and electrical displacement are uniform, i.e., \( \tau_{zx} = i \tau_{zx} = \tau_{zx} = i \tau_{zx}, D_x = i D_y = D_x^\infty - i D_y^\infty \). Obviously, the result is expected.

5. Stress and electrical field concentrations

The microstructure induces microscopic stress and electrical field fluctuations. The dependence of microscopic stress and electrical field concentrations on the microstructural pattern are important in design, manufacture and use of piezoelectric composites. To improve mechanical and electrical strength of materials and structures, we generally go in quest for alleviating stress and electrical field concentrations. To build sensitive sensors, however, we may need a high electrical field concentration. The present work provides an exact and effective method of predicting the stress and electrical field in the microstructure, which makes it possible to understand deeply the interesting coupling phenomenon of mechanical and electrical fields.

To demonstrate the efficiency and accuracy of the present method, first a comparison of the present solution with the FEM calculations is made.

Example 1. The computational object is a pure elastic boron/epoxy composite with a square array of fibers subjected to an antiplane mechanical load \( \tau_{zx}^\infty = q_1 \) at infinity. The shear moduli of the constituent materials are \( G^l = 172.37 \text{ GPa}, G^M = 1.5322 \text{ GPa} \), where the superscripts I and M refer to the fiber (boron) and matrix (epoxy), respectively. Examine the stress field fluctuations in a unit cell.

Computations show that to guarantee the series converge, the terms used in Eq. (48) should increase with the increase of the fiber volume fraction \( \lambda \). When \( \lambda = 0.1 (R \approx 0.357a) \), it is enough to take five terms, whereas when \( \lambda \) goes up to 0.75 (\( R \approx 0.977a \)) for a square array of fibers, it is needed to take fifty terms.

In FEM analysis, a special attention is given on how to approach the periodicity condition. To examine the finite boundary influence, we take 1, 3 \times 3, 5 \times 5, 7 \times 7 (repeated) cell or cells as the computational object, respectively. The homogeneous traction boundary condition \( \tau_{zx} = \tau_{zx}^\infty = q_1 \) is imposed. The computational results are collected from the central (fundamental) cell with side length \( 2a \) and fiber radius \( R = 0.8a \) as shown in Fig. 2, where Points 1, 2, 3, 4 are four stress computational points.
The results for the dimensionless stress $\tau_{xz}/q_1$ at the four points are listed in Table 1. With the increase of the distance from the outer boundary, the stress distribution well converges to the result by the present method. It is seen that whereas the outer boundary traction is uniform, the stress on the inner cell boundary is non-uniform. The stress fluctuations in microstructures depend on the microstructural parameters. In this example, the finite boundary influence can be ignored as the distance from the cell where the computational results are collected, to the nearest boundary is larger than thrice the cell size. This distance will be adopted in FEM analysis later.

Now we examine the stress and electrical field concentrations in piezoelectric fiber composites.

**Example 2.** A square array of piezoelectric fibers in an infinite elastic matrix subjected to a purely mechanical load $\tau_{xz}^\infty = q_1$ at infinity.

Take $C^I_{44} = 35.3$ GPa, $d^I_{11} = 15.1$ nC/N m$^2$ for piezoelectric fibers, and $e^M_{15} = 0$, $d^M_{11} = 0$ for the elastic matrix. In this case, numerical results show that the maximum value of the electrical field in the $x$-direction in fibers appears at the fiber boundary point $A$ (refer to Fig. 3). The variation of $E_{x,A}$ (the electrical field in the $x$-direction at point $A$) with the fiber piezoelectric modulus $e^I_{15}$ for various values of $\delta = C^M_{44}/C^I_{44}$ and $\lambda$ (fiber volume fraction) are depicted in Figs. 4a–d.

From Figs. 4a–d, a non-monotonic dependence of the induced electrical field in the fiber on the piezoelectric modulus $e^I_{15}$ is observed. With the increase of $|e^I_{15}|$ from zero, $|E_{x,A}/q_1|$ first increases, then decreases, where $|\cdot|$ denotes the absolute value. A maximum can be found. At the same time it is seen that the dependence of $|E_{x,A}/q_1|$ on $\lambda$ is also non-monotonic. With the increase of $\lambda$ from zero, $|E_{x,A}/q_1|$ first goes down, then goes up. These data may be useful for optimizing the performance of piezoelectric composites.

<table>
<thead>
<tr>
<th>Computational points</th>
<th>Finite element method results</th>
<th>Present results</th>
</tr>
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<tbody>
<tr>
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<td>3 × 3 cells</td>
</tr>
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<td>1.6888</td>
</tr>
<tr>
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</tr>
<tr>
<td>3</td>
<td>1.0200</td>
<td>0.9140</td>
</tr>
<tr>
<td>4</td>
<td>0.9871</td>
<td>0.6008</td>
</tr>
</tbody>
</table>

Fig. 3. The computational point $A$ of electrical field and stress.
For a piezoelectric composite, another main concern is its strength. Numerical results show that the maximum stress in the fiber appears also at the point \( A \) (Fig. 3). Define the stress concentration factor \( \beta \) in the fiber:

\[
\beta = \frac{\tau_{zz,A}}{q_1}.
\]  

(62)

The variation of \( \beta \) with the piezoelectric coefficient \( e_{15}^f \) are depicted in Fig. 5a–d for various values of the volume fraction \( \lambda \) and \( \delta = C_{44}^M/C_{44}^f \) (\( C_{44}^f = 35.3 \) GPa, \( d_{11}^f = 15.1 \) nC/V m²): (a) \( \lambda = 0.1 \), (b) \( \lambda = 0.3 \), (c) \( \lambda = 0.5 \) and (d) \( \lambda = 0.7 \).

For a piezoelectric composite, another main concern is its strength. Numerical results show that the maximum stress in the fiber appears also at the point \( A \) (Fig. 3). Define the stress concentration factor \( \beta \) in the fiber:

\[
\beta = \frac{\tau_{zz,A}}{q_1}.
\]  

(62)

The variation of \( \beta \) with the piezoelectric coefficient \( e_{15}^f \) are depicted in Fig. 5a–d for various values of the volume fraction \( \lambda \) and \( \delta = C_{44}^M/C_{44}^f \).

Lastly, it should be pointed out that the fiber interaction also gives rise to non-uniform stress and electrical fields in the \( y \)-direction. However, their values are small and they are not discussed here.

6. Prediction of effective electroelastic moduli

Effective electroelastic moduli play an important role in the design of piezoelectric composites. For a transversely isotropic piezoelectric composite under antiplane shear, the generalized stress vector \( \Sigma \) and the generalized strain vector \( \mathbf{Z} \) can be written as
Then the constitutive equations (3) and (4) can be written in matrix form

\[ \Sigma = CZ \]  

or

\[ Z = C^{-1} \Sigma, \]  

where \( C^{-1} \) is the inverse of \( C \): 

\[ C = \begin{bmatrix} C_{44} & -e_{15} \\ e_{15} & d_{11} \end{bmatrix}. \]
For a two-phase piezoelectric composite, the averaged generalized stress and strain can be expressed as

\[ \Sigma = \lambda \Sigma_1 + (1 - \lambda) \Sigma_2, \]  
\[ \bar{Z} = \lambda \bar{Z}_1 + (1 - \lambda) \bar{Z}_2, \]

where \( \lambda \) denotes the fiber volume fraction, the subscripts 1 and 2 refer to the fiber and the matrix, respectively.

The averaged generalized stress and strain can be expressed as

\[ \Sigma = \frac{k \Sigma_1}{1 + \frac{1}{C_0 k}} \Sigma_2; \]  
\[ \bar{Z} = \frac{k \bar{Z}_1}{1 + \frac{1}{C_0 k}} \bar{Z}_2; \]

where \( k \) denotes the fiber volume fraction, the subscripts 1 and 2 refer to the fiber and the matrix, respectively.

For the doubly periodic problem under consideration, we can calculate the average stress and strain in the fundamental cell \( P_{00} \).

According to the averaged field theorem

\[ \Sigma = C_e \bar{Z}, \]

where \( C_e \) is the effective electroelastic moduli. In this section, the overbar denotes averaging.

From Eqs. (67)–(69), it is seen that

\[ (C_e^{-1} - C_2^{-1})\Sigma = \lambda (C_1^{-1} - C_2^{-1})\Sigma_1 \]

Letting \( \Sigma^\infty \) denote a uniform far-field condition and noting that the average values of disturbing stress and electrical displacement are zeros, we have

\[ \Sigma = \Sigma^\infty. \]

After \( \Sigma_1 \) is determined by the present method, \( C_e^{-1} \) can be determined by Eq. (70). The inverse of \( C_e^{-1} \) leads to the effective electroelastic moduli \( C_{44}^e, e_{15}^e, d_{11}^e \).

It is of practical interest to compare the results predicted by the present method with the FEM results and the experimental data.

**Example 3.** The computational object is the same as the one in Example 1. A comparison of the effective longitudinal shear modulus predicted by the present method, the FEM and the experiment is shown in Table 2, where the fiber array in the present method and the FEM is square.

Numerical results show that the predictions for the effective longitudinal shear modulus by the present method are in excellent agreement with those by the FEM. This fact is expected since from Eq. (70), the accuracy of the effective modulus depends on that of stress computations. From Section 5, the stresses computed by the two methods excellently verified each other. To save space, only the data at two points of the volume fraction close to touching cylinders (most difficult cases in numerical computations) are given in Table 2.

As for the comparison with the experimental results, an apparent discrepancy is observed. The same observations are also found in the literature. For example, Chamis (1989) pointed out: “A noticeable exception is the shear modulus. This modulus is difficult to measure accurately; this may account for apparent discrepancies” between the theoretical predictions and experimental data. Perhaps, the accurate experimental measure remains an open problem.

In the following, two examples are given about predictions of the effective electroelastic moduli for composites containing square and hexagonal arrays (Fig. 6) of piezoelectric fibers.

**Example 4.** A PZT-7A piezoelectric fiber/epoxy composite with the fiber material constants \( C_{44}^f = 25.4 \) GPa, \( e_{15}^f = 9.2 \) C/m², \( d_{11}^f = 4.071 \) nC/V m and the matrix material constants \( C_{44}^M = 1.8 \) GPa, \( e_{15}^M = 0, \) \( d_{11}^M = 0.03717 \) nC/V m.

In terms of the exact results of the generalized stress vector \( \Sigma_1 \), Eq. (70) produces exact predictions of effective electroelastic moduli. The variations of effective electroelastic moduli with the fiber volume fraction \( \lambda \) for

<table>
<thead>
<tr>
<th>Volume fraction ( \lambda )</th>
<th>Present method</th>
<th>Finite element method</th>
<th>Experimental data*</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>10.570</td>
<td>10.562</td>
<td>12.21</td>
</tr>
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<td>0.75</td>
<td>16.966</td>
<td>16.964</td>
<td>16.75</td>
</tr>
</tbody>
</table>

* Refer to Whitney and Riley (1966).
the square and hexagonal arrays of piezoelectric fibers are depicted in Fig. 7. It is seen that the effective electroelastic moduli for a square array are larger than those for a hexagonal array while the volume fraction $k$ is fixed. This difference is more considerable for a large value of $k$. The reason seems to be that the distribution of fibers in the hexagonal array is “more even” than that in the square array.

On other hand, the micromechanics methods have been used to estimate the effective properties of composites. They are all approximate methods which relate the macroscopic effective properties of composites to the microstructural parameters in a statistical sense. Generally, the micromechanics methods cannot reflect the influence of the fiber array. It is of interest to make a comparison between the present exact predictions for the particular fiber arrays and micromechanical results.

As well known, the dilute, self-consistent, Mori–Tanaka, differential and generalized self-consistent methods are the micromechanics methods which have been extensively used (for example, see Dunn and Taya, 1993; Jiang et al., 2003). The essential assumption in the dilute model is that a single inclusion is embedded in an infinite matrix subjected to a far-field loading. The essential assumption in the self-consistent model is that a single inclusion is embedded in an infinite equivalent medium of the composite. Obviously, the dilute model ignores the inclusion interaction, while the self-consistent model overestimates it. The two model traditionally received criticism on their accuracy. As an incremental form of the self-consistent method, the differential method does not exhibit the intuitively unacceptable results of the self-consistent method, however, it may lead to non-unique solutions. The key assumption of the Mori–Tanaka method is that a single inclusion is embedded in an infinite matrix subjected to an applied far-field equal to the as-yet-unknown average stress (strain) field in the matrix, and the method considerably improved the accuracy of the dilute method. The generalized self-consistent method (GSCM) is a more sophisticated one, and the method is based on a three-phase model: an inclusion is embedded in a finite matrix, which in turn is embedded in an infinite composite with the as-yet-unknown effective moduli. Generally, the results predicted by the GSCM are in good agreement with the experiment data. A comparison of results predicted by the present method and micromechanics methods are shown in Fig. 7. It is seen that the results predicted by the GSCM are almost in exact agreement with those for the hexagonal array of piezoelectric fibers predicted by the present method. It appears that the GSCM reflects “the idealized even distribution” of inclusions and in the doubly periodic arrays the hexagonal array approaches most “the idealized even distribution”. It should be pointed out that a comprehensive discussion about the influence of fiber arrays on effective properties has been made by Pettermann and Suresh (2000). Their investigation and the present work can verify and complement each other. The emphasis of the present work is on the effective electroelastic properties corresponding to plane electrical field and antiplane shear and on a comparison with the GSCM and other micromechanics method. By the way it be mentioned that the results predicted by the Mori–Tanaka method are very close to those by the generalized self-consistent method and they are not shown.

**Example 5.** A porous piezoelectric ceramic PZH-5 with the electroelastic constants $C_{44}^M = 21.1$ GPa, $e_{15}^M = 12.3$ C/m$^2$, $d_{11}^M = 8.107$ nC/V m.

Consider two arrays of holes, i.e., the square and hexagonal arrays; two cases, i.e., without fillings (hole parameters $C_{44}^l = 0$, $e_{15}^l = 0$, $d_{11}^l = 8.85 \times 10^{-3}$ nC/V m) and filled with the polymer ($C_{44}^l = 0.64$ GPa, $e_{15}^l = 0$, $d_{11}^l = 0.0797$ nC/V m). The results of the effective electroelastic moduli are listed in Tables 3–5. For a comparison with the micromechanics method, the results predicted by the GSCM are also listed in the Tables. It is seen that the effective electroelastic moduli for the square array are smaller than those for
Fig. 7. A comparison of effective electroelastic moduli for square and hexagonal arrays of piezoelectric fibers with results by various micromechanics methods (Fiber (PZT-7A)): $C_{44}^I = 25.4 \text{ GPa}$, $e_{15}^I = 9.2 \text{ C/m}^2$, $d_{11}^i = 4.071 \text{ nC/V m}$; Matrix (epoxy): $C_{44}^M = 1.8 \text{ GPa}$, $e_{15}^M = 0$, $d_{11}^M = 0.03717 \text{ nC/V m}$: (a) effective elastic modulus $C_{44}^e$, (b) effective piezoelectric modulus $e_{15}^e$ and (c) effective dielectric modulus $d_{11}^e$. 
the hexagonal array while the hole volume fraction $\lambda$ is fixed, and the results for the hexagonal array are almost in exact agreement with those by the GSCM.

7. Conclusions

For composites with a doubly periodic parallelogrammic array of piezoelectric fibers under a far-field anti-plane shear coupled with an inplane electrical field, a rigorous analytical method is developed by introducing the concepts of eigenstrain and eigen-electrical-field integrated with the new results for the doubly quasi-periodic Riemann boundary problem. This method can provide benchmark results for mechanically and electrically coupled composites and may be useful in the design of smart materials and structures.

A comparison of results predicted by the present method, the FEM and experimental data is made and good agreement is observed, which demonstrates the efficiency and accuracy of the present method.

To improve mechanical and electrical strength, it is desirable to alleviate stress and electrical field concentrations, but sometimes we may need a high electrical field concentration for the construction of sensitive sensors. From the numerical results it is found that the stress and electrical field exhibit a non-monotonic and complex dependence on the electroelastic properties of material constituents and the fiber array parameters, which is of practical importance in the assessment of the performance and in the optimization of piezoelectric composites.

The present method provides an exact solution for the effective electroelastic moduli of such composites with idealized doubly periodic array of piezoelectric fibers. It is found that the effective electroelastic moduli

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Effective elastic modulus $C_{44}^{e}$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>For empty holes</td>
</tr>
<tr>
<td></td>
<td>Square</td>
</tr>
<tr>
<td>0.1</td>
<td>17.2635</td>
</tr>
<tr>
<td>0.3</td>
<td>11.3429</td>
</tr>
<tr>
<td>0.5</td>
<td>6.85021</td>
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<tr>
<td>0.6</td>
<td>4.85971</td>
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<table>
<thead>
<tr>
<th>Table 4</th>
<th>Effective piezoelectric modulus $e_{15}^{e}$ (C/m²)</th>
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<tr>
<td>$\lambda$</td>
<td>For empty holes</td>
</tr>
<tr>
<td></td>
<td>Square</td>
</tr>
<tr>
<td>0.1</td>
<td>10.0636</td>
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<tr>
<td>0.2</td>
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<td>6.61223</td>
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<tr>
<td>0.5</td>
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<table>
<thead>
<tr>
<th>Table 5</th>
<th>Effective dielectric modulus $d_{11}^{e}$ (nC/V m)</th>
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</thead>
<tbody>
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<td>$\lambda$</td>
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<tr>
<td></td>
<td>Square</td>
</tr>
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<tr>
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<td>5.40849</td>
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<tr>
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<tr>
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<td>2.64020</td>
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<tr>
<td>0.6</td>
<td>1.87630</td>
</tr>
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</table>
of the piezoelectric fiber composites predicted by the generalized self-consistent method are very close to the exact results for the composites with a hexagonal array of piezoelectric fibers.

Acknowledgements

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Appendix A. Derivation of the general solutions (36) and (37)

If a function \( f(z) \) is single-valued and analytic in the entire plane, satisfying the conditions \( f(z + 2\omega_j) = f(z) + \omega_j \) \( (j = 1, 2) \), then the function is called an additive doubly quasi-periodic analytic function, where \( 2\omega_1, 2\omega_2 \) are its periods, \( \omega_1, \omega_2 \) are its addends, and the only singularities of \( f(z) \) are poles. If \( \omega_j = 0 \) \( (j = 1, 2) \), \( f(z) \) is called doubly periodic function.

The function \( f(z) \) have the following property: If \( f(z) \) has no singularity, then \( \omega_2/\omega_1 = \omega_1/\omega_2 \) and \( f(z) = C + \mu z \), where \( C \) is a constant and \( \mu = \omega_1/2\omega_j \) \( (j = 1, 2) \).

Referring to Section 2.1 and Fig. 1, a doubly quasi-periodic Riemann boundary value problem is described as follows:

\[
F^+(t) - F^-(t) = g(t), \quad t \in L,
\]

(A.1)

where \( F(z) \) has no singularity in \( S^\pm \), \( g(z) \) is an additive doubly quasi-periodic function with addends \( g_1, g_2 \).

Use the notation \([z]_0\) to represent the point in \( P_{00} \) congruent to \( z \) \( \text{mod} \ 2\omega_j \) and denote \( g([I]_0) = g_d(t) \) which is doubly periodic. If we set

\[
F_0(z) = \begin{cases} 
F^+(z) - mg_1 - ng_2 & \text{when } z = [z]_0 + 2m\omega_1 + 2n\omega_2 \in S^+, \\
F^-(z) & \text{when } z \in S^{-}
\end{cases}
\]

(A.2)

then

\[
F_0^+(t) - F_0^-(t) = g_0(t), \quad t \in L,
\]

(A.3)

where \( F_0(z) \) is still additively doubly quasi-periodic and \( F_0^\pm(z) \) have the same addends. Therefore, the function

\[
\Psi_0(z) = F_0(z) - \frac{1}{2\pi i} \int_{L_0} g_0(t) \zeta(t - z) \, dt = F_0(z) - \frac{1}{2\pi i} \int_{L_0} g(t) \zeta(t - z) \, dt
\]

(A.4)

has no jump on \( L \) and no singularity in \( S^\pm \) but is doubly quasi-periodic, where \( \zeta(z) \) is Wierestrass Zeta function.

According to the property mentioned above,

\[
\Psi_0(z) = C_0 + C_1 z,
\]

(A.5)

where \( C_0, C_1 \) are the constants. Thus, we obtain finally

\[
F(z) = \begin{cases} 
C_0 + C_1 z + mg_1 + ng_2 + \frac{1}{2\pi i} \int_{L_0} g(t) \zeta(t - z) \, dt & \text{when } z = [z]_0 + 2m\omega_1 + 2n\omega_2 \in S^+, \\
C_0 + C_1 z + \frac{1}{2\pi i} \int_{L_0} g(t) \zeta(t - z) \, dt & \text{when } z \in S^{-}
\end{cases}
\]

(A.6)

In \( P_{00} \) \( (S_0) \), \( m = n = 0 \), we have

\[
F(z) = C_0 + C_1 z + \frac{1}{2\pi i} \int_{L_0} g(t) \zeta(t - z) \, dt, \quad z \in S_0.
\]

(A.7)
Appendix B. Derivation of Eq. (38)

Wirerestrass Zeta function \( \zeta(z) \) is an additive quasi-elliptic function

\[
\zeta(z) = \frac{1}{z} + \Sigma' \left( \frac{1}{z - \Omega_{mn}} + \frac{1}{\Omega_{mn}} + \frac{z}{\Omega_{mn}^2} \right),
\]

where \( \Omega_{mn} = 2m\omega_1 + 2n\omega_2 \) and \( \Sigma' \) denotes summation for all \( m, n = 0, \pm 1, \pm 2, \ldots \) except \( m = n = 0 \). It has one simple pole in every parallelogram.

Consider function \( \zeta(z^* - z) \), where \( z^* \) is the argument. When \( z \in S^0_+ \), \( \zeta(z^* - z) \) is analytic in the region \( S^0_+ \) and its boundary \( L_0 \). Noting Eq. (34) and according to Cauchy’s integral theorem, we have

\[
\frac{1}{2\pi i} \int_{L_0} w_0(t)\zeta(t - z) dt = \frac{1}{2\pi i} \int_{L_0} \frac{1}{2} \left( \sum_{k=1}^{\infty} A_k t^k + \sum_{k=1}^{\infty} A_k R^{2k} \frac{1}{t^k} \right) \zeta(t - z) dt
\]

\[
= \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k - 1)!} \frac{1}{2} \frac{R^{2k}}{t^k} (z^* - z),
\]  

(B.2)

When \( z \in S^0_+ \), \( \zeta(z^* - z) \) is analytic in region \( S^0_+ \) and its boundary \( L_0 \) except a simple pole at \( z \), we can rewrite \( \zeta(z^* - z) \) as follows:

\[
\zeta(z^* - z) = \frac{1}{z^* - z} + \left[ \zeta(z^* - z) - \frac{1}{z^* - z} \right].
\]  

(B.3)

Then \( \zeta(z^* - z) - \frac{1}{z^* - z} \) is analytic in \( S^0_+ \) and its boundary \( L_0 \), and we have

\[
\frac{1}{2\pi i} \int_{L_0} w_0(t)\zeta(t - z) dt = \frac{1}{2\pi i} \int_{L_0} \frac{1}{2} \left( \sum_{k=1}^{\infty} A_k t^k + \sum_{k=1}^{\infty} A_k R^{2k} \frac{1}{t^k} \right) \left\{ \frac{1}{t - z} + \left[ \zeta(t - z) - \frac{1}{t - z} \right] \right\} dt
\]

\[
= \frac{1}{2} \left[ \sum_{k=1}^{\infty} A_k z^k + \sum_{k=1}^{\infty} \frac{(-1)^k}{(k - 1)!} \frac{1}{2} \frac{R^{2k}}{t^k} \left( \zeta(z) - \frac{1}{z} \right)^{(k - 1)} \right].
\]  

(B.4)

Now, Eq. (38) can be obtained immediately.

References


