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Direct and Inverse Addition in Convex Analysis and Applications

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Inspired simultaneously from the theory of network synthesis and from the theory of means for positive operators, we introduce the notions of direct and inverse addition of order p for pairs of convex sets in a locally convex topological linear space. In this way a general formalism is introduced which allows us, on the one hand, to recover the operations of series and parallel addition derived from network connections as well as the notions of arithmetic and harmonic means for positive operators and, on the other hand, to recover the main binary operations on convex sets appearing in Convex Analysis. © 1990 Academic Press, Inc.

1. INTRODUCTION

In studying the series and parallel connection of two resistors in an electrical network, Erickson [9] was led to introduce a couple of dual operations called series and parallel addition. In order to consider also the electrical connection of multiports, Anderson and Duffin [1] extended these operations

$$\begin{aligned}&(A, B) \mapsto A + B\\&(A, B) \mapsto A \sqcup B = (A^{-1} + B^{-1})^{-1}\end{aligned} \tag{1.2}$$

from the scalar case $A, B \in \mathbb{R}_+^*$ to the case in which A and B are symmetric positive definite matrices. They gave, in fact, a more general expression for $A \square B$ allowing the possibility of singular matrices A and B. This is not a minor point since in the generalized setting, the possibility of a short circuit in some of the components of each multiport is not excluded. In the literature two other streams of generalizations for these binary operations are discussed. On the one hand, Fillmore and Williams [12] extended these operations to the class of bounded positive linear operators on a Hilbert space. In the same vein the works of Anderson and Trapp [4] and

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Morley [26], among others, must be considered. They gave equivalent formulations of the parallel addition in the infinite dimensional setting mentioned above. Anderson, Morly, and Trapp [2] initated the study of the parallel sum of nonlinear subdifferentials of convex functions in Hilbert spaces. Passty [29] pushed through this program and considered the natural extension to nonlinear monotone operators. On the other hand, it is worth mentioning a different path of generalizations for the notions of series and parallel addition. These operations have an interpretation relating them to the notions of arithmetic and harmonic means, rather than as operations derived from network connections. Recall that the arithmetic and the harmonic means, on the class of bounded linear strictly positive operators, are the binary operations defined by

$$(A, B) \mapsto \tfrac{1}{2}(A + B)$$

and

$$(A, B) \mapsto [\tfrac{1}{2}(A^{-1} + B^{-1}]^{-1},$$

respectively. As quoted by Kubo and Ando [20], these operations are, of course, the same as in (1.2) except for the normalization factor $\frac{1}{2}$. By using an axiomatic approach these authors introduced the notion of mean for pair of operators like the ones mentioned above. In their general formalism they recover, as a particular case, the arithmetic and the harmonic mean, as well as the geometric mean considered by Pusz and Woronowicz [31]. Nevertheless they pushed their generalization in a direction which excludes the power mean of order p,

$$(A, B) \mapsto [\tfrac{1}{2}(A^p + B^p)]^{1/p},$$

studied by Bhagwat and Subramanian [6].

In this paper we go beyond the framework proper to the theory of network connections, as well as the one proper to the theory of means for positive operators. Consequently, rather than series and parallel addition or arithmetic and harmonic mean, we refer to this pair of dual operations simply by direct and inverse addition. Inspired by both theories, we introduce the notions of direct and inverse addition of order p for pairs of convex sets in a locally convex topological linear space and for pairs of positive extended real-valued functions defined over such a space. These notions appear to be extremely fruitful from a theoretical viewpoint, since they set up a framework from which a general theory for these dual operations can be derived. What is more surprising is the fact that these operations allow us to recover the main binary operations appearing in Convex Analysis by choosing appropriately the order p. The title of this paper has not been chosen arbitrarily. Throughout all this work the author confesses a special predilection for Convex Analysis and the reader will be confronted at each moment with the tools proper to this theory. The direct and the inverse addition of order p appear to be interesting also from an applied viewpoint, as shown by means of some examples. We claim that the field of applications of these notions can be considerably enlarged.

The plan of this paper is described below.

Section 2. Introduction and study of the direct and inverse addition of order p for pairs of convex sets. Without loss of generality we restrict our attention to the convex sets containing the origin.

Section 3. Introduction and study of the direct and inverse addition of order p for pairs of positive extended real-valued functions.

Section 4. Discussion on the preservation of the closedness of the direct sum $(A, B) \mapsto A \oplus_p B$ and the inverse sum $(A, B) \mapsto A \Box_p B$.

Section 5. Characterization of the support functions of the sets $A \oplus_p B$ and $A \Box_p B$.

Section 6. Study of the duality between the direct and the inverse addition. This is done by characterizing the polar sets of $A \oplus_p B$ and $A \Box_p B$ and by establishing some polarity relationships.

Section 7. Application in the theory of second-order subdifferentials for convex functions. Calculus rules are derived for the computation of the second-order subdifferential of the sum and the infimal-convolution of two convex functions. These rules are given in terms of the direct and the inverse addition for pairs of convex sets.

Section 8. Application to the algebra of ellipsoids. We characterize the ellipsoids associated to the series and parallel sum of two symmetric positive semidefinite matrices. These operations derived from network connections are interpreted in terms of the direct and the inverse addition for pairs of convex sets.

For the reader's convenience, a partial list of notations is provided below.

- $\bar{G}$ closure of G
- co G convex hull of G
- G^0 polar set of G
- 0^+G recession cone of G
- λG scalar multiple (by λ) of G
- Ψ_G indicatrice function of G
- Ψ_G^* support function of G
- f^* conjugate or Fenchel transform of f
- f^{**} biconjugate of f
- $f0^+$ recession function of f
- dom f effective domain of f

2. Two Families of Operations on the Class of Convex Sets

Unless we specify the contrary, (X, X^*) will be a couple of locally convex topological linear spaces in duality by means of the bilinear form $\langle \cdot, \cdot \rangle : X \times X^* \to \mathbb{R}$ (see [7, p. 48]). We shall henceforth assume X and X^* have each been supplied with a topology compatible with this duality [7, p. 67], so that each one can be identified with the space of continuous linear functions on the other. All questions of closure, continuity, and boundedness refer to these given topologies. The class of convex subsets of X is preserved by a rich variety of binary operations, such as the intersection, the convex hull, and the Minkowski addition. We begin this section by introducing two new families of binary operations that preserve the convexity and that include the above examples as particular cases. In this paper we restrict, however, our attention to the class K of all convex subsets of X containing the origin:

$$K = \{G \subset X / G \text{ convex}, 0 \in G\}.$$

It is in this setting that we shall consider further some applications. In the following definition and throughout this paper, the pair (p, q) of numbers in $[1, \infty]$ verifies the conjugacy relationship $p^{-1} + q^{-1} = 1$. The couples $(1, \infty)$ and $(\infty, 1)$ are not excluded, since we adopt the convention $\infty^{-1} = 0$. For a vector $\lambda = (\lambda_1, \lambda_2)$ in $\mathbb{R}^2$ we shall write $\lambda \geqslant 0$ when both components λ_1 and λ_2 are nonnegative. Recall that the norm $\|\lambda\|_q$ of such a vector is given by

$$\|\lambda\|_q = \begin{cases} [(\lambda_1)^q + (\lambda_2)^q]^{1/q} & \text{if } q \neq \infty \\ \operatorname{Max}\{\lambda_1, \lambda_2\} & \text{if } q = \infty. \end{cases}$$

Definition 2.1. Let A and B be two sets in K. The direct and the inverse sum of order p of A and B are respectively the sets

$$A \oplus_p B = \bigsqcup \{\lambda_1 A + \lambda_2 B / \lambda \geqslant 0, \|\lambda\|_q = 1\}$$

and

$$A \square_p B = \bigsqcup \{\lambda_1 A \cap \lambda_2 B / \lambda \geqslant 0, \|\lambda\|_q = 1\}.$$

Note that when G belongs to K, the function $\alpha \in \mathbb{R}_+ \mapsto \alpha G$ verifies the monotony condition

$$\alpha_1 \leqslant \alpha_2 \Rightarrow \alpha_1 G \subset \alpha_2 G, \qquad \forall \alpha_1, \alpha_2 \in \mathbb{R}_+.$$

This shows that over the class K the operations $\oplus_p$ and $\square_p$ can be defined equivalently by

$$A \oplus_p B = \bigsqcup \{\lambda_1 A + \lambda_2 B / \lambda \geqslant 0, \|\lambda\|_q \leqslant 1\}$$

and

$$A \,\Box_p\, B = \bigsqcup \{\lambda_1 A \cap \lambda_2 B / \lambda \geqslant 0, \|\lambda\|_q \leqslant 1\},$$

respectively. The following proposition shows that with the particular choices $p = 1$ and $p = \infty$, we recover the main binary operations on convex sets appearing in Convex Analysis.

PROPOSITION 2.2. *Let A and B be in K. Then*

(a)	$A \oplus_1 B = A + B,$	*Minkowski or direct sum,*
(b)	$A \oplus_\infty B = \mathrm{co}[A \cup B],$	*convex hull,*
(c)	$A \,\Box_1\, B = A \cap B,$	*intersection,*
(d)	$A \,\Box_\infty\, B = A \# B,$	*inverse sum.*

Proof.

$$\text{(a) } A \oplus_1 B = \bigcup \{\lambda_1 A + \lambda_2 B / \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \operatorname{Max}\{\lambda_1, \lambda_2\} = 1\}$$
$$= 1A + 1B.$$

(b) $A \oplus_\infty B = \bigsqcup \{\lambda_1 A + \lambda_2 B / \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\}$. But the set on the right hand side of the above equality coincides with the convex hull of $A \sqcup B$. For a proof of this fact, see [27, Sect. 5.2].

(c) Analogous to (a).

(d) $A \,\Box_\infty\, B = \bigsqcup \{\lambda_1 A \cap \lambda_2 B / \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\}$. This is just the definition of the inverse sum $A \# B$ of A and B, as given by Rockafellar in [34, Sect. 3]. ∎

In the next theorem it is shown that on the class K, $\oplus_p$ and $\Box_p$ are indeed internal composition laws.

THEOREM 2.3. *If A and B are convex sets in X containing the origin, then so are their direct sum $A \oplus_p B$ and their inverse sum $A \,\Box_p\, B$.*

Proof. If $0 \in A \cap B$, it is evident that $A \oplus_p B$ and $A \,\Box_p\, B$ contain the origin. Despite the fact that, in general, the union of convex sets is not necessarily convex, the following lemma shows that the binary operations $\oplus_p$ and $\Box_p$ preserve the convexity. It suffices to apply this lemma with the choice $\Lambda = \{\lambda \in \mathbb{R}^2 / \lambda \geqslant 0, \|\lambda\|_q \leqslant 1\}$. ∎

LEMMA 2.4. *Let Λ be a nonempty convex subset of $\mathbb{R}^2$. If A and B are convex subsets of X, then so are the sets*

$$C = \bigsqcup \{\lambda_1 A + \lambda_2 B / (\lambda_1, \lambda_2) \in \Lambda\}$$

and

$$D = \bigsqcup \{\lambda_1 A \cap \lambda_2 B/(\lambda_1, \lambda_2) \in \Lambda\}.$$

Proof. (a) Convexity of the set C. Assume that x_1 and x_2 belong to C, that is to say, that they can be represented in the form

$$\begin{aligned} x_1 &= \lambda_1^1 a_1 + \lambda_2^1 b_1 \\ x_2 &= \lambda_1^2 a_2 + \lambda_2^2 b_2 \end{aligned} \tag{2.1}$$

for suitables a_1 and a_2 in A, b_1 and b_2 in B, and vectors $\lambda^1 = (\lambda_1^1, \lambda_2^1)$ and $\lambda^2 = (\lambda_1^2, \lambda_2^2)$ in Λ. We need to prove that any convex combination

$$x = \mu_1 x_1 + \mu_2 x_2 \qquad (\mu_1 > 0, \mu_2 > 0, \mu_1 + \mu_2 = 1)$$

of x_1 and x_2 belongs to C too. From (2.1) we deduce the equality

$$x = \mu_1 \lambda_1^1 a_1 + \mu_2 \lambda_1^2 a_2 + \mu_1 \lambda_2^1 b_1 + \mu_2 \lambda_2^2 b_2. \tag{2.2}$$

If we define

$$\begin{aligned} \lambda_1 &= \mu_1 \lambda_1^1 + \mu_2 \lambda_1^2 \\ \lambda_2 &= \mu_1 \lambda_2^1 + \mu_2 \lambda_2^2 \end{aligned}$$

and

$$\begin{aligned} a &= \frac{\mu_1 \lambda_1^1}{\lambda_1} a_1 + \frac{\mu_2 \lambda_1^2}{\lambda_1} a_2 \\ b &= \frac{\mu_1 \lambda_2^1}{\lambda_2} b_1 + \frac{\mu_2 \lambda_2^2}{\lambda_2} b_2, \end{aligned}$$

we can write

$$\begin{aligned} \lambda_1 a &= \mu_1 \lambda_1^1 a_1 + \mu_2 \lambda_1^2 a_2 \\ \lambda_2 b &= \mu_1 \lambda_2^1 b_1 + \mu_2 \lambda_2^2 b_2 \end{aligned}$$

and therefore

$$x = \lambda_1 a + \lambda_2 b. \tag{2.3}$$

We see that a belongs to A, since it is a convex combination of $a_1 \in A$ and $a_2 \in A$. Analogously $b \in B$. For proving that $x \in C$, it suffices then to show that the vector $\lambda = (\lambda_1, \lambda_2)$ belongs to Λ. But this follows from the convexity of Λ and the fact that

$$(\lambda_1, \lambda_2) = \mu_1(\lambda_1^1, \lambda_2^1) + \mu_2(\lambda_1^2, \lambda_2^2),$$

i.e., λ is a convex combination of λ^1 and λ^2.

(b) Convexity of the set D. The proof is exactly the same as in part (a). We only need to change (2.1), (2.2), and (2.3) by

$$x_1 = \lambda_1^1 a_1 = \lambda_2^1 b_1$$
$$x_2 = \lambda_1^2 a_2 = \lambda_2^2 b_2, \tag{2.1)'$$

$$x = \mu_1 \lambda_1^1 a_1 + \mu_2 \lambda_1^2 a_2 = \mu_1 \lambda_2^1 b_1 + \mu_2 \lambda_2^2 b_2, \tag{2.2)'$$

and

$$x = \lambda_1 a = \lambda_2 b, \tag{2.3)'$$

respectively. ∎

3. Direct and Inverse Addition of Order p on the Class of Positive Functions

We introduce now the notions of direct and inverse addition of order p for positive extended real-valued functions defined on X^*. In the next definition the norm

$$\| \mu_1, \mu_2 \|_p = \begin{cases} [(\mu_1)^p + (\mu_2)^p]^{1/p} & \text{if } p \neq \infty \\ \operatorname{Max}\{\mu_1, \mu_2\} & \text{if } p = \infty \end{cases}$$

of a vector (μ_1, μ_2) in $[0, \infty] \times [0, \infty]$ must be interpreted as equal to ∞ if either μ_1 or μ_2 is equal to ∞.

Definition 3.1. The direct and the inverse sum of order p of the functions $f\colon X^* \to [0, \infty]$ and $g\colon X^* \to [0, \infty]$ are the positive extended real-valued functions, which assign to each element $x^* \in X^*$ respectively the values

$$[f \oplus_p g](x^*) = \| f(x^*), g(x^*) \|_p$$

and

$$[f \,\square_p\, g](x^*) = \operatorname*{Inf}_{x_1^* + x_2^* = x^*} \| f(x_1^*), g(x_2^*) \|_p.$$

With the choices $p = 1$ and $p = \infty$ in the above definition, we recover the main functional operations appearing in Convex Analysis. We get, namely, the addition

$$x^* \in X^* \mapsto [f \oplus_1 g](x^*) = f(x^*) + g(x^*),$$

the maximum

$$x^* \in X \mapsto [f \oplus_\infty g](x^*) = \operatorname{Max}\{f(x^*), g(x^*)\},$$

and the infimal-convolution (cf. Moreau [25, Sect. 3] or Laurent [21, Sect. 6.5])

$$x^* \in X^* \mapsto [f \square_1 g](x^*) = \operatorname*{Inf}_{x_1^* + x_2^* = x^*} \{f(x_1^*) + g(x_2^*)\}.$$

For the sake of completeness, we mention too a different kind of infimal-convolution operation:

$$x^* \in X^* \mapsto [f \square_\infty g]]x^*) = \operatorname*{Inf}_{x_1^* + x_2^* = x^*} \operatorname{Max}\{f(x_1^*), g(x_2^*)\}.$$

Even if this notion has known a less extensive use than the previous one, it deserves at least some attention. Now, coming back to the general case in which p is an arbitrary number in $[1, \infty]$, we shall prove that $\oplus_p$ and $\square_p$ are indeed internal composition laws on the class of positive convex extended real-valued functions. Before establishing the functional version of Theorem 2.3, it is convenient for us to recall a simple fact that will be extensively used in what follows: since p and q verify the conjugacy relationship $p^{-1} + q^{-1} = 1$, the norms $\| \ \|_p$ and $\| \ \|_q$ are dual to each other. This means, in particular, that the norm $\|\mu\|_p$ of a vector $\mu = (\mu_1, \mu_2)$ in $[0, \infty] \times [0, \infty]$ admits the representation

$$\|\mu\|_p = \sup\{\lambda_1 \mu_1 + \lambda_2 \mu_2 / \lambda \geqslant 0, \|\lambda\|_q = 1\}. \tag{3.1}$$

THEOREM 3.2. *If f and g are positive convex extended real-valued functions defined on X^*, then so are their direct sum $f \oplus_p g$ and their inverse sum $f \square_p g$.*

Proof. (a) Convexity of the the direct sum. The representation (3.1) of the norm $\| \ \|_p$ shows that the function

$$x \mapsto [f \oplus_p g](x) = \sup_{\substack{\lambda \geqslant 0 \\ \|\lambda\|_q = 1}} \{\lambda_1 f(x) + \lambda_2 g(x)\}$$

is convex, since it is defined as the supremum of the family $\{\lambda_1 f + \lambda_2 g / \lambda \geqslant 0, \|\lambda\|_q = 1\}$ of convex functions.

(b) Convexity of the inverse sum. The inverse sum $f \square_p g$ can be represented as the image of the function

$$(x_1^*, x_2^*) \in X^* \times X^* \mapsto H(x_1^*, x_2^*) = \| f(x_1^*), g(x_2^*)\|_p$$

under the linear operator $A: X^* \times X^* \to X^*$ defined by $A(x_1^*, x_2^*) = x_1^* + x_2^*$. For proving the convexity of the function

$$x^* \mapsto (f \,\square_p\, g)(x^*) = \operatorname*{Inf}_{A(x_1^*, x_2^*) = x^*} H(x_1^*, x_2^*),$$

it suffices then to show that H is convex. Like in part (a), we use the representation (3.1) of the norm $\| \ \|_p$ and we write

$$H(x_1^*, x_2^*) = \sup_{\substack{\lambda \geqslant 0 \\ \|\lambda\|_q = 1}} \{\lambda_1 f(x_1^*) + \lambda_2 g(x_2^*)\}.$$

Now we remark that H is convex since it is defined as the supremum of the convex functions

$$(x_1^*, x_2^*) \mapsto \lambda_1 f(x_1^*) + \lambda_2 g(x_2^*).$$

In this way the proof is complete. ∎

4. On the Preservation of Closedness

We have seen in Section 2 that $\oplus_p$ and $\square_p$ were internal composition laws on the class K of all convex subsets of X containing the origin. Nevertheless the operation $\oplus_p$ does not preserve in general the closedness—i.e., even if A and B are two closed sets in K, it does not follow necessarily that $A \oplus_p B$ is a closed set too. Simple conditions for the preservation of closedness under various operations, like the Minkowski addition and the convex hull, have been deduced from the theory of recession cones. Recall that the recession cone 0^+G of a nonempty convex set G in X is defined as the set of directions $y \in X$ such that the ray $\{x + ty / t \geqslant 0\}$ is contained in G for every $x \in G$. If the convex set G is closed, then 0^+G is the "upper limit" of λG as $\lambda \to 0^+$, i.e.,

$$0^+G = \limsup_{\lambda \to 0^+} \lambda G = \bigcap_{\varepsilon > 0} \overline{\bigcup_{0 < \lambda < \varepsilon} \lambda G},$$

where $\bar{C}$ denotes the closure of C. Other equivalent formulations of 0^+G can be found in [33, Theorem 2A]. If G is a closed set in K, then the recession cone of G obviously takes the simpler form

$$0^+G = \bigcap_{\lambda > 0} \lambda G.$$

The scalar multiple notation 0^+G is particularly helpful in the writing of various algebraic formulae. It also suggests to interpret the scalar multiplication of G by 0 as the set 0^+G instead of merely $0 \cdot G = \{0\}$. In this way we get slightly different versions for the direct and the inverse sum of order p of the sets $A, B \in K$:

$$\begin{aligned} A \mathbin{\bar{\oplus}}_p B &= \bigsqcup \{\lambda_1 A + \lambda_2 B / \lambda \geqslant 0^+, \|\lambda\|_q = 1\} \\ A \mathbin{\bar{\square}}_p B &= \bigsqcup \{\lambda_1 A \cap \lambda_2 B / \lambda \geqslant 0^+, \|\lambda\|_q = 1\}. \end{aligned} \tag{4.1}$$

In the above formulae, the notation $\lambda \geqslant 0^+$ means that 0^+A is substituted for $0A$ when $\lambda_1 = 0$ and 0^+B is substituted for $0B$ when $\lambda_2 = 0$. These modifications are, of course, superfluous when we are dealing with bounded sets. The recession cone 0^+G of a bounded convex set G indeed reduces to $0G = \{0\}$. The introduction of the operations $\bar{\oplus}_p$ and $\bar{\square}_p$ is justified by several reasons. For instance, the set $A \mathbin{\bar{\oplus}}_p B$ can be closed even if $A \oplus_p B$ does not have this property, but the converse does not hold true. Choquet [8, Corollary 6] and Rokafellar [34, Theorem 19.6] exhibit some conditions on A and B ensuring, for example, that the set

$$A \mathbin{\bar{\oplus}}_\infty B = \bigsqcup \{\lambda_1 A + \lambda_2 B / \lambda_1 \geqslant 0^+, \lambda_2 \geqslant 0^+, \lambda_1 + \lambda_2 = 1\}$$

coincides with the closed convex hull of A and B and therefore

$$A \mathbin{\bar{\oplus}}_\infty B = \overline{A \oplus_\infty B}.$$

The operations $\bar{\oplus}_p$ and $\bar{\square}_p$ seem to be more convenient than $\oplus_p$ and $\square_p$ when we are dealing with closed sets and we are interested in preserving the closedness. Both versions for the direct and inverse addition are compared in the next proposition.

PROPOSITION 4.1. *Let A and B be two sets in K. Then the inclusions*

$$A \oplus_p B \subset A \mathbin{\bar{\oplus}}_p B \subset \overline{A \oplus_p B} \tag{4.2}$$

and

$$A \square_p B \subset A \mathbin{\bar{\square}}_p B \subset \overline{A \square_p B} \tag{4.3}$$

hold true.

Proof. The proof of the case $p = 1$ presents no difficulty. From the fact that

$$\{0\} \subset 0^+G \subset G, \qquad \forall G \in K$$

we conclude

$$A+B=A \oplus_1 B=A \bar{\oplus}_1 B \subset \overline{A+B}$$

and

$$A \cap B=A \square_1 B=A \bar{\square}_1 B \subset \overline{A \cap B}.$$

Let us consider then $p>1$. Since 0^+A and 0^+B contain the origin, the sets $A \bar{\oplus}_p B$ and the $A \bar{\square}_p B$ include $A \oplus_p B$ and $A \square_p B$, respectively. Let us show first that $A \bar{\oplus}_p B$ is included in the closure of $A \oplus_p B$. Let x be in $A \bar{\oplus}_p B$. Therefore

$$x \in \lambda_1 A+\lambda_2 B \tag{4.4}$$

for some $\lambda=(\lambda_1, \lambda_2) \geqslant 0^+$ such that $\|\lambda\|_q=1$. If $\lambda_1>0$ and $\lambda_2>0$, then x belongs evidently to $A \oplus_p B$ and the proof is ended. Let us consider the more interesting case in which $\lambda_1=0^+$ and $\lambda_2=1$. In such a case we can write

$$x=x_1+x_2 \tag{4.5}$$

with $x_1 \in 0^+ A$ and $x_2 \in B$. If for all $k>1$ we put

$$\lambda_1^k=\left(\frac{1}{k}\right)^{1/q}>0$$

$$\lambda_2^k=\left(1-\frac{1}{k}\right)^{1/q}>0,$$

then the sequence $\lambda^k=(\lambda_1^k, \lambda_2^k)$ verifies $\|\lambda^k\|_q=1$ and

$$x=\lim_k\left[\lambda_1^k\left(\frac{\lambda_2^k}{\lambda_1^k} x_1\right)+\lambda_2^k x_2\right]. \tag{4.6}$$

Since $x_1 \in 0^+A$ and A contains the origin, we deduce that

$$\frac{\lambda_2^k}{\lambda_1^k} x_1 \in A, \qquad \forall k>1.$$

This proves that x is the limit of a sequence of points in $A \oplus_p B$ and therefore belongs to $\overline{A \oplus_p B}$. The proof of the case $\lambda_1=1$ and $\lambda_2=0^+$ is analogous. In what concerns the proof of the inclusion $A \bar{\square}_p B \subset \overline{A \square_p B}$, we proceed as before but we need to change (4.4), (4.5), and (4.6) by

$$x \in \lambda_1 A \cap \lambda_2 B \tag{4.4$'$}$$

$$x=x_1=x_2, \tag{4.5$'$}$$

and

$$x = \lim_k \left[\lambda_1^k \left(\frac{\lambda_2^k}{\lambda_1^k} x_1 \right) \right] = \lim_k \, [\lambda_2^k x_2], \tag{4.6}$$

respectively. ∎

By imposing additional assumptions on the sets A and B, some of the inclusions stated in the above proposition become, in fact, equalities. For instance, the following result shows that $\bar{\square}_p$ corresponds to the "closed" version of $\square_p$ or, more precisely, that $\bar{\square}_p$ preserves the closedness.

PROPOSITION 4.2. *Let A and B be two closed sets in K. Then the following equality holds*:

$$A \, \bar{\square}_p \, B = \overline{A \, \square_p \, B}.$$

Proof. Let $\bar{x}$ be the limit of a net $\{x_i\}_{i \in I}$ in $A \, \square_p \, B$. We can write then

$$x_i \in \lambda_1^i A \cap \lambda_2^i B, \qquad \forall i \in I$$

with $\lambda^i = (\lambda_1^i, \lambda_2^i) \geqslant 0$ such that $\|\lambda^i\|_q \leqslant 1$.

The compactness of the set $\Lambda_q = \{\lambda \in \mathbb{R}^2 / \lambda \geqslant 0, \ \|\lambda\|_q \leqslant 1\}$ ensures the existence of a cluster point $\bar{\lambda} \in \Lambda_q$ of the net $\{\lambda^i\}_{i \in I}$ and therefore the existence of a subnet $\{\lambda^j\}_{j \in J}$ of $\{\lambda^i\}_{i \in I}$ converging to $\bar{\lambda}$. It is a simple matter to check that if A and B are closed set in K, then the condition

$$x_j \in \lambda_1^j A \cap \lambda_2^j B, \qquad \forall j \in I$$

$$\bar{x} = \lim \{x_j\}_{j \in I}, \qquad \bar{\lambda} = \lim \{\lambda^j\}_{j \in J}$$

implies

$$\bar{x} \in \bar{\lambda}_1^+ A \cap \bar{\lambda}_2^+ B,$$

where

$$\lambda^+ G = \begin{cases} \lambda G & \text{if } \lambda > 0 \\ 0^+ G & \text{if } \lambda = 0. \end{cases}$$

Note that the above implication corresponds to the closedness of the multifunction

$$\lambda \in \Lambda_q \mapsto \lambda_1^+ A \cap \lambda_2^+ B$$

at the point $\bar{\lambda}$. We conclude in this way that $\bar{x} \in A \, \bar{\square}_p \, B$ and hence we have proven the remaining inclusion $\overline{A \, \square_p \, B} \subset A \, \bar{\square}_p \, B$. ∎

It is natural to try to establish an analogous version of Proposition 4.2 for the operation $\bar{\oplus}_p$. However, this case requires handling with more care. The example

$$A = \mathbb{R}_- \times \{0\}$$
$$B = (-1, -1) + \{(x, y) \in \mathbb{R}^2 / x > 0, y \geqslant x^{-1}\}$$

shows that

$$A \bar{\oplus}_\infty B = \{(x, y) \in \mathbb{R}^2 / y > -1\}$$

can be nonclosed even if A and B are closed sets in K. It is not the purpose of this paper to explore deeper in this matter. For the sake of completeness we establish here at least the following result.

PROPOSITION 4.3. *Let A and B be two closed sets in K. If at least one of them is a bounded set, then the following equality holds*:

$$A \bar{\oplus}_p B = \overline{A \oplus_p B}.$$

Proof. Analogous to the proof of Proposition 4.2. The boundedness of either A or B is used to ensure the closedness of the multifunction

$$\lambda \in \Lambda_q \mapsto \lambda_1^+ A + \lambda_2^+ B. \quad \blacksquare$$

5. THE SUPPORT FUNCTIONS OF $A \oplus_p B$ AND $A \square_p B$

Recall that the support function of a nonempty subset G of X is, by definition, the extended real-valued function $\Psi_G^*: X^* \to \mathbb{R} \cup \{\infty\}$ which assigns to each element x^* of X^* the value

$$\Psi_G^*(x^*) = \sup_{x \in G} \langle x, x^* \rangle.$$

It follows at once from the definition that Ψ_G^* is proper (i.e., not identically equal to ∞), sublinear (i.e., subadditive and positively homogeneous), and lower-semicontinuous. Of fundamental importance to us is the fact that the above properties characterize the functions on X^* which are support functions of nonempty subsets of X. This result, due to Hörmander [18] in its full generality, implies that if we perform on the couple (Ψ_A^*, Ψ_B^*) a functional operation which preserves the above properties, we get the support function of a new set related with A and B. (This set is not necessarily unique, but so is its closed convex hull.) As an example of such a functional operation, we can point out the sum

$$\Psi_A^* + \Psi_B^* = \Psi_{A+B}^*$$

and the maximum

$$\max\{\Psi_A^*, \Psi_B^*\} = \Psi_{\mathrm{co}[A \cup B]}^*.$$

Which kind of operations do we need to perform on the couple (Ψ_A^*, Ψ_B^*) in order to get the support functions of the sets $A \oplus_p B$ and $A \square_p B$? Or equivalently, how can we characterize the support functions of $A \oplus_p B$ and $A \square_p B$ in terms of Ψ_A^* and Ψ_B^*?

If A and B are two subsets of X containing the origin, then their support functions are extended real-valued and positive. Therefore, Definition 3.1 can be applied with the choices $f = \Psi_A^*$ and $g = \Psi_B^*$. At once from its definition and from Theorem 3.2 it follows that the direct sum $\Psi_A^* \oplus_p \Psi_B^*$ is a proper sublinear lower-semicontinuous function. It is then the support function of some nonempty subset of X. The following theorem answers the first part of the question we are concerned with in this section.

THEOREM 5.1. *Let A and B be in K. Then the direct sum of order p of the support functions Ψ_A^* and Ψ_B^* is equal to the support function of the direct sum of order p of A and B, i.e.,*

$$\Psi_{A \oplus_p B}^* = \Psi_A^* \oplus_p \Psi_B^*.$$

Proof. For notational convenience we shall use also the symbol $\Psi^*(\cdot; G)$ for denoting the support function of a set $G \subset X$. Let x^* be an arbitrary element of X^*. Applying the standard calculus rules on support functions, we obtain the equalities

$$\begin{aligned}\Psi_{A \oplus_p B}^*(x^*) &= \Psi^*\left(x^*; \bigsqcup \{\lambda_1 A + \lambda_2 B / \lambda \geqslant 0, \|\lambda\|_q = 1\}\right)\\ &= \sup\{\Psi^*(x^*; \lambda_1 A + \lambda_2 B) / \lambda \geqslant 0, \|\lambda\|_q = 1\}\\ &= \sup\{\lambda_1 \Psi_A^*(x^*) + \lambda_2 \Psi_B^*(x^*) / \lambda \geqslant 0, \|\lambda\|_q = 1\}.\end{aligned}$$

But due to the equality (3.1), this last term is equal to $\|\Psi_A^*(x^*), \Psi_B^*(x^*)\|_p$. The case in which either $\Psi_A^*(x^*) = \infty$ or $\Psi_B^*(x^*) = \infty$ is, of course, not excluded. In such a situation both functions $\Psi_{A \oplus_p B}^*$ and $\Psi_A^* \oplus_p \Psi_B^*$ assign to x^* the value ∞. ∎

Let us consider now the second part of the question we are concerned with in this section. Directly from its definition and from Theorem 3.2, it follows that the inverse sum $\Psi_A^* \square_p \Psi_B^*$ is a proper sublinear function defined on X^*. Since the operation $\Box_p$ does not preserve in general the lower-semicontinuity, we are led to take the lower-semicontinuous hull or closure of $\Psi_A^* \Box_p \Psi_B^*$. This classical procedure (see, for example, [21, Sect. 6.2]) gives us in this case a new function

$$x^* \mapsto [\overline{\Psi_A^* \Box_p \Psi_B^*}](x^*) = \liminf_{y^* \to x^*} [\Psi_A^* \Box_p \Psi_B^*](y^*)$$

which fulfills all the requirements in Hörmander's theorem to be a support function of a nonempty subset of X. After this short prelude, we present the second part of a diptych having Theorem 5.1 as the first component.

THEOREM 5.2. *Let A and B be two closed sets in K. Then the lower-semicontinuous hull of the inverse sum of order p of the support functions Ψ_A^* and Ψ_B^* is equal to the support function of the inverse sum of order p of A and B, i.e.,*

$$\Psi_{A \sqsubset_p B}^* = \overline{\Psi_A^* \Box_p \Psi_B^*}.$$

The proof of this theorem is an excellent exercise on the use of the so-called Fenchel transform or conjugate of a function. Recall that the conjugate of a proper function $f: X \to \mathbb{R} \cup \{\infty\}$ is a new proper function $f^*: X^* \to \mathbb{R} \cup \{\infty\}$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}, \qquad \forall x^* \in X^*.$$

This justifies the notation attributed to the support function Ψ_G^*, which is the conjugate of the indicatrice function of G:

$$x \in X \mapsto \Psi_G(x) = \begin{cases} 0 & \text{if } x \in G \\ \infty & \text{if } x \notin G. \end{cases}$$

Of course, we can apply the operation of conjugacy on f^* in order to get the bi-conjugate $f^{**} = (f^*)^*$ of f. Recall also that the recession function $f0^+$ of f is the one which assigns to each element $x \in X$ the value

$$[f0^+](x) = \sup\{\langle x, x^* \rangle / x^* \in \operatorname{dom} f^*\},$$

where

$$\operatorname{dom} f^* = \{x^* \in X^* / f^*(x^*) < \infty\}.$$

Without further ado, we use in the next proof some properties and calculus rules associated to the notions mentioned above. The books of Aubin and Ekeland [5, Sect. 1] and Laurent [21, Sect. 6] cover the ones we need.

Proof of Theorem 5.2. For all x in X, the following equalities hold:

$$\begin{aligned}[\Psi_A^* \square_p \Psi_B^*]^*(x) &= \sup_{x^* \in X^*} \{\langle x, x^* \rangle - \inf_{x_1^* + x_2^* = x^*} \| \Psi_A^*(x_1^*), \Psi_B^*(x_2^*) \|_p\} \\ &= \sup_{x^* \in X^*} \sup_{x_1^* + x_2^* = x^*} \{\langle x, x^* \rangle - \| \Psi_A^*(x_1^*), \Psi_B^*(x_2^*) \|_p\} \\ &= \sup_{x_1^*, x_2^* \in X^*} \{\langle x, x_1^* \rangle + \langle x, x_2^* \rangle - \| \Psi_A^*(x_1^*), \Psi_B^*(x_2^*) \|_p\} \\ &= \sup_{(x_1^*, x_2^*) \in M} \{\langle x, x_1^* \rangle + \langle x, x_2^* \rangle - \| \Psi_A^*(x_1^*), \Psi_B^*(x_2^*) \|_p\},\end{aligned}$$

where $M = b(A) \times b(B)$ is the cartesian product of the barrier cones $b(A) = \text{dom}\, \Psi_A^*$ and $b(B) = \text{dom}\, \Psi_B^*$ of A and B, respectively. By using the representation (3.1) of the norm $\| \; \|_p$, we get

$$\begin{aligned}\| \Psi_A^*(x_1^*), \Psi_B^*(x_2^*) \|_p &= \sup\{\lambda_1 \Psi_A^*(x_1^*) + \lambda_2 \Psi_B^*(x_2^*) / \lambda \geq 0, \| \lambda \|_q = 1\} \\ &= \sup_{\lambda \in \Lambda} \{\lambda_1 \Psi_A^*(x_1^*) + \lambda_2 \Psi_B^*(x_2^*)\},\end{aligned}$$

with $\Lambda = \{\lambda \in \mathbb{R}^2 / \geq 0, \| \lambda \|_q \leq 1\}$. Therefore

$$[\Psi_A^* \square_p \Psi_B^*]^*(x) = \sup_{(x_1^*, x_2^*) \in M} \inf_{\lambda \in \Lambda} L_x((x_1^*, x_2^*), \lambda),$$

where L_x is a finite function over $M \times \Lambda$ defined by

$$L_x((x_1^*, x_2^*), \lambda) = \langle x, x_1^* \rangle + \langle x, x_2^* \rangle - \lambda_1 \Psi_A^*(x_1^*) - \lambda_2 \Psi_B^*(x_2^*).$$

Now it is important to note that in the above minimax formulation of $[\Psi_A^* \square_p \Psi_B^*]^*(x)$, it is possible to exchange the order of the supremum and the infimum. This fact can be justified by using one of the numerous minimax theorems existing in the literature. In the present case we can choose either a version due to Sion [37, Corollary 3.3] or one due to Fan [10, Theorem 2]. It is a simple matter to check that L_x and the sets M and Λ verify all the hypotheses invoked in these two versions. Therefore we write

$$[\Psi_A^* \square_p \Psi_B^*]^*(x) = \inf_{\lambda \in \Lambda} \sup_{(x_1^*, x_2^*) \in M} L_x((x_1^*, x_2^*), \lambda)$$

and consequently

$$[\Psi_A^* \square_p \Psi_B^*]^*(x) = \inf_{\lambda \in \Lambda} [r_{A, \lambda_1}(x) + r_{B, \lambda_2}(x)], \tag{5.1}$$

where

$$r_{A,\lambda_1}(x) = \sup_{x_1^* \in b(A)} \{\langle x, x_1^* \rangle - \lambda_1 \Psi_A^*(x_1^*)\}$$

$$r_{B,\lambda_2}(x) = \sup_{x_2^* \in b(B)} \{\langle x, x_2^* \rangle - \lambda_2 \Psi_B^*(x_2^*)\}.$$

A straightforward calculation shows that

$$r_{A,\lambda_1}(x) = \begin{cases} [\Psi_A 0^+](x) & \text{if } \lambda_1 = 0 \\ \Psi_{\lambda_1 A}(x) & \text{if } \lambda_1 > 0 \end{cases}$$

and

$$r_{B,\lambda_2}(x) = \begin{cases} [\Psi_B 0^+](x) & \text{if } \lambda_2 = 0 \\ \Psi_{\lambda_2 B}(x) & \text{if } \lambda_2 > 0. \end{cases}$$

Now, if we apply the Fenchel transform on both sides of the equality (5.1), we get

$$[\Psi_A^* \square_p \Psi_B^*]^{**}(x^*) = \sup_{\lambda \in \Lambda} [r_{A,\lambda_1} + r_{B,\lambda_2}]^*(x^*)$$

and therefore

$$\overline{[\Psi_A^* \square_p \Psi_B^*]}(x^*) = \sup_{\lambda \in \Lambda} [r_{A,\lambda_1} + r_{B,\lambda_2}]^*(x^*).$$

By using the calculus rules on conjugate and recession functions we deduce that

$$(r_{A,\lambda_1} + r_{B,\lambda_2})^* = \begin{cases} \Psi^*_{\lambda_1 A \cap \lambda_2 B} & \text{if } \lambda_1 > 0, \lambda_2 > 0 \\ \Psi^*_{0^+ A \cap \lambda_2 B} & \text{if } \lambda_1 = 0, \lambda_2 > 0 \\ \Psi^*_{\lambda_1 A \cap 0^+ B} & \text{if } \lambda_1 > 0, \lambda_2 = 0 \\ \Psi^*_{0^+ A \cap 0^+ B} & \text{if } \lambda_1 = 0, \lambda_2 = 0. \end{cases}$$

Hence, $\overline{\Psi_A^* \square_p \Psi_B^*}$ is the support function of the set $A \bar{\square}_p B$. But due to Proposition 4.1, this set has the same support function as the inverse sum $A \square_p B$. The proof is then complete. ∎

6. The Polar Sets of $A \oplus_p B$ and $A \square_p B$

Let G be a nonempty subset of X. The polar of G is a closed convex subset of X^* containing the origin defined by

$$G^0 = \{x^* \in X^* / \forall x \in G, \langle x, x^* \rangle \leqslant 1\}$$

or, equivalently, by

$$G^0 = \{x^* \in X^* / \Psi_G^*(x^*) \leqslant 1\}.$$

The polarity correspondence $G \mapsto G^0$ enjoys a rich variety of properties on the class K of all convex subsets of X containing the origin. For instance, it allows us to relate different binary operations on K, like the intersection and the convex hull, as is shown by the formulae

$$\begin{aligned} [\mathrm{co}(A \cup B)]^0 &= A^0 \cap B^0, \qquad &\forall A, B \in K \\ [\bar{A} \cap \bar{B}]^0 &= \overline{\mathrm{co}(A^0 \cup B^0)}, \qquad &\forall A, B \in K. \end{aligned} \tag{6.1}$$

Note that each one of the above formulae can be obtained from the other, just by exchanging the roles of the couples (A, B) and (A^0, B^0) and by taking into acount that the bipolar $G^{00} = (G^0)^0$ of a set G in K is equal to the closure $\bar{G}$ of G. In this section it is shown that the polarity correspondence $G \mapsto G^0$ give us a new light in the understanding of the connection between the direct and the inverse addition. We shall see that a polarity relationship like (6.1) also holds for the operations $\oplus_p$ and $\Box_q$. As a matter of fact, the formulae (6.1) can be obtained as a particular case of the more general polarity theorem which we present immediately.

THEOREM 6.1. *Let A and B be two convex sets in X containing the origin. Then the following polarity relationships hold*:

$$[A \oplus_p B]^0 = \overline{A^0 \Box_q B^0}$$
$$[\bar{A} \Box_q \bar{B}]^0 = \overline{A^0 \oplus_p B^0}.$$

In particular, with the choices $(p, q) = (1, \infty)$ and $(p, q) = (\infty, 1)$ we get respectively

$$[A + B]^0 = \overline{A^0 \# B^0}$$
$$[\bar{A} \# \bar{B}]^0 = \overline{A^0 + B^0}$$

and

$$[\mathrm{co}(A \cup B)]^0 = A^0 \cap B^0$$
$$[\bar{A} \cap \bar{B}]^0 = \overline{\mathrm{co}(A^0 \cup B^0)}.$$

Proof. In the case in which $p = \infty$ and $q = 1$, the present theorem states the validity of the well-known formulae (6.1). We point out Ref. [36, Sect. 4.1] for the reader interested in a proof. Note that in this case the closure operation on $A^0 \cap B^0$ is superfluous. We prove therefore only the case in which p and q belong to $]1, \infty[$ and the case in which $p=1$ and $q=\infty$.

(i) Assume $p, q \in]1, \infty[$. Since $A \oplus_p B$ belongs to K, we can write

$$\overline{A \oplus_p B} = [A \oplus_p B]^{00},$$
$$= \{x^* \in X^* / \Psi^*_{A \oplus_p B}(x^*) \leqslant 1\}^0$$

and, due to Theorem 5.1, we obtain

$$\overline{A \oplus_p B} = \{x^* \in X^* / [\Psi^*_A \oplus_p \Psi^*_B](x^*) \leqslant 1\}^0.$$

Hence

$$\overline{A \oplus_p B} = \{x^* \in X^* / (f_1 + f_2)(x^*) \leqslant p^{-1}\}^0, \tag{6.2}$$

where $f_1, f_2 \colon X^* \to [0, \infty]$ are defined by

$$f_1(x^*) = p^{-1}[\Psi^*_A(x^*)]^p, \qquad \forall x^* \in X^*$$
$$f_2(x^*) = p^{-1}[\Psi^*_B(x^*)]^p, \qquad \forall x^* \in X^*.$$

Now we remark that $f_1 + f_2$ is a lower-semicontinuous proper convex function positively homogeneous of degree p. This fact allows us to compute the polar set in (6.2) by using a calculus rule due to Rockafellar [34, Corollary 15.3.2]. This rule was stated in a finite dimensional setting, but can be extended to our more general framework. So, we obtain the equality

$$\overline{A \oplus_p B} = \{x \in X / (f_1 + f_2)^* (x) \leqslant q^{-1}\}$$

and therefore

$$\overline{A \oplus_p B} = \{x \in X / [\overline{f_1^* \,\square\, f_2^*}](x) \leqslant q^{-1}\}.$$

where $\square$ denotes the infimal-convolution operation. A straightforward calculation shows that the Fenchel transforms of f_1 and f_2 are given by

$$f_1^*(x_1) = q^{-1}[\Psi^*_{A^0}(x_1)]^q, \qquad \forall x_1 \in X$$
$$f_2^*(x_2) = q^{-1}[\Psi^*_{B^0}(x_2)]^q, \qquad \forall x_2 \in X$$

and hence

$$[\overline{f_1^* \,\square\, f_2^*}](x) = \liminf_{z \to x} \inf_{x_1 + x_2 = z} \{q^{-1}[\Psi^*_{A_0}(x_1)]^q + q^{-1}[\Psi^*_{B_0}(x_2)]^q\}$$
$$= q^{-1}\, \{\liminf_{z \to x} \inf_{x_1 + x_2 = z} \| \Psi^*_{A^0}(x_1), \Psi^*_{B^0}(x_2) \|_q\}^q$$
$$= q^{-1}\{[\Psi^*_{A^0} \,\square_q\, \Psi^*_{B^0}](x)\}^q.$$

We get in this way the equality

$$\overline{A \oplus_p B} = \{x \in X/[\overline{\Psi^*_{A^0} \square_q \Psi^*_{B^0}}](x) \leqslant 1\}$$

and, due to Theorem 4.2, we conclude

$$\begin{aligned}\overline{A \oplus_p B} &= \{x \in X/\Psi^*(x; A^0 \square_q B^0) \leqslant 1\} \\ &= [A^0 \square_q B^0]^0.\end{aligned}$$

By computing the polar on both sides of the above equality we get

$$[A \oplus_p B]^0 = [\overline{A \oplus_p B}]^0 = [A^0 \square_q B^0]^{00} = \overline{A^0 \square_q B^0},$$

whereas the second formula in the statement of Theorem 6.1 is obtained by exchanging the roles played by the couples (A, B) and (A^0, B^0).

(ii) Assume $p = 1$ and $q = \infty$. We start with the set of equalities

$$\begin{aligned}[A + B]^0 &= \{x^* \in X^*/\Psi^*_{A+B}(x^*) \leqslant 1\} \\ &= \{x^* \in X^*/\Psi^*_A(x^*) + \Psi^*_B(x^*) \leqslant 1\}.\end{aligned}$$

Now we recall the equivalence between the inequality

$$\Psi^*_A(x^*) + \Psi^*_B(x^*) \leqslant 1$$

and the existence of two nonnegative numbers λ_1 and λ_2 adding up to 1 and such that

$$\begin{aligned}\Psi^*_A(x^*) &\leqslant \lambda_1 \\ \Psi^*_B(x^*) &\leqslant \lambda_2.\end{aligned}$$

This equivalence is a direct consequence of the fact that the addition + is isotonic with respect to the usual order relation in $\mathbb{R}$ (cf. Moreau [25, Sect. 2.6]). Therefore we can write

$$[A + B]^0 = \bigsqcup \{S_{\lambda_1}(\Psi^*_A) \cap S_{\lambda_2}(\Psi^*_B)/\lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\}, \quad (6.3)$$

where

$$S_\lambda(\Psi^*_G) = \{x^* \in X^*/\Psi^*_G(x^*) \leqslant \lambda\}$$

is the level set at λ of the function Ψ^*_G. Without difficulty it can be proved that

$$S_\lambda(\Psi^*_G) = \begin{cases} \lambda G^0 & \text{if } \lambda > 0 \\ 0^+ G^0 & \text{if } \lambda = 0 \end{cases}$$

when G is a set in K. This shows that the equality (6.3) takes the form

$$[A+B]^0 = A^0 \bar{\Box}_\infty B^0 = \bigsqcup \{\lambda_1 A^0 \cap \lambda_2 B^0 / \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\}$$

with 0^+A^0 substituted for $0 \cdot A^0$ when $\lambda_1 = 0$ and 0^+B^0 substituted for $0 \cdot B^0$ when $\lambda_2 = 0$. Taking into account Proposition 4.2 we get finally

$$[A+B]^0 = \overline{A^0 \Box_\infty B^0} = \overline{A^0 \# B^0}.$$

Exchanging the roles of the couples (A, B) and (A^0, B^0) we get the last equality

$$[\bar{A} \# \bar{B}]^0 = \overline{A^0 + B^0}$$

and we complete the proof. ∎

We end this section by stating without proof two immediate corollaries of Theorem 6.1.

COROLLARY 6.2. *Let A and B be two bounded closed convex sets in X containing the origin. Then the sets $A \oplus_p B$ and $\overline{A^0 \Box_q B^0}$ are polar to each other.*

COROLLARY 6.3. *Let A and B be two closed convex sets in X containing the origin in their interior. Then the sets $\overline{A \Box_q B}$ and $A^0 \oplus_p B^0$ are polar to each other.*

7. CALCULUS RULES FOR THE SECOND-ORDER SUBDIFFERENTIAL OF A CONVEX FUNCTION

In [15] Hiriart-Urruty introduced the so-called second-order subdifferential of a real-valued convex function defined on the finite dimensional space $\mathbb{R}^n$. This notion has been further developed by the author [35], namely, by dealing with an extended real-valued proper convex function defined on a locally convex space X, like the one in this paper. We recall now in few lines this notion and their main properties and we point out Refs. [15, 35, 17] for a more complete discussion.

Let $f: X \to \mathbb{R} \cup \{\infty\}$ be a proper convex function continuous at the point $\bar{x} \in X$. It is well known that in such a case the directional derivative of f at $\bar{x}$

$$h \mapsto f'(\bar{x}; h) = \lim_{t \to 0^+} t^{-1}[f(\bar{x}+th) - f(\bar{x})]$$

exists and is finite. We shall say that f is twice directionally differentiable at $\bar{x}$ if the second-order directional derivative of f at $\bar{x}$

$$h \mapsto f''(\bar{x}; h) = \lim_{t \to 0^+} \frac{2}{t} \left[\frac{f(\bar{x}+th)-f(\bar{x})}{t} - f'(\bar{x}; h) \right]$$

exists and is finite. Note that the above limit is necessarily nonnegative since the function f is convex. The so-called subdifferential $\partial f(\bar{x})$ of f at $\bar{x}$ is the set of elements in X^* verifying

$$f(x) \geqslant f(\bar{x}) + \langle x - \bar{x}, x^* \rangle, \qquad \forall x \in X$$

or, what is equivalent,

$$\partial f(\bar{x}) = \{x^* \in X^* / \langle h, x^* \rangle \leqslant f'(\bar{x}; h), \forall h \in X\}.$$

In what concerns the second-order subdifferential $\partial^2 f(\bar{x})$ of f at $\bar{x}$, we need to know only that this set can be given by

$$\partial^2 f(\bar{x}) = \{z^* \in X^* / \langle h, z^* \rangle \leqslant \sqrt{f''(\bar{x}; h)}, \forall h \in X\} \tag{7.1}$$

when the function f is continuous and twice directionally differentiable at $\bar{x}$. We recall the main properties of this set in the next proposition (cf. [35, Sect. B.1] or [15, Proposition 2] in the finite dimensional setting).

PROPOSITION 7.1. *Let $f: X \to \mathbb{R} \cup \{\infty\}$ be a proper convex function which is continuous and twice directionally differentiable at $\bar{x}$. Then*

(a) *the second-order subdifferential $\partial^2 f(\bar{x})$ of f at $\bar{x}$ is a closed convex subset of X^* containing the origin,*

(b) *the support function of $\partial^2 f(\bar{x})$ is equal to the biconjugate of $\sqrt{f''(\bar{x}; \cdot)}$, i.e.,*

$$\Psi^*_{\partial^2 f(\bar{x})} = [\sqrt{f''(\bar{x}; \cdot)}\,]^{**}.$$

In particular, the set $\partial^2 f(\bar{x})$ is bounded (since the function $f''(\bar{x}; \cdot)$ is assumed to be finite).

The aim of this section is to derive expressions of the second-order subdifferential of a function which has been built up from other functions whose properties are better known. We shall give, namely, calculus rules for the computation of the second-order subdifferential of the sum $f+g$ and the infimal-convolution $f \,\square\, g$ of the proper convex functions f and g. It is at this point where the notions of direct and inverse addition of order 2 will play an important role. They allow us to present these calculus rules in a more elegant way and to have a better understanding of them. Recall then that the direct and inverse sums of order 2 of the sets A and B in K are by definition

$$A \oplus_2 B = \bigsqcup \{\lambda_1 A + \lambda_2 B / \lambda \geqslant 0, \|\lambda\|_2 = 1\}$$

and

$$A \square_2 B = \bigsqcup \{\lambda_1 A \cap \lambda_2 B / \lambda \geqslant 0, \|\lambda\|_2 = 1\}$$

respectively. It is immediate to see that these sets admit also the characterization

$$A \oplus_2 B = \bigsqcup \{\sqrt{\lambda_1}\, A + \sqrt{\lambda_2}\, B / \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\}$$

and

$$A \square_2 B = \bigsqcup \{\sqrt{\lambda_2}\, A \cap \sqrt{\lambda_2}\, B / \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\}.$$

Addition. Let us begin now by presenting an expression of the second-order subdifferential of the sum of two convex functions. As a first step we state the next lemma.

LEMMA 7.2. *Let $f, g: X \to \mathbb{R} \cup \{\infty\}$ be two proper convex functions which are continuous and twice directionally differentiable at the point $\bar{x}$. Then so is their sum $f+g$, which has a second-order directional derivative $(f+g)''(\bar{x}; \cdot)$ at $\bar{x}$ verifying the equality*

$$\sqrt{(f+g)''(\bar{x}; h)} = \operatorname*{Max}_{\substack{\lambda_1 \geqslant 0,\, \lambda_2 \geqslant 0 \\ \lambda_1 + \lambda_2 = 1}} \{\sqrt{\lambda_1}\sqrt{f''(\bar{x}; h)} + \sqrt{\lambda_2}\sqrt{g''(\bar{x}; h)}\}, \qquad \forall h \in X. \tag{7.2}$$

Proof. A direct calculation shows that the second-order directional derivative $(f+g)''(\bar{x}; \cdot)$ of $f+g$ at $\bar{x}$ is given by

$$(f+g)''(\bar{x}; h) = f''(\bar{x}; h) + g''(\bar{x}; h), \qquad \forall h \in X.$$

We conclude the desired result by taking the square root on both sides of the above equality and by applying the general formula

$$\sqrt{a+b} = \operatorname{Max}\{\sqrt{\lambda_1}\sqrt{a} + \sqrt{\lambda_2}\sqrt{b} / \lambda_1 \geqslant 0, \lambda_2 \geqslant 0, \lambda_1 + \lambda_2 = 1\} \qquad \forall a, b \in [0, \infty], \tag{7.3}$$

with the choices $a = f''(\bar{x}; h)$ and $b = g''(\bar{x}; h)$. The formula (7.3) does not seem to be well known, although it can be easily derived from the equality (3.1). ∎

Recall that the subdifferential of the sum of two proper convex functions verifies

$$\partial(f+g)(\bar{x}) \supset \partial f(\bar{x}) + \partial g(\bar{x}),$$

and that the above inclusion becomes an equality

$$\partial(f+g)(\bar{x}) = \partial f(\bar{x}) + \partial g(\bar{x})$$

if, for instance, the functions f and g are continuous at the point $\bar{x}$. In what concerns the second-order subdifferential of $f+g$ at $\bar{x}$, we establish now the following calculus rule. We improve previous results of Hiriart-Urruty and the author [16, 35], who gave only some lower and upper estimations of this set.

THEOREM 7.3. *Let f and g be as in Lemma 7.2. We have then the inclusion*

$$\partial^2(f+g)(\bar{x}) \supset \partial^2 f(\bar{x}) \oplus_2 \partial^2 g(\bar{x}).$$

The equality

$$\partial^2(f+g)(\bar{x}) = \partial^2 f(\bar{x}) \oplus_2 \partial^2 g(\bar{x}) \tag{7.4}$$

holds if, moreover, $f''(\bar{x}, \cdot)$ and $g''(\bar{x}; \cdot)$ *are convex functions.*

Proof. From Proposition 7.1 we know that

$$\Psi^*_{\partial^2 g(\bar{x})} = \sqrt{f''(\bar{x}; \cdot)}^{**} \leqslant \sqrt{f''(\bar{x}; \cdot)}$$
$$\Psi^*_{\partial^2 g(\bar{x})} = \sqrt{g''(\bar{x}; \cdot)}^{**} \leqslant \sqrt{g''(\bar{x}; \cdot)}$$

and therefore, due to the previous lemma, we can write for all $h \in X$

$$\begin{aligned}\sqrt{(f+g)''(\bar{x}; h)} &\geqslant \operatorname*{Max}_{\substack{\lambda_1 \geqslant 0,\, \lambda_2 \geqslant 0 \\ \lambda_1+\lambda_2=1}} \{\sqrt{\lambda_1}\, \Psi^*_{\partial^2 f(\bar{x})}(h) + \sqrt{\lambda_2}\, \Psi^*_{\partial^2 g(\bar{x})}(h)\} \\ &= \operatorname*{Max}_{\substack{\lambda_1 \geqslant 0,\, \lambda_2 \geqslant 0 \\ \lambda_1+\lambda_2=1}} \Psi^*(h; \sqrt{\lambda_1}\, \partial^2 f(\bar{x}) + \sqrt{\lambda_2}\, \partial^2 g(\bar{x})) \\ &= \Psi^*(h; \partial^2 f(\bar{x}) \oplus_2 \partial^2 g(\bar{x})).\end{aligned}$$

From this we get the inequality

$$\Psi^*(h; \partial^2(f+g)(\bar{x})) \geqslant \Psi^*(h; \partial^2 f(\bar{x}) \oplus_2 \partial^2 g(\bar{x})) \qquad \forall h \in X$$

and hence the inclusion stated in this theorem. If we assume additionally that $f''(\bar{x};\cdot)$ and $g''(\bar{x};\cdot)$ are convex functions, then so is $(f+g)''(\bar{x};\cdot)$ and we have

$$\Psi^*_{\partial^2 f(\bar{x})} = \sqrt{f''(\bar{x};\cdot)}$$
$$\Psi^*_{\partial^2 g(\bar{x})} = \sqrt{g''(\bar{x};\cdot)}$$
$$\Psi^*_{\partial^2(f+g)(\bar{x})} = \sqrt{(f+g)''(\bar{x};\cdot)}.$$

Proceeding as in the first part, we get the equality

$$\Psi^*(h; \partial^2(f+g)(\bar{x})) = \Psi^*(h; \partial^2 f(\bar{x}) \oplus_2 \partial^2 g(\bar{x})) \qquad \forall h \in X$$

and therefore

$$\partial^2(f+g)(\bar{x}) = \overline{\text{co}}\,[\partial^2 f(\bar{x}) \oplus_2 \partial^2 g(\bar{x})].$$

But in this case the above closed convex hull operation is superfluous. We know already that $\partial^2 f(\bar{x}) \oplus_2 \partial^2 g(\bar{x})$ is convex, and we see that this set is also closed because $\partial^2 f(\bar{x})$ and $\partial^2 g(\bar{x})$ are two closed bounded sets. ▮

Remark 7.4. The inclusion stated in the above theorem can be strict if the functions $f''(\bar{x};\cdot)$ and $g''(\bar{x};\cdot)$ are not convex. Consider, for instance, the point $\bar{x} = (0, 0) \in \mathbb{R}^2$ and the functions $f, g: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \text{Max}\{\tfrac{1}{2}(x_1)^2 + \tfrac{1}{2}(x_2)^2 + x_1, 0\}$$
$$g(x_1, x_2) = \text{Max}\{\tfrac{1}{2}(x_1)^2 + \tfrac{1}{2}(x_2)^2 - x_1, 0\}.$$

A straightforward calculation show that

$$\partial^2(f+g)(\bar{x}) = \{x \in \mathbb{R}^2 / \|x\|_2 \leqslant 1\}$$

includes strictly the set

$$\partial^2 f(\bar{x}) \oplus_2 \partial^2 g(\bar{x}) = [-1, 1] \times \{0\}.$$

Infimal-Convolution. We shall consider now the problem of estimating the second-order subdifferential of the infimal-convolution

$$x \mapsto (f \,\Box\, g)(x) = \underset{x_1 + x_2 = x}{\text{Inf}} \{f(x_1) + g(x_2)\}$$

of the proper convex functions $f, g: X \to \mathbb{R} \cup \{\infty\}$. We shall assume that there exists a couple $(\bar{x}_1, \bar{x}_2)$ verifying

$$\bar{x}_1 + \bar{x}_2 = \bar{x},\ f(\bar{x}_1) + g(\bar{x}_2) = \underset{x_1 + x_2 = \bar{x}}{\text{Inf}} \{f(x_1) + g(x_2)\} \in \mathbb{R}.$$

In such a case the infimal-convolution is said to be exact at $\bar{x} = \bar{x}_1 + \bar{x}_2$ and, as is well known, the subdifferential of $f \Box g$ at $\bar{x}$ is given by

$$\partial(f \Box g)(\bar{x}) = \partial f(\bar{x}_1) \cap \partial g(\bar{x}_2). \tag{7.5}$$

Several criteria can be used for ensuring the existence of such a couple $(\bar{x}_1, \bar{x}_2)$. The reader can refer, for instance, to the book of Laurent [21, Sect. 7.5] or to the original works of Moreau [24, 25], Fenchel [11], and Rockafellar [32]. We shall assume in what follows that f and g are continuous and twice directionally differentiable at the points $\bar{x}_1$ and $\bar{x}_2$, respectively. As far as we know, stronger assumptions need to be made on f and g in order to ensure that $f \Box g$ is continuous and twice directionally differentiable at $\bar{x}$. We will not discuss here the ones we require and we shall just assume that $f \Box g$ verifies the above condition. In terms of their support functions, the sets in (7.5) verify

$$\Psi^*_{\partial(f \Box g)(\bar{x})} = \overline{\Psi^*_{\partial f(\bar{x}_1)} \Box \Psi^*_{\partial g(\bar{x}_2)}}$$

and therefore

$$(f \Box g)'(\bar{x}; \cdot) = \overline{f'(\bar{x}_1; \cdot) \Box g'(\bar{x}_2; \cdot)}. \tag{7.6}$$

Now what we need to know is how to compare $(f \Box g)''(\bar{x}; \cdot)$ in terms of $f''(\bar{x}_1; \cdot)$ and $g''(\bar{x}_2; \cdot)$, in order to get an estimation of $\partial^2(f \Box g)(\bar{x})$ in terms of $\partial^2 f(\bar{x}_1)$ and $\partial^2 g(\bar{x}_2)$. Looking at the equality (7.6) one is tempted to write

$$(f \Box g)''(\bar{x}; \cdot) = \overline{f''(\bar{x}_1; \cdot) \Box g''(\bar{x}_2; \cdot)}, \tag{7.7}$$

just by changing first- by second-order directional derivatives. As a matter of fact this temptation is well justified in many cases. We shall give further in an auxiliary remark some sufficient conditions ensuring the validity of the above equality. Without further ado we present now the sister version of Theorem 7.3, which allows us to compute the second-order subdifferential $\partial^2(f \Box g)(\bar{x})$.

THEOREM 7.5. *Assume that the infimal-convolution $f \Box g$ of the proper convex functions $f, g: X \to \mathbb{R} \cup \{\infty\}$ is exact at $\bar{x} = \bar{x}_1 + \bar{x}_2$. Let f, g, and $f \Box g$ be continuous and twice directionally differentiable at the points $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}$, respectively. Under these general conditions, if we assume that $f''(\bar{x}_1, \cdot)$ and $g''(\bar{x}_2; \cdot)$ are convex and that equality (7.7) holds, then we can write*

$$\partial^2(f \Box g)(\bar{x}) = \partial^2 f(\bar{x}_1) \Box_2 \partial^2 g(\bar{x}_2).$$

Proof. Under the hypothesis of this theorem, $(f \square g)''(\bar{x}; \cdot)$ is a convex finite function and

$$\Psi^*_{\partial^2(f \square g)(\bar{x})} = \sqrt{(f \square g)''(\bar{x}; \cdot)}.$$

Since we are assuming that the equality (7.7) holds, this last function is equal to the closure of $\sqrt{f''(\bar{x}_1; \cdot) \square g''(\bar{x}_2; \cdot)}$. Now, we see that for all $h \in X$,

$$\begin{aligned}
&[\operatorname*{Inf}_{h_1+h_2=h} f''(\bar{x}_1; h_1) + g''(\bar{x}_2; h_2)]^{1/2} \\
&\quad = \operatorname*{Inf}_{h_1+h_2=h} [[\Psi^*_{\partial^2 f(\bar{x}_1)}(h_1)]^2 + [\Psi^*_{\partial^2 g(\bar{x}_2)}(h_2)]^2]^{1/2} \\
&\quad = [\Psi^*_{\partial^2 f(\bar{x}_1)} \square_2 \Psi^*_{\partial^2 g(\bar{x}_2)}](h),
\end{aligned}$$

and hence

$$\sqrt{(f \square g)''(\bar{x}; \cdot)} = \overline{\Psi^*_{\partial^2 f(\bar{x}_1)} \square_2 \Psi^*_{\partial^2 g(\bar{x}_2)}}.$$

We obtain in this way the equality

$$\Psi^*(\cdot; \partial^2(f \square g)(\bar{x})) = \Psi^*(\cdot; \partial^2 f(\bar{x}_1) \square_2 \partial^2 g(\bar{x}_2)),$$

from which the desired result follows. ▮

For the sake of completeness we give here an auxiliary remark which allows us to check the hypotheses of the above theorem. Since we want to present sufficient conditions which are easily to handle, we do not pay care to the loss of generality, although these coditions can be weakened in order to recover a more general setting.

Remark 7.6. If we assume that the infimal-convolution $f \square g$ of the differentiable convex functions $f, g: \mathbb{R}^n \to \mathbb{R}$ is exact at $\bar{x} = \bar{x}_1 + \bar{x}_2$ and that f, g, and $f \square g$ are twice directionally differentiable at the points $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}$, respectively, then the functions $f''(\bar{x}_1; \cdot)$ and $g''(\bar{x}_2; \cdot)$ are convex and the equality (7.7) holds.

8. Direct and Inverse Addition of Ellipsoids and Network Synthesis

In a 1969 paper [1] of Anderson and Duffin we can read : "the connection of resistors in series and parallel is a familiar concept from elementary network theory. If two resistors having resistances A and B are connected in series the joint resistance is $S = A + B$, and if they are in parallel, the joint resistance is $P = (A^{-1} + B^{-1})^{-1} = AB/(A+B)$. These two methods of combining resistance are then called series and parallel addition." Of course, it is tacitly assumed in the above that A and B are positive numbers. However we do not need to exclude the case $A=0$ or $B=0$, since in such a situation—a short circuit—we can let $P=0$. A few lines below, in the same paper, Anderson and Duffin extended these operations to symmetric positive semidefinite $n \times n$ matrices A and B in order to deal with series and parallel connections of n-ports. They defined the series sum of A and B as the ordinary sum $A+B$ and the parallel sum of A and B as

$$A \square B = A(A+B)^{+} A, \tag{8.1}$$

where C^{+} denotes the Moore–Penrose generalized inverse of C.

The equality (8.1) reduces of course to the more familiar form

$$A \square B = (A^{-1} + B^{-1})^{-1}$$

if the matrices A and B are nonsingular. Since then, and even now, a great deal of effort has been made in order to give equivalent or more general formulations of this last operation. Interesting contributions can be found for instance in the papers of Fillmore and Williams [12], Anderson and Schreiberg [3], Anderson and Trapp [4], Pekarev and Smul'jan [30], Nishio and Ando [28], Morley [26], Kubo [19], Passty [29], and Mazure [23] among others.

As proved in [1, Lemmas 2 and 4] the series and parallel additions are indeed internal composition laws on the class of symmetric positive semidefinite $n \times n$ matrices. Contrarily to the notation $A: B$ which has been the most extensively used in the literature, we denote the matrix $A(A+B)^{+} B$ by $A \square B$ for emphasizing the fact that this matrix is obtained by performing an infimal-convolution operation (i.e., an inverse addition of order 1). This is made clear by recalling a variational property of $A \square B$ given by Anderson and Trapp [4] which, as shown by Morley [26], can be used to define the parallel addition (even for positive semidefinite linear operators on a Hilbert space):

$$\langle [A \square B]\, x, x \rangle = \operatorname*{Inf}_{x_1 + x_2 = x} \{ \langle A x_1, x_1 \rangle + \langle B x_2, x_2 \rangle \}, \qquad \forall x \in \mathbb{R}^n. \tag{8.2}$$

In terms of the positive (convex) quadratic form

$$x \mapsto q_c(x) = \tfrac{1}{2} \langle Cx, x \rangle$$

associated to a symmetric positive semidefinite matrix C, the variational property (8.2) takes the simpler form

$$q_{A \square B} = q_A \square q_B.$$

In her recent thesis [23] and in a related paper [22], Mazure adopted this approach for showing in an elegant way how the properties of the parallel sum of matrices are then deduced from those of the infimal-convolution of convex functions. Nevertheless, in this paper we want to follow a different path, left unexplored as far as we known. Instead of positive quadratic forms we consider a geometric concept. There is a one-to-one correspondence between the set of symmetric positive semidefinite $n \times n$ matrices and the set of ellipsoids with center 0 in $\mathbb{R}^n$. We can consider, for instance, the correspondence

$$C \mapsto E(C) = C^{1/2}(S),$$

where $S = \{x \in \mathbb{R}^n / \langle x, x \rangle \leqslant 1\}$ is the closed unit ball in $\mathbb{R}^n$. We see then that the operations of series and parallel addition of matrices can be established, in an equivalent way, by considering ad hoc operations on the set of ellipsoids. We are then led to recognize the kind of geometric operations we need to perform on the ellipsoids $E(A)$ and $E(B)$ in order to get $E(A+B)$ and $E(A \square B)$. Note that from a geometric point of view, $E(A)$ and $E(B)$ are closed onvex sets in $\mathbb{R}^n$ containing the origin. They are necessarily bounded because their support functions, given by

$$h \mapsto \Psi^*_{E(A)}(h) = \sqrt{\langle Ah, h \rangle}$$
$$h \mapsto \Psi^*_{E(B)}(h) = \sqrt{\langle Bh, h \rangle},$$

are finite. These sets do not contain necessarily the origin in their interior, since the matrices A or B could be singular. The possibility of degenerate or "flat" ellipsoids is therefore not excluded. With all that in mind we can state the next theorem, which answers the question we are concerned with in this section.

THEOREM 8.1. *Let A and B be two symmetric semidefinite $n \times n$ matrices. Then*

(a) *the ellipsoid $E(A+B)$ associated to the series sum of A and B is equal to the direct sum of order two of the elipsoids $E(A)$ and $E(B)$ associated to A and B, respectively,*

$$E(A+B) = E(A) \oplus_2 E(B), \tag{8.3}$$

(b) *the ellipsoid $E(A \square B)$ associated to the parallel sum of A and B is equal to the inverse sum of order two of the ellipsoids $E(A)$ and $E(B)$ associated to A and B, respectively,*

$$E(A \square B) = E(A) \square_2 E(B). \tag{8.4}$$

First Proof of Theorem 8.1. Equality (8.3) is obtained by applying Theorem 7.3 with $f=q_A$, $g=q_B$ and an arbitrary choice for $\bar{x}$. Let us take $\bar{x}=0$ for instance. It is a simple matter to check that

$$\begin{aligned}\partial^2 f(\bar{x}) &= \{z \in \mathbb{R}^n / \langle h, z\rangle \leqslant \sqrt{\langle Ah, h\rangle}, \forall h \in \mathbb{R}^n\}\\ &= \{z \in \mathbb{R}^n / \langle h, z\rangle \leqslant \Psi^*_{E(A)}(h), \forall h \in \mathbb{R}^n\}\\ &= E(A)\end{aligned}$$

and similarly

$$\partial^2 g(\bar{x}) = E(B)$$

$$\partial^2 (f+g)(\bar{x}) = E(A+B).$$

In order to deduce the equality (8.4), we of course need to apply Theorem 7.5. As before we take $f=q_A$, $g=q_B$ and we choose $\bar{x}$ arbitrarily. Let $\bar{x}=0$ for instance. Note that the infimal-convolution $q_A \square q_B$ is exact at $\bar{x}=\bar{x}_1+\bar{x}_2$ with $\bar{x}_1=0$ and $\bar{x}_2=0$. Taking into account Remark 7.6, we see that all the hypotheses mentioned in Theorem 7.5 are satisfied. Of course in this case we have

$$\partial^2 f(\bar{x}_1) = E(A)$$

$$\partial^2 g(\bar{x}_2) = E(B)$$

and

$$\partial^2 (f \square g)(\bar{x}) = E(A \square B). \quad \blacksquare$$

Second Proof of Theorem 8.1. We prove the same theorem by using a methodology which seems to be quite different, but that in fact is intimately connected with the previous one. Instead of using the calculus rules for computing the second-order subdifferentials $\partial^2(f+g)(\bar{x})$ and $\partial^2(f \square g)(\bar{x})$, we use some calculus rules on the approximate subdifferentials $\partial_\varepsilon(f+g)(\bar{x})$ and $\partial_\varepsilon(f \square g)(\bar{x})$ established by Hiriart-Urruty [13, 14]. Recall that the approximate or ε-subdifferential $\partial_\varepsilon l(z)$ at $z \in \mathbb{R}^n$ of a convex function $l: \mathbb{R}^n \to \mathbb{R}$ is given by

$$\partial_\varepsilon l(z) = \{x^* \in \mathbb{R}^n / l(x) \geqslant l(z) + \langle x-z, x^*\rangle - \varepsilon, \forall x \in \mathbb{R}^n\},$$

whereas the approximate or ε-directional derivative $l'_\varepsilon(z; \cdot)$ at z of l is its support function:

$$h \mapsto l'_\varepsilon(z; h) = \Psi^*(h; \partial_\varepsilon l(z)).$$

Under suitable assumptions made on f and g (as usual, among other hypotheses, the infimal-convolution $f \square g$ is assumed to be exact at $\bar{x} = \bar{x}_1 + \bar{x}_2$), the following formulae hold true:

$$(f+g)'_\varepsilon(\bar{x}; h) = \operatorname*{Max}_{\substack{\varepsilon_1 \geqslant 0,\, \varepsilon_2 \geqslant 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{f'_{\varepsilon_1}(\bar{x}; h) + g'_{\varepsilon_2}(\bar{x}; h)\} \tag{8.5}$$

$$(f \square g)'_\varepsilon(\bar{x}; h) = \operatorname*{Max}_{\substack{\varepsilon_1 \geqslant 0,\, \varepsilon_2 \geqslant 0 \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \operatorname*{Inf}_{h_1 + h_2 = h} \{f'_{\varepsilon_1}(\bar{x}_1; h_1) + g'_{\varepsilon_2}(\bar{x}_2; h_2)\} \tag{8.6}$$

or, what is equivalent,

$$\partial_\varepsilon(f+g)(\bar{x}) = \bigsqcup \{\partial_{\varepsilon_1} f(\bar{x}) + \partial_{\varepsilon_2} g(\bar{x}) / \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0, \varepsilon_1 + \varepsilon_2 = \varepsilon\} \tag{8.7}$$

$$\partial_\varepsilon(f \square g)(\bar{x}) = \bigsqcup \{\partial_{\varepsilon_1} f(\bar{x}_1) \cap \partial_{\varepsilon_2} g(\bar{x}_2) / \varepsilon_1 \geqslant 0, \varepsilon_2 \geqslant 0, \varepsilon_1 + \varepsilon_2 = \varepsilon\}. \tag{8.8}$$

We have written all these formulae in order to exhibit their astonishing similarity with the definitions of direct and inverse addition for functions and sets given in Sections 3 and 2, respectively. As in the previous proof, we choose now $f = q_A$, $g = q_B$, and $\bar{x} = 0$. Taking into account that the ε-subdifferential at 0 of a positive quadratic form q_c is given by

$$\partial_\varepsilon q_c(\bar{x}) = \sqrt{2\varepsilon}\, E(C),$$

it is a simple exercise to derive the desired equalities (8.3) and (8.4) starting from (8.7) and (8.8), respectively. ∎

We conclude this section by pointing out that a couple of formulae, (8.3) and (8.4), are in fact dual to each other. More precisely, each formula can be deduced from the other by using the polarity relationships established in Theorem 6.1.

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