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On the asymptotics of polynomial interpolation to $|x|^\alpha$ at the Chebyshev nodes

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Abstract

In this paper, we discuss asymptotic relations for the approximation of $|x|^\alpha$, $\alpha > 0$ in $L_\infty[-1,1]$ by Lagrange interpolation polynomials based on the zeros of the Chebyshev polynomials of first kind.

Keywords: Lagrange interpolation; Bernstein constant; Chebyshev nodes

1. Introduction

Let $n \in \mathbb{N}_0$, π_n be the space of all algebraic polynomials of degree at most n with real coefficients, $L_\infty[-1,1]$ the space of all continuous real-valued functions on $[-1,1]$ equipped with the supremum norm $\|f\|_{L_\infty[-1,1]} = \sup_{x\in[-1,1]} |f(x)|$ and let $f_\alpha(x) = |x|^\alpha$ for some fixed $\alpha > 0$.

The problem of approximation of $|x|$ on the interval $[-1,1]$ started at the beginning of the 20th century, when polynomial approximations to this prototype of a non-smooth function attracted the interest of Lebesgue, Jackson and Bernstein. In the year 1913, Bernstein [1] investigated the best polynomial approximation for $|x|$ and established the following remarkable and difficult result. There exists a positive finite constant B_1 with

$$\lim_{n\to\infty} \min_{p_n\in\pi_n} n\,\|f_1 - p_n\|_{L_\infty[-1,1]} = B_1.$$

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The precise value of B_1 is still unknown and its determination seems to be very difficult. Varga and Carpenter [14] showed in 1985, by means of high-precision numerical computations that $B_1 = 0.28017+\varepsilon$ where $|\varepsilon| \leq 4\cdot 10^{-6}$. In the year 1938, Bernstein published in [3] a remarkable result for an analogue asymptotic expression for $|x|^\alpha$, $\alpha > 0$. He showed that for each $\alpha > 0$,

$$\lim_{n\to\infty} \min_{p_n\in\pi_n} n^\alpha \, \| f_\alpha - p_n\|_{L_\infty[-1,1]} = B_\alpha$$

holds for some finite and, apart of even integer values for α, also positive constant B_α, the so called Bernstein constant(s) depending on the size α. Moreover, Bernstein [3, formula 47] obtained the bounds

$$\frac{1}{\pi}\left|\sin\frac{\pi\alpha}{2}\right|\Gamma(\alpha)\left(1-\frac{1}{\alpha-1}\right) \leq B_\alpha \leq \frac{1}{\pi}\left|\sin\frac{\pi\alpha}{2}\right|\Gamma(\alpha), \quad \alpha > 2,$$

from which we may deduce the asymptotic behavior of B_α when $\alpha \to \infty$. Here Γ denotes the usual Gamma function. There is not a single value of α, apart from the trivial cases when α is an even integer, for which B_α is explicitly known. Varga and Carpenter [15] computed numerical approximations for B_α for different relevant values of α. Unfortunately, while we know these highly accurate estimates for B_α, no one has succeeded in finding a closed form expression in terms of hypergeometric functions and/or integrals, which exactly fits with the computed data. The question arises what type of formulas would stand behind the mystery of B_α?

From the Chebyshev alternation theorem we simply deduce that for each integer n the best approximating polynomial to $|x|^\alpha$ out of π_n can be represented as an interpolating polynomial with (unknown) consecutive nodes in $[-1, 1]$. In finding a constructive method for approximating the Bernstein constants and/or the best approximating polynomials it seems natural to study the interpolation process for different node systems like the zeros of certain orthogonal polynomials. One may not expect that a specific choice for such a node system would lead us into an instant range close to the Bernstein constant. But we can find out what type of formulas will be generated by the interpolation process itself for these node systems and hopefully these formulas may turn out to be a part of a closed form expression for the Bernstein constants.

The interpolation process for $|x|^\alpha$ was first and extensively studied in 1937 by Bernstein in his Russian monograph [2] for the (modified) Chebyshev system

$$x_0 = 0, \qquad x_j = \cos\frac{\left(j-\frac{1}{2}\right)}{2n}, \quad j = 1, 2, \ldots, 2n, \tag{1.1}$$

where the x_j, $j = 1, \ldots, 2n$ are the zeros of the Chebyshev polynomial T_{2n} of first kind, defined by $T_n(x) = \cos(n \arccos x)$ and $x_0 = 0$ is an additional choice, but not a zero of T_{2n}, in order to obtain the corresponding interpolating polynomial $P_{2n}^{(1)}$ of degree at most $2n$ to $|x|^\alpha$. The formulas obtained by Bernstein revealed a first estimate for the asymptotic behavior for the error function and gave a weaker version than the subsequent quoted asymptotic formula (1.2). An estimate, valid for all integers $n \in \mathbb{N}$, was obtained in [12]: let $\alpha \in \left(0, \frac{2}{3}\right) \cup \{1\}$ then one has

$$(2n)^\alpha \left\| f_\alpha - P_{2n}^{(1)} \right\|_{L_\infty[-1,1]} \leq 2\left(\frac{2}{3}\right)^{1-\alpha}.$$

In a prominent paper from 2002, Ganzburg [4] established, among others, the following remarkable limit relation. For all $\alpha > 0$ one has

$$\lim_{n\to\infty}(2n)^{\alpha}\left\|f_{\alpha}-P_{2n}^{(1)}\right\|_{L_{\infty}[-1,1]}=\frac{4}{\pi}\left|\sin\frac{\pi\alpha}{2}\right|\int_{0}^{\infty}\frac{t^{\alpha-1}}{e^{t}+e^{-t}}dt. \tag{1.2}$$

An extension of this asymptotic relation to a complex α was obtained in [5, Theorem 1]. It is worth mentioning that the integral on the right side of (1.2) occurs also in different connections and variants within the approximation of $|x|^{\alpha}$ by interpolating polynomials, for instance in the study of best polynomial approximations to $|x|^{\alpha}$ [3, formulas 2,27 and 42] and surprisingly also in equidistant(!) polynomial interpolation; see [13].

The question arises why one should not select the more natural node system consisting of the $2n+1$ zeros of T_{2n+1}, since in this case $x=0$ is always a zero of T_{2n+1}? In fact, I counted only one paper, see [17], dealing with this node set for the approximation of $|x|^{\alpha}$, where it is shown that the order of approximation attains the Jackson order, i.e.

$$\|f_{\alpha}-P_{n}\|_{L_{\infty}[-1,1]}=O(1)\frac{1}{n^{\alpha}},\quad 0<\alpha<1,\ n\in\mathbb{N},$$

where the polynomials P_n are the corresponding interpolating polynomials to f_{α} based on the zeros of T_n. In establishing a limit relation like (1.2), one is confronted with two significant problems. First, to give a construction of a strong asymptotic formula for the error function and second to calculate the supremum norm from this. It turns out, when using the modified Chebyshev system (1.1), that the second step is surprisingly comfortable to handle, since the supremum takes its values at the right end of the interval (even for all integers n) and is quite easy to calculate. The situation dramatically changes when using the zeros of T_{2n+1}. Even the first step, the construction of an asymptotic formula for the error function, though following the general method based on Bernstein [2], requires stronger arguments, while at the same time, the precise determination of the supremum norm seems to be rather difficult and, at the moment, is not completely solved.

2. Results

We prove the following.

Theorem 1. *Let $2n>\alpha>0$ and $P_{2n}^{(2)}$ be the unique Lagrange interpolation polynomial out of π_{2n} to $|x|^{\alpha}$ on $[-1,1]$ based on the zeros of T_{2n+1}, i.e.*

$$x_j=\cos\frac{\left(j-\frac{1}{2}\right)\pi}{2n+1},\quad j=1,2,\ldots,2n+1,\ n\in\mathbb{N}.$$

(Obviously x_{n+1} equals zero, $n\in\mathbb{N}$.) Then, for all $x\in[-1,1]$, we have

$$(2n)^{\alpha}\left(|x|^{\alpha}-P_{2n}^{(2)}(x)\right)=(-1)^{n}\frac{4}{\pi}\sin\frac{\pi\alpha}{2}\frac{T_{2n+1}(x)}{(2n+1)x}\cdot\int_{0}^{\infty}\frac{t^{\alpha}x^{2}}{\left(x^{2}+\left(\frac{t}{2n}\right)^{2}\right)\left(e^{t}-e^{-t}\right)}dt+o(1),\quad n\to\infty,$$

where $o(1)$ is independent of x.

From this result we further obtain the following.

Corollary 2. *For $\alpha > 0$ and $P_{2n}^{(2)}$ to be defined as in Theorem 1 we have*

$$\limsup_{n\to\infty} (2n)^{\alpha} \left\| f_\alpha - P_{2n}^{(2)} \right\|_{L_\infty[-1,1]} \le \frac{4}{\pi} \left|\sin\frac{\pi\alpha}{2}\right| \int_0^\infty \frac{t^\alpha}{e^t - e^{-t}} dt.$$

Remark 3. Based on numerical computations it seems plausible that there exists a constant $C_\alpha < \frac{4}{\pi}$, depending on α, such that

$$\lim_{n\to\infty} (2n)^{\alpha} \left\| f_\alpha - P_{2n}^{(2)} \right\|_{L_\infty[-1,1]} = C_\alpha \left|\sin\frac{\pi\alpha}{2}\right| \int_0^\infty \frac{t^\alpha}{e^t - e^{-t}} dt.$$

Remark 4. Note that the integral on the right-hand side of Corollary 2 can also be represented through a series, see [4, p. 196], by

$$\int_0^\infty \frac{t^\alpha}{e^t - e^{-t}} dt = \Gamma(\alpha+1) \sum_{j=0}^{\infty} \frac{1}{(2j+1)^{\alpha+1}}, \quad \alpha > 0.$$

The organization of the paper is as follows. In Section 3 we establish a representation for the interpolation formula for $|x|^\alpha$ valid for arbitrary node systems. To this end we follow the general method established by Bernstein [2]. Since many technical details are missing in [2] we refer the interested reader also to [4] which provides the most extensive description for this topic. To keep the paper selfcontained as much as possible, we pass in this section through the major steps and give details only when we transfer the formulas into the context of our notation which might be slightly different from that used in [4]. In Section 4 we are going to develop a strong asymptotics for the error formula from which we later deduce Theorem 1 and Corollary 2.

3. A formula for the error function

Lemma 5. *Let $n > s > 0$ and $-1 \le y_0 < \cdots < y_n \le 1$. Let $P_n \in \pi_n$ be the interpolation polynomial to $(1-y)^s$ on $[-1,1]$ at the node system $\{y_j : j = 0, \ldots, n\}$. Then, for any $y \in [-1,1]$,*

$$(1-y)^s - P_n(y) = -\frac{1}{\pi} v_n(y) \sin\pi s \int_1^\infty \frac{(t-1)^s}{v_n(t)(t-y)} dt, \tag{3.1}$$

where $v_n(y) = (y-y_0)\cdots(y-y_n)$.

Proof. Let $a > 1$ and M, ε_0 be positive numbers with $M > a > a - 2\varepsilon_0 > 1$ and $M \ge 2$. Next, let $0 < \varepsilon < \varepsilon_0$ be arbitrary and let $P_{n,a}$ be the interpolating polynomial to $(a-y)^s$ on $[-1,1]$ at the nodes $y_0, \ldots, y_n$. Then, by the error formula for Hermite interpolation (see [8,16]),

$$(a-y)^s - P_{n,a}(y) = \frac{v_n(y)}{2\pi i} \lim_{M\to\infty} \lim_{\varepsilon\to 0} \int_{C_{M,\varepsilon}} \frac{(a-z)^s}{v_n(z)(z-y)} dz,$$

where $C_{M,\varepsilon} = C_{M,\varepsilon}^{(1)} \cup C_{M,\varepsilon}^{(2)} \cup C_{\varepsilon}^{(3)} \cup C_{M,\varepsilon}^{4}$ is a contour in $\mathbb{C}$, oriented in a positive sense, where

$$C_{M,\varepsilon}^{(1)} = \left\{ z : |z| = M, \arcsin\frac{\varepsilon}{M} \le |\arg z| \le \pi \right\},$$
$$C_{M,\varepsilon}^{(2)} = \left\{ z = x - i\varepsilon : a \le x \le \sqrt{M^2 - \varepsilon^2} \right\},$$

$$C_{\varepsilon}^{(3)} = \left\{ z : |z-a| = \varepsilon, \frac{\pi}{2} \le |\arg z| \le \pi \right\},$$
$$C_{M,\varepsilon}^{(4)} = \left\{ z = x + i\varepsilon : a \le x \le \sqrt{M^2 - \varepsilon^2} \right\}.$$

Now, following [4], for M large and ε small, the contour integral contributes only on the segments $C^{(2)}$ and $C^{(4)}$. Further, using the limit relation

$$\lim_{\varepsilon \to 0} \left[(a - t - i\varepsilon)^s - (a - t + i\varepsilon)^s \right] = -2i\,(t-a)^s \sin \pi s, \quad \forall t \ge a,$$

a routine calculation leads us to

$$(a-y)^s - P_{n,a}(y) = -\frac{1}{\pi} v_n(y) \sin \pi s \int_a^{\infty} \frac{(t-a)^s}{v_n(t)\,(t-y)} dt, \tag{3.2}$$

which holds for all $a > 1, n > s > 0$ and $-1 \le y \le 1$. Finally, letting $a \to 1^+$ in (3.2) and taking account of $\lim_{a \to 1^+} P_{n,a} = P_n$, we obtain (3.1). □

Lemma 6. *Let $2n > \alpha > 0$ and $0 \le x_0 < \cdots < x_n \le 1$. $Q_{2n} \in \pi_{2n}$ denotes the interpolation polynomial to $|x|^\alpha$ at the node system $\left\{ \pm x_j : j = 0, \ldots, n \right\}$. Then, for any $x \in [-1, 1]$,*

$$|x|^\alpha - Q_{2n}(x) = -2^{-\frac{\alpha}{2}} \frac{1}{\pi} \sin \frac{\pi \alpha}{2} w_n\left(1 - 2x^2\right) \int_1^{\infty} \frac{(t-1)^{\frac{\alpha}{2}}}{w_n(t)\left(t - 1 + 2x^2\right)} dt, \tag{3.3}$$

where $w_n(x) = \left(x - 1 + 2x_0^2\right) \cdots \left(x - 1 + 2x_n^2\right)$.

Proof. First note that the node system consists of a maximum of $2n + 2$ elements which are symmetrically distributed around the origin. Since f_α is an even function and Q_{2n} is unique in π_{2n+1} it follows that Q_{2n} is even and thus Q_{2n} has degree at most $2n$. Next, with $\alpha = 2s$, $y_{n-j} = 1 - 2x_j^2$ for $j = 0, \ldots, n$ and substituting $y = 1 - 2x^2$ in (3.1) we conclude that

$$|x|^\alpha - 2^{-\frac{\alpha}{2}} P_n\left(1 - 2x^2\right) = \frac{-2^{-\frac{\alpha}{2}}}{\pi} \sin \frac{\pi \alpha}{2} v_n\left(1 - 2x^2\right) \int_1^{\infty} \frac{(t-1)^{\frac{\alpha}{2}}}{v_n(t)\left(t - 1 + 2x^2\right)} dt.$$

Now, by definition of y_j, $v_n(x) = \prod_{j=0}^{n} \left(x - y_{n-j}\right) = \prod_{j=0}^{n} \left(x - 1 + 2x_j^2\right) = w_n(x)$ and a routine argument shows that $2^{-\frac{\alpha}{2}} P_n\left(1 - 2x^2\right)$ equals the interpolation polynomial $Q_{2n}(x)$. □

Remark 7. By uniqueness, see [9], of the best approximating polynomial p_{2n}^* out of π_{2n} to $|x|^\alpha$ it follows that a constructive solution for p_{2n}^* can be produced via the solution for the Minmax problem,

$$\min_{0 \le x_0 < \cdots < x_n \le 1} \max_{0 \le x \le 1} \left| w_n\left(1 - 2x^2\right) \int_1^{\infty} \frac{(t-1)^{\frac{\alpha}{2}}}{w_n(t)\left(t - 1 + 2x^2\right)} dt \right|.$$

Thus a good choice of nodes would be achieved when, at the same time, w_n is as small as possible inside the interval $[-1, 1]$ and grows as rapidly as possible outside of it when weighted by the other factors inside the integral.

Lemma 8. *Let $n \in \mathbb{N}_0$ and $t_j^{(2n+1)} = t_j = \cos \frac{\left(j-\frac{1}{2}\right)\pi}{2n+1}$ for $j = 1, 2, \ldots, 2n+1$ be the zeros of T_{2n+1}. Further, let $x_j = t_{n+1-j}$ for $j = 0, 1, \ldots, n$ and w_n to be defined as in Lemma 6. Then*

(a) $\left\{\pm x_j : j = 0, 1, \ldots, n\right\} = \left\{t_j : j = 1, 2, \ldots, 2n+1\right\}$,
(b) $w_n\left(1-2x^2\right) = (-1)^{n+1} 2^{1-n} x T_{2n+1}(x)$, *for* $-1 \le x \le 1$,
(c) $w_n(x) = 2^{-n}\left(T_{n+1}(x) - T_n(x)\right)$, *for* $x \ge 1$,

where, in (c)*, the definition of the polynomials T_n has to be extended in the usual way, by $T_n(z) = \frac{1}{2}\left(\left(z+\sqrt{z^2-1}\right)^n + \left(z-\sqrt{z^2-1}\right)^n\right)$, $z \in \mathbb{C}$.*

Proof. The proofs for (a) and (b) are easy. We prove only (c). First

$$\begin{aligned} w_n(x) &= \prod_{j=0}^{n}\left(x-1+2x_j^2\right) = \prod_{j=0}^{n}\left(x-1+2t_{n+1-j}^2\right) \\ &= \prod_{j=1}^{n+1}\left(x-1+2t_j^2\right) = (x-1)\prod_{j=1}^{n}\left(x-1+2t_j^2\right). \end{aligned}$$

Using the identity $\cos 2\alpha = 2\cos^2\alpha - 1$, we get

$$\begin{aligned} w_n(x) &= (x-1)\prod_{j=1}^{n}\left(x+\cos\frac{(2j-1)\pi}{2n+1}\right) \\ &= (x-1)\prod_{j=1}^{n}\left(x-\cos\frac{2j\pi}{2n+1}\right) = (x-1)\frac{1}{2^n}W_n(x), \end{aligned}$$

where W_n is the Chebyshev polynomial of 4th kind. For details, we refer the reader to [11, Chapter 1.2.3]. From [11, formula 1.18] we see that W_n can be represented as a sum of Chebyshev polynomials of 2nd kind, i.e. $W_n(x) = U_n(x) + U_{n-1}(x)$, from which we further deduce that

$$\begin{aligned} w_n(x) &= \frac{1}{2^n}(x-1)\left[U_n(x) - xU_{n-1}(x) + xU_{n-1}(x) + U_{n-1}(x)\right] \\ &= \frac{1}{2^n}(x-1)\left[T_n(x) + (x+1)U_{n-1}(x)\right] \\ &= \frac{1}{2^n}\left[(x-1)T_n(x) - \left(1-x^2\right)U_{n-1}(x)\right] \\ &= \frac{1}{2^n}\left[xT_n(x) - T_n(x) - \left(xT_n(x) - T_{n+1}(x)\right)\right] \\ &= 2^{-n}\left[T_{n+1}(x) - T_n(x)\right], \end{aligned}$$

where we have used [7, formulas 8.941,3 and 4]. □

Now, combining Lemmas 6 and 8, we arrive at the following.

Corollary 9. *Let $2n > \alpha > 0$. Then, for $-1 \le x \le 1$,*

$$\begin{aligned} |x|^\alpha - P_{2n}^{(2)}(x) = (-1)^n 2^{1-\frac{\alpha}{2}} \frac{1}{\pi} \sin\frac{\pi\alpha}{2} x T_{2n+1}(x) \\ \cdot \int_1^\infty \frac{(t-1)^{\frac{\alpha}{2}}}{\left[T_{n+1}(t) - T_n(t)\right]\left(t-1+2x^2\right)} dt. \end{aligned} \tag{3.4}$$

At this point it should be mentioned that a different and more general approach to identities like (3.4) was developed by Ganzburg [6] and Lubinsky [10].

4. The asymptotics

We now investigate the asymptotic properties of the integral on the right side of Eq. (3.4). Using the substitution $t = \frac{1}{2}\left(z + z^{-1}\right)$, we obtain

$$\begin{aligned} I_n(x) &:= \int_1^\infty \frac{(t-1)^{\frac{\alpha}{2}}}{\left(t-1+2x^2\right)\left[T_{n+1}(t) - T_n(t)\right]} dt \\ &= 2^{1-\frac{\alpha}{2}} \int_1^\infty \frac{(z-1)^{\alpha+1}(z+1)}{z^{1+\frac{\alpha}{2}}\left((z-1)^2 + 4x^2 z\right)\left[z^{n+1} + z^{-(n+1)} - z^n - z^{-n}\right]} dz. \end{aligned}$$

By some routine arguments, substituting $z = 1 + \frac{t}{n}$, we get

$$\begin{aligned} I_n(x) &= 2^{2-\frac{\alpha}{2}} n^{-\alpha-2} \int_0^\infty \left(\frac{t^{1+\alpha}\left(1+\frac{t}{2n}\right)}{\left(1+\frac{t}{n}\right)^{2+\frac{\alpha}{2}}\left(4x^2 + \frac{t^2}{n(n+t)}\right)} \right. \\ &\quad \left. \cdot \frac{1}{\left[\left(1+\frac{t}{n}\right)^{n+1} + \left(1+\frac{t}{n}\right)^{-(n+1)} - \left(1+\frac{t}{n}\right)^n - \left(1+\frac{t}{n}\right)^{-n}\right]} \right) dt \\ &= 2^{2-\frac{\alpha}{2}} n^{-\alpha-1} \int_0^\infty \frac{t^\alpha\left(1+\frac{t}{2n}\right) dt}{\left(1+\frac{t}{n}\right)^{2+\frac{\alpha}{2}}\left(4x^2 + \frac{t^2}{n(n+t)}\right)\left[\left(1+\frac{t}{n}\right)^n - \left(1+\frac{t}{n}\right)^{-(n+1)}\right]} dt. \end{aligned} \tag{4.1}$$

Combining (3.4) and (4.1), we obtain for $2n > \alpha > 0$, $x \in [-1, 1]$,

$$\begin{aligned} |x|^\alpha - P_{2n}^{(2)}(x) &= (-1)^n 2^{3-\alpha} \frac{1}{\pi} \sin\frac{\pi\alpha}{2} \frac{T_{2n+1}(x)}{(2n+1)x} \frac{2n+1}{n} n^{-\alpha} \\ &\quad \cdot \int_0^\infty \frac{t^\alpha\left(1+\frac{t}{2n}\right)x^2}{\left(1+\frac{t}{n}\right)^{1+\frac{\alpha}{2}}\left(4x^2 + \frac{t^2}{n(n+t)}\right)\left[\left(1+\frac{t}{n}\right)^{n+1} - \left(1+\frac{t}{n}\right)^{-n}\right]} dt. \end{aligned} \tag{4.2}$$

The further analysis requires some preparatory work. Let $n \in \mathbb{N}$ and $t \geq 0$. From the well known inequality $x - \frac{x^2}{2} \leq \log(1+x) \leq x$, $x \geq 0$, we obtain

$$e^t e^{-\frac{t^2}{2n}} \leq \left(1 + \frac{t}{n}\right)^n \leq e^t. \tag{4.3}$$

Now, define $g_n(t) = \left(e^t - e^{-t}\right)\sqrt{n} - te^t$. An easy argument reveals that g_n is increasing for $t \in \left[0, \sqrt{n} - 1\right]$ and thus $0 = g_n(0) \leq g_n(t)$, from which we get

$$\frac{t}{n} e^t \leq \frac{e^t - e^{-t}}{\sqrt{n}}, \quad n \geq 1,\ t \in \left[0, \sqrt{n} - 1\right]. \tag{4.4}$$

Similarly, let $h_n(t) = 1 + \frac{t}{n} e^{2t} - e^{\frac{t^2}{n}}$. Again, we conclude that h_n is increasing for $t \in [0, 2n]$ and thus $0 = h_n(0) \leq h_n(t)$, from which we obtain

$$\frac{t}{n} e^t e^{-\frac{t^2}{2n}} - e^{-t} e^{\frac{t^2}{2n}} \geq -e^{-t} e^{-\frac{t^2}{2n}}, \quad n \geq 1,\ t \in [0, 2n]. \tag{4.5}$$

Combining (4.3)–(4.5) we verify that

$$\begin{aligned}\left(1+\frac{t}{n}\right)^{n+1}-\left(1+\frac{t}{n}\right)^{-n} &\le \left(1+\frac{t}{n}\right)e^t-\left(1+\frac{t}{n}\right)^{-n}\\ &\le e^t+\frac{t}{n}e^t-e^{-t}\le \left(e^t-e^{-t}\right)\left(1+\frac{1}{\sqrt{n}}\right)\end{aligned}$$

and

$$\begin{aligned}\left(1+\frac{t}{n}\right)^{n+1}-\left(1+\frac{t}{n}\right)^{-n} &\ge \left(1+\frac{t}{n}\right)e^t e^{-\frac{t^2}{2n}}-\left(1+\frac{t}{n}\right)^{-n}\\ &\ge e^t e^{-\frac{t^2}{2n}}+\frac{t}{n}e^t e^{-\frac{t^2}{2n}}-e^{-t}e^{\frac{t^2}{2n}}\\ &\ge \left(e^t-e^{-t}\right)e^{-\frac{t^2}{2n}},\end{aligned}$$

where both estimates are valid for $t \in [0, n^{\frac{1}{3}}]$ and $n \ge 10$, since in this case we have $n^{\frac{1}{3}} \le \sqrt{n}-1 \le 2n$. Now, using the elementary inequalities

$$\begin{aligned}&1-x\le\frac{1}{1+x}, && x\ge 0,\\ &\frac{1}{1-x}\le 1+2x, && 0\le x\le\frac{1}{2},\\ &e^x\le\frac{1}{1-x}, && x<1,\end{aligned}\tag{4.6}$$

some routine arguments give a first substantial result for the error analysis,

$$\frac{1-n^{-\frac{1}{3}}}{e^t-e^{-t}}\le\frac{1}{\left(1+\frac{t}{n}\right)^{n+1}-\left(1+\frac{t}{n}\right)^{-n}}\le\frac{1+n^{-\frac{1}{3}}}{e^t-e^{-t}},\tag{4.7}$$

valid for $t \in \left[0, n^{\frac{1}{3}}\right]$ and $n \ge 10$. Using again (4.6), we proceed further to obtain the bounds

$$\begin{aligned}\frac{1}{4x^2+\frac{t^2}{n^2}}\le\frac{1}{4x^2+\frac{t^2}{n(n+t)}}&\le\frac{1}{4x^2+\frac{t^2}{n^2}\frac{1}{1+n^{-\frac{2}{3}}}}\\ &\le\frac{1}{4x^2+\frac{t^2}{n^2}\left(1-n^{-\frac{2}{3}}\right)}\le\frac{1}{\left(4x^2+\frac{t^2}{n^2}\right)\left(1-n^{-\frac{2}{3}}\right)}\\ &\le\frac{1}{4x^2+\frac{t^2}{n^2}}\left(1+2n^{-\frac{2}{3}}\right),\quad t\in\left[0,n^{\frac{1}{3}}\right],\ n\ge 3.\end{aligned}$$

From the previous estimates we further obtain

$$\begin{aligned}\frac{1+\frac{t}{2n}}{\left(1+\frac{t}{n}\right)^{1+\frac{\alpha}{2}}\left(4x^2+\frac{t^2}{n(n+t)}\right)}&\le\frac{1+\frac{1}{2n^{\frac{2}{3}}}}{4x^2+\frac{t^2}{n^2}}\left(1+2n^{-\frac{2}{3}}\right)\\ &\le\frac{1}{4x^2+\frac{t^2}{n^2}}\left(1+4n^{-\frac{1}{3}}\right).\end{aligned}\tag{4.8}$$

Next, using a convexity argument, we get for $t \in \left[0, n^{\frac{1}{3}}\right]$,

$$\frac{1+\frac{t}{2n}}{\left(1+\frac{t}{n}\right)^{1+\frac{\alpha}{2}}\left(4x^2+\frac{t^2}{n(n+t)}\right)} \geq \frac{1}{\left(1+n^{-\frac{2}{3}}\right)^{1+\frac{\alpha}{2}}\left(4x^2+\frac{t^2}{n^2}\right)}$$
$$\geq \frac{1}{4x^2+\frac{t^2}{n^2}}\left(1-\left(1+\frac{\alpha}{2}\right)n^{-\frac{1}{3}}\right). \tag{4.9}$$

Now, splitting the integral from (4.2) into two parts by $\int_0^\infty = \int_0^{n^{\frac{1}{3}}} + \int_{n^{\frac{1}{3}}}^\infty$ and combining (4.7)–(4.9), we arrive at

$$\int_0^{n^{\frac{1}{3}}} \frac{t^\alpha\left(1+\frac{t}{2n}\right)x^2}{\left(1+\frac{t}{n}\right)^{1+\frac{\alpha}{2}}\left(4x^2+\frac{t^2}{n(n+t)}\right)\left[\left(1+\frac{t}{n}\right)^{n+1}-\left(1+\frac{t}{n}\right)^{-n}\right]}dt$$
$$= (1+\varepsilon_1(n))\int_0^{n^{\frac{1}{3}}} \frac{t^\alpha x^2}{\left(4x^2+\frac{t^2}{n^2}\right)\left(e^t-e^{-t}\right)}dt, \tag{4.10}$$

which holds for $x \in [-1,1]$ with $|\varepsilon_1(n)| \leq Cn^{-\frac{1}{3}}$ for some constant C, independent of n. Now, we turn to $\int_{n^{\frac{1}{3}}}^\infty$. For $x \in [-1,1]$, we have

$$0 \leq \frac{x^2}{4x^2+\frac{t^2}{n(n+t)}} \leq \frac{n(n+t)}{(2n+t)^2}$$

and, by substituting $z = 1+\frac{t}{n}$, some routine arguments show that

$$0 \leq \int_{n^{\frac{1}{3}}}^\infty \frac{t^\alpha\left(1+\frac{t}{2n}\right)x^2}{\left(1+\frac{t}{n}\right)^{1+\frac{\alpha}{2}}\left(4x^2+\frac{t^2}{n(n+t)}\right)\left[\left(1+\frac{t}{n}\right)^{n+1}-\left(1+\frac{t}{n}\right)^{-n}\right]}dt$$
$$\leq \frac{1}{2}n^{1+\alpha}\int_{1+n^{-\frac{2}{3}}}^\infty \frac{z^{\frac{\alpha}{2}-1}}{z^{n+1}-z^{-n}}dz. \tag{4.11}$$

We now consider the function $g_n(z) = z^{2n+1}-z^{2n}-1$, $n \geq 1$ and $z \geq 1$. Our next task is to show that $g_n(z) \geq 0$ for all $z \geq 1+n^{-\frac{2}{3}}$, at least when n becomes sufficiently large, i.e. $n \geq n_0$. It is easy to see that g_n is increasing for $z \geq 1$. Using (4.3) and (4.6), we estimate

$$g_n(z) \geq g_n\left(1+n^{-\frac{2}{3}}\right) = \left(1+\frac{n^{\frac{1}{3}}}{n}\right)^{2n} n^{-\frac{2}{3}} - 1$$
$$\geq \frac{e^{2n^{\frac{1}{3}}}\left(1-n^{-\frac{1}{3}}\right)}{n^{\frac{2}{3}}} - 1 \geq 0, \quad \forall n \geq n_0.$$

Combining this fact together with (4.3) and (4.11), we conclude that

$$0 \leq \frac{1}{2}n^{1+\alpha}\int_{1+n^{-\frac{2}{3}}}^\infty \frac{z^{\frac{\alpha}{2}-1}}{z^{n+1}-z^{-n}}dz$$

$$\leq \frac{1}{2} n^{1+\alpha} \int_{1+n^{-\frac{2}{3}}}^{\infty} z^{\frac{\alpha}{2}-1-n} dz \leq \frac{n^{1+\alpha}}{2n-\alpha} 2^{\alpha} \frac{1}{\left(1+n^{-\frac{2}{3}}\right)^{n}}$$

$$\leq D n^{\alpha} e^{-n^{\frac{1}{3}}}, \tag{4.12}$$

for some constant D, independent of n. Now, taking account of $e^t - e^{-t} \geq \frac{1}{4} e^t$ for $t \geq n^{\frac{1}{3}} \geq 1$, we obtain for $x \in [-1, 1]$,

$$\begin{aligned} 0 &\leq \int_{n^{\frac{1}{3}}}^{\infty} \frac{t^{\alpha} x^2}{\left(4x^2 + \frac{t^2}{n^2}\right)\left(e^t - e^{-t}\right)} dt \\ &\leq \frac{1}{4} \int_{n^{\frac{1}{3}}}^{\infty} \frac{t^{\alpha}}{e^t - e^{-t}} \leq \int_{n^{\frac{1}{3}}}^{\infty} t^{\alpha} e^{-t} dt \\ &= \Gamma\left(\alpha + 1, n^{\frac{1}{3}}\right), \end{aligned}$$

where $\Gamma(\cdot, \cdot)$ denotes the incomplete gamma function, defined by $\Gamma(\beta, z) = \int_z^{\infty} t^{\beta-1} e^{-t} dt$. By the well known asymptotics, see [7, formula 8.357],

$$\Gamma(\beta, z) = z^{\beta-1} e^{-z} (1 + o(1)), \quad z \to \infty,$$

we obtain for $x \in [-1, 1]$ the estimates

$$0 \leq \int_{n^{\frac{1}{3}}}^{\infty} \frac{t^{\alpha} x^2}{\left(4x^2 + \frac{t^2}{n^2}\right)\left(e^t - e^{-t}\right)} dt \leq E n^{\alpha} e^{-n^{\frac{1}{3}}}, \tag{4.13}$$

for some constant E, independent of n. Finally, combining (4.2), (4.10)–(4.13), we arrive at

$$\begin{aligned} &\int_0^{\infty} \frac{t^{\alpha}\left(1 + \frac{t}{2n}\right) x^2}{\left(1 + \frac{t}{n}\right)^{1+\frac{\alpha}{2}} \left(4x^2 + \frac{t^2}{n(n+t)}\right) \left[\left(1 + \frac{t}{n}\right)^{n+1} - \left(1 + \frac{t}{n}\right)^{-n}\right]} dt \\ &= \int_0^{n^{\frac{1}{3}}} + \int_{n^{\frac{1}{3}}}^{\infty} \\ &= (1 + \varepsilon_1(n)) \int_0^{n^{\frac{1}{3}}} \frac{t^{\alpha} x^2}{\left(4x^2 + \frac{t^2}{n^2}\right)\left(e^t - e^{-t}\right)} dt + \varepsilon_2(n) \\ &= (1 + \varepsilon_1(n)) \left[\int_0^{\infty} - \int_{n^{\frac{1}{3}}}^{\infty}\right] + \varepsilon_2(n) \\ &= (1 + \varepsilon_1(n)) \left[\int_0^{\infty} \frac{t^{\alpha} x^2}{\left(4x^2 + \frac{t^2}{n^2}\right)\left(e^t - e^{-t}\right)} dt + \varepsilon_3(n)\right] + \varepsilon_2(n), \end{aligned}$$

with $|\varepsilon_1(n)| \leq C n^{-\frac{1}{3}}$, $|\varepsilon_2(n)| \leq D n^{\alpha} e^{-n^{\frac{1}{3}}}$ and $|\varepsilon_3(n)| \leq E n^{\alpha} e^{-n^{\frac{1}{3}}}$. Thus, for $x \in [-1, 1]$, we have shown the asymptotics

$$\int_0^\infty \frac{t^\alpha \left(1+\frac{t}{2n}\right) x^2}{\left(1+\frac{t}{n}\right)^{1+\frac{\alpha}{2}} \left(4x^2+\frac{t^2}{n(n+t)}\right)\left[\left(1+\frac{t}{n}\right)^{n+1}-\left(1+\frac{t}{n}\right)^{-n}\right]} dt$$
$$= \int_0^\infty \frac{t^\alpha x^2}{\left(4x^2+\frac{t^2}{n^2}\right)\left(e^t-e^{-t}\right)} dt + o(1), \quad n \to \infty, \tag{4.14}$$

where $o(1)$ is independent of x.

Lemma 10. *For $n \in \mathbb{N}$ and $x \in [-1, 1]$, we have*

$$\left|\frac{T_{2n+1}(x)}{(2n+1)x}\right| \le 1.$$

Proof. First, an easy induction argument reveals

$$|\sin my| \le m |\sin y|, \quad m \in \mathbb{N}, \ y \in \mathbb{R}.$$

Now, for $x \in [-1, 1]$ let $y = \arcsin x$. It then follows that $x = \sin y = \cos\left(\frac{\pi}{2} - y\right)$. Thus

$$\begin{aligned} T_{2n+1}(x) &= \cos\left[(2n+1)\arccos x\right] \\ &= \cos\left[(2n+1)\left(\frac{\pi}{2} - y\right)\right] \\ &= (-1)^n \sin\left[(2n+1)\arcsin x\right]. \end{aligned}$$

Then

$$\begin{aligned} \left|\frac{T_{2n+1}(x)}{(2n+1)x}\right| &= \left|\frac{\sin\left[(2n+1)\arcsin x\right]}{(2n+1)x}\right| \\ &= \left|\frac{\sin\left[(2n+1)y\right]}{(2n+1)\sin y}\right| \le 1. \quad \square \end{aligned}$$

From Lemma 10 combined with formulas (4.2) and (4.14) we finally arrive at Theorem 1. It remains to establish Corollary 2. First we note that the error function is even on the interval $[-1, 1]$. So it is further sufficient to restrict ourselves to the case $x \in [0, 1]$. Now, we investigate the asymptotic behavior of the integral in Theorem 1 by providing the following bounds.

Lemma 11. *For $n \in \mathbb{N}$ and $x \in [0, 1]$, we have*

$$\begin{aligned} \left(\frac{2nx}{2nx+\sqrt{2}}\right)^{1+\alpha} \int_0^\infty \frac{t^\alpha}{e^t-e^{-t}} dt &\le \int_0^\infty \frac{t^\alpha x^2}{\left(x^2+\left(\frac{t}{2n}\right)^2\right)\left(e^t-e^{-t}\right)} dt \\ &\le \int_0^\infty \frac{t^\alpha}{e^t-e^{-t}} dt. \end{aligned}$$

Proof. The upper bound is easy to establish. Also the lower bound holds obviously for the case $x = 0$. For the remaining case $0 < x \le 1$, by substituting $u = t\left(1 + \frac{1}{\sqrt{2}nx}\right)$, a routine argument shows that

$$\int_0^\infty \frac{t^\alpha}{\left(e^t-e^{-t}\right)\left[1+\left(\frac{t}{2nx}\right)^2\right]} dt = \frac{1}{2}\int_0^\infty \frac{t^\alpha}{\sinh(t)\left[1+\left(\frac{t}{2nx}\right)^2\right]} dt$$

$$\geq \frac{1}{2} \int_0^\infty \frac{t^\alpha}{\sinh\left(t\left(1+\frac{1}{\sqrt{2nx}}\right)\right)} dt$$
$$= \frac{1}{2} \left(\frac{2nx}{2nx+\sqrt{2}}\right)^{1+\alpha} \int_0^\infty \frac{u^\alpha}{\sinh(u)} du. \quad \square$$

From Lemma 11 an easy consequence is the following.

Corollary 12. *For any fixed $x \in [0, 1]$,*

$$\int_0^\infty \frac{t^\alpha x^2}{\left(x^2+\left(\frac{t}{2n}\right)^2\right)\left(e^t-e^{-t}\right)} dt \overset{n\to\infty}{\longrightarrow} \left\{ \begin{array}{ll} 0 & x=0, \\ \int_0^\infty \frac{t^\alpha}{e^t-e^{-t}} dt & x \neq 0. \end{array} \right\}.$$

Finally, combining Lemma 10, Corollary 12 and Theorem 1, we arrive at Corollary 2 and so we are done. We conclude with the following.

Remark 13. From Theorem 1 combined with Lemma 11 we can now analyze in more detail the problem of the precise determination of the supremum norm in the relevant interval $[0, 1]$. While $|T_{2n+1}(x)/(2n+1)x|$ is a decreasing function in the interval $[0, z_0]$ (here z_0 denotes the smallest positive zero of T_{2n+1}), at the same time

$$\int_0^\infty \frac{t^\alpha x^2}{\left(x^2+\left(\frac{t}{2n}\right)^2\right)\left(e^t-e^{-t}\right)} dt$$

is increasing in x. It seems plausible that the exact location of the supremum takes place for a certain sequence of positive numbers $(x_n)_{n\in\mathbb{N}}$ with $x_n \to 0$ (with order $1/n$) as $n \to \infty$.

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