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Stable Viscosity Matrices for Systems of Conservation Laws

ANDREW MAJDA*,†

Department of Mathematics, University of California, Berkeley, California 94720

AND

ROBERT L. PEGO*,‡

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

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A natural class of appropriate viscosity matrices for strictly hyperbolic systems of conservation laws in one space dimension, $u_t + f(u)_x = 0$, $u \in R^m$, is studied. These matrices are *admissible* in the sense that small-amplitude shock wave solutions of the hyperbolic system are shown to be limits of smooth traveling wave solutions of the parabolic system $u_t + f(u)_x = \nu(Du_x)_x$ as $\nu \to 0$ if D is in this class. The class is determined by a linearized stability requirement: The Cauchy problem for the equation $u_t + f'(u_0)\, u_x = \nu D u_{xx}$ should be well posed in L^2 uniformly in ν as $\nu \to 0$. Previous examples of inadmissible viscosity matrices are accounted for through violation of the stability criterion.

1. INTRODUCTION

The simplest discontinuous solutions of the $m \times m$ system of hyperbolic conservation laws

$$u_t + f(u)_x = 0, \qquad u \in R^m \tag{1.1}$$

are the shock wave solutions defined by

$$u(x, t) = \begin{cases} u_L, & x < st \\ u_R, & x > st \end{cases} \tag{1.2}$$

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where u_{L} and u_{R} are constant vectors which together with the constant s satisfy the Rankine–Hugoniot conditions

$$-s(u_{\mathrm{R}}-u_{\mathrm{L}})+f(u_{\mathrm{R}})-f(u_{\mathrm{L}})=0 \tag{1.3}$$

and a suitable strict entropy condition: Lax's shock inequalities in the genuinely nonlinear case (see (1.15) below), and Liu's strict condition (E) in the general case (see Section 3).

We assume that the system (1.1) is strictly hyperbolic. Thus, if $A(u)=\partial f/\partial u$ is the $m\times m$ Jacobian matrix, $A(u)$ has m distinct real eigenvalues, ordered $\lambda_1(u)<\lambda_2(u)<\cdots<\lambda_m(u)$ with corresponding right and left eigenvectors $r_j(u)$ and $l_k(u)$ for $j, k=1,..., m$, satisfying

$$\begin{gathered} A(u)\, r_j=\lambda_j r_j, \qquad l_k A(u)=\lambda_k l_k \\ l_k\cdot r_j=\delta_{kj}. \end{gathered} \tag{1.4}$$

An eigenvalue $\lambda_j(u)$ is called genuinely nonlinear if $\nabla\lambda_j\cdot r_j(u)$ never vanishes.

In 1959 Gelfand introduced the following problem: Show that a discontinuous solution of the form (1.2) (so satisfying (1.3)) is the limit of special smooth solutions $u^\nu=U((x-st)/\nu)$ of a reasonable parabolic system

$$u_t^\nu+f(u^\nu)_x=\nu(D(u^\nu)\,u_x^\nu)_x \qquad \text{as} \quad \nu\to 0. \tag{1.5}$$

if (and only if) the entropy condition is satisfied. In particular, one should determine the class of viscosity matrices $D(u)$ for which the above is true: such a matrix is called *admissible*. The smooth traveling wave solution $U((x-st)/\nu)$ is called a viscous shock profile.

The existence of the traveling wave $U((x-st)/\nu)$ having the desired limit (1.2) requires that, with $\xi=(x-st)/\nu$, $U(\xi)$ should satisfy the $m\times m$ system of nonlinear ODE's

$$D(U)\,U_\xi=-s(U-u_{\mathrm{L}})+f(U)-f(u_{\mathrm{L}}) \tag{1.6}$$

for $-\infty<\xi<\infty$ together with the boundary conditions

$$\lim_{\xi\to-\infty} U(\xi)=u_{\mathrm{L}}, \qquad \lim_{\xi\to+\infty} U(\xi)=u_{\mathrm{R}}. \tag{1.7}$$

That is, the autonomous system of ODE's (1.6) should admit a trajectory connecting the critical point u_{L} on the left to the critical point u_{R} on the right.

Several authors have established sufficient conditions ensuring the existence of the connecting orbit for (1.6), (1.7) under various assumptions, typically including either $m=2$ or $|u_{\mathrm{L}}-u_{\mathrm{R}}|$ small (weak shocks), and either genuine nonlinearity or $D(u) \equiv I$ (see [4, 1–3, 12]). A notable exception is Mock's more recent paper [13], which finds a broad class of matrices $D(u)$ admissible for general m and strong shocks, making global assumptions of genuine nonlinearity and the existence of a convex entropy function. On the other hand, Conley and Smoller [1] have discovered puzzling examples of constant positive definite matrices D which are inadmissible, so that for some family of shock waves the system of ODE's (1.6) fails to admit a trajectory satisfying (1.7). Moreover, these examples exist in a simple context, that arising when (1.1) represents the equations of isentropic gas dynamics

$$\begin{aligned} \tau_t - v_x &= 0 \\ v_t + p(\tau)_x &= 0 \end{aligned} \tag{1.8}$$

where one assumes that $p'(\tau) < 0$, $p''(\tau) > 0$. The system (1.8) is also called the *p-system.*

The natural requirement of parabolicity imposed by Gelfand on (1.5) means that the eigenvalues of $D(u_0)$ always have positive real parts. Equivalently, if $\kappa_j(\xi)$, $j = 1,..., m$, are the eigenvalues of the matrix symbol

$$P(\xi) = -i\xi A(u_0) - \xi^2 \nu D(u_0) \tag{1.9}$$

obtained from linearizing (1.5) at a constant state u_0, then for some $\delta > 0$ depending on u_0 these eigenvalues satisfy

$$\operatorname{Re} \kappa_j(\xi) \leqslant -\delta\, |\xi|^2 \qquad \text{for} \quad |\xi| \text{ large}, \ j = 1,..., m. \tag{1.10}$$

Our central objective in this paper is to introduce a very natural algebraic requirement on the linearized system from (1.5) beyond the condition (1.10)—a condition on the viscosity matrix $D(u_0)$ we call *strict stability.* Given the concept of strict stability, the bulk of this paper (Sections 2–4) is devoted to two goals: (1) To explain the examples of inadmissibility in [1], and link the mechanisms of inadmissibility with quantitative violation of the strict stability condition. (2) To elucidate the close relationship between strict stability and necessary and sufficient conditions for admissibility.

The requirement of strict stability is motivated by the following considerations. The main interest in the problem of finding viscous shock profiles as described in (1.5)–(1.7) is to investigate in a special case the limit as $\nu \to 0$ of solutions of (1.5). The constant states, u_0, are special solutions of these diffusion equations. Linearization of (1.5) around the constant state u_0 yields the linear equation

$$\begin{aligned} v_t^\nu + A(u_0)\, v_x^\nu &= \nu D(u_0)\, v_{xx}^\nu, \qquad \nu \geqslant 0 \\ v^\nu(x, 0) &= v_0(x) \end{aligned} \tag{1.11}$$

so that a natural minimum requirement for any viscosity matrix in (1.5) is that the solution v^ν of (1.11) converges to v^0 for any initial data v_0. This is true for all $v_0(x) \in L^2(R)$ if and only if for any $T > 0$, there is a fixed constant $C(T)$ so that with $S^\nu(t)\, v_0 = v^\nu$,

$$\max_{\substack{0<\nu<1 \\ 0 \leqslant t \leqslant T}} \|S^\nu(t)\, v_0\|_{L^2} \leqslant C(T)\, \|v_0\|_{L^2}. \tag{1.12}$$

That is, the initial value problem (1.11) is required to be *uniformly well posed* in L^2 as $\nu \to 0$. At any given value u_0, there is a set of $m \times m$ matrices $D(u_0)$, the uniformly stable matrices, $S(u_0)$, guaranteeing (1.12). However, this set of matrices, $S(u_0)$, is a bit too large since the boundary of $S(u_0)$, $\partial S(u_0)$, includes $D \equiv 0$ as well as $m \times m$ matrices for which the solutions of (1.11) have a purely dispersive character (see Section 4). The set of *strictly stable viscosity matrices* at the point u_0 is the *interior* of the set $S(u_0)$. The strictly stable viscosity matrices at u_0 admit the following algebraic characterization (see Section 2 and [14] for further results):

$$\begin{array}{l} \text{A viscosity matrix is strictly stable if} \\ \text{and only if there exists a } \delta > 0 \text{ so that} \\ \text{the eigenvalues } \kappa_j(\xi),\ 1 \leqslant j \leqslant m, \text{ for} \\ \text{the symbol } P(\xi) = -A(u_0)\, i\xi - \xi^2 D(u_0) \text{ satisfy} \\ \operatorname{Re} \kappa_j(\xi) \leqslant -\delta\, |\xi|^2 \text{ for } \textit{all } \xi \in R. \end{array} \tag{1.13}$$

Looking back at (1.10), we see that the condition of strict stability strengthens the requirement of parabolicity to an algebraic stability condition valid for all $\xi \in R^1$ and not just for sufficiently high wavenumbers. Thus, one objective here is to study the existence of viscous shock profiles for diffusion matrices satisfying (1.13) for every value of u_0—we call these the strictly stable viscosity matrices in the remainder of this paper. Our second objective is to explain the inadmissibility examples from [1] and to identify the concrete mechanisms of inadmissibility through violation of the strict stability conditions in (1.13).

In Section 2, first we develop and discuss a variety of necessary and sufficient criteria for strict stability for general $m \times m$ systems. We then examine the implications of strict stability regarding the linearized structure of the ODE's (1.6) at the critical points, u_{L}, u_{R} for a general system. For a k-shock solution of (1.1) satisfying Lax's entropy inequalities,

$$\lambda_k(u_{\mathrm{L}}) > s > \lambda_k(u_{\mathrm{R}}), \qquad \lambda_{k+1}(u_{\mathrm{R}}) > s > \lambda_{k-1}(u_{\mathrm{L}}) \tag{1.14}$$

we verify there that for the ODE in (1.6),

$$\begin{array}{l} \text{the dimension of the unstable manifold at } u_{\mathrm{L}} \text{ is } m-k+1 \\ \text{the dimension of the stable manifold at } u_{\mathrm{R}} \text{ is } k. \end{array} \tag{1.15}$$

The following is an immediate consequence of this simple fact:

COROLLARY 1. *All the explicit inadmissible viscosity matrices for the system in* (1.10) *constructed in* [1] (*via Theorem* 3.2 *or* 5.2 *in* [1]) *are* not *uniformly stable at either* u_L *or* u_R.

Thus, the natural requirement of strict stability discussed in (1.11)–(1.13) is violated in these explicit examples of inadmissibility. Finally, in Section 2 we discuss how the criteria for strict stability simplify for 2×2 systems ($m = 2$). These results are important in our discussion of the isentropic gas dynamics equations in Section 4.

In Section 3 we study the existence of weak shock profiles for general $m \times m$ systems without making any assumption of genuine nonlinearity or the existence of convex entropies. In this generality, only the identity matrix was previously proved admissible [3]. With the assumption of genuine nonlinearity, matrices near the identity were proved admissible by Conley and Smoller in [2]. Conley and Smoller also reproved a theorem of Kulikovskii that if (1.1) can be put into the form

$$h(u)_t + g(u)_x = 0 \tag{1.16}$$

where $h = \nabla H$ and $g = \nabla G$ are gradients, then any positive definite matrix $P(u)$ inducing a viscous perturbation *of (1.16)* is admissible for weak shocks. (One observation of Mock in his treatment of strong shocks in [13] is that the form (1.16) can be achieved when a convex entropy exists for the system (1.1).)

Our Theorem 3.1 essentially *characterizes* matrices admissible for weak k-shocks. An immediate consequence is that *strictly stable viscosity matrices are always admissible for weak shocks.* That is, we have

COROLLARY 2. *Assume that* $D(u_0)$ *is strictly stable at* u_0 *for the* $m \times m$ *system* (1.1). *Then there is a fixed neighborhood of* u_0 *such that for any weak solution* (1.2) *of* (1.1) (*satisfying* (1.3)) *with* u_L *and* u_R *in that neighborhood,* u_L *and* u_R *can be connected in* (1.6) *by a viscous shock profile satisfying* (1.7) *if and only if the weak solution* (1.2) *satisfies Liu's strict entropy condition* (*see Section* 3).

Our proof of this theorem, based on the center manifold theorem, is quite simple and was motivated by the work of Kopell and Howard [8], which might be applied in the genuinely nonlinear case. However, some new observations are needed to handle the general (nongenuinely nonlinear) case.

On the other hand, from our necessary conditions for admissibility, we obtain the following for 2×2 systems.

COROLLARY 3. *Suppose $m=2$. If the 2×2 viscosity matrix $D(u)$ is not stable at u_0, necessarily it is inadmissible for all weak k-shocks with either $k=1$ or $k=2$.*

In Section 4 we present new examples of inadmissible viscosity matrices in the large for the p-system (1.8) where no connection of u_L and u_R is possible, satisfying (1.7) or the reverse. In these examples, violation of strict stability is linked directly to Hopf bifurcation in a way which elucidates the mechanism behind the nonconstructive Theorem 5.3 in [1].

The above results might lead us to guess:

> Strictly stable viscosity matrices, for reasonable 2×2 genuinely nonlinear systems, are always admissible for any shock solutions of (1.1) satisfying Lax's shock inequalities.

In Theorem 4.2 we identify broad classes of strictly stable $D(u)$ which are admissible for all (strong) shocks of the p-system (1.8). We mention here some special results:

COROLLARY 4. *Consider the p-system (1.8), and assume $p'(\tau)\to 0$ as $\tau\to\infty$, $p'(\tau)\to -\infty$ as $\tau\to 0$.*

(1) *Any constant strictly stable viscosity matrix D for (1.8) is globally admissible.*

(2) *Given any viscosity matrix D_0 strictly stable at a fixed point $u_0\in R^2$, there is a smooth strictly stable $D(u)$ such that $D(u_0)=D_0$ and $D(u)$ is globally admissible.*

Our proofs make use of Lyapunov functions obtained from convex entropies in a way suggested by Mock's work [13]. However, our results for the p-system do not follow from Mock's theorem in [13]. His theorem does not apply here, since global growth conditions important in his treatment do not hold.

Despite these positive results, the conjecture above fails. We can construct a (rapidly varying) strictly stable $D(u)$ for the system (1.8) which is inadmissible for any *given* shock. However, the mechanism of inadmissibility is subtle and very different from those discussed earlier.

Also in Section 4 we present an example with $D(u)\in\partial S(u)$ for all u (stable but not strictly stable) for the p-system (1.8) for which (1.5) is a dispersive system and (1.6) is a conservative system admitting a first integral, and no shock profiles.

Finally, we remark that the choice of the L^2-norm for determining a class of linearly stable viscosity matrices is not the only natural choice.

Other natural norms for shock wave theory such as the L^1 or BV norms might single out a smaller class of stable viscosity matrices which perhaps admit shock profiles with more special structure.

2. The Algebraic Structure Implied by Stable Viscosity Matrices

If we apply Fourier transforms and Plancherel's theorem to (1.13), we conclude that a viscosity matrix is uniformly stable if and only if the uniform bound

$$\max_{\substack{0 \leqslant t \leqslant T \\ 0 < \nu, \xi \in R}} \left| \exp \frac{t}{\nu} P(\xi) \right| \leqslant C(T) \tag{2.1}$$

is satisfied, where $P(\xi) = -\xi^2 D(u_0) - i\xi A(u_0)$. In Appendix A, we prove the following algebraic characterization of the strictly stable viscosity matrices, using the Kreiss matrix theorem:

Theorem 2.1. *The following are equivalent for a viscosity matrix $D(u)$ for the $m \times m$ system in* (1.5):

(1) *$D(u)$ is strictly stable at u_0.*

(2) *The eigenvalues, $\kappa_j(\xi)$, $j = 1, \ldots, m$, of $P(\xi)$ satisfy* $\operatorname{Re} \kappa_j(\xi) \leqslant -\delta_0 |\xi|^2$ *for some fixed $\delta_0 > 0$ and* all $\xi \in R$.

(3) *The following three conditions are satisfied:*

(i) *The system in* (1.5) *is parabolic, i.e., the eigenvalues of $D(u_0)$ have positive real part.*

(ii) $l_k D r_k(u_0) > 0$, $k = 1, \ldots, m$.

(iii) *The symbol $P(\xi)$ has no purely imaginary eigenvalues for $\xi \neq 0$.*

Corollary 2.2. *A simple sufficient condition guaranteeing strict stability at u_0 is the following: There is a positive definite symmetric matrix, $M(u_0)$, so that $MA(u_0)$ is symmetric and $MD(u_0)$ is positive definite (perhaps not symmetric).*

Proof of 2.2. If $(P(\xi) - \kappa) z = 0$, then $\operatorname{Re}(\kappa) z^* M z + \xi^2 \operatorname{Re} z^* M D z = 0$, so that criterion (2) of Theorem 2.1 is satisfied.

It seems worthwhile to discuss the criteria above a bit more, and in particular to compare the class of strictly stable viscosity matrices defined here to be the class of admissible matrices found by Mock [13]. Recall that an *entropy* for the system of conservation laws (1.1) is a function $E(u)$ such that for some $Q(u)$ (the entropy flux) we have

$$\nabla E \cdot A(u) = \nabla Q(u). \tag{2.2}$$

Smooth solutions of (1.1) also satisfy $E(u)_t + Q(u)_x = 0$. The Hessian of $E(u)$ symmetrizes (1.1), i.e., $\nabla^2 E \cdot A(u)$ is symmetric (differentiate above). One of Mock's main assumptions in [13] was the existence of a globally defined uniformly convex entropy for the system (1.1), so $\nabla^2 E(u)$ is uniformly positive definite. In addition to some global assumptions of genuine nonlinearity and growth at infinity necessary in his treatment, his main local assumption guaranteeing admissibility for a viscosity matrix $D(u)$ was that $\nabla^2 E \cdot D(u)$ be positive definite. Thus applying the corollary with $M = \nabla^2 E$, Mock's admissible viscosity matrices are strictly stable at each point. (A converse of sorts to this statement holds when $m = 2$. See below.)

Convex entropies exist for most *physical* systems (1.1), though they may not have the global properties Mock requires. But when $m > 2$, in general systems (1.1) do not admit *any* entropies. Since (1.1) is strictly hyperbolic in one space dimension, however, many positive definite symmetrizers $M(u)$ for $A(u)$ still exist. In fact, these may be characterized as follows:

LEMMA 2.3. *M is positive definite symmetric and $MA(u_0)$ is symmetric if and only if for some matrix L of left eigenvectors of $A(u_0)$, with $LA = \operatorname{diag}(\lambda_1, \ldots, \lambda_m) L$, we have $M = L^T L$.*

Proof. Fix any matrix L_0 of left eigenvectors of $A(u_0)$, and let $R_0 = L_0^{-1}$ be the corresponding matrix of right eigenvectors. Our hypotheses imply that $R_0^T M R_0 L_0 A R_0$ is symmetric, and $R_0^T M R_0$ is positive definite symmetric. Since $L_0 A R_0$ is diagonal with distinct eigenvalues it follows that $R_0^T M R_0 = S^2$ where S is a positive diagonal matrix. Then $M = L^T L$ where $L = S L_0$.

The criterion of the corollary is therefore equivalent to the requirement that

$$LDR(u_0) \text{ be positive definite} \tag{2.3}$$

for some choice of L and R with $LR = I$ and $LAR(u_0)$ diagonal. Although the set of strictly stable matrices D satisfying this criterion is large, it is worth pointing out that when $m > 2$, that set does not account for all strictly stable matrices. (For examples, see the end of this section.)

Next in this section we use Theorem 2.1 together with a structural lemma about strictly stable matrices to determine the linearized structure of the ODE's (1.6) at the critical points u_L, u_R. If u_0 is any critical point of (1.6), the stable (unstable) manifold M_- (M_+) of (1.6) at u_0 is tangent to the invariant subspace of the linearization of (1.6) at u_0,

$$D^{-1}(A(u_0)-sI)=Q(u_0,s) \tag{2.4}$$

corresponding to eigenvalues with negative (positive) real parts, and

$$\begin{aligned}\dim M_-(u_0) &= \text{number of eigenvalues of } Q(u_0,s)\\ &\qquad \text{with negative real parts}\\ \dim M_+(u_0) &= \text{number of eigenvalues of } Q(u_0,s)\\ &\qquad \text{with positive real parts.}\end{aligned} \tag{2.5}$$

We have the following general fact:

THEOREM 2.4. *Suppose $D(u)$ is a strictly stable $m\times m$ viscosity matrix for* (1.1). *For any k-shock satisfying Lax's entropy inequalities,*

$$\lambda_k(u_L)>s>\lambda_k(u_R)$$
$$\lambda_{k+1}(u_R)>s>\lambda_{k-1}(u_L)$$

it follows that

$$\begin{aligned}\dim M_-(u_R) &= k\\ \dim M_+(u_L) &= m-k+1.\end{aligned} \tag{2.6}$$

Before proving Theorem 2.2, we note that Corollary 1 of the Introduction follows immediately in the following fashion: For shocks moving with positive wave speed for the p-system (1.8), $k=2$, so that applying Theorem 2.2 we have

$$\dim M_-(u_R)=2, \qquad \dim M_+(u_L)=1$$

for strictly stable viscosity matrices. On the other hand, all the inadmissibility criteria in Theorem 3.2 and Theorem 5.2 of [1] imply that necessarily

$$\dim M_-(u_R)=1, \qquad \dim M_+(u_L)=2.$$

Thus the inadmissible viscosity matrices constructed through these criteria cannot be strictly stable. Similar remarks apply for shocks in the p-system with $s<0$, and for applications of the results in [1] for general 2×2 systems.

Proof of 2.4. Condition (3)(iii) of Theorem 2.1 implies that $Q(u_{\mathrm{L}},s)$

and $Q(u_R, s)$, which are nonsingular, cannot have any purely imaginary eigenvalues $i\xi$. Since 2.4 is obvious if $D(u) \equiv I$, the result follows if $D(u)$ can be smoothly deformed to the identity matrix through the set of strictly stable viscosity matrices. But that is the content of the following:

LEMMA 2.5. *The set of strictly stable viscosity matrices is star-shaped with respect to the identity matrix.*

Proof. Let D be strictly stable, so $\operatorname{Re} \kappa \leqslant -\delta\xi^2$ if $-\xi^2 D - i\xi A - \kappa$ is singular. Suppose $-\xi^2(tD + (1-t)I) - i\xi A - \tilde{\kappa}$ is singular, $0 < t < 1$. Then $-(\xi t)^2 D - i(\xi t) A - t(\tilde{\kappa} + \xi^2(1-t))$ is singular, so $\operatorname{Re}(\tilde{\kappa} + \xi^2(1-t))\, t \leqslant -\delta(\xi t)^2$, or $\operatorname{Re} \tilde{\kappa} \leqslant -(\delta t + (1-t))\, \xi^2$.

(The authors thank the referee for suggesting this proof of 2.4.)

Strict Stability for 2×2 *Systems*

The criteria above for strict stability simplify when $m = 2$. For one thing, the condition of Corollary 2.2 is necessary as well as sufficient when $m = 2$. We have

PROPOSITION 2.6. *The following are necessary and sufficient conditions that* $D(u)$ *be strictly stable at* u_0 *when* $m = 2$.

(1) *There exists a positive definite symmetric matrix* M, *such that* $MA(u_0)$ *is symmetric and* $MD(u_0)$ *is positive definite (possibly not symmetric).*

(2) D^{-1} *is strictly stable at* u_0.

(3) (i) $\det D(u_0) > 0$, *and*

(ii) $l_k D r_k(u_0) > 0$ *for* $k = 1$ *and* 2.

This proposition follows without difficulty from 2.1 and the following, which implies in addition that the symmetrizer M may be chosen smoothly wherever A is smooth. Below, assume that $A(u)$ is smooth in some domain, and $R_0(u)$ is a fixed smooth right eigenvector matrix with $L_0(u)$ the corresponding left eigenvector matrix ($L_0 R_0 = I$, $L_0 A R_0$ diagonal).

PROPOSITION 2.7. *Assume* $m = 2$. *Let* $D(u)$ *be a smooth* 2×2 *matrix. Then* $D(u)$ *is strictly stable at each point if and only if there exists a smooth positive diagonal matrix* $S(u)$, *such that if* $R(u) = R_0 S(u)$, $L = R^{-1}$, *then* $LDR(u)$ *is positive definite for all* u.

Here the *only if* part is the key fact, and this follows from the lemma below.

LEMMA 2.8. *Suppose* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *satisfies* $a, d > 0$, $ad - bc > 0$. *For*

$-1<\alpha<1$, *let* $S_\alpha = \operatorname{diag}(\sqrt{1-\alpha}, \sqrt{1+\alpha})$. *Then there exist* α_- *and* α_+, $-1 \leqslant \alpha_- < \alpha_+ \leqslant 1$, *depending smoothly on* A, *such that for any* α *with* $\alpha_- < \alpha < \alpha_+$, *the scaled matrix* $S_\alpha^{-1} A S_\alpha$ *is positive definite.*

Proof. $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ is positive definite if and only if $\alpha, \delta > 0$ and $(\beta+\gamma)^2 - 4\alpha\delta < 0$. Now $S_\alpha^{-1} A S_\alpha = \left(\begin{smallmatrix} a & b\lambda \\ c/\lambda & d \end{smallmatrix}\right)$, $\lambda = ((1+\alpha)/(1-\alpha))^{1/2}$. For $-1<\alpha<1$, this is positive definite exactly when

$$Q(\alpha) \equiv [b(1+\alpha) + c(1-\alpha)]^2 - 4ad(1-\alpha^2) < 0.$$

To prove the lemma, we shall show that $Q<0$ between distinct roots α_- and α_+ of $Q(\alpha)$ with $-1 \leqslant \alpha_- < \alpha_+ \leqslant 1$. We compute

$$\begin{aligned} Q(-1) &= 4c^2 \geqslant 0, & Q(1) &= 4b^2 \geqslant 0 \\ Q(0) &= (b+c)^2 - 4ad, & Q'(0) &= 2(b+c)(b-c) \\ \tfrac{1}{2}Q'' &= 4ad + (b-c)^2 = 4(ad-bc) + (b+c)^2 > 0. \end{aligned}$$

The minimum of $Q(\alpha)$ is thus attained at $\alpha_0 = -Q'(0)/Q''$. Either $|b-c| \leqslant |b+c|$ or vice versa, so $|\alpha_0| < 1$. The discriminant of Q is

$$\begin{aligned} \tfrac{1}{4}Q'(0)^2 - \tfrac{1}{2}Q''Q(0) &= (b+c)^2(b-c)^2 + (4ad - (b+c)^2)(4ad + (b-c)^2) \\ &= 4ad(4ad - 4bc) > 0. \end{aligned}$$

So $Q(\alpha)$ has distinct real roots in the interval $[-1, 1]$, which are then smooth functions of the coefficients.

As a final observation about 2×2 systems, we note that Mock's condition for admissibility discussed above is *locally* equivalent to strict stability when $m=2$:

PROPOSITION 2.9. *Assume* $A(u)$ *is smooth in an open domain and* $D(u)$ *is smooth. Then* $D(u)$ *is strictly stable at* u_0 *if and only if there exists a smooth convex entropy* $E(u)$ *for* (1.1) *defined in a neighborhood of* u_0, *such that* $\nabla^2 ED(u_0)$ *is positive definite.*

Proof. We may choose a left eigenvector matrix L_0 for $A(u_0)$ such that $L_0^T L_0 D(u_0)$ is positive definite, by 2.8. The existence of an entropy $E(u)$ in a neighborhood of u_0 such that $\nabla^2 E(u_0) = L_0^T L_0$ is a result of Lax [11, see Theorem 3.2].

As promised, we now present examples of strictly stable viscosity matrices which do not satisfy the condition of Corollary 2.2 for *any* symmetrizer M. It is not difficult to see that it suffices to give examples in the case that $m=3$ and $A = \operatorname{diag}(-\lambda, 0, 1)$ where $\lambda \geqslant 1$. When A is diagonal, any symmetrizer M must be a positive diagonal matrix, by 2.3. Our object then is to present a matrix D satisfying conditions (3)(i)–(3)(iii) of

Theorem 2.1 such that MD is not positive definite for any positive diagonal M. It suffices to specify D^{-1} such that D satisfies (3)(i)–(3)(iii) and D^{-1} has a negative diagonal element, so $D^{-1}M$ is not positive definite for any positive diagonal M. We will choose D^{-1} of the form

$$D^{-1}=\begin{bmatrix}-1 & b & 2\\ -1 & a & 0\\ -1 & 0 & 1\end{bmatrix}$$

where b is specifically related to a with $b>a>1$ and a is sufficiently large, depending on λ. The three principle invariants of D^{-1} are

$$g_1=\operatorname{tr} D^{-1}=a, \qquad g_3=\det D^{-1}=b+a$$
$$g_2=D_{11}^{-1}+D_{22}^{-1}+D_{33}^{-1}=a+1+(b-a)=b+1.$$

The characteristic polynomial is $\det(D-xI)=-x^3+g_1x^2-g_2x+g_3$. Note that the determinant and three principal minors are all positive, so that the inverse D has positive diagonal, fulfilling (3)(ii). Since g_1, g_2, g_3 are positive with $g_1 g_2>g_3$, one may verify that the eigenvalues of D^{-1} have positive real part, which ensures (3)(i).

Condition (3)(iii) is the most difficult one to check. We note that $-\xi^2D-i\xi A-i\tau$ is nonsingular for all $\xi\neq 0$ and all real τ if and only if $D^{-1}(A-\tau)$ has no nonzero imaginary eigenvalues for all real τ. The principal invariants of $B=D^{-1}(A-\tau)$ are polynomials in τ and take the form

$$\begin{aligned}
i_1&=\operatorname{tr} B=-a\tau+\lambda+1\\
i_2&=(b+1)\,\tau^2+c\tau-\lambda \qquad \text{where} \quad c=\lambda b-(\lambda+1)\,a+(\lambda-1)\\
i_3&=-(b+a)(\tau^3+(\lambda-1)\,\tau^2-\lambda\tau).
\end{aligned}$$

A necessary and sufficient condition for B to have a nonzero imaginary eigenvalue is that $i_1 i_2=i_3$ and $i_2>0$ (look at the characteristic polynomial). Thus, define $P(\tau)=i_3-i_1 i_2$. $P(\tau)$ is cubic, so must have at least one real root, but we will be able to restrict b and a in a way that guarantees that $P(\tau)$ is monotone and its only real root lies in an interval where $i_2<0$. The discriminant of i_2 is $c^2+4\lambda(b+1)>c^2$. Below we will see that $c>0$, so $i_2(\tau)<0$ in the interval $[-c/(b+1), 0]$. Now

$$P(\tau)=(a-1)\,b\tau^3+p_2\tau^2+p_1\tau+\lambda(\lambda+1)$$

where

$$\begin{aligned}
p_2&=ac-(b+a)(\lambda-1)-(b+1)(\lambda+1)\\
p_1&=\lambda b-(\lambda+1)\,c.
\end{aligned}$$

Require that $p_2 = 0$. This means that

$$b = \left(\frac{\lambda+1}{\lambda}\right)\left(\frac{a^2+1}{a-2}\right) = \frac{\lambda+1}{\lambda}\left(a+2+\frac{5}{a-2}\right)$$

and also implies $c > 0$. Note that $P(0) > 0$. We claim that if a is large, then

$$-cp_1 + \lambda(\lambda+1)(b+1) < 0$$

which implies $P(-c/(b+1)) < 0$ and $p_1 > 0$, so $P(\tau)$ is monotone. For the proof, we estimate, say, for $a > 7$,

$$(\lambda+1)\, a < \lambda b < \lambda(b+1) < (\lambda+1)(a+4)$$
$$(\lambda+1)^2\, a < \lambda(b+a) < (\lambda+1)^2(a+4).$$

Using the relation $p_2 = 0$ we obtain $(\lambda+1)^2 < c < 2(\lambda+1)^2$. Then for a large,

$$\lambda(\lambda+1)(b+1) < (\lambda+1)^2(a+4) < (\lambda+1)^3(a-c) < cp_1.$$

Now condition (3)(iii) holds, so D is strictly stable.

3. Admissibility in General for Weak k-Shocks

We begin this section by defining Liu's strict condition (E). We consider the structure of the *Hugoniot set* of pairs of vectors (u_L, u_R) satisfying

$$f(u_L) - f(u_R) - s(u_L - u_R) = 0 \tag{3.1}$$

for some wave speed s. Fixing u_L, the local structure of the set of states u_R satisfying (3.1) is well known (see [3] and [10]). In some neighborhood of u_L this set consists of m curves, $\tilde{u}^k(\rho)$, $k = 1, ..., m$, passing through u_L with corresponding shock speeds $s^k(\rho)$ for $k = 1, ..., m$ satisfying

$$\begin{aligned} &\tilde{u}^k(0) = u_L, && s^k(0) = \lambda_k(u_L) \\ &\frac{d\tilde{u}^k}{d\rho}(0) = r_k(u_L), && \frac{ds^k}{d\rho}(0) = \frac{1}{2}(\nabla\lambda_k \cdot r_k)(u_L) \\ &\rho = l_k(u_L)\cdot(\tilde{u}^k(\rho) - u_L). \end{aligned} \tag{3.2}$$

Liu's *strict entropy condition* for the k-shock wave in (1.2) with $u_R = \tilde{u}^k(\rho_R)$ is

$$s^k(\rho) > s = s^k(\rho_R) \qquad \text{for} \quad \rho \text{ between zero and } \rho_R. \tag{E$_s$}$$

If $\lambda_k(u)$ is genuinely nonlinear and $|u_L - u_R|$ is small, this condition is equivalent to Lax's shock inequalities.

Corollary 2 of the Introduction is an immediate consequence of Theorem 2.1(3) and the main result of this section to be described below. Before stating this result, we remark that in the special case where u and $f(u)$ are scalars, Liu's (strict) entropy condition reduces to Oleinik's familiar (strict) condition E; furthermore, we invite the reader to check by explicit quadrature that for the scalar parabolic equation

$$u_t + f(u)_x = \nu u_{xx}$$

u_L, u_R can be connected by a viscous shock profile if and only if Oleinik's (strict) condition E is satisfied—this fact indicates that the weak shock theorem stated below is sharp in general.

Our main result here is the following:

THEOREM 3.1. *Fix $u_0 \in R^m$ and k, $1 \leqslant k \leqslant m$. Assume $\lambda_k(u)$ is not linearly degenerate in any neighborhood of u_0. Assume that $D(u_0)$ satisfies the non-degeneracy conditions*:

(i) *$D(u_0)$ is nonsingular.*

(ii) $l_k D r_k(u_0) \neq 0$.

(iii) *$[-\xi^2 D + i\xi(A - \lambda_k)](u_0)$ is nonsingular for all real $\xi \neq 0$.*

Then the following are equivalent:

(1) $l_k D r_k(u_0) > 0$ $[l_k D r_k(u_0) < 0]$.

(2) *D is locally [in]admissible for k-shocks in a neighborhood of u_0. That is, there exists $\delta > 0$ so that for any u_L and u_R in $B_\delta(u_0) = \{u \mid |u - u_0| < \delta\}$ satisfying the Rankine–Hugoniot relations for some speed $s = s^k(\rho_R)$, then a shock profile lying in $B_\delta(u_0)$ exists connecting u_L to u_R [u_R to u_L] if and only if Liu's strict entropy condition* (E)$_s$ *is satisfied. In any case, at most one trajectory $u(\xi)$ of* (1.6) *connecting u_R and u_L exists which remains in $B_\delta(u_0)$ for all ξ real.*

Proof of Corollary 1.3. If a parabolic 2×2 viscosity matrix $D(u)$ is not uniformly stable at u_0, then necessarily, from Proposition 2.6,

$$\text{either} \quad l_1 D r_1(u_0) < 0$$
$$\text{or} \quad l_2 D r_2(u_0) < 0.$$

With these inequalities, we apply the inadmissibility criterion from Theorem 3.1 to either the 1-waves or 2-waves to deduce Corollary 3.

Theorem 3.1 is proved in two steps. First, for all u_L near u_0 and s near $\lambda_k(u_0)$ we reduce the connection problem for the system (1.6) to that for a scalar ODE locally, by employing the center manifold theorem with the nondegeneracy conditions (i)–(iii). That is, a *curve* is constructed, locally invariant for (1.6), which contains all the critical points of (1.6), for any u_L, in a fixed neighborhood of u_0. In the second step, this one-dimensional flow is analyzed: Critical points on the invariant curve are points $\tilde{u}_k(\rho)$ on the Hugoniot curve for u_L having $s^k(\rho) = s$. The stability of the rest point u_L in the flow is determined by the sign of $l_k D r_k(u_L)(\lambda_k(u_L) - s)$. For a shock satisfying $(E)_s$, $s^k(0) = \lambda_k(u_L) \geqslant s$. Degenerate cases are treated by continuity, using the center manifold.

Step 1. Extend the system (1.6) by introducing the parameters $v = u_L$ and s as additional variables; then (1.6) may be written

$$\begin{aligned} u_\xi &= D^{-1}(u)[f(u) - f(v) - s(u - v)] \\ v_\xi &= 0 \\ s_\xi &= 0. \end{aligned} \tag{3.3}$$

Our analysis will be based on the construction of a *center manifold* for (3.3) at the critical point $(u, v, s) = (u_0, u_0, \lambda_k(u_0))$. The center manifold theorem (Kelley [6, Theorem 3]) says:

THEOREM. *Suppose that a system of ordinary differential equations may be written as*

$$\begin{aligned} x' &= Ax + \tilde{X}(x, y, z) \\ y' &= By + \tilde{Y}(x, y, z) \\ z' &= Cz + \tilde{Z}(x, y, z) \end{aligned}$$

where A, B, and C are constant square matrices whose eigenvalues have positive, zero, and negative real parts, respectively, and $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$ are C^r $(r \geqslant 2)$ and vanish along with their first derivatives at $(x, y, z) = 0$.

Then there exists a locally invariant manifold for this system,

$$M^* = \{(x, y, z) \mid |y| < \delta, x = u^*(y), z = w^*(y)\}$$

where u^* and w^* are C^r functions defined for $|y| < \delta$ for some δ sufficiently small, and vanishing with their first derivatives at $y = 0$.

The center manifold need not be unique, but the following uniqueness property for trajectories does hold: The center manifold (parametrized by y) may be taken to be the intersection of a center-stable manifold (parametrized by y and z) and a center-unstable manifold (parametrized by y and x). Then any trajectory which lies in a small neighborhood $B_\delta(0)$ for all time must lie on this center manifold. This property follows from this fact, stated in Kelley [7]: If a trajectory starts in a small neighborhood $B_\delta(0)$ at a point not on the center-stable manifold, then it must leave $B_\delta(0)$ at some positive time.

Let us now apply the center manifold theorem. Without loss of generality, we may assume $u_0=0$, $\lambda_k(u_0)=0$. For convenience, we introduce $w=u-v$ and the vector $W=(w, v, s)^t$ in R^{2m+1}. We write (3.3) in the form

$$W_\xi = T(W). \tag{3.4}$$

To apply the center manifold theorem, it suffices to describe two invariant subspaces for the linearization dT at the critical point 0: algebraic eigenspaces corresponding to groups of eigenvalues with zero and nonzero real parts, respectively. In block form on $R^m \times R^m \times R$, we calculate

$$dT(0) = \begin{bmatrix} D^{-1}A(0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The characteristic equation for $dT(0)$ may be written

$$\lambda^{m+1} \det(A - \lambda D)_{u=0} = 0.$$

We claim that the nondegeneracy conditions imply that the algebraic eigenspace for eigenvalues with zero real part is simply the *kernel* of $dT(0)$. Indeed, condition (iii) means that $dT(0)$ has no nonzero imaginary eigenvalues.

Claim. The algebraic eigenspace for $dT(0)$ for the eigenvalue 0 is equal to $\ker dT(0)$ if and only if $l_k D r_k(0) \neq 0$.

Proof. This eigenspace is larger than $\ker dT(0)$ if and only if range $dT(0) \cap \ker dT(0)$ is nontrivial. This occurs when there exists v in R^m such that $D^{-1}A \cdot v = r_k$, or $Dr_k = A \cdot v$, whence $l_k D r_k = 0$. If $l_k D r_k \neq 0$, no such v exists.

Thus, let $Y = \ker dT(0)$, $X = \text{range}\, dT(0)$. X and Y are complementary invariant subspaces for $dT(0)$ comprising the algebraic eigenspaces for eigenvalues with nonzero and zero real parts, respectively. Further explicit decomposition of R^{2m+1} is unnecessary. Applying the center manifold theorem, we have:

PROPOSITION 3.2. *Assume that D satisfies the nondegeneracy conditions* *at $u_0 = 0$ with $\lambda_k(u_0) = 0$. Then there exists $\delta > 0$ and a C^r function g: $Y \to X$ defined on $B_\delta(0) \cap Y$ so that*

(1) *$M^* = \{x + y \in R^{2m+1} \mid x = g(y)\}$ is a locally invariant manifold for Eq.* (3.4).

(2) *$g(0) = 0$ and $dg(0) = 0$. Thus M^* is tangent to Y at* 0.

(3) *Any trajectory of* (3.4) *which lies in $B_\delta(0)$ for all ξ lies in M^*. In particular, critical points in $B_\delta(0)$ lie in M^*.*

We now describe how this center manifold reduces the connection problem for the system (1.6) to one dimension. Observe that $Y = \ker dT(0)$ is spanned by the $m+2$ vectors $(r_k, 0, 0)$, $(0, 0, 1)$, and $(0, r_j, 0)$, $j = 1, \ldots, m$.

PROPOSITION 3.3. *Assume that $D(u_0)$ satisfies the nondegeneracy conditions* (i)–(iii). *Then there exists $\delta > 0$ so that if $|u_L - u_0| + |s - \lambda_k(u_0)| < \delta$, there is a locally invariant curve $u(\eta, u_L, s)$ for the system* (1.6) *containing u_L and any point u_R in $B_\delta(u_0)$ satisfying the Rankine–Hugoniot relations $f(u_R) - f(u_L) - s(u_R - u_L) = 0$.*

Proof. Define a line in Y parametrized by $y(\eta) = (\eta r_k, u_{\mathrm{L}}, s)$. The curve $W(\eta) = y(\eta) + g(y(\eta))$ lies in M^* while $|y(\eta)| < \delta$. Since g maps into range $dT(0)$, we may write $g(y(\eta)) = (\tilde{g}(\eta, u_{\mathrm{L}}, s), 0, 0)$. Under Eq. (3.4), the v and s components of $W = (w, v, s)$ remain invariant. Hence the curve $W(\eta)$ is the intersection of two locally invariant manifolds, so is locally invariant. Returning to the (u, v, s) coordinates of (3.3), we obtain an invariant curve for (1.6) parametrized by

$$u(\eta, u_{\mathrm{L}}, s) = u_{\mathrm{L}} + \eta r_k(u_0) + \tilde{g}(\eta, u_{\mathrm{L}}, s)$$

so long as

$$|y(\eta)| = |\eta r_k| + |u_{\mathrm{L}} - u_0| + |s - \lambda_k(u_0)| < \delta.$$

If u_{R} is in $B_\delta(u_0)$ and $f(u_{\mathrm{R}}) - f(u_{\mathrm{L}}) - s(u_{\mathrm{R}} - u_{\mathrm{L}}) = 0$, then the point $(u_{\mathrm{R}} - u_{\mathrm{L}}, u_{\mathrm{L}}, s)$ is a critical point of $T(W)$, so lies in M^*, and therefore $u_{\mathrm{R}} - u_{\mathrm{L}} = \eta_{\mathrm{R}} r_k + \tilde{g}(\eta_{\mathrm{R}}, u_{\mathrm{L}}, s)$ for some η_{R}.

The flow on the invariant curve $u(\eta, u_{\mathrm{L}}, s)$ is now determined by a scalar ODE for $\eta(\xi)$,

$$\eta_\xi = F(\eta, u_{\mathrm{L}}, s) \tag{3.5}$$

where F is C^r, determined by the equation

$$D(u)\, u_\eta F(\eta, u_{\mathrm{L}}, s) = f(u) - f(u_{\mathrm{L}}) - s(u - u_{\mathrm{L}}) \tag{3.6}$$

where $u = u(\eta, u_{\mathrm{L}}, s)$. From the uniqueness property (3) of Proposition 3.2, two critical points u_L, u_R in $B_\delta(u_0)$ are connected, left to right, by a trajectory in $B_\delta(u_0)$ if and only if $\eta_L = 0$ and η_R are connected, left to right, by a trajectory of (3.5).

Step 2. We proceed to analyze the flow (3.5). Two critical points $\eta_L = 0$ and η_R are connected, left to right, by a trajectory of (3.5) if and only if $\operatorname{sgn} F(\eta, u_L, s) = \operatorname{sgn} \eta_R$ for η between 0 and η_R. From (3.6) we obtain

$$\begin{aligned} F(\eta, u_L, s)\, l_k(u_L)\, D(u)\, u_\eta &= l_k(u_L)[f(u) - f(u_L) - s(u - u_L)] \\ F_\eta(0, u_L, s)\, l_k(u_L)\, D(u_L)\, u_\eta &= (\lambda_k(u_L) - s)\, l_k(u_L)\, u_\eta(0, u_L, s). \end{aligned} \tag{3.7}$$

We assume δ is so small that for each u_L in $B_\delta(u_0)$ the Hugoniot curves $\tilde{u}^k(\rho, u_L)$ in $B_\delta(u_0)$ are as described at the beginning of this section. The invariant curve $u(\eta, u_L, s)$ intersects the Hugoniot curve $\tilde{u}^k(\rho, u_L)$ just when η is a critical point of (3.5). We define a correspondence between η and ρ (given u_L and s) by

$$\rho(\eta) = l_k(u_L)(u(\eta, u_L, s) - u_L), \qquad \text{so} \quad \rho_\eta = l_k(u_L)\, u_\eta.$$

LEMMA 3.4. *If δ is sufficiently small, then if $|u_L - u_0| + |s - \lambda_k(u_0)| < \delta$, we have*:

(1) $\operatorname{sgn} l_k(u_L)\, D(u)\, u_\eta = \operatorname{sgn} l_k Dr_k(u_0)$ *and* $\operatorname{sgn} l_k(u_L)\, u_\eta = 1$ *in* $B_\delta(u_0)$. *So ρ increases with η.*

(2) *$F(\eta, u_L, s) = 0$ if and only if $s^k(\rho(\eta)) = s$ or $\eta = 0$.*

(3) *For all η between 0 and η_0,*

$$\operatorname{sgn} F(\eta, u_L, s) \operatorname{sgn} l_k Dr_k(u_0) = \operatorname{sgn} \eta(s^k(\rho(\eta)) - s) \tag{3.8}$$

provided $s^k(\rho(\eta)) - s$ is of one sign between 0 and η_0.

Using (3.8) we may complete the proof of Theorem (3.1). Assume u_L and $u_R = \tilde{u}^k(\rho_R, u_L)$ satisfy the Rankine–Hugoniot relations with $s = s^k(\rho_R)$, and assume Liu's strict entropy condition $(E)_s$ holds. Then $u_R = u(\eta_R, u_L, s)$ for some η_R, and $\rho_R = l_k(u_L)(u_R - u_L) = \rho(\eta_R)$. By (3.8) and condition $(E)_s$,

$$\operatorname{sgn} F(\eta, u_L, s) \operatorname{sgn} l_k Dr_k(u_0) = \operatorname{sgn} \eta_R.$$

for all η between 0 and η_R. So a trajectory of the flow (3.5) connects $\eta_L = 0$ and η_R, left to right, if and only if $l_k Dr_k(u_0) > 0$.

If $l_k Dr_k(u_0) > 0$ and u_R is as above, but the entropy condition is not satisfied, then either $s^k(\rho) = s$ for some ρ between 0 and ρ_R, whence a critical point separates 0 and η_R in (3.8), or else $s^k(\rho) < s$ for all ρ between 0 and ρ_R. Then (3.8) implies that a trajectory of (3.5) connects η_R on the left to $\eta_L = 0$ on the right. In either case, no trajectory of (1.6) lying in $B_\delta(u_0)$ can connect u_L on the left to u_R on the right.

Proof of Proposition 3.4. Part (1) follows from continuity and the fact that $u_\eta(0, u_0, \lambda_k(u_0)) = r_k(u_0)$, since $d\tilde{g}(0, u_0, \lambda_k(u_0)) = 0$. For part (2), if δ is sufficiently small and $\eta \neq 0$, then $F(\eta, u_L, s) = 0$ if and only if $u(\eta, u_L, s)$ lies on the kth Hugoniot curve for u_L, so $u(\eta, u_L, s) = \tilde{u}^k(\rho, u_L)$ for some ρ, and $s^k(\rho, u_L) = s$. But then $\rho = l_k(u_L)(\tilde{u}^k(\rho) - u_L) = \rho(\eta)$.

We shall establish part (3) in the case that $s^k(\rho(\eta)) > s$ for η between 0 and $\eta_0 > 0$, and $l_k Dr_k(u_0) > 0$ (remaining cases are similar). First, $\lambda_k(u_L) = s^k(0) \geqslant s$. Then $\lambda_k(u_L) > \tilde{s}$ for any $\tilde{s} < s$, so $F_\eta(0, u_L, \tilde{s}) > 0$ by (2.5). If $\tilde{s}$ is close to s, then $\rho(\frac{1}{2}\eta_0, u_L, \tilde{s}) < \rho(\eta_0, u_L, s)$, so $F(\eta, u_L, \tilde{s}) > 0$ for η between 0 and $\frac{1}{2}\eta_0$. (Since $s^k(\rho(\eta, u_L, \tilde{s})) > \tilde{s}$, it cannot vanish by part (2).) Letting $\tilde{s}$ increase to s we get $F(\eta, u_L, s) > 0$ for η between 0 and $\frac{1}{2}\eta_0$. (Again, $F(\eta, u_L, s)$ cannot vanish for η between 0 and η_0 by part (2).)

4. Admissible Viscosities for the p-System

Besides the basic conditions $p'(\tau) < 0$, $p''(\tau) > 0$, for the p-system in (1.8), in this section we assume additionally that

$$\begin{aligned} -p'(\tau) &\to \infty \qquad \text{as} \quad \tau \to 0 \\ -p'(\tau) &\to 0 \qquad \text{as} \quad \tau \to \infty. \end{aligned} \tag{4.1}$$

These conditions simplify many of the statements below. Suitable modifications of these results when (4.1) is not satisfied we leave for the interested reader to verify. With $u_L = (\tau_L, v_L)$, $u_R = (\tau_R, v_R)$, the Hugoniot relations from (1.2) imply that

$$\begin{gathered} -(v_L - v_R) = s(\tau_L - \tau_R), \qquad s = \pm\left(\frac{p(\tau_L) - p(\tau_R)}{-(\tau_L - \tau_R)}\right)^{1/2} \\ p(\tau_L) - p(\tau_R) = s(v_L - v_R). \end{gathered} \tag{4.2}$$

The back shocks (1-shocks) are those waves moving with speed $s < 0$ and satisfying the entropy inequality

$$-c(\tau_L) > s > -c(\tau_R) \tag{4.3}$$

while the front shocks (2-shocks) are those waves moving with speed $s > 0$ and satisfying the entropy inequality

$$c(\tau_L) > s > c(\tau_R). \tag{4.4}$$

Here $c(\tau)=(-dp/d\tau)^{1/2}$ is the Lagrangian sound speed. As a consequence of Galilean invariance, the front and back shocks should correspond under spatial reflection. Indeed, this is the case and one can easily verify that

$$\begin{aligned}&(\tau_L, v_L), (\tau_R, v_R) \text{ define a front shock moving with}\\ &\text{speed } s>0 \text{ satisfying (4.2) and (4.4) if and only if}\\ &(\tilde{\tau}_L, \tilde{v}_L), (\tilde{\tau}_R, \tilde{v}_R) \text{ define a back shock moving with}\\ &\text{speed } -s<0 \text{ satisfying (4.2) and (4.3) where}\\ &(\tilde{\tau}_L, \tilde{v}_L)=(\tau_R, -v_R), (\tilde{\tau}_R, \tilde{v}_R)=(\tau_L, -v_L).\end{aligned} \tag{4.5}$$

For the p-system, one right eigenvector matrix is

$$R(\tau)=\begin{pmatrix}1 & 1\\ c(\tau) & -c(\tau)\end{pmatrix} \tag{4.6}$$

with corresponding left eigenvector matrix

$$L(\tau)=\frac{1}{2}\begin{bmatrix}1 & c^{-1}\\ 1 & -c^{-1}\end{bmatrix}. \tag{4.7}$$

Here we study the (non)existence of viscous profiles for parabolic perturbations of (1.8) where the diffusion matrix D is given by

$$D(\tau, v)=\begin{bmatrix}d_{11} & d_{12}\\ d_{21} & d_{22}\end{bmatrix}, \qquad \begin{aligned}&d_{11}+d_{22}>0\\ &d_{11}d_{22}-d_{12}d_{21}>0\end{aligned} \tag{4.8}$$

and the matrix entries are smoothly varying functions of (τ, v). From (4.6)–(4.8) we compute

$$\begin{aligned}2(LDR)_{11}&=(cd_{12}+c^{-1}d_{21})+d_{11}+d_{22}\\ 2(LDR)_{22}&=-(cd_{12}+c^{-1}d_{21})+d_{11}+d_{22}\end{aligned} \tag{4.9}$$

therefore, from (3) of Proposition 2.1, we conclude that $D(\tau, v)$ *is strictly stable at* (τ, v) *if and only if*

$$|c(\tau)\,d_{12}+c(\tau)^{-1}\,d_{21}|<d_{11}+d_{22}. \tag{4.10}$$

Admissible and Inadmissible Viscosity Matrices for Weak Shocks

The result below is an immediate corollary of Theorem 3.1 and the proof of Corollary 1.3.

THEOREM 4.1. *Consider an arbitrary state* (τ_0, v_0) *and assume* $(d_{11}+d_{22})\pm(c(\tau_0)\,d_{12}+c(\tau_0)^{-1}\,d_{21})\neq 0$. *Then*

(1) *D is admissible for both front and back weak shocks in a neighborhood of (τ_0, v_0) if and only if D is strictly stable at (τ_0, v_0).*

(2) *Assume D is not strictly stable at (τ_0, v_0).*

(A) *If $cd_{12} + c^{-1}d_{21} < -(d_{11} + d_{22})$ at (τ_0, v_0), D is inadmissible for all weak back shocks but D is admissible for all weak front shocks in a neighborhood of (τ_0, v_0).*

(B) *If $cd_{12} + c^{-1}d_{21} > d_{11} + d_{22}$ at (τ_0, v_0), D is admissible for all weak back shocks but D is inadmissible for all weak front shocks in a neighborhood of (τ_0, v_0).*

We make the following two remarks which are easy consequences of Theorem 4.1.

Remark 1. If D is a *constant* diffusion matrix, then D is admissible for all weak shocks in a small neighborhood of *all* points (τ_0, v_0) if and only if D is diagonal, i.e.,

$$D = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix}, \qquad d_{11} > 0, \quad d_{22} > 0.$$

This fact follows easily from Theorem 4.1 since admissibility requires

$$|c(\tau)\, d_{11} + c(\tau)^{-1}\, d_{22}| \leqslant d_{11} + d_{22}$$

and we use (4.1) with $\tau \downarrow 0$, $\tau \uparrow \infty$ to justify the above remark.

Remark 2. There are *never* any examples of diffusion matrices D for the p-system which are inadmissible for *all* (front and back) weak shocks. The announced example in the introduction of [1] and described at the end of that paper is inadmissible for all front shocks but admissible for weak back shocks at least (this remark corrects a small error in [1]).

Hopf Bifurcation and Inadmissibility in the Large

The examples of inadmissibility described through Theorem 3.1 are not the only ones which occur in the large. One of the critical points can also be encircled by a periodic (or homoclinic) orbit leading to oscillatory behavior and preventing connection of the critical points. This possibility was pointed out in Theorem 5.3 of [1] through a nonconstructive argument. Here we link the appearance of such periodic orbits quantitatively with violation of the strict stability condition. A more subtle and quite different example of inadmissibility through oscillatory behavior for strictly stable diffusion matrices is discussed later in this section.

To be specific, we assume that there is a fixed point (τ_0, v_0) where $D(\tau_0, v_0)$ is not uniformly stable (see (4.10)) and in fact,

$$d_{11}+d_{22}<d_{12}c(\tau_0)+d_{21}c(\tau_0)^{-1}. \tag{4.11}$$

We consider front shocks with $(\tau_R, v_R)=(\tau_0, v_0)$ and the shock speed, s, as a bifurcation parameter with $s>c(\tau_0)$. From (4.2) and (4.4) it is easy to see that given (τ_R, v_R) and $s>c(\tau_R)$ there is a unique (τ_L, v_L) defining a front shock. From (4.1) s varies over $(c(\tau_R), \infty)$ so that given (4.11), there is a critical shock wave speed $s_0>c(\tau_0)$ with

$$\frac{s_0}{c(\tau_0)}(d_{11}+d_{22})=c(\tau_0)\,d_{12}+c(\tau_0)^{-1}\,d_{21}. \tag{4.12}$$

We compute that

$$\operatorname{tr}\left(D^{-1}\begin{pmatrix}-s & -1\\ -c^2 & -s\end{pmatrix}\right)=\frac{c}{\det D}\left[-\frac{s}{c}(d_{11}+d_{22})+cd_{12}+c^{-1}d_{21}\right]. \tag{4.13}$$

Therefore, from (4.12), (4.13) at the critical point $(\tau_R, v_R)=(\tau_0, v_0)$ and for $|s-s_0|<\delta$, $D^{-1}(A-sI)$ has nonzero complex conjugate eigenvalues, $\lambda(s)$, $\bar{\lambda}(s)$ with

$$\begin{aligned}&\operatorname{Re}\lambda(s)>0, && s_0-\delta<s<s_0\\ &\operatorname{Re}\lambda(s)<0, && s_0<s<s_0+\delta\\ &\operatorname{Re}\lambda(s_0)=0\end{aligned} \tag{4.14}$$

and also $\operatorname{Re}\lambda'(s)\,|_{s=s_0}\neq 0$ so by Hopf's bifurcation theorem,

$$\begin{aligned}&(\tau_R, v_R)\text{ is encircled by a small amplitude periodic}\\ &\text{orbit for either } s_0-\delta'<s<s_0 \text{ or } s_0<s<s+\delta'\end{aligned} \tag{4.15}$$

and this viscosity matrix is necessarily inadmissible for the corresponding front shock. The same phenomenon at (τ_0, v_0) occurs for back shocks provided that the strict stability condition is violated through

$$d_{12}c(\tau_0)+d_{21}c(\tau_0)^{-1}<-(d_{11}+d_{22})$$

rather than (4.11).

By looking back at Theorems 2.1 and 2.2, the reader can see that the above argument for the p-system illustrates in a special case a very general link between violation of strict stability and occurrence of small-amplitude periodic orbits bifurcating from critical points for (1.6) for general $m\times m$ systems and guaranteeing inadmissibility when $m=2$; however, to keep our discussion brief, we do not develop this here in detail beyond the above example.

Admissibility in the Large

Here we exhibit a reasonably wide class of strictly stable viscosity matrices which are admissible for *all* shocks of the p-system. In particular, we prove Corollary 4 of the Introduction.

THEOREM 4.2. *Suppose conditions* (4.1) *hold, and suppose* $D(\tau, v)$ *is a smooth strictly stable viscosity matrix for the p-system such that*:

(a) $D(\tau, v)$ *is constant exterior to a compact region* Ω *in the half plane* $\tau > 0$.

(b) *For some fixed* λ, $\left[\begin{smallmatrix} c(\tau)^2 & \lambda \\ \lambda & 1 \end{smallmatrix}\right] \cdot D(\tau, v)$ *is positive definite in* Ω.

Then $D(\tau, v)$ *is admissible for all shocks of the p-system.*

Remark. For any *fixed* (τ, v), there does exist λ, $|\lambda| < c(\tau)$, such that $\left(\begin{smallmatrix} c^2 & \lambda \\ \lambda & 1 \end{smallmatrix}\right) \cdot D(\tau, v)$ is positive definite. To see this, consider the matrix of left eigenvectors for the p-system defined by

$$L_\alpha = \begin{bmatrix} \sqrt{1+\alpha} & 0 \\ 0 & \sqrt{1-\alpha} \end{bmatrix} \cdot \begin{bmatrix} c & 1 \\ c & -1 \end{bmatrix}.$$

It follows from Proposition 2.7 that for certain α in the interval $(-1, 1)$, the matrix $L_\alpha D L_\alpha^{-1}$, hence $L_\alpha^T L_\alpha D$, is positive definite. But

$$L_\alpha^T L_\alpha = 2 \begin{pmatrix} c^2 & c\alpha \\ c\alpha & 1 \end{pmatrix}.$$

Corollary 4 of the Introduction follows immediately from the theorem and remark above. For if D_0 is strictly stable at a fixed (τ_0, v_0), then for some λ, $\left(\begin{smallmatrix} c^2(\tau) & \lambda \\ \lambda & 1 \end{smallmatrix}\right) \cdot D_0$ is positive definite at $\tau = \tau_0$, so also for τ in a small neighborhood Ω of (τ_0, v_0). Let $\psi(\tau, v)$ be a function such that $\psi = 1$ at (τ_0, v_0), $\psi \equiv 0$ outside Ω, and let $D(\tau, v) = \psi D_0 + (1 - \psi) I$. Theorem 4.2 applies, yielding Corollary 4.

To put our proof of 4.2 in context, recall that a 2×2 system such as the p-system admits many entropies $E(u)$. In fact, for the p-system (2.2) reduces to the one equation $E_{\tau\tau} - c^2(\tau) E_{vv} = 0$. We consider two special solutions: $E_0(\tau, v) = v^2/2 + P(\tau)$, where $P'(\tau) = p(\tau)$, and $\tilde{E}(\tau, v) = \tau v$. We define $E_\lambda = E_0 + \lambda \tilde{E}$, and note $\nabla^2 E_\lambda = \left(\begin{smallmatrix} c^2 & \lambda \\ \lambda & 1 \end{smallmatrix}\right)$. Hypothesis (b) of 4.2 simply says that $\nabla^2 E_\lambda D$ is positive definite in Ω, and implies that E_λ itself is convex in Ω.

Proof of 4.2. Fix a front shock with $u_L = (\tau_L, v_L)$, $u_R = (\tau_R, v_R)$ and s satisfying (4.4) so that $\tau_R > \tau_L$, $v_L > v_R$. For the system

$$D(u)\, u_x = \begin{bmatrix} -s(\tau - \tau_R) - (v - v_R) \\ p(\tau) - p(\tau_R) - s(v - v_R) \end{bmatrix} \equiv V(u) \tag{4.16}$$

we shall show a trajectory exists connecting u_L on the left to u_R on the right. From Theorem 2.4, u_R is a stable node for this system and u_L is a saddle point.

We now invoke some results of Conley and Smoller [1] and claim: If no periodic or homoclinic orbit exists encircling u_R, then one branch of the unstable manifold of u_L approaches u_R as $x \to \infty$. The results of [1] which are pertinent are a classification theorem for flows in the plane with two critical points, one a saddle, one a node (Lemma 4.1), and the existence of an "isolating disk" for the system (4.16) with a constant diffusion matrix (Lemma 5.1), which traps some branch of an invariant manifold of u_L inside it (Lemma 4.2). In the present situation, the isolating disk of Lemma 5.1 may be constructed to contain Ω in its interior, since $D(u)$ is constant exterior to Ω. We now introduce two functions

$$\begin{aligned} \Lambda_0 &= p(\tau)(s(\tau - \tau_R) + v - v_R) - s(v^2/2 + P(\tau) - v(p(\tau_R) - sv_R)) \\ \tilde{\Lambda} &= P(\tau) - \tau p(\tau_R) - s(\tau - \tau_R)(v - v_R) - (v^2/2 - vv_R) \end{aligned} \tag{4.17}$$

and define $\Lambda_\lambda = \Lambda_0 + \lambda \tilde{\Lambda}$. This Λ_λ coincides with the functional Λ which appears in Mock [13] if the entropy is taken to be E_λ. It has the property that

$$\nabla \Lambda_\lambda = \begin{pmatrix} c^2(\tau) & \lambda \\ \lambda & 1 \end{pmatrix} V(u) = \nabla^2 E_\lambda \cdot V(u).$$

Along any trajectory of (4.17), then,

$$\Lambda_\lambda(u)_x = V(u) \cdot \nabla^2 E_\lambda D^{-1} V(u).$$

So in any region where $\nabla^2 E_\lambda D$ is positive definite, Λ_λ is increasing along trajectories.

Note that from the first remark of Section 4, $D(\tau, v)$ is diagonal exterior to Ω. It follows that Λ_0 increases along trajectories exterior to Ω, and that Λ_λ increases along trajectories in the strip $0 < \tau < \tau_1$ where $c(\tau_1) = |\lambda|$. (This strip contains Ω.) The phase portrait of (4.17) for D diagonal is shown in Fig. 1.

We now consider two cases. First, suppose $\tau_R > \tau_1$. Then it is clear that no periodic or homoclinic orbit encircling u_R can exist, for it would have to

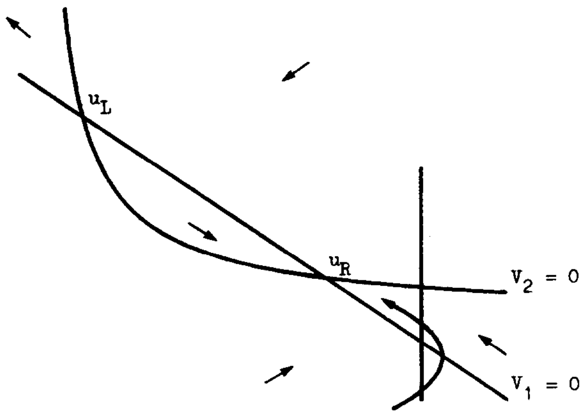

FIG. 1. Phase portrait of (4.16) for diagonal D.

intersect the vertical line $\tau = \tau_R$ in two places, but Fig. 1 shows that this is impossible for a trajectory in the region $\tau \geqslant \tau_R$.

In the second case, $\tau_R < \tau_1$, no periodic or homoclinic orbit encircling u_R could lie entirely to the left of the line $\tau = \tau_1$, for Λ_λ increases on trajectories there. The remaining possibility is that part of the encircling orbit lay to the right of the line $\tau = \tau_1$. It can only do so if that part is one connected piece as indicated in Fig. 1. Let $u_- = (\tau_1, v_-)$ be the point of entry, $u_+ = (\tau_1, v_+)$ the point of exit. Then

$$\Lambda_0(u_+) - \Lambda_0(u_-) > 0 > \Lambda_\lambda(u_+) - \Lambda_\lambda(u_-)$$

if the encircling orbit exists. Since $\lambda = c(\tau_1)$, this implies

$$c(\tau_1)\, |\tilde{\Lambda}(u_+) - \tilde{\Lambda}(u_-)| > \Lambda_0(u_+) - \Lambda_0(u_-).$$

We will show this cannot hold, so the encircling orbit cannot exist. Now

$$\tilde{\Lambda}(u_+) - \tilde{\Lambda}(u_-) = (v_+ - v_-)(v_1 - (v_+ + v_-)/2)$$

where $v_1 - v_R = -s(\tau_1 - \tau_R)$. With $v_2 - v_R = (p(\tau_1) - p(\tau_R))/s$, observe that $v_- < v_1 < v_+ < v_2$. Also

$$\Lambda_0(u_+) - \Lambda_0(u) = s(v_+ - v_-)(v_2 - (v_+ + v_-)/2).$$

Then $v_2 - (v_+ + v_-)/2 > |v_1 - (v_+ + v_-)/2|$ and $s > c(\tau_R) > c(\tau_1)$, so the inequalities above cannot hold, concluding the proof of Theorem 4.2 for front shocks. For back shocks, replace v by $-v$, x by $-x$ in (4.16), reducing the connection problem to that for a front shock.

An Example: Strictly Stable, but Inadmissible in the Large

We give here a construction which shows that, despite the positive results above, the local condition of strict stability is not quite sufficient for global admissibility.

PROPOSITION 4.3. *Fix a front shock (τ_L, v_L), (τ_R, v_R) with $c(\tau_L) > s > c(\tau_R) > 0$. There exists a smooth choice of $D(\tau, v)$, strictly stable at each point, such that the system of ODE's* (4.16) *for the p-system admits a closed periodic orbit encircling (τ_R, v_R), excluding (τ_L, v_L), so no trajectory can connect the two. We may choose $D \equiv I$ outside an annular region containing the periodic orbit.*

Our proof relies on a construction which characterizes what the vector field in (1.6) can be for arbitrary smooth, strictly stable $D(u)$ when $m = 2$. Assume $R(u) = (r_1, r_2)$ is a smooth matrix of right eigenvectors of $A(u)$. Two vectors w and $\tilde{w}$ are said to be *not of opposite sign* if there exists a positive diagonal matrix S so that $w^T S \tilde{w} > 0$.

LEMMA 4.4. *Assume $m = 2$. Suppose nonzero vectors w and $\tilde{w}$ are not of opposite sign. Then there exists a matrix $\tilde{D}(u, w, \tilde{w})$, strictly stable at u and smooth in u, w, and $\tilde{w}$, with $\tilde{D}(u, w, \tilde{w}) = I$ when $w = \tilde{w}$, such that $\tilde{D} R(u) w = R(u) \tilde{w}$. Conversely, if D is strictly stable at u and w is any nonzero 2-vector, then w and $LDR(u) w$ are not of opposite sign ($L = R^{-1}$).*

The proof of this lemma is supplied in Appendix B.

Proof of 4.3. Our procedure is as follows:

(a) We will exhibit a vector field $V_1(u)$ in an annular region encircling (τ_R, v_R) excluding (τ_L, v_L) which admits periodic orbits, and is also such that the vectors $LV(u)$, $LV_1(u)$ are not of opposite sign at each point $u = (\tau, v)$.

(b) Then V and V_1 are patched together by a partition of unity: Take a function $\psi(u)$ which is 1 on a periodic orbit and 0 outside the annular region, and let $\tilde{V} = \psi V_1 + (1 - \psi) V$.

(c) One easily checks that LV and $L\tilde{V}$ are not of opposite sign everywhere. Simply take $D^{-1}(u) = \tilde{D}(u, LV(u), L\tilde{V}(u))$. Then $D^{-1} V = \tilde{V}$ and 4.3 follows.

It remains to perform step (a). Because $u_R = (\tau_R, v_R)$ is a stable node for the vector field $V(u)$, the map $u \to w$ given by

$$w = -LV(u)$$

is locally invertible at $w=0$ by the inverse function theorem. In the w-plane we consider the family of curves $w(\theta; r_0)$ in polar coordinates,

$$r(\theta, r_0) = (|\cos\theta - \theta_0|^p + |\sin\theta - \theta_0|^p)^{-1/p} \cdot r_0$$

for r_0 small, p and θ_0 fixed, $1 < p < 2$, $0 < \theta_0 < \pi/4$ (see Fig. 2). This curve is the boundary of a ball in the p-metric $\|w\|_p = (|w_1|^p + |w_2|^p)^{1/p}$ rotated through an angle θ_0. It has the property that for each θ, the vectors $-w(\theta)$, $\dot{w}(\theta)$ are not of opposite sign. For r_0 sufficiently small, we may then obtain closed curves $u(\theta; r_0)$ encircling u_R via the isomorphism $u \leftrightarrow w$ defined above. Then

$$\dot{w}(\theta) = \frac{-\partial(LV)}{\partial u} \dot{u}(\theta) = \left(\begin{bmatrix} s + c(\tau) & 0 \\ 0 & s - c(\tau) \end{bmatrix} + \mathcal{O}(r_0) \right) L\dot{u}(\theta; r_0).$$

There is a smooth choice of positive diagonal $S(\theta)$ (see Appendix B) so that

$$-w^T S \dot{w}(\theta) \geqslant C r_0^2 \qquad \text{for all } \theta,\ C > 0 \text{ constant.}$$

With $\tilde{S} = S \cdot \operatorname{diag}(s + c, s - c)$, $c = c(\tau(\theta; r_0))$ we have

$$LV(u(\theta))\, \tilde{S} L\dot{u}(\theta) \geqslant c r_0^2 - \mathcal{O}(r_0^3) > 0 \qquad \text{if} \quad r_0 \text{ is small.}$$

Therefore it suffices to take $V_1 = \dot{u}(\theta; r_0)$ in step (a).

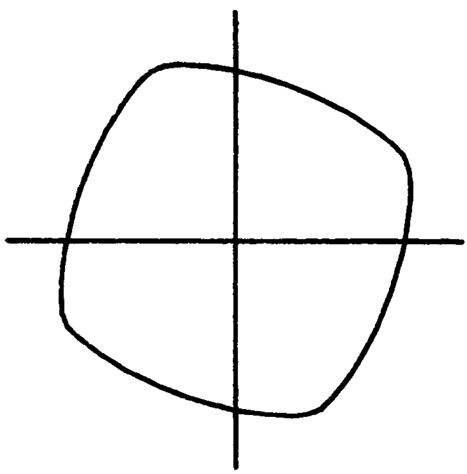

FIGURE 2

The Admissibility Criteria of Conley and Smoller and Strict Stability

Here we remark that under very general circumstances, application of the admissibility criteria of Conley and Smoller from [1] to one front shock and one back shock associated with a given value of τ automatically forces the diffusion matrix to satisfy a stronger requirement than strict stability. For simplicity in exposition we only state these results below for the paired front and back shocks described in (4.5), related through reflectional symmetry. We have

PROPOSITION 4.5. *Assume the admissibility criterion of Theorem* 3.2 *of* [1] *applies to both the front shock and back shock described in* (4.5) *for a fixed shock speed* $s > 0$. *Also assume the diffusion matrix* D *satisfies* $D(\tau_R, v_R) = D(\tau_R, -v_R)$ (*in particular any constant* D *always satisfies this requirement*). *Then* D *is strictly stable at* $(\tau_R, \pm v_R)$ *and satisfies*

$$|c(\tau_R)\, d_{12} + c(\tau_R)^{-1}\, d_{21}| \leqslant \frac{c(\tau_R)}{s}(d_{11} + d_{22}) < d_{11} + d_{22}.$$

To apply Theorem 3.2 of [1] to a given front shock connecting (τ_L, v_L) to (τ_R, v_R) with speed $s > 0$ requires that at (τ_R, v_R)

$$d_{12} s < d_{22}$$

$$\frac{s}{c^2} d_{21} < d_{11}$$

so that

$$d_{12} c + d_{21} c^{-1} < \frac{c}{s}(d_{11} + d_{22}).$$

Similarly, connecting $(\tau_R, -v_R)$ to $(\tau_L, -v_L)$ with wave speed $-s$ requires by the same conditions that at $(\tau_R, -v_R)$

$$d_{12} c + d_{21} c^{-1} > -\frac{c}{s}(d_{11} + d_{22}).$$

Since the entropy condition guarantees $c(\tau_R) < s$, the conclusion of Proposition 4.5 follows.

We also have

PROPOSITION 4.6. *Assume the admissibility criterion of Theorem* 5.2 *of* [1] *applies to both the front shock and back shock described in* (4.5) *for a* *fixed shock speed $s>0$. Also assume the diffusion matrix D satisfies $D(\tau_L, v_L)=D(\tau_L, -v_L)$. Then D is strictly stable at $(\tau_L, \pm v_L)$ and satisfies*

$$|c(\tau_L)\, d_{12}+c(\tau_L)^{-1}\, d_{21}| \leqslant \frac{s}{c(\tau_L)}\,(d_{11}+d_{22}).$$

First, we apply the admissibility criterion of Theorem 5.2 from [1] for the shock moving with speed $s>0$; this requires that the trace of the matrix in (4.13) is negative at (τ_L, v_L) so that

$$cd_{12}+c^{-1}d_{21} \leqslant \frac{s}{c(\tau_L)}\,(d_{11}+d_{22}).$$

Similarly, applying this criterion to the shock from (4.5) with speed $-s$ requires that the trace of the matrix in (4.13) is negative at $(\tau_L, -v_L)$ so that

$$cd_{12}+c^{-1}d_{21} > \frac{-s}{c(\tau_L)}\,(d_{11}+d_{22})$$

and these two inequalities together with the entropy condition $c(\tau_L)>s$ imply the conclusion of Proposition 4.6.

An Inadmissible Matrix on the Boundary of the Strictly Stable Viscosities

We consider the explicit choice of the matrix D given by

$$D(\tau)=\begin{bmatrix} 0 & 1 \\ -c^2(\tau) & 0 \end{bmatrix}.$$

This matrix is associated with purely dispersive wave propagation for (1.5) since $LDR(\tau)$ is a skew symmetric matrix. Furthermore, D is on the boundary of the set of stable viscosity matrices since $D(\tau)$ is the limit as $\varepsilon \downarrow 0$ of

$$D^{\varepsilon}(\tau)=\begin{bmatrix} \varepsilon & 1 \\ -c^2(\tau) & \varepsilon \end{bmatrix}$$

and $D^{\varepsilon}(\tau)$ is strictly stable because (4.10) is satisfied.

Fixing any shock (τ_L, v_L), (τ_R, v_R), s, it is easy to check that the function $\Lambda_0(\tau, v)$ of (4.17) is *constant* along trajectories of (4.16). Since $\Lambda_0(\tau_L, v_L) \neq \Lambda_0(\tau_R, v_R)$, no connection is possible.

APPENDIX A: PROOF OF THEOREM 2.1

We begin by developing some necessary criteria for a viscosity matrix D to be stable, i.e., for $D \in S(u_0)$. From (2.1) it follows that

$$\max_{\substack{0 \leqslant t < \infty \\ \xi \in R}} |e^{tP(\xi)}| \leqslant C \tag{A.1}$$

for some fixed constant C, where $P(\xi) = -\xi^2 D - i\xi A$. This estimate can hold only if the eigenvalues of $P(\xi)$ have nonpositive real part for all real ξ. Using this principle, we can establish:

PROPOSITION A.1. *Assume D is stable at u_0 ($D \in S(u_0)$). Then*

(1) *The eigenvalues of D have nonnegative real part.*

(2) $l_k D r_k(u_0) \geqslant 0$ *for* $k = 1,..., m$.

(3) *For any eigenvalue $\kappa_j(\xi)$ of $P(\xi)$,*

$$\operatorname{Re} \kappa_j(\xi) \leqslant 0, \qquad j = 1,..., m.$$

Proof. From the discussion above, (3) is immediate. Define, for convenience,

$$B(\theta) = D \sin \theta + iA \cos \theta.$$

Eigenvalues $\mu_j(\theta)$ of $B(\theta)$ are related to eigenvalues $\kappa_j(\xi)$ of $P(\xi)$ by

$$\mu_j(\theta) \cdot (\tan \theta / \cos \theta) = -\kappa_j(\tan \theta) \qquad \text{for} \quad \theta \neq 0, \ -\pi/2 < \theta < \pi/2.$$

From (3), and using continuity,

$$(\operatorname{sgn} \theta) \operatorname{Re} \mu_j(\theta) \geqslant 0, \qquad -\pi/2 \leqslant \theta \leqslant \pi/2, \ j = 1,..., m. \tag{A.2}$$

Setting $\theta = \pi/2$ we obtain (1). For (2), observe $B(0) = iA$ has distinct imaginary eigenvalues. Therefore, for small θ there exist smooth eigenvalues $\mu_k(\theta)$ and eigenvectors $R_k(\theta)$, with $\mu_k(0) = i\lambda_k(u_0)$, $R_k(0) = r_k(u_0)$, satisfying

$$(B(\theta) - \mu_k(\theta)) R_k(\theta) = 0.$$

Differentiate and set $\theta = 0$ ($B'(0) = D$). Then dot with $l_k(u_0)$. We obtain

$$l_k D r_k(u_0) = \mu_k'(\theta). \tag{A.3}$$

Part (2) now follows using (A.2).

Proof of Theorem 2.1. We shall argue that (1) implies (2) implies (3) implies (1). Assume that D is in the interior of $S(u_0)$. Then for some $\delta_0 > 0$, $D - \delta_0 I$ is stable, i.e., in $S(u_0)$. The eigenvalues $\tilde{\kappa}_j(\xi)$ of $\tilde{P}(\xi) = -\xi^2(D - \delta_0 I) - i\xi A$ then satisfy

$$\operatorname{Re} \tilde{\kappa}_j(\xi) = \operatorname{Re} \kappa_j(\xi) + \delta_0 \xi^2 \leqslant 0$$

establishing (2).

It is not hard to show that if some part of condition (3) does not hold, then (2) cannot hold, by using scaling arguments as in the proof of Proposition A.1. Since the conditions in (3) are open conditions, to complete the proof it remains only to show that if conditions (3)(i)–(3)(iii) are satisfied, then D is stable, i.e., in $S(u_0)$. We will make use of one part of the Kreiss matrix theorem.

THEOREM (Kreiss, 1959 [9]). *Let a family of $m \times m$ square matrices be given. A necessary and sufficient condition that $C_1 > 0$ exist so that*

$$\|e^{tA}\| \leqslant C_1$$

for all $t \geqslant 0$ and all A in the family, is:

(K3) *There exist constants C_{31} and C_{32} and a matrix $S(A)$ for each A in the family, with* $\max(\|S\|, \|S^{-1}\|) \leqslant C_{31}$, *so that*

$$SAS^{-1} = \begin{bmatrix} \mu_1 & b_{12} & \cdots & b_{1m} \\ 0 & \mu_2 & \cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \mu_m \end{bmatrix}$$

is upper triangular, with

$$\operatorname{Re} \mu_1 \leqslant \cdots \leqslant \operatorname{Re} \mu_m \leqslant 0 \tag{A.4}$$

and

$$|b_{ij}| \leqslant C_{32} \, |\operatorname{Re} \mu_j| \qquad \textit{for} \quad i < j. \tag{A.5}$$

Assume D satisfies conditions (3)(i)–(3)(iii). We shall verify the condition (K3) for the family of matrices $\{P(\xi) \mid \xi \in R\}$, or what is the same, because of a positive scaling factor, for the bounded family $\{-\sin\theta B(\theta) \mid -\pi/2 \leqslant \theta \leqslant \pi/2\}$.

We begin by using condition (ii), constructing a suitable S for θ near 0. From the proof of A.1, the matrix $B(\theta)$ may be diagonalized by a matrix $R(\theta)$ of right eigenvectors for $|\theta|$ small (so $S^{-1} = R(\theta)$, and $b_{ij} = 0$) and its distinct eigenvalues $\mu_k(\theta)$ satisfy $\mu_k'(\theta) = l_k Dr_k(u_0) > 0$, so for $|\theta| < \theta_0$, we have $-\sin\theta\mu_k(\theta) < 0$.

For $\theta_0 \leqslant |\theta| \leqslant \pi/2$, we may choose a unitary matrix $S(\theta)$ which puts $-\sin\theta B(\theta)$ into upper triangular form satisfying (A.4), by Schur's theorem (or see Richtmyer and Morton [15, p. 77]). The eigenvalues $\tilde{\mu}_j(\theta)$ of $-\sin\theta B(\theta)$ are continuous and never touch the imaginary axis for $\theta_0 \leqslant |\theta| < \pi/2$ by conditions (i) and (iii). Therefore $|\mathrm{Re}\,\tilde{\mu}_j(\theta)| > \mu_0 > 0$ for $\theta_0 \leqslant |\theta| \leqslant \pi/2$. But the off-diagonal elements of $S(-\sin\theta B)\,S^{-1}(\theta)$ are uniformly bounded, since S is unitary and $B(\theta)$ bounded. So (K3) holds, and Theorem 2.1 is established.

APPENDIX B: PROOF OF LEMMA 4.4

Consider the converse part first. If D is strictly stable, there exists a positive diagonal $\tilde{S}$ such that $\tilde{S}^{-1}LDR\tilde{S}$ is positive definite, by 2.7. Then if w is nonzero,

$$0 < (\tilde{S}^{-1}w)^T\,\tilde{S}^{-1}LDR\tilde{S}(\tilde{S}^{-1}w) = w^T\tilde{S}^{-2}LDRw$$

so w and $LDRw$ are not of opposite sign.

Now assume w and $\tilde{w}$, nonzero, are not of opposite sign. Then there exists a rotation matrix $Q(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, with $|\theta| < \pi/2$, and a constant $c > 0$, such that $\tilde{w} = cQ(\theta)\,Sw$. With $L\tilde{D}R = cQ(\theta)\,S$, $\tilde{D}$ is strictly stable, by 2.6(3), since $\cos\theta > 0$. In order to show that $\tilde{D}$ may be chosen smoothly, it suffices to show that S, depending on w, $\tilde{w}$, may be chosen smoothly.

Let $S(t) = \mathrm{diag}(t, 2-t)$. We shall show that t may be chosen as a smooth function of the angles $\theta = \arg w$, $\tilde{\theta} = \arg\tilde{w}$ in the proper subdomain of the torus $S^1 \times S^1$. In Fig. 3 below, the torus is divided into 16 square patches $(k(\pi/2), (k+1)(\pi/2)) \times (\tilde{k}(\pi/2), (\tilde{k}+1)(\pi/2))$ indicating regions in which w

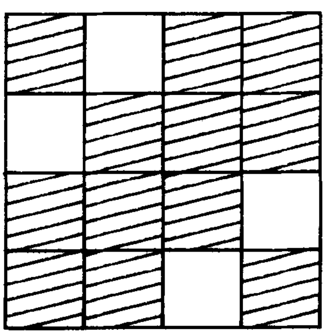

FIG. 3. Domain of $t(\theta, \tilde{\theta})$ in $S^1 \times S^1$.

and $\tilde{w}$ lie in given quadrants. The domain is the open, connected set indicated by shading.

Define $t(\theta, \tilde{\theta})$ as follows: If $(\theta, \tilde{\theta})$ lies in a square on the main diagonal (so $k=\tilde{k}$, i.e., w and $\tilde{w}$ lie in the same quadrant), define $t(\theta, \tilde{\theta})=1$, so $S(t)=I$. Consider a particular patch P off the main diagonal, $P=(0, \pi/2)\times(\pi/2, \pi)$. Let $w, \tilde{w}$ be given, with $(\theta, \tilde{\theta})\in P$. In order that $w^T S\tilde{w}>0$, we must have $t(\theta, \tilde{\theta})<t_c$ where $\tilde{w}^T S(t_c)\, w=0$, i.e., $\tilde{w}_1 w_1 t_c+(2-t_c)\,\tilde{w}_2 w_2=0$ or

$$\frac{t_c}{2-t_c}=-\tan\tilde{\theta}\tan\theta\equiv T(\theta, \tilde{\theta}).$$

Thus $t_c=2T/(1+T)$. Simply take $t(\theta, \tilde{\theta})=t(T)$ as any C^∞ function of T on the interval $(0, \infty)$ such that $t(T)<t_c$ and $t(T)=1$ for T sufficiently large. Other patches off the main diagonal are treated similarly, so a smooth $t(\theta, \tilde{\theta})$ may be defined as required. This concludes the proof of Lemma 4.4.

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