Adinkra 'color' confinement in exemplary off-shell constructions of 4D, $\mathcal{N} = 2$ supersymmetry representations

S. James Gates Jr.[a] **and Kory Stiffler**[b]

[a] *Center for String and Particle Theory, Department of Physics, University of Maryland,*
College Park, MD 20742-4111 U.S.A.

[b] *Department of Chemistry, Physics, and Astronomy, Indiana University Northwest,*
Gary, Indiana 46408 U.S.A.

E-mail: gatess@wam.umd.edu, kmstiffl@iun.edu

ABSTRACT: Evidence is presented in some examples that an adinkra quantum number, χ_o (arXiv: 0902.3830 [hep-th]), seems to play a role with regard to off-shell 4D, $\mathcal{N} = 2$ SUSY similar to the role of color in QCD. The vanishing of this adinkra quantum number appears to be a condition required for when two off-shell 4D, $\mathcal{N} = 1$ supermultiplets form an off-shell 4D, $\mathcal{N} = 2$ supermultiplet. We also explicitly comment on a deformation of the Lie bracket and anti-commutator operators that has been extensively and implicitly used in our work on "Garden Algebras" adinkras, and codes.

Contents

1 Introduction

For a number of years, we have been developing a "Garden Algebra", Adinkras, and Codes [1]–[13] approach for providing a deeper understanding of puzzling aspects of why and how supersymmetric *off-shell* representation theory works as it does. The basis of this is our conjecture that *off-shell* supersymmetrical representation theory should share as many as possible features with the representation theory of Lie compact algebras, but must be distinctive is some ways. Let us use su(3) as an exemplar of the former.

For su(3) it is well accepted that the fundamental representations, provided by quark triplets and anti-quark triplets, give the basic degrees of freedom from which to understand all representations of baryonic matter. In the work of ref. [18], it was proposed that there exist fundamental objects, given the names 'cis-adinkras' and 'trans-adinkras', which play a similar role in the context of 4D, $\mathcal{N} = 1$ supermultiplet representations. However, within the work of [19], it was shown that there is a degeneracy in the 'trans-adinkras', that can be recognized by considering the representation theory of the permutation group $\mathcal{S}_4$ and which is embedded in all adinkras with more than four colors. This allows the imposition of an intrinsic class structure on adinkras and these classes become relevant for defining how adinkras are related to the higher dimensional supermultiplets.

So the final result of our analysis is that there are three distinct off-shell adinkra classes, which can be identified with the 4D, $\mathcal{N} = 1$ chiral, vector and tensor supermultiplets, that are the irreducible supersymmetry or SUSY equivalent to quarks. Just as all hadronic matter can be regarded as composites of p quark triplets and q anti-quark triplets (here p and q are simply integers), our research suggests all *off-shell* 4D, $\mathcal{N} = 1$ supermultiplets may be regarded as composites of p chiral valise adinkras, q vector valise adinkras and r tensor valise adinkras (where p, q, and r are integers).

Although the Quark Model is now well accepted as being of fundamental importance to describing hadronic matter and its interactions, it is often forgotten that one of the major reasons the Quark Model was initially accepted had to do with the discovery of the Ω^- composed of three strange quarks and first seen in 1964. This particle had been predicted by the Quark Model prior to being seen in the laboratory and was a true 'smoking gun' indicating the validity of the Quark Model as an accurate description of physics in Nature. Furthermore, analysis of the statistics of composites in the Quark Model led to the discovery of color [14–17], the fiftieth anniversary of which has most recently been celebrated.

In this work, we provide more compelling evidence to support the assertion about an adinkra-based model of *off-shell* supersymmetric 4D, $\mathcal{N} = 1$ representations. Of course, since SUSY has not yet been seen in the laboratory, our evidence must perforce be purely mathematical. An assertion such as we have made ought to have implications for the structure of off-shell SUSY representations that go beyond the simple one-dimensional context used to discover the three foundational adinkras. We demonstrate one such implication in this work.

2 A mathematical background question

Some time ago, the topic of q-deformation of the usual Lie bracket

$$[A,\,B] = AB - BA \quad \rightarrow \quad [A,\,B]_q = AB - q\,BA\,, \tag{2.1}$$

where $-1 \leq q \leq 1$, was subject to numbers of studies (see for example [20, 21]). We have not formally commented previously, but in some ways the work of [1–3] can be interpreted in a similar manner. One can imagine two operators defined by

$$
\begin{aligned}
[A,\,B]_{q\mathcal{GR}>} &= A\,(B^{\mathrm{T}}) - q\,B\,(A^{\mathrm{T}})\,,\\
[A,\,B]_{q\mathcal{GR}<} &= (A^{\mathrm{T}})\,B - q\,(B^{\mathrm{T}})\,A\,,
\end{aligned}
\tag{2.2}
$$

acting on matrices A and B. The quantities (A^{T}) and (B^{T}) correspond to the respective transposed matrices. This bracket (for $q = -1$) has shown up as part of the mathematical foundation of the structures we call the 'Garden Algebras'. It is an interesting question (to which we do not possess an answer) as whether our use of such brackets can be extended in other ways? One such possibility would be to ask whether such a bracket admits analogs of Lie algebras?

To use matrices in such a construction we would begin with some set $\{\mathcal{G}\}$ with N elements denoted by $g_1, g_2, \ldots, g_N$ and impose upon them the conditions

$$
\begin{aligned}
[A,\,B]_{q\mathcal{GR}>} &= i\,f^{>}_{A\,B}{}^{C}\,h^{>}_{C}\,,\\
[A,\,B]_{q\mathcal{GR}<} &= i\,f^{<}_{A\,B}{}^{C}\,h^{<}_{C}\,,
\end{aligned}
\tag{2.3}
$$

where $h^{>}_{C}$ and $h^{<}_{C}$ are other matrices and $f^{>}_{A\,B}{}^{C}$ and $f^{<}_{A\,B}{}^{C}$ are analogous to structure constants. One other feature of the "q$\mathcal{GR}$" brackets in (2.2) is that they permit non-diagonal matrices to be used in their calculations. Thus, if A is a $\mathrm{d}_L \times \mathrm{d}_R$ and B is a $\mathrm{d}_R \times \mathrm{d}_L$ matrix, then $h^{>}_{C}$ will be a $\mathrm{d}_L \times \mathrm{d}_L$ matrix and $h^{<}_{C}$ will be a $\mathrm{d}_R \times \mathrm{d}_R$ matrix. We have long used these properties on our previous works investigating Garden Algebras, adinkras, and codes.

3 Building $\mathcal{N}\!=\!2$ supermultiplets from $\mathcal{N}\!=\!1$ supermultiplets

The basic idea of 'Garden Algebras' is very simple. The bracket operations above are used to impose upon the N elements Clifford algebra-like conditions

$$[g_A,\,g_B]_{(-1)\mathcal{GR}>} = 2\,\delta_{A\,B}\,\mathrm{I}\,, \qquad [g_A,\,g_B]_{(-1)\mathcal{GR}<} = 2\,\delta_{A\,B}\,\mathrm{I}\,, \tag{3.1}$$

where I is the identity map and both equations are valid for all values of A and B. When the elements satisfy these conditions, we say $g_1, g_2, \ldots, g_N$ forms a "Garden Algebra". The set $\{\mathcal{G}\}$ can be subject to the more stringent requirement that it form a group. However, this is not a requirement. There is no a priori choices made for the group $\mathcal{G}$. The question of whether all groups allow non-vanishing solutions to these conditions is unknown. However, when $\mathcal{G}$ is picked to be one of the orthogonal groups O(d), these have been found to be important for the representations of space-time supersymmetry realized off-shell.

The Garden Algebras (GA's) go back to the oldest part of our system of analysis [1–3] and when coupled with the SUSY Holography conjecture [4] assert *all* supermultiplets that are off-shell and possess no central charges must be representations to which the 0-brane reduction leads to a set of matrices (L_I) that satisfy

$$
(L_I)_i{}^{\hat{j}} (R_J)_{\hat{j}}{}^k + (L_J)_i{}^{\hat{j}} (R_I)_{\hat{j}}{}^k = 2\, \delta_{IJ}\, \delta_i{}^k ,
$$
$$
(R_J)_{\hat{i}}{}^j (L_I)_j{}^{\hat{k}} + (R_I)_{\hat{i}}{}^j (L_J)_j{}^{\hat{k}} = 2\, \delta_{IJ}\, \delta_{\hat{i}}{}^{\hat{k}} ,
\tag{3.2}
$$
$$
(R_I)_{\hat{j}}{}^k\, \delta_{ik} = (L_I)_i{}^{\hat{k}}\, \delta_{\hat{j}\hat{k}} ,
\tag{3.3}
$$

which we have denoted as the "$\mathcal{GR}(d, N)$ Algebras". Here the indices have ranges that correspond to $I, J, \cdots = 1, \ldots, N$; $i, j, \cdots = 1, \ldots, d$; and $\hat{i}, \hat{j}, \cdots = 1, \ldots, d$ for some integers N, and d. For this paper, we mostly consider the cases of $N = 8$ (for 4D, $\mathcal{N} = 2$) and $N = 4$ (for 4D, $\mathcal{N} = 1$) 1d, SUSY.

However, there are closely related algebraic structures that we denote as the "$\mathcal{GR}(d_L, d_R, N)$ Algebras" that satisfy

$$
(L_I)_i{}^{\hat{j}} (R_J)_{\hat{j}}{}^k + (L_J)_i{}^{\hat{j}} (R_I)_{\hat{j}}{}^k = 2\, \delta_{IJ}\, \delta_i{}^k + \Delta_{IJi}{}^k ,
$$
$$
(R_J)_{\hat{i}}{}^j (L_I)_j{}^{\hat{k}} + (R_I)_{\hat{i}}{}^j (L_J)_j{}^{\hat{k}} = 2\, \delta_{IJ}\, \delta_{\hat{i}}{}^{\hat{k}} + \widehat{\Delta}_{IJ\hat{i}}{}^{\hat{k}} ,
\tag{3.4}
$$
$$
(R_I)_{\hat{j}}{}^k\, \delta_{ik} = (L_I)_i{}^{\hat{k}}\, \delta_{\hat{j}\hat{k}} .
\tag{3.5}
$$

Here the indices have ranges that correspond to $I, J, \cdots = 1, \ldots, N$; $i, j, \cdots = 1, \ldots, d_L$; and $\hat{i}, \hat{j}, \cdots = 1, \ldots, d_R$ for some integers N, d_L, and d_R and for some quantities $\Delta_{IJi}{}^k$ and $\widehat{\Delta}_{IJ\hat{i}}{}^{\hat{k}}$. Past experience [18] has shown us that when there are off-shell central charges present in the higher dimensional theory, these cast 'shadows' in the 1D models in the form of the non-vanishing values of the quantities $\Delta_{IJi}{}^k$ and $\widehat{\Delta}_{IJ\hat{i}}{}^{\hat{k}}$.

The strategy of this section is to start with some well-known 4D, $\mathcal{N} = 1$ supermultiplets to explore the possibility of constructing 4D, $\mathcal{N} = 2$ supermultiplets. The reason this works conceptually is described below.

Let some 4D, $\mathcal{N} = 1$ supermultiplet denoted by $\{\mathcal{A}\}$ possess an action $S_1(\mathcal{A}|\mathcal{A})$ quadratic in its fields and invariant under the action of an off-shell SUSY operator D_a. This means that

$$
D_a \left[S_1(\mathcal{A}|\mathcal{A}) \right] = 0 ,
\tag{3.6}
$$

up to total derivative terms, and it is off-shell if the condition

$$
\left\{ D_a , D_b \right\} = i\, 2(\gamma^\mu)_{ab}\, \partial_\mu ,
\tag{3.7}
$$

is satisfied on all fields without regard to any field equations. A second such 4D, $\mathcal{N} = 1$ supermultiplet, with the same number of degrees of freedom, denoted by $\{\mathcal{B}\}$ will possess its own invariant action $S_2(\mathcal{B}|\mathcal{B})$ that satisfies the same property. Denoting the SUSY operator above by D_a^1, it follows that D_a^1 satisfies

$$
D_a^1 \left[S_1(\mathcal{A}|\mathcal{A}) + S_2(\mathcal{B}|\mathcal{B}) \right] = 0 ,
\tag{3.8}
$$

from its linearity. However, this statement guarantees that a second invariance generated by an operator D_a^2 satisfying

$$D_a^2 \left[S_1(\mathcal{A}|\mathcal{A}) + S_2(\mathcal{B}|\mathcal{B}) \right] = 0 \,, \tag{3.9}$$

must also exist. The realization of this second operator is such that it maps the bosons in the $\{\mathcal{A}\}$ supermultiplet into the fermions of the $\{\mathcal{B}\}$ (using the same equations as were the case of the fermions in the $\mathcal{A}$-multiplet) and maps the fermions in the $\{\mathcal{A}\}$ supermultiplet into the bosons of the $\{\mathcal{B}\}$ (using the same equations as were the case of the fermions in the $\mathcal{B}$-multiplet) that allowed the realization of D_a^1.

So we have a second fermionic invariance, but is it an on-shell or off-shell supersymmetry? To answer this requires calculating the anticommutator algebra of D_a^1. and D_a^2 on all the component fields. There is nothing in the above construction that guarantees that the two fermionic generators must form an extended off-shell 4D, $\mathcal{N} = 2$ supersymmetry algebra and one must check on a case-by-case basis. In the following, we will carry out such checks using the familiar 4D, $\mathcal{N} = 1$ chiral, vector, and tensor supermultiplets to play the roles of $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$. We will show that an interesting dichotomy emerges.

3.1 Building an $\mathcal{N} = 2$ supermultiplet from chiral + chiral $\mathcal{N} = 1$ supermultiplets

Among the first discussions of a 4D, $\mathcal{N} = 2$ supermultiplet containing only propagating field of spin-1/2 or less is the work by Fayet [23] in which there appears a citation to a work by Wess [24] as providing the initial discussion of the 'W-F hypermultiplet'. For the sake of completeness we review these results.

The transformation laws for the W-F hypermultiplet containing the Chiral-Chiral multiplet combination are (our notational conventions can be found in the [18])

$$
\begin{aligned}
&D_a^i A = (\sigma^3)^{ij} \psi_a^j \,, && D_a^i B = i(\gamma^5)_a{}^b \psi_b^i \,, \\
&D_a^i F = (\sigma^3)^{ij} (\gamma^\mu)_a{}^b \partial_\mu \psi_b^j \,, && D_a^i G = i(\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \psi_b^i \,, \\
&D_a^i \tilde{A} = (\sigma^1)^{ij} \psi_a^j \,, && D_a^i \tilde{B} = -(\sigma^2)^{ij} (\gamma^5)_a{}^b \psi_b^j \,, \\
&D_a^i \tilde{F} = (\sigma^1)^{ij} (\gamma^\mu)_a{}^b \partial_\mu \psi_b^j \,, && D_a^i \tilde{G} = -(\sigma^2)^{ij} (\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \psi_b^j \,, \\
&D_a^i \psi_b^j = i(\sigma^3)^{ij} \big((\gamma^\mu)_{ab} \partial_\mu A - C_{ab} F \big) + \delta^{ij} \big(-(\gamma^5 \gamma^\mu)_{ab} \partial_\mu B + (\gamma_{ab}^5) G \big) \\
&\qquad\quad + i(\sigma^1)^{ij} \big((\gamma^\mu)_{ab} \partial_\mu \tilde{A} - C_{ab} \tilde{F} \big) + i(\sigma^2)^{ij} \big(-(\gamma^5 \gamma^\mu)_{ab} \partial_\mu \tilde{B} + (\gamma_{ab}^5) \tilde{G} \big) \,,
\end{aligned}
\tag{3.10}
$$

where $i = 1, 2$ labels the two supersymmetries, and

$$(\sigma^0)^{ij} = \delta^{ij}. \tag{3.11}$$

The following Lagrangian is invariant with respect to these transformations:

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2} \partial_\mu A \partial^\mu A - \frac{1}{2} \partial_\mu \tilde{A} \partial^\mu \tilde{A} - \frac{1}{2} \partial_\mu B \partial^\mu B - \frac{1}{2} \partial_\mu \tilde{B} \partial^\mu \tilde{B} \\
&+ \frac{1}{2} F^2 + \frac{1}{2} \tilde{F}^2 + \frac{1}{2} G^2 + \frac{1}{2} \tilde{G}^2 + \frac{1}{2} i(\gamma^\mu)^{cd} \psi_c^i \partial_\mu \psi_d^i \,,
\end{aligned}
\tag{3.12}
$$

which is easily seen to be the direct sum of the $\mathcal{N} = 1$ invariant Lagrangians for the separate (A, ψ_c^1, F) chiral supermultiplet and the $(\widetilde{A}, \psi_c^2, \widetilde{F})$ chiral supermultiplet. Direct calculation yields the following algebra:

$$
\begin{aligned}
\{D_a^i, D_b^j\} A &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu A + i(\sigma^2)^{ij} 2i C_{ab} \tilde{F} \\
\{D_a^i, D_b^j\} \tilde{A} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \tilde{A} - i(\sigma^2)^{ij} 2i C_{ab} F \\
\{D_a^i, D_b^j\} B &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu B + i(\sigma^2)^{ij} 2i C_{ab} \tilde{G} \\
\{D_a^i, D_b^j\} \tilde{B} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \tilde{B} - i(\sigma^2)^{ij} 2i C_{ab} G \\
\{D_a^i, D_b^j\} F &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu F + i(\sigma^2)^{ij} 2i C_{ab} \Box \tilde{A} \\
\{D_a^i, D_b^j\} \tilde{F} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \tilde{F} - i(\sigma^2)^{ij} 2i C_{ab} \Box A \\
\{D_a^i, D_b^j\} G &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu G + i(\sigma^2)^{ij} 2i C_{ab} \Box \tilde{B} \\
\{D_a^i, D_b^j\} \tilde{G} &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \tilde{G} - i(\sigma^2)^{ij} 2i C_{ab} \Box B \\
\{D_a^i, D_b^j\} \psi_c^k &= \delta^{ij} 2i(\gamma^\mu)_{ab} \partial_\mu \psi_c^k - (\sigma^2)^{ij}(\sigma^2)^{kr} 2i C_{ab} (\gamma^\mu)_c{}^d \partial_\mu \psi_d^r .
\end{aligned}
\tag{3.13}
$$

From the results in eqs. (3.13) we can see there must be an additional symmetry of the Lagrangian with respect to the variations:

$$
\delta A = P\tilde{F}, \qquad\qquad \text{with} \quad \delta\tilde{F} = -P\Box A \tag{3.14}
$$

$$
\delta\tilde{A} = PF, \qquad\qquad \text{with} \quad \delta F = -P\Box\tilde{A} \tag{3.15}
$$

$$
\delta B = P\tilde{G}, \qquad\qquad \text{with} \quad \delta\tilde{G} = -P\Box B \tag{3.16}
$$

$$
\delta\tilde{B} = PG, \qquad\qquad \text{with} \quad \delta G = -P\Box\tilde{B} \tag{3.17}
$$

$$
\delta\psi_c^k = P(\sigma^2)^{kr}(\gamma^\mu)_c{}^d \partial_\mu \psi_d^r , \tag{3.18}
$$

where P is a constant parameter. In eqs. (3.13) is the composition of the two supersymmetry variation parameter according to

$$
P \equiv \varepsilon_i^a \varepsilon_j^b (\sigma^2)^{ij} C_{ab} , \tag{3.19}
$$

where ε_i^a is an infinitesimal Grassmann spinor.

When the fields $(A, \widetilde{A}, B, \widetilde{B}, F, \widetilde{F}, G, \widetilde{G}, \psi_a^r)$ satisfy their equations of motion, all the variations in (3.14)–(3.18) vanish. As well, the 'extra terms' in the anticommutators of (3.13) also vanish. So the symmetry generated by these variations are only non-trivial off the mass shell. This led to these being named as "off-shell central charges".

We now dimensionally reduce to an eight by eight adinkra by considering all fields to have only temporal dependence. As in [25], we identify

$$
\begin{aligned}
\psi_1^1 &= i\Psi_1, & \psi_2^1 &= i\Psi_2, & \psi_3^1 &= i\Psi_3, & \psi_4^1 &= i\Psi_4, \\
\psi_1^2 &= i\Psi_5, & \psi_2^2 &= i\Psi_6, & \psi_3^2 &= i\Psi_7, & \psi_4^2 &= i\Psi_8, \\
\Phi_1 &= A, & \Phi_2 &= B, & \partial_0\Phi_3 &= F, & \partial_0\Phi_4 &= G, \\
\Phi_5 &= \tilde{A}, & \Phi_6 &= \tilde{B}, & \partial_0\Phi_7 &= \tilde{F}, & \partial_0\Phi_8 &= \tilde{G},
\end{aligned}
\tag{3.20}
$$

and define
$$
D_I = \begin{cases} D_I^1 & 1 \leq I \leq 4 \\ D_{I-4}^2 & 5 \leq I \leq 8 \,, \end{cases}
\tag{3.21}
$$

whereupon the supersymmetric transformations reduce to

$$
D_I \Phi_j = i(L_I)_{j\hat{k}} \Psi_{\hat{k}} \,, \qquad D_I \Psi_{\hat{k}} = (R_I)_{\hat{k}j} \partial_0 \Phi_j \,.
\tag{3.22}
$$

The explicit form of the matrices in these equations are given in appendix A. These matrices satisfy the orthogonal relationship

$$
\mathbf{L}_I = (\mathbf{R}_I)^{-1} = (\mathbf{R}_I)^T.
\tag{3.23}
$$

The Chromocharacters are defined as

$$
\begin{aligned}
\left(\varphi^{(p)}\right)_{I_1 J_1 \ldots I_p J_p} &= \mathrm{Tr}\left\{ L_{I_1}(L_{J_1})^T \cdots L_{I_p}(L_{J_p})^T \right\} \\
\left(\tilde{\varphi}^{(p)}\right)_{I_1 J_1 \ldots I_p J_p} &= \mathrm{Tr}\left\{ (L_{I_1})^T L_{J_1} \cdots (L_{I_p})^T L_{J_p} \right\},
\end{aligned}
\tag{3.24}
$$

with first order chromocharacters given by:

$$
(\varphi^{(1)})_{I_1 J_1} = (\tilde{\varphi}^{(1)})_{I_1 J_1} = 8\delta_{I_1 J_1}
\tag{3.25}
$$

and second order characters given by:

$$
\begin{aligned}
\left(\varphi^{(2)}\right)_{I_1 J_1 I_2 J_2} = 8\big(&\delta_{I_1 J_1}\delta_{I_2 J_2} + (\sigma^0 \otimes \sigma^3 \otimes \sigma^2)_{I_1 J_1}(\sigma^0 \otimes \sigma^3 \otimes \sigma^2)_{I_2 J_2} \\
&+ (\sigma^3 \otimes \sigma^2 \otimes \sigma^0)_{I_1 J_1}(\sigma^3 \otimes \sigma^2 \otimes \sigma^0)_{I_2 J_2} \\
&+ (\sigma^3 \otimes \sigma^1 \otimes \sigma^2)_{I_1 J_1}(\sigma^3 \otimes \sigma^1 \otimes \sigma^2)_{I_2 J_2} \\
&+ (\sigma^2 \otimes \sigma^0 \otimes \sigma^0)_{I_1 J_1}(\sigma^2 \otimes \sigma^0 \otimes \sigma^0)_{I_2 J_2} \\
&+ (\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{I_1 J_1}(\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{I_2 J_2} \\
&+ (\sigma^1 \otimes \sigma^2 \otimes \sigma^0)_{I_1 J_1}(\sigma^1 \otimes \sigma^2 \otimes \sigma^0)_{I_2 J_2} \\
&+ (\sigma^1 \otimes \sigma^1 \otimes \sigma^2)_{I_1 J_1}(\sigma^1 \otimes \sigma^1 \otimes \sigma^2)_{I_2 J_2}\big) \\[4pt]
\left(\tilde{\varphi}^{(2)}\right)_{I_1 J_1 I_2 J_2} = 8\big(&\delta_{I_1 J_1}\delta_{I_2 J_2} + (\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{I_1 J_1}(\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{I_2 J_2} \\
&+ (\sigma^0 \otimes \sigma^0 \otimes \sigma^2)_{I_1 J_1}(\sigma^0 \otimes \sigma^0 \otimes \sigma^2)_{I_2 J_2} \\
&+ (\sigma^0 \otimes \sigma^2 \otimes \sigma^3)_{I_1 J_1}(\sigma^0 \otimes \sigma^2 \otimes \sigma^3)_{I_2 J_2} \\
&+ (\sigma^0 \otimes \sigma^2 \otimes \sigma^1)_{I_1 J_1}(\sigma^0 \otimes \sigma^2 \otimes \sigma^1)_{I_2 J_2} \\
&+ (\sigma^2 \otimes \sigma^3 \otimes \sigma^0)_{I_1 J_1}(\sigma^2 \otimes \sigma^3 \otimes \sigma^0)_{I_2 J_2} \\
&+ (\sigma^2 \otimes \sigma^1 \otimes \sigma^3)_{I_1 J_1}(\sigma^2 \otimes \sigma^1 \otimes \sigma^3)_{I_2 J_2} \\
&+ (\sigma^2 \otimes \sigma^1 \otimes \sigma^1)_{I_1 J_1}(\sigma^2 \otimes \sigma^1 \otimes \sigma^1)_{I_2 J_2}\big) \,.
\end{aligned}
\tag{3.26}
$$

Finally the L-matrices and R-matrices that arise in the case of combining two 4D $\mathcal{N} = 1$ chiral supermultiplets in an attempt to derive a 4D $\mathcal{N} = 2$ supermultiplet satisfy (3.4) where,

$$
\Delta_{IJ i}{}^k = -2\,(\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{IJ}(\sigma^2 \otimes \sigma^2 \otimes \sigma^0)_i{}^k,
\tag{3.27}
$$

and

$$
\widehat{\Delta}_{IJ \hat{i}}{}^{\hat{k}} = -2\,(\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{IJ}(\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{\hat{i}}{}^{\hat{k}}.
\tag{3.28}
$$

3.2 Building a $\mathcal{N} = 2$ supermultiplet from chiral + vector $\mathcal{N} = 1$ supermultiplets

The same work by Fayet [23] also introduced the now familiar 4D, $\mathcal{N} = 2$ 'vector multiplet'. From these we derive the following realization of for the D-algebra,

$$
\begin{aligned}
\mathrm{D}_a^1 A &= \psi_a \,, \\
\mathrm{D}_a^1 B &= i \, (\gamma^5)_a{}^b \, \psi_b \,, \\
\mathrm{D}_a^1 \psi_b &= i \, (\gamma^\mu)_{ab} \, \partial_\mu A - (\gamma^5 \gamma^\mu)_{ab} \, \partial_\mu B - i \, C_{ab} \, F + (\gamma^5)_{ab} G \,, \\
\mathrm{D}_a^1 F &= (\gamma^\mu)_a{}^b \, \partial_\mu \psi_b \,, \\
\mathrm{D}_a^1 G &= i \, (\gamma^5 \gamma^\mu)_a{}^b \, \partial_\mu \psi_b \,,
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
\mathrm{D}_a^1 A_\mu &= (\gamma_\mu)_a{}^b \, \lambda_b \,, \\
\mathrm{D}_a^1 \lambda_b &= - i \, \frac{1}{4} \big([\gamma^\mu , \, \gamma^\nu] \big)_{ab} \, (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\gamma^5)_{ab} \, \mathrm{d} \,, \\
\mathrm{D}_a^1 \mathrm{d} &= i \, (\gamma^5 \gamma^\mu)_a{}^b \, \partial_\mu \lambda_b \,,
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
\mathrm{D}_a^2 A &= \lambda_a \,, \\
\mathrm{D}_a^2 B &= i \, (\gamma^5)_a{}^b \, \lambda_b \,, \\
\mathrm{D}_a^2 \lambda_b &= i \, (\gamma^\mu)_{ab} \, \partial_\mu A - (\gamma^5 \gamma^\mu)_{ab} \, \partial_\mu B - i \, C_{ab} \, F - (\gamma^5)_{ab} G \,, \\
\mathrm{D}_a^2 F &= (\gamma^\mu)_a{}^b \, \partial_\mu \lambda_b \,, \\
\mathrm{D}_a^2 G &= - i \, (\gamma^5 \gamma^\mu)_a{}^b \, \partial_\mu \lambda_b \,,
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
\mathrm{D}_a^2 A_\mu &= -(\gamma_\mu)_a{}^b \, \psi_b \,, \\
\mathrm{D}_a^2 \psi_b &= i \, \frac{1}{4} \big([\gamma^\mu , \, \gamma^\nu] \big)_{ab} \, (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\gamma^5)_{ab} \, \mathrm{d} \,, \\
\mathrm{D}_a^2 \mathrm{d} &= i \, (\gamma^5 \gamma^\mu)_a{}^b \, \partial_\mu \psi_b \,,
\end{aligned}
\tag{3.32}
$$

that are equivalent to an invariance, up to total derivatives, of the Lagrangian

$$
\begin{aligned}
\mathcal{L} = {}& -\frac{1}{2} \partial_\mu A \partial^\mu A - \frac{1}{2} \partial_\mu B \partial^\mu B - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mathrm{d}^2 \\
& + \frac{1}{2} i (\gamma^\mu)^{bc} \lambda_b \partial_\mu \lambda_c + \frac{1}{2} i (\gamma^\mu)^{bc} \psi_b \partial_\mu \psi_c \,,
\end{aligned}
\tag{3.33}
$$

where

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,.
\tag{3.34}
$$

The transformation laws satisfy the algebra

$$
\begin{aligned}
\{\mathrm{D}_a^i, \mathrm{D}_b^j\} \chi &= 2 i \delta^{ij} (\gamma^\mu)_{ab} \chi \,, \\
\{\mathrm{D}_a^i, \mathrm{D}_b^j\} A_\nu &= 2 i \delta^{ij} (\gamma^\mu)_{ab} F_{\mu\nu} + i (\sigma^2)^{ij} \big(2 i C_{ab} \partial_\nu A - 2 (\gamma^5)_{ab} \partial_\nu B \big) \,,
\end{aligned}
\tag{3.35}
$$

where

$$
\chi = \{A, B, F, G, \mathrm{d}, \psi_c, \lambda_c\} \,.
\tag{3.36}
$$

Note, the algebra closes up to gauge transformations. We now dimensionally reduce to an eight by eight adinkra by considering all fields to have only temporal dependence. As in [18], we choose the gauge

$$A_0 = 0 \,, \tag{3.37}$$

and identify

$$
\begin{aligned}
&\psi_1 = i\Psi_1\,, &&\psi_2 = i\Psi_2\,, &&\psi_3 = i\Psi_3\,, &&\psi_4 = i\Psi_4\,, \\
&\lambda_1 = i\Psi_5\,, &&\lambda_2 = i\Psi_6\,, &&\lambda_3 = i\Psi_7\,, &&\lambda_4 = i\Psi_8\,, \\
&\Phi_1 = A\,, &&\Phi_2 = B\,, &&\partial_0\Phi_3 = F\,, &&\partial_0\Phi_4 = G\,, \\
&\Phi_5 = A_1\,, &&\Phi_6 = A_2\,, &&\Phi_7 = A_3\,, &&\partial_0\Phi_8 = \mathrm{d}\,,
\end{aligned} \tag{3.38}
$$

and define $\mathrm{D_I}$ as in eq. (3.21) whereupon the supersymmetric transformations reduce to the familiar form, eq. (3.22), where now the adinkra matrices are given in appendix B and the $(\mathrm{R_I})_{\hat{k}i}$ are given by the orthogonal relationship, eq. (3.23), as in section 3.1. The adinkra matrices satisfy the Garden Algebra in (3.2). The first and second order chromocharacters are given by

$$(\varphi^{(1)})_{\mathrm{I_1 J_1}} = (\tilde{\varphi}^{(1)})_{\mathrm{I_1 J_1}} = 8\delta_{\mathrm{I_1 J_1}}\,, \tag{3.39}$$

and

$$\left(\varphi^{(2)}\right)_{\mathrm{I_1 J_1 I_2 J_2}} = \left(\tilde{\varphi}^{(2)}\right)_{\mathrm{I_1 J_1 I_2 J_2}} = 8(\delta_{\mathrm{I_1 J_1}}\delta_{\mathrm{I_2 J_2}} + \delta_{\mathrm{I_1 J_2}}\delta_{\mathrm{I_2 J_1}} - \delta_{\mathrm{I_1 I_2}}\delta_{\mathrm{J_1 J_2}})\,. \tag{3.40}$$

3.3 Building an $\mathcal{N}=2$ supermultiplet from chiral + tensor $\mathcal{N}=1$ supermultiplets

The "4D, $\mathcal{N}=2$ tensor" multiplet (also known as the O(2) multiplet) was first introduced by Wess [24] and in our notation has a set of transformation laws of the form

$$
\begin{aligned}
\mathrm{D}_a^1 A &= \psi_a\,, \\
\mathrm{D}_a^1 B &= i\,(\gamma^5)_a{}^b\,\psi_b\,, \\
\mathrm{D}_a^1 \psi_b &= i\,(\gamma^\mu)_{ab}\,\partial_\mu A - (\gamma^5\gamma^\mu)_{ab}\,\partial_\mu B - i\,C_{ab}\,F + (\gamma^5)_{ab}G\,, \\
\mathrm{D}_a^1 F &= (\gamma^\mu)_a{}^b\,\partial_\mu\psi_b\,, \\
\mathrm{D}_a^1 G &= i\,(\gamma^5\gamma^\mu)_a{}^b\,\partial_\mu\psi_b\,,
\end{aligned} \tag{3.41}
$$

$$
\begin{aligned}
\mathrm{D}_a^1 \varphi &= \chi_a\,, \\
\mathrm{D}_a^1 B_{\mu\nu} &= -\frac{1}{4}\big([\gamma_\mu\,,\,\gamma_\nu]\big)_a{}^b\,\chi_b\,, \\
\mathrm{D}_a^1 \chi_b &= i\,(\gamma^\mu)_{ab}\,\partial_\mu\varphi - (\gamma^5\gamma^\mu)_{ab}\,\epsilon_\mu{}^{\rho\sigma\tau}\partial_\rho B_{\sigma\tau}\,,
\end{aligned} \tag{3.42}
$$

$$
\begin{aligned}
\mathrm{D}_a^2 A &= -\chi_a\,, \\
\mathrm{D}_a^2 B &= i\,(\gamma^5)_a{}^b\,\chi_b\,, \\
\mathrm{D}_a^2 \chi_b &= -i\,(\gamma^\mu)_{ab}\,\partial_\mu A - (\gamma^5\gamma^\mu)_{ab}\,\partial_\mu B - i\,C_{ab}\,F + (\gamma^5)_{ab}G\,, \\
\mathrm{D}_a^2 F &= (\gamma^\mu)_a{}^b\,\partial_\mu\chi_b\,, \\
\mathrm{D}_a^2 G &= i\,(\gamma^5\gamma^\mu)_a{}^b\,\partial_\mu\chi_b\,,
\end{aligned} \tag{3.43}
$$

– 9 –

$$\mathrm{D}_a^2 \varphi = \psi_a \,,$$

$$\mathrm{D}_a^2 B_{\mu\nu} = \frac{1}{4} \big([\gamma_\mu \,, \gamma_\nu] \big)_a{}^b \, \psi_b \,, \tag{3.44}$$

$$\mathrm{D}_a^2 \psi_b = i \, (\gamma^\mu)_{a\,b} \, \partial_\mu \varphi + (\gamma^5 \gamma^\mu)_{a\,b} \, \epsilon_\mu{}^{\rho\,\sigma\,\tau} \partial_\rho B_{\sigma\,\tau} \,,$$

derived from the supersymmetry invariance, up to total derivatives, of the Lagrangian

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2} \partial_\mu A \partial^\mu A - \frac{1}{2} \partial_\mu B \partial^\mu B - \frac{1}{3} H_{\mu\nu\alpha} H^{\mu\nu\alpha} - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi \\
&+ \frac{1}{2} i (\gamma^\mu)^{bc} \chi_b \partial_\mu \chi_c + \frac{1}{2} i (\gamma^\mu)^{bc} \psi_b \partial_\mu \psi_c + \frac{1}{2} F^2 + \frac{1}{2} G^2,
\end{aligned} \tag{3.45}$$

where

$$H_{\mu\nu\alpha} = \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\alpha\mu} + \partial_\alpha B_{\mu\nu} \,. \tag{3.46}$$

We find the algebra closes up to gauge transformations on $B_{\mu\nu}$:

$$\begin{aligned}
\{\mathrm{D}_a^i, \mathrm{D}_b^j\} X &= 2i \delta^{ij} (\gamma^\mu)_{ab} X \,, \\
\{\mathrm{D}_a^i, \mathrm{D}_b^j\} B_{\mu\nu} &= 2i \delta^{ij} (\gamma^\alpha)_{ab} H_{\alpha\mu\nu} \\
&+ i (\gamma_\mu \partial_\nu - \gamma_\nu \partial_\mu)_{ac} \big[(\sigma^1)^{ij} \delta_b{}^c A + (\sigma^2)^{ij} (\gamma^5)_b{}^c B - (\sigma^3)^{ij} \delta_b{}^c \varphi \big] \,,
\end{aligned} \tag{3.47}$$

where

$$X = \{A, B, F, G, \varphi, \psi_c, \chi_c\} \,. \tag{3.48}$$

We now dimensionally reduce to an eight by eight adinkra by considering all fields to have only temporal dependence. As in [18], we choose the gauge

$$B_{0i} = 0 \tag{3.49}$$

and identify

$$\begin{array}{llll}
\psi_1 = i \Psi_1 \,, & \psi_2 = i \Psi_2 \,, & \psi_3 = i \Psi_3 \,, & \psi_4 = i \Psi_4 \,, \\
\chi_1 = i \Psi_5 \,, & \chi_2 = i \Psi_6 \,, & \chi_3 = i \Psi_7 \,, & \chi_4 = i \Psi_8 \,, \\
\Phi_1 = A \,, & \Phi_2 = B \,, & \partial_0 \Phi_3 = F \,, & \partial_0 \Phi_4 = G \,, \\
\Phi_5 = \varphi \,, & \Phi_6 = 2 B_{12} \,, & \Phi_7 = 2 B_{23} \,, & \Phi_8 = 2 B_{31} \,, \tag{3.50}
\end{array}$$

using again the definition (3.21) for D_I, the transformation rules can be cast into the form, eq. (3.22), with the adinkra matrices as in appendix C with $(\mathrm{R}_\mathrm{I})_{\hat{k}i}$ once again given by the orthogonality relationship, eq. (3.23). These matrices satisfy the Garden Algebra, (3.2), and have the first and second order chromocharacters, eq. (3.39) and eq. (3.40), the same as for the chiral-vector multiplet of section 3.2.

3.4 Building an $\mathcal{N} = 2$ supermultiplet from vector + vector $\mathcal{N} = 1$ supermultiplets

The three constructs discussed previously in this section are well known. Their starting point may be regarded as utilizing the three 4D, $\mathcal{N} = 1$ chiral, vector, and tensor supermultiplets as building blocks for models with a higher degree of extended SUSY. However,

a little thought reveals there are more similar constructions to explore. Since there are three distinct 4D, $\mathcal{N} = 1$ building blocks, there should be 6 ways in which one can attempt to realize 4D, $\mathcal{N} = 2$ multiplets. We direct our attention to these other possibilities in this and the next subsection.

For the case of two 4D, $\mathcal{N} = 1$ vector multiplets, we introduce the transformation laws for this system as

$$
\begin{aligned}
\mathrm{D}_a^i \, A_\mu &= (\gamma_\mu)_a{}^b \, b^{ij} \lambda_b^j \,, \\
\mathrm{D}_a^i \, \tilde{A}_\mu &= (\gamma_\mu)_a{}^b \, a^{ij} \lambda_b^j \,, \\
\mathrm{D}_a^i \lambda_b^j &= b^{ij} \left(-i \frac{1}{4} \big([\gamma^\mu, \, \gamma^\nu]\big)_{ab} (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\gamma^5)_{a\,b} \, \mathrm{d} \right) \\
&\quad + a^{ij} \left(-i \frac{1}{4} \big([\gamma^\mu, \, \gamma^\nu]\big)_{ab} (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu) + (\gamma^5)_{a\,b} \, \tilde{\mathrm{d}} \right) \\
\mathrm{D}_a^i \, \mathrm{d} &= i \, (\gamma^5 \gamma^\mu)_a{}^b \, b^{ij} \, \partial_\mu \lambda_b^j \,, \\
\mathrm{D}_a^i \, \tilde{\mathrm{d}} &= i \, (\gamma^5 \gamma^\mu)_a{}^b a^{ij} \, \partial_\mu \lambda_b^j \,,
\end{aligned}
\tag{3.51}
$$

where $i, j = 1, 2$ and

$$
\begin{aligned}
a^{ij} &= \cos a_0 \, (\sigma^1)^{ij} + i \sin a_0 \, (\sigma^2)^{ij} \,, \\
b^{ij} &= \cos b_0 \, \delta^{ij} + \sin b_0 \, (\sigma^3)^{ij} \,.
\end{aligned}
\tag{3.52}
$$

The transformation laws (3.51) lead to an invariance of the Lagrangian

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} i (\gamma^\mu)^{bc} \lambda_b^j \partial_\mu \lambda_c^j + \frac{1}{2} \mathrm{d}^2 + \frac{1}{2} \tilde{\mathrm{d}}^2 \,,
\tag{3.53}
$$

where

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \qquad \tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \,,
\tag{3.54}
$$

and satisfy the algebra

$$
\begin{aligned}
\{\mathrm{D}_a^i, \mathrm{D}_b^j\} A_\nu = \ &2i\big(\delta^{ij} + (\sigma^3)^{ij} \sin 2b_0\big)(\gamma^\mu)_{ab} F_{\mu\nu} \\
&+ 2i(\sigma^1)^{ij} \cos(a_0 + b_0)(\gamma^\mu)_{ab} \tilde{F}_{\mu\nu} \\
&- i(\sigma^2)^{ij} \sin(a_0 - b_0) \epsilon_\nu{}^{\mu\alpha\beta} (\gamma^5 \gamma_\mu)_{ab} \tilde{F}_{\alpha\beta} \\
&- 2i \sin(a_0 - b_0)(\sigma^2)^{ij} \sin(a_0 - b_0)(\gamma^5 \gamma_\nu)_{ab} \tilde{\mathrm{d}} \,,
\end{aligned}
$$

$$
\begin{aligned}
\{\mathrm{D}_a^i, \mathrm{D}_b^j\} \tilde{A}_\nu = \ &2i\big(\delta^{ij} + (\sigma^3)^{ij} \sin 2b_0\big)(\gamma^\mu)_{ab} \tilde{F}_{\mu\nu} \\
&+ 2i(\sigma^1)^{ij} \cos(a_0 + b_0)(\gamma^\mu)_{ab} F_{\mu\nu} \\
&+ i(\sigma^2)^{ij} \sin(a_0 - b_0) \epsilon_\nu{}^{\mu\alpha\beta} (\gamma^5 \gamma_\mu)_{ab} F_{\alpha\beta} \\
&+ 2i \sin(a_0 - b_0)(\sigma^2)^{ij} \sin(a_0 - b_0)(\gamma^5 \gamma_\nu)_{ab} \mathrm{d} \,,
\end{aligned}
$$

$$
\begin{aligned}
\{\mathrm{D}_a^i, \mathrm{D}_b^j\} \lambda_c^k = \ &\big(2i \delta^{ij} \delta^{km} + i(\cos 2a_0 + \cos 2b_0)(\sigma^1)^{ij}(\sigma^1)^{km})(\gamma^\mu)_{ab} \partial_\mu \lambda_c^m \\
&+ i(\cos 2a_0 - \cos 2b_0)(\sigma^2)^{ij}(\sigma^2)^{km} \big((\gamma^5 \gamma^\mu)_{ab}(\gamma^5)_c{}^d \\
&\hspace{4cm} + C_{ab}(\gamma^\mu)_c{}^d + (\gamma^5)_{ab}(\gamma^5 \gamma^\mu)_c{}^d\big) \partial_\mu \lambda_d^m \\
&+ \text{terms proportional to } \sin 2a_0 \text{ and } \sin 2b_0 \,,
\end{aligned}
$$

$$\left\{ \mathrm{D}_a^i , \mathrm{D}_b^j \right\} \mathrm{d} = 2i \left(\delta^{ij} + \sin 2b_0 \, (\sigma^3)^{ij} \right) (\gamma^\mu)_{ab} \partial_\mu \mathrm{d}$$
$$+ 2i (\sigma^1)^{ij} \cos(a_0 + b_0) (\gamma^\mu)_{ab} \partial_\mu \tilde{\mathrm{d}}$$
$$+ 2i (\sigma^2)^{ij} \sin(a_0 - b_0) (\gamma^5 \gamma^\nu)_{ab} \partial^\mu (\partial_\nu \tilde{A}_\mu - \partial_\mu \tilde{A}_\nu) \,,$$

$$\left\{ \mathrm{D}_a^i , \mathrm{D}_b^j \right\} \tilde{\mathrm{d}} = 2i \left(\delta^{ij} + \sin 2a_0 \, (\sigma^3)^{ij} \right) (\gamma^\mu)_{ab} \partial_\mu \tilde{\mathrm{d}}$$
$$+ 2i (\sigma^1)^{ij} \cos(a_0 + b_0) (\gamma^\mu)_{ab} \partial_\mu \mathrm{d}$$
$$- 2i (\sigma^2)^{ij} \sin(a_0 - b_0) (\gamma^5 \gamma^\nu)_{ab} \partial^\mu (\partial_\nu A_\mu - \partial_\mu A_\nu) \,. \tag{3.55}$$

To have the canonical SUSY relationship to the momentum operator on the right hand side for the bosons forces

$$a_0 = m \frac{\pi}{2} \,, \qquad b_0 = n \frac{\pi}{2} \,, \qquad m, n \text{ integers} \,, \tag{3.56}$$

which makes

$$a^{ij} = \cos m \frac{\pi}{2} \, (\sigma^1)^{ij} + i \sin m \frac{\pi}{2} \, (\sigma^2)^{ij} \,,$$
$$b^{ij} = \cos n \frac{\pi}{2} \, \delta^{ij} + \sin n \frac{\pi}{2} \, (\sigma^3)^{ij} \,. \tag{3.57}$$

Defining

$$c_1 \equiv \cos \left(\frac{(m+n)\pi}{2} \right) \,,$$
$$s_1 \equiv \sin \left(\frac{(m-n)\pi}{2} \right) \,, \tag{3.58}$$
$$c_{2\pm} \equiv \cos m\pi \pm \cos n\pi \,,$$

the algebra becomes

$$\left\{ \mathrm{D}_a^i , \mathrm{D}_b^j \right\} A_\nu = 2i \delta^{ij} (\gamma^\mu)_{ab} F_{\mu\nu} + 2i c_1 (\sigma^1)^{ij} (\gamma^\mu)_{ab} \tilde{F}_{\mu\nu}$$
$$- i s_1 (\sigma^2)^{ij} \epsilon_\nu{}^{\mu\alpha\beta} (\gamma^5 \gamma_\mu)_{ab} \tilde{F}_{\alpha\beta} - 2i s_1 (\sigma^2)^{ij} (\gamma^5 \gamma_\nu)_{ab} \tilde{\mathrm{d}} \,,$$

$$\left\{ \mathrm{D}_a^i , \mathrm{D}_b^j \right\} \tilde{A}_\nu = 2i \delta^{ij} (\gamma^\mu)_{ab} \tilde{F}_{\mu\nu} + 2i c_1 (\sigma^1)^{ij} (\gamma^\mu)_{ab} F_{\mu\nu}$$
$$+ i s_1 (\sigma^2)^{ij} \epsilon_\nu{}^{\mu\alpha\beta} (\gamma^5 \gamma_\mu)_{ab} F_{\alpha\beta} + 2i s_1 (\sigma^2)^{ij} (\gamma^5 \gamma_\nu)_{ab} \mathrm{d} \,,$$

$$\left\{ \mathrm{D}_a^i , \mathrm{D}_b^j \right\} \lambda_c^k = \left(2i \delta^{ij} \delta^{km} + i c_{2+} (\sigma^1)^{ij} (\sigma^1)^{km} \right) (\gamma^\mu)_{ab} \partial_\mu \lambda_c^m$$
$$+ i c_{2-} (\sigma^2)^{ij} (\sigma^2)^{km} \left((\gamma^5 \gamma^\mu)_{ab} (\gamma^5)_c{}^d + C_{ab} (\gamma^\mu)_c{}^d \right.$$
$$\left. + (\gamma^5)_{ab} (\gamma^5 \gamma^\mu)_c{}^d \right) \partial_\mu \lambda_d^m \,,$$

$$\left\{ \mathrm{D}_a^i , \mathrm{D}_b^j \right\} \mathrm{d} = 2i \delta^{ij} (\gamma^\mu)_{ab} \partial_\mu \mathrm{d} + 2i c_1 (\sigma^1)^{ij} (\gamma^\mu)_{ab} \partial_\mu \tilde{\mathrm{d}}$$
$$+ 2i s_1 (\sigma^2)^{ij} (\gamma^5 \gamma^\nu)_{ab} \partial^\mu (\partial_\nu \tilde{A}_\mu - \partial_\mu \tilde{A}_\nu) \,,$$

$$\left\{ \mathrm{D}_a^i , \mathrm{D}_b^j \right\} \tilde{\mathrm{d}} = 2i \delta^{ij} (\gamma^\mu)_{ab} \partial_\mu \tilde{\mathrm{d}} + 2i c_1 (\sigma^1)^{ij} (\gamma^\mu)_{ab} \partial_\mu \mathrm{d}$$
$$- 2i s_1 (\sigma^2)^{ij} (\gamma^5 \gamma^\nu)_{ab} \partial^\mu (\partial_\nu A_\mu - \partial_\mu A_\nu) \,. \tag{3.59}$$

We now dimensionally reduce to an eight by eight adinkra by considering all fields to have only temporal dependence. We choose the gauge

$$A_0 = \tilde{A}_0 = 0 \,, \tag{3.60}$$

and define

$$\begin{array}{llll}
\lambda_1^1 = i\Psi_1\,, & \lambda_2^1 = i\Psi_2\,, & \lambda_3^1 = i\Psi_3\,, & \lambda_4^1 = i\Psi_4\,, \\
\lambda_1^2 = i\Psi_5\,, & \lambda_2^2 = i\Psi_6\,, & \lambda_3^2 = i\Psi_7\,, & \lambda_4^2 = i\Psi_8\,, \\
\Phi_1 = A_1\,, & \Phi_2 = A_2\,, & \Phi_3 = A_3\,, & \partial_0\Phi_4 = \mathrm{d}\,, \\
\Phi_5 = \tilde{A}_1\,, & \Phi_6 = \tilde{A}_2\,, & \Phi_7 = \tilde{A}_3\,, & \partial_0\Phi_8 = \tilde{\mathrm{d}}\,,
\end{array} \tag{3.61}$$

and D_I as before, eq. (3.21). The transformation laws reduce to the ever-now-more familiar form, eq. (3.22), now with the L-matrices and R-matrices given in appendix D. These matrices satisfy the orthogonality relationship, eq. (3.23), and the algebra of (3.4) where

$$\begin{aligned}
\Delta_{\mathrm{IJ}i}{}^k = {}& 2c_1(\sigma^1\otimes\sigma^0\otimes\sigma^0)_{\mathrm{IJ}}(\sigma^1\otimes\sigma^0\otimes\sigma^0)_i{}^k \\
& + 2s_1(\sigma^2\otimes\sigma^0\otimes\sigma^2)_{\mathrm{IJ}}(\sigma^2\otimes\sigma^2\otimes\sigma^3)_i{}^k \\
& - 2s_1(\sigma^2\otimes\sigma^2\otimes\sigma^3)_{\mathrm{IJ}}(\sigma^2\otimes\sigma^0\otimes\sigma^2)_i{}^k \\
& - 2s_1(\sigma^2\otimes\sigma^2\otimes\sigma^1)_{\mathrm{IJ}}(\sigma^2\otimes\sigma^2\otimes\sigma^1)_i{}^k,
\end{aligned} \tag{3.62}$$

and

$$\begin{aligned}
\widehat{\Delta}_{\mathrm{IJ}\hat{i}}{}^{\hat{k}} = {}& c_{2+}(\sigma^1\otimes\sigma^0\otimes\sigma^0)_{\mathrm{IJ}}(\sigma^1\otimes\sigma^0\otimes\sigma^0)_{\hat{i}}{}^{\hat{k}} \\
& + c_{2-}(\sigma^2\otimes\sigma^3\otimes\sigma^2)_{\mathrm{IJ}}(\sigma^2\otimes\sigma^3\otimes\sigma^2)_{\hat{i}}{}^{\hat{k}} \\
& - c_{2-}(\sigma^2\otimes\sigma^2\otimes\sigma^0)_{\mathrm{IJ}}(\sigma^2\otimes\sigma^2\otimes\sigma^0)_{\hat{i}}{}^{\hat{k}} \\
& + c_{2-}(\sigma^2\otimes\sigma^1\otimes\sigma^2)_{\mathrm{IJ}}(\sigma^2\otimes\sigma^1\otimes\sigma^2)_{\hat{i}}{}^{\hat{k}}.
\end{aligned} \tag{3.63}$$

The first order chromocharacters are as in eq. (3.39). For *no* values of the integers n and m can the second order chromocharacters be made to satisfy eq. (3.40). This is not a surprise as $n_c \neq n_t$ for this system. The second order chromocharacters take a complicated form, similar to that of the chiral-chiral system in eq. (3.26), though even more complicated, and less enlightening. We have not completely worked out this formula, but we reiterate that we have proved by direct calculation that it *can not* for any values of n and m take the form of eq. (3.40).

3.5 Building an $\mathcal{N}=2$ supermultiplet from tensor + tensor $\mathcal{N}=1$ supermultiplets

For the case of two 4D, $\mathcal{N}=1$ tensor multiplets, we introduce the transformation laws for this system as

$$H_{\mu\nu\alpha} \equiv \partial_\mu B_{\nu\alpha} + \partial_\alpha B_{\mu\nu} + \partial_\nu B_{\alpha\mu}\,, \qquad \tilde{H}_{\mu\nu\alpha} \equiv \partial_\mu \tilde{B}_{\nu\alpha} + \partial_\alpha \tilde{B}_{\mu\nu} + \partial_\nu \tilde{B}_{\alpha\mu}\,, \tag{3.64}$$

the Lagrangian for this multiplet is

$$\mathcal{L} = -\frac{1}{3}H_{\mu\nu\alpha}H^{\mu\nu\alpha} - \frac{1}{3}\tilde{H}_{\mu\nu\alpha}\tilde{H}^{\mu\nu\alpha} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\partial_\mu\tilde{\varphi}\partial^\mu\tilde{\varphi} + i\frac{1}{2}(\gamma^\mu)^{ab}\lambda_a^i\partial_\mu\lambda_b^i\,, \tag{3.65}$$

which is an invariant of the transformation laws

$$\mathrm{D}_a^i \varphi = b^{ij}\lambda_a^j \,,$$

$$\mathrm{D}_a^i \tilde{\varphi} = a^{ij}\lambda_a^j \,,$$

$$\mathrm{D}_a^i B_{\mu\nu} = -\frac{1}{4}\big([\gamma_\mu,\gamma_\nu]\big)_a^{\ b} b^{ij}\lambda_b^j \,,$$

$$\mathrm{D}_a^i \tilde{B}_{\mu\nu} = -\frac{1}{4}\big([\gamma_\mu,\gamma_\nu]\big)_a^{\ b} a^{ij}\lambda_b^j \,, \tag{3.66}$$

$$\mathrm{D}_a^i \lambda_b^j = a^{ij}\big(i(\gamma^\mu)_{ab}\partial_\mu\tilde{\varphi} - (\gamma^5\gamma_\mu)_{ab}\epsilon^{\mu\rho\sigma\tau}\partial_\rho\tilde{B}_{\sigma\tau}\big)$$
$$+ b^{ij}\big(i(\gamma^\mu)_{ab}\partial_\mu\varphi - (\gamma^5\gamma_\mu)_{ab}\epsilon^{\mu\rho\sigma\tau}\partial_\rho B_{\sigma\tau}\big) \,.$$

For canonical bosonic momentum in the algebra, eqs. (3.56) and (3.57) are once again forced on us and the algebra is

$$\{\mathrm{D}_a^i, \mathrm{D}_b^j\}\varphi = 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\varphi + 2ic_1(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{\varphi}$$
$$- i\frac{2}{3}s_1(\sigma^2)^{ij}\epsilon_{\mu\nu\alpha\beta}(\gamma^5\gamma^\mu)_{ab}\tilde{H}^{\nu\alpha\beta} \,,$$

$$\{\mathrm{D}_a^i, \mathrm{D}_b^j\}\tilde{\varphi} = 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\tilde{\varphi} + 2ic_1(\sigma^1)^{ij}(\gamma^\mu)_{ab}\partial_\mu\varphi$$
$$+ i\frac{2}{3}s_1(\sigma^2)^{ij}\epsilon_{\mu\nu\alpha\beta}(\gamma^5\gamma^\mu)_{ab}H^{\nu\alpha\beta} \,,$$

$$\{\mathrm{D}_a^i, \mathrm{D}_b^j\}B_{\mu\nu} = 2i\delta^{ij}(\gamma^\alpha)_{ab}H_{\alpha\mu\nu} - i\delta^{ij}(\gamma_{[\mu})_{|ab|}\partial_{\nu]}\varphi$$
$$- ic_1(\sigma^1)^{ij}(\gamma_{[\mu})_{|ab|}\partial_{\nu]}\tilde{\varphi} + is_1(\sigma^2)^{ij}\epsilon_{\mu\nu}^{\ \ \alpha\beta}(\gamma^5\gamma_\alpha)_{ab}\partial_\beta\tilde{\varphi}$$
$$+ 2ic_1(\sigma^1)^{ij}(\gamma^\alpha)_{ab}\tilde{H}_{\alpha\mu\nu} + i\frac{1}{3}s_1(\sigma^2)^{ij}(\gamma^5\gamma_{[\mu})_{|ab|}\epsilon_{\nu]\rho\alpha\beta}\tilde{H}^{\rho\alpha\beta} \,,$$

$$\{\mathrm{D}_a^i, \mathrm{D}_b^j\}\tilde{B}_{\mu\nu} = 2i\delta^{ij}(\gamma^\alpha)_{ab}\tilde{H}_{\alpha\mu\nu} - i\delta^{ij}(\gamma_{[\mu})_{|ab|}\partial_{\nu]}\tilde{\varphi}$$
$$- ic_1(\sigma^1)^{ij}(\gamma_{[\mu})_{|ab|}\partial_{\nu]}\varphi - is_1(\sigma^2)^{ij}\epsilon_{\mu\nu}^{\ \ \alpha\beta}(\gamma^5\gamma_\alpha)_{ab}\partial_\beta\varphi$$
$$+ 2ic_1(\sigma^1)^{ij}(\gamma^\alpha)_{ab}H_{\alpha\mu\nu} - i\frac{1}{3}s_1(\sigma^2)^{ij}(\gamma^5\gamma_{[\mu})_{|ab|}\epsilon_{\nu]\rho\alpha\beta}H^{\rho\alpha\beta} \,,$$

$$\{\mathrm{D}_a^i, \mathrm{D}_b^j\}\lambda_c^k = i\big(2\delta^{ij}\delta^{kl} + c_{2+}(\sigma^1)^{ij}(\sigma^1)^{kl}\big)(\gamma^\mu)_{ab}\partial_\mu\lambda_c^l$$
$$- ic_{2-}(\sigma^2)^{ij}(\sigma^2)^{kl}\big((\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c^{\ d} + C_{ab}(\gamma^\mu)_c^{\ d}$$
$$- (\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c^{\ d}\big)\partial_\mu\lambda_d^l \,. \tag{3.67}$$

Choosing the gauge

$$B_{0i} = 0 = \tilde{B}_{0i} = 0 \,, \tag{3.68}$$

defining

$$B_{0i} = 0 = \tilde{B}_{0i} = 0 \,, \tag{3.69}$$

and

$$\begin{array}{llll}
\Phi_1 = \varphi\,, & \Phi_2 = 2B_{12}\,, & \Phi_3 = 2B_{23}\,, & \Phi_4 = 2B_{31}\,, \\
\Phi_5 = \tilde{\varphi}\,, & \Phi_2 = 2\tilde{B}_{12}\,, & \Phi_3 = 2\tilde{B}_{23}\,, & \Phi_4 = 2\tilde{B}_{31}\,, \\
i\Psi_1 = \lambda_1^1\,, & i\Psi_2 = \lambda_2^1\,, & i\Psi_3 = \lambda_3^1\,, & i\Psi_4 = \lambda_4^1\,, \\
i\Psi_5 = \lambda_1^2\,, & i\Psi_6 = \lambda_2^2\,, & i\Psi_7 = \lambda_3^2\,, & i\Psi_8 = \lambda_4^2\,,
\end{array} \tag{3.70}$$

and considering only temporal dependence of the fields reduces the transformation laws to eq. (3.22) with the D_I identifications as in eq. (3.21). The adinkra matrices in this basis are given in appendix E and satisfy the algebra of eq. (3.4) with

$$
\begin{aligned}
\Delta_{\mathrm{IJ}i}{}^{k} = {} & 2c_1(\sigma^1 \otimes \sigma^0 \otimes \sigma^0)_{\mathrm{IJ}}(\sigma^1 \otimes \sigma^0 \otimes \sigma^0)_i{}^{k} \\
& - 2s_1(\sigma^2 \otimes \sigma^0 \otimes \sigma^2)_{\mathrm{IJ}}(\sigma^2 \otimes \sigma^2 \otimes \sigma^1)_i{}^{k} \\
& - 2s_1(\sigma^2 \otimes \sigma^2 \otimes \sigma^3)_{\mathrm{IJ}}(\sigma^2 \otimes \sigma^0 \otimes \sigma^2)_i{}^{k} \\
& - 2s_1(\sigma^2 \otimes \sigma^2 \otimes \sigma^1)_{\mathrm{IJ}}(\sigma^2 \otimes \sigma^2 \otimes \sigma^3)_i{}^{k},
\end{aligned}
\tag{3.71}
$$

and

$$
\begin{aligned}
\widehat{\Delta}_{\mathrm{IJ}\hat{\imath}}{}^{\hat{k}} = {} & c_{2+}(\sigma^1 \otimes \sigma^0 \otimes \sigma^0)_{\mathrm{IJ}}(\sigma^1 \otimes \sigma^0 \otimes \sigma^0)_{\hat{\imath}}{}^{\hat{k}} \\
& - c_{2-}(\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{\mathrm{IJ}}(\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{\hat{\imath}}{}^{\hat{k}} \\
& - c_{2-}(\sigma^2 \otimes \sigma^2 \otimes \sigma^0)_{\mathrm{IJ}}(\sigma^2 \otimes \sigma^2 \otimes \sigma^0)_{\hat{\imath}}{}^{\hat{k}} \\
& - c_{2-}(\sigma^2 \otimes \sigma^1 \otimes \sigma^2)_{\mathrm{IJ}}(\sigma^2 \otimes \sigma^1 \otimes \sigma^2)_{\hat{\imath}}{}^{\hat{k}}.
\end{aligned}
\tag{3.72}
$$

The chromocharacters are *exactly the same* as for the $\mathcal{N} = 2$ vector + vector multiplet. The most crucial result of these calculations is that for the tensor + tensor multiplet the second order chromocharacters *can not* take the form of eq. (3.40) for any values of the integers n and m.

3.6 Building an $\mathcal{N} = 2$ supermultiplet from vector + tensor $\mathcal{N} = 1$ supermultiplets

This supermultiplet has been discussed since the pioneering work of [26, 27]. The Lagrangian for this multiplet is

$$
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{3}H_{\mu\nu\alpha}H^{\mu\nu\alpha} - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + i\frac{1}{2}(\gamma^\mu)^{ab}\lambda_a^i\partial_\mu\lambda_b^i + \frac{1}{2}\mathrm{d}^2,
\tag{3.73}
$$

which is an invariant of the transformation laws

$$
\begin{aligned}
\mathrm{D}_a^i\varphi &= b^{ij}\lambda_a^j, \\
\mathrm{D}_a^iA_\mu &= (\gamma_\mu)_a{}^b a^{ij}\lambda_b^j, \\
\mathrm{D}_a^iB_{\mu\nu} &= -\frac{1}{4}\big([\gamma_\mu, \gamma_\nu]\big)_a{}^b b^{ij}\lambda_b^j, \\
\mathrm{D}_a^i\lambda_b^j &= a^{ij}\left(-\frac{i}{4}\big([\gamma^\mu, \gamma^\nu]\big)_{ab}F_{\mu\nu} + (\gamma^5)_{ab}\mathrm{d}\right) \\
& \quad + b^{ij}\big(i(\gamma^\mu)_{ab}\partial_\mu\varphi - (\gamma^5\gamma_\mu)_{ab}\epsilon^{\mu\rho\sigma\tau}\partial_\rho B_{\sigma\tau}\big), \\
\mathrm{D}_a^i\mathrm{d} &= i(\gamma^5\gamma^\mu)_a{}^b a^{ij}\partial_\mu\lambda_b^j.
\end{aligned}
\tag{3.74}
$$

For the canonical bosonic momentum term to appear in the algebra, eqs. (3.56) and (3.57) are once again forced on us and the algebra is

$$
\{D_a^i, D_b^j\}\varphi = 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu\varphi + 2is_1(\sigma^2)^{ij}(\gamma^5)_{ab}\partial_\mu d
$$
$$
\quad - ic_1(\sigma^1)^{ij}(\gamma^\mu\gamma^\nu)_{ab}F_{\mu\nu}\,,
$$

$$
\{D_a^i, D_b^j\}A_\nu = 2i\delta^{ij}(\gamma^\mu)_{ab}F_{\mu\nu} + 2ic_1(\sigma^1)^{ij}(\gamma^\alpha\gamma^\mu)_{ab}H_{\alpha\mu\nu}
$$
$$
\quad - i\frac{2}{3}s_1(\sigma^2)^{ij}(\gamma^5)_{ab}\epsilon_{\nu\alpha\beta\mu}H^{\alpha\beta\mu} + ic_1(\sigma^1)^{ij}\left([\gamma_\nu,\gamma^\mu]\right)_{ab}\partial_\mu\varphi
$$
$$
\quad - 2s_1C_{ab}(\sigma^2)^{ij}\partial_\nu\varphi\,,
$$

$$
\{D_a^i, D_b^j\}B_{\mu\nu} = 2i\delta^{ij}(\gamma^\alpha)_{ab}H_{\alpha\mu\nu} - s_1(\sigma^2)^{ij}C_{ab}F_{\mu\nu}
$$
$$
\quad + i\frac{1}{2}s_1(\sigma^2)^{ij}(\gamma^5)_{ab}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} + i\frac{1}{2}c_1(\sigma^1)^{ij}\left([\gamma_{[\mu},\gamma^\alpha]\right)_{|ab|}F_{\nu]\alpha}
$$
$$
\quad + \frac{1}{2}c_1(\sigma^1)^{ij}\left(\gamma^5[\gamma_\mu,\gamma_\nu]\right)_{ab}d - i\delta^{ij}(\gamma_{[\mu})_{|ab|}\partial_{\nu]}\varphi\,,
$$

$$
\{D_a^i, D_b^j\}\lambda_c^k = i\left(2\delta^{ij}\delta^{kl} + c_{2+}(\sigma^1)^{ij}(\sigma^1)^{kl}\right)(\gamma^\mu)_{ab}\partial_\mu\lambda_c^l
$$
$$
\quad + ic_{2+}(\sigma^2)^{ij}(\sigma^2)^{kl}\left((\gamma^5\gamma^\mu)_{ab}(\gamma^5)_c{}^d + C_{ab}(\gamma^\mu)_c{}^d\right)\partial_\mu\lambda_d^l
$$
$$
\quad + ic_{2-}(\sigma^2)^{ij}(\sigma^2)^{kl}(\gamma^5)_{ab}(\gamma^5\gamma^\mu)_c{}^d\partial_\mu\lambda_d^l\,,
$$

$$
\{D_a^i, D_b^j\}d = 2i\delta^{ij}(\gamma^\mu)_{ab}\partial_\mu d - 2c_1(\sigma^1)^{ij}(\gamma^5\gamma^\mu\gamma^\nu)_{ab}\partial^\alpha H_{\alpha\mu\nu}
$$
$$
\quad - 2is_1(\sigma^2)^{ij}(\gamma^5)_{ab}\partial_\mu\partial^\mu\varphi\,. \tag{3.75}
$$

Choosing the gauge

$$
B_{0i} = A_0 = 0\,, \tag{3.76}
$$

defining

$$
\begin{array}{llll}
\Phi_1 = \varphi\,, & \Phi_2 = 2B_{12}\,, & \Phi_3 = 2B_{23}\,, & \Phi_4 = 2B_{31}\,, \\[4pt]
\Phi_5 = A_1\,, & \Phi_2 = A_2\,, & \Phi_3 = A_3\,, & \Phi_4 = \int dt\, d\,, \\[4pt]
i\Psi_1 = \lambda_1^1\,, & i\Psi_2 = \lambda_2^1\,, & i\Psi_3 = \lambda_3^1\,, & i\Psi_4 = \lambda_4^1\,, \\[4pt]
i\Psi_5 = \lambda_1^2\,, & i\Psi_6 = \lambda_2^2\,, & i\Psi_7 = \lambda_3^2\,, & i\Psi_8 = \lambda_4^2\,, \tag{3.77}
\end{array}
$$

and considering only temporal dependence of the fields reduces the transformation laws to eq. (3.22) with the D_I identifications as in eq. (3.21). The adinkra matrices in this basis are given in appendix F. They satisfy the orthogonality relationship, eq. (3.23), and the algebra of eq. (3.4) with $\Delta_{IJi}{}^k$ and $\widehat{\Delta}_{IJ\hat{i}}{}^{\hat{k}}$ given by

$$
\Delta_{IJi}{}^k = 2c_1(\sigma^1\otimes\sigma^0\otimes\sigma^3)_{IJ}(\Delta_1^{(VT)})_i{}^k
$$
$$
\quad - 2c_1(\sigma^1\otimes\sigma^0\otimes\sigma^1)_{IJ}(\Delta_2^{(VT)})_i{}^k
$$
$$
\quad - 2c_1(\sigma^1\otimes\sigma^2\otimes\sigma^2)_{IJ}(\Delta_3^{(VT)})_i{}^k
$$
$$
\quad - 2s_1(\sigma^2\otimes\sigma^2\otimes\sigma^0)_{IJ}(\Delta_4^{(VT)})_i{}^k\,, \tag{3.78}
$$

where the quantities $\Delta_1^{(VT)}$, etc. are defined by

$$
\Delta_1^{(VT)} = \begin{bmatrix} 0 & (8)_b(1324) \\ (1)_b(1423) & 0 \end{bmatrix},
$$
$$
\Delta_2^{(VT)} = \begin{bmatrix} 0 & (13)_b(34) \\ (13)_b(34) & 0 \end{bmatrix},
$$
$$
\Delta_3^{(VT)} = \begin{bmatrix} 0 & (11)_b(12) \\ (11)_b(12) & 0 \end{bmatrix},
$$
$$
\Delta_4^{(VT)} = \begin{bmatrix} 0 & (1)_b(1423) \\ (8)_b(1324) & 0 \end{bmatrix},
$$

(3.79)

where we have used the Boolean Factor notation of [19] to indicate locations of minus signs for permutation matrices defined as, for instance:

$$
(8)_b(1324) \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \qquad (11)_b(12) \equiv \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
$$

(3.80)

We also have

$$
\begin{aligned}
\widehat{\Delta}_{\mathrm{IJ}\hat{i}}{}^{\hat{k}} = {} & c_{2+}(\sigma^1 \otimes \sigma^0 \otimes \sigma^0)_{\mathrm{IJ}}(\sigma^1 \otimes \sigma^0 \otimes \sigma^0)_{\hat{i}}{}^{\hat{k}} \\
& + c_{2+}(\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{\mathrm{IJ}}(\sigma^2 \otimes \sigma^3 \otimes \sigma^2)_{\hat{i}}{}^{\hat{k}} \\
& + c_{2+}(\sigma^2 \otimes \sigma^1 \otimes \sigma^2)_{\mathrm{IJ}}(\sigma^2 \otimes \sigma^1 \otimes \sigma^2)_{\hat{i}}{}^{\hat{k}} \\
& - c_{2-}(\sigma^2 \otimes \sigma^2 \otimes \sigma^0)_{\mathrm{IJ}}(\sigma^2 \otimes \sigma^2 \otimes \sigma^0)_{\hat{i}}{}^{\hat{k}}.
\end{aligned}
$$

(3.81)

The first order chromocharacters satisfy eq. (3.39). The second order chromocharacters *can not* take the form of eq. (3.40) for any values of the integers n and m. In summary, none of the $\mathcal{N}=2$ supermultiplets from the list of chiral + chiral, vector + vector, and tensor + tensor, nor vector+tensor have second order chromocharacters as given in eq. (3.40).

3.7 Summary of building $\mathcal{N}=2$ multiplets from $\mathcal{N}=1$ multiplets

To recapitulate the results of this section, we have seen by explicit construction that there are six possible pairings of 4D, $\mathcal{N} = 1$ chiral, vector, and tensor multiplets that may be taken as starting points in an attempt to construct 4D, $\mathcal{N} = 2$ supermultiplets that are:

(a.) completely off-shell (i.e. require no a priori differential constraints imposed on any fields), and

(b.) require no off-shell central charges.

However, the result of this study is that only two combinations:

(a.) chiral + vector, and

(b.) chiral + tensor,

satisfy the required conditions stated immediately above.

The following two questions seem important to ask. "Why do the results work out in this way?" "What is it that distinguishes two of the six possible starting points from the others?" Simply reporting these results does nothing to reveal what deeper mathematical structures impose these results.

If we use only the conventional and traditional approaches to analyzing these results, there is no simple and elegant way (at least known to these authors) to answer these questions. The situation is vaguely analogous to looking at the quark model and asking, "Which composite systems of quarks occur as an observable baryons?" The answer is well known, "All observable baryonic composite states must have vanishing color quantum number." In the next section, we will argue that the adinkra-based model of off-shell SUSY representations provides a remarkably elegant and simple answer to the questions above and does so in a manner similar to the confinement of QCD color.

4 Adinkra 'color-like' confinement rules for 4D, $\mathcal{N} = 1$ reps within off-shell $\mathcal{N} = 2$ supermultiplets

The survey of the attempts of building off-shell $\mathcal{N} = 2$ supermultiplets from off-shell $\mathcal{N} = 1$ supermultiplets shows there appears to be a 'super-selection-like rule' that governs the $\mathcal{N} = 1$ content of the $\mathcal{N} = 2$ extended supermultiplets.

In the work of [18], it was argued that the chromocharacters associated 4D, $\mathcal{N} = 1$ supermultiplets generally take the form

$$
\begin{aligned}
\varphi^{(2)}{}_{\text{IJKL}}(\mathcal{N} = 1) &= 4\,(n_c + n_t)\big[\delta_{\text{IJ}}\delta_{\text{KL}} - \delta_{\text{IK}}\delta_{\text{JL}} + \delta_{\text{IL}}\delta_{\text{JK}}\big] + 4\,\chi_{\text{o}}\,\epsilon_{\text{IJKL}}\,,\\
\chi_{\text{o}} &= (n_c - n_t)\,,
\end{aligned}
\tag{4.1}
$$

and we notice the appearance of the Levi-Civita tensor is allowed because 4D, $\mathcal{N} = 1$ supersymmetric multiplets have only four supercharges and thus require only $O(4)$ symmetry of their chromocharacters. It follows that the quantity χ_{o} ('Kye-Oh') can be abstracted from

$$
\chi_{\text{o}} = \frac{1}{4 \cdot 4!}\,\epsilon^{\text{IJKL}}\,\varphi^{(2)}{}_{\text{IJKL}}(\mathcal{N} = 1)\,.
\tag{4.2}
$$

In every system investigated in section three, if one begins with off-shell $\mathcal{N} = 1$ supermultiplets and uses these as a basis for constructing off-shell $\mathcal{N} = 2$ supermultiplets, the latter will not be off-shell and free of central charges unless $\Sigma(n_c - n_t) = 0$, where the sum is taken over the $\mathcal{N} = 1$ supermultiplets.

This observation is a very explicit demonstration of the utility of the adinkra-based view that has been developed in a number of our past works. Taking the adinkra approach [1]–[13], one is naturally led to the existence of n_c, and n_t. Below we will give a simple explanation on why this super-selection-like rule must appear in all supermultiplets that arise in the adinkra approach. In the process, we will show that the adinkra-based approach thus leads to a new and effective tool, which is obscured in more conventional approaches, for understanding fundamental aspects of SUSY representation theory in four dimensions.

Let us consider the chromocharacter in (3.24) for the case of $p = 2$. All our previous works suggests that the chromocharacters possess $SO(N)$ symmetry, i.e. $SO(8)$ symmetry for our considerations. This is a very powerful assertion and we will now argue that it is the cause of the proposed super-selection-like rule. Due to $SO(8)$ symmetry, the form of the second order chromocharacter in this case must be

$$\varphi^{(2)}{}_{\mathrm{I}_1\,\mathrm{J}_1\,\mathrm{I}_2\,\mathrm{J}_2}(\mathcal{N}=2) \propto \left[\delta_{\mathrm{I}_1\,\mathrm{J}_1}\delta_{\mathrm{I}_2\,\mathrm{J}_2} - \delta_{\mathrm{I}_1\,\mathrm{I}_2}\delta_{\mathrm{J}_1\,\mathrm{J}_2} + \delta_{\mathrm{I}_1\,\mathrm{J}_2}\delta_{\mathrm{I}_2\,\mathrm{J}_1} \right] , \tag{4.3}$$

where we have used the properties of the $(\mathrm{L_I})$-matrices to arrive at this conclusion. This must be true for the chromocharacters associated with supermultiplets that possess 4D, $\mathcal{N} = 2$ supersymmetry by our SUSY holography conjecture.

For the $\mathcal{N} = 2$ chromocharacter, the indices I_1, I_2, J_1, and J_2 take on values $1, \ldots, 8$ while for the $\mathcal{N} = 1$ chromocharacter, the indices I, J, K, and L take on values $1, \ldots, 4$. So in order to compare these two chromocharacter formulae, one must perform a projection of $SO(8)$ down to $SO(4)$. However, such a projection will *never* create a term proportional to the Levi-Civita tensor.

Thus, these considerations lead to the conclusion that for all 4D, $\mathcal{N} = 1$ supermultiplets that occur as sub-supermultiplets within off-shell 4D, $\mathcal{N} = 2$ supermultiplets, the value of χ_o when summed over the $\mathcal{N} = 1$ sub-supermultiplets must vanish. It is very satisfying to see that this formal argument is in agreement with the explicit calculations performed in the previous sections.

5 Seeing 'Kye-Oh' in 4D, $\mathcal{N}=1$ supermultiplets without 0-brane reduction

In this section, we will make an observation about the determination of χ_o that shows its value on these three supermultiplets can be found *without* actually carrying out 0-brane reduction. We find this is an interesting result as it will show that χ_o can be directly determined by a calculation in four dimensions.

In this section, we are going to use the conventions of *Superspace* where two-component Weyl spinors have been our tradition. To facilitate this, we first establish a dictionary between the conventions of [18] and *Superspace* [39]. Using the former we have

$$
\begin{aligned}
(\gamma^0)_a{}^b &= i(\sigma^3 \otimes \sigma^2)_a{}^b , & (\gamma^1)_a{}^b &= (\mathbf{I}_2 \otimes \sigma^1)_a{}^b , \\
(\gamma^2)_a{}^b &= (\sigma^2 \otimes \sigma^2)_a{}^b , & (\gamma^3)_a{}^b &= (\mathbf{I}_2 \otimes \sigma^3)_a{}^b , \\
(\gamma^5)_a{}^b &= -(\sigma^1 \otimes \sigma^2)_a{}^b ,
\end{aligned}
$$

$$
C_{ab} \equiv -i(\sigma^3 \otimes \sigma^2)_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \rightarrow C_{ab} = -C_{ba} . \tag{5.1}
$$

The inverse spinor metric is defined by the condition $C^{ab}C_{ac} = \delta_c{}^b$.

The chiral projection operators $(\mathbf{P}_\pm)$ are defined by

$$(\mathbf{P}_\pm)_a{}^b = \frac{1}{2}\left[(\mathbf{I}_4)_a{}^b \pm (\gamma^5)_a{}^b \right] , \tag{5.2}$$

which implies

$$(\mathbf{P}_+)_a{}^b \mathrm{D}_b = \frac{1}{2} \begin{bmatrix} (\mathrm{D}_1 + i\,\mathrm{D}_4) \\ (\mathrm{D}_2 - i\,\mathrm{D}_3) \\ (\mathrm{D}_3 + i\,\mathrm{D}_2) \\ (\mathrm{D}_4 - i\,\mathrm{D}_1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (\mathrm{D}_1 + i\,\mathrm{D}_4) \\ (\mathrm{D}_2 - i\,\mathrm{D}_3) \\ i\,(\mathrm{D}_2 - i\,\mathrm{D}_3) \\ -i\,(\mathrm{D}_1 + i\,\mathrm{D}_4) \end{bmatrix}, \tag{5.3}$$

$$(\mathbf{P}_-)_a{}^b \mathrm{D}_b = \frac{1}{2} \begin{bmatrix} (\mathrm{D}_1 - i\,\mathrm{D}_4) \\ (\mathrm{D}_2 + i\,\mathrm{D}_3) \\ (\mathrm{D}_3 - i\,\mathrm{D}_2) \\ (\mathrm{D}_4 + i\,\mathrm{D}_1) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (\mathrm{D}_1 - i\,\mathrm{D}_4) \\ (\mathrm{D}_2 + i\,\mathrm{D}_3) \\ -i\,(\mathrm{D}_2 + i\,\mathrm{D}_3) \\ i\,(\mathrm{D}_1 - i\,\mathrm{D}_4) \end{bmatrix}. \tag{5.4}$$

Let us define

$$D_A = \frac{1}{\sqrt{2}}\,(\mathrm{D}_1 + i\,\mathrm{D}_4), \qquad D_B = \frac{1}{\sqrt{2}}\,(\mathrm{D}_2 - i\,\mathrm{D}_3),$$

$$\overline{D}_{\dot{A}} = -\frac{1}{\sqrt{2}}\,(\mathrm{D}_1 - i\,\mathrm{D}_4), \qquad \overline{D}_{\dot{B}} = -\frac{1}{\sqrt{2}}\,(\mathrm{D}_2 + i\,\mathrm{D}_3), \tag{5.5}$$

here the subscripts A, B, $\dot{A}$ and $\dot{B}$ are understood to be labels, not indices taking on multiple values. We next derive the form of the super algebra generated by these four spinorial derivative operators. We find

$$\begin{aligned} \{D_A,\, D_A\} &= \frac{1}{2}\left[\{\mathrm{D}_1,\, \mathrm{D}_1\} - \{\mathrm{D}_4,\, \mathrm{D}_4\} + i2\,\{\mathrm{D}_1,\, \mathrm{D}_4\} \right] \\ &= i\left[(\gamma^\mu)_{1\,1}\partial_\mu - (\gamma^\mu)_{4\,4}\partial_\mu \right] - 2\,(\gamma^\mu)_{1\,4}\partial_\mu, \end{aligned}$$

$$\begin{aligned} \{D_A,\, D_B\} &= \frac{1}{2}\left[\{\mathrm{D}_1,\, \mathrm{D}_2\} + \{\mathrm{D}_3,\, \mathrm{D}_4\} + i\,\{\mathrm{D}_2,\, \mathrm{D}_4\} - i\,\{\mathrm{D}_1,\, \mathrm{D}_3\} \right] \\ &= i\left[(\gamma^\mu)_{1\,2}\partial_\mu + (\gamma^\mu)_{3\,4}\partial_\mu \right] - (\gamma^\mu)_{2\,4}\partial_\mu + (\gamma^\mu)_{1\,3}\partial_\mu, \end{aligned}$$

$$\begin{aligned} \{D_B,\, D_B\} &= \frac{1}{2}\left[\{\mathrm{D}_2,\, \mathrm{D}_2\} - \{\mathrm{D}_3,\, \mathrm{D}_3\} - i2\,\{\mathrm{D}_2,\, \mathrm{D}_3\} \right] \\ &= i\left[(\gamma^\mu)_{2\,2}\partial_\mu - (\gamma^\mu)_{3\,3}\partial_\mu \right] + 2\,(\gamma^\mu)_{2\,3}\partial_\mu, \end{aligned}$$

$$\begin{aligned} \{D_A,\, \overline{D}_{\dot{B}}\} &= -\frac{1}{2}\left[\{\mathrm{D}_1,\, \mathrm{D}_2\} - \{\mathrm{D}_3,\, \mathrm{D}_4\} + i\,\{\mathrm{D}_2,\, \mathrm{D}_4\} + i\,\{\mathrm{D}_1,\, \mathrm{D}_3\} \right] \\ &= -i\left[(\gamma^\mu)_{1\,2}\partial_\mu - (\gamma^\mu)_{3\,4}\partial_\mu \right] - (\gamma^\mu)_{2\,4}\partial_\mu - (\gamma^\mu)_{1\,3}\partial_\mu, \end{aligned}$$

$$\begin{aligned} \{D_A,\, \overline{D}_{\dot{A}}\} &= -\frac{1}{2}\left[\{\mathrm{D}_1,\, \mathrm{D}_1\} + \{\mathrm{D}_4,\, \mathrm{D}_4\} \right] \\ &= -i\left[(\gamma^\mu)_{1\,1}\partial_\mu + (\gamma^\mu)_{4\,4}\partial_\mu \right], \end{aligned}$$

$$\begin{aligned} \{D_B,\, \overline{D}_{\dot{B}}\} &= -\frac{1}{2}\left[\{\mathrm{D}_2,\, \mathrm{D}_2\} + \{\mathrm{D}_3,\, \mathrm{D}_3\} \right] \\ &= -i\left[(\gamma^\mu)_{2\,2}\partial_\mu + (\gamma^\mu)_{3\,3}\partial_\mu \right]. \end{aligned} \tag{5.6}$$

It is a straightforward exercise to show that given the representation for the gamma matrices we use further imply

$$(\gamma^0)_{ab} = (\mathbf{I}_2 \otimes \mathbf{I}_2)_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (\gamma^1)_{ab} = (\sigma^3 \otimes \sigma^3)_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(\gamma^2)_{ab} = (\sigma^1 \otimes \mathbf{I}_2)_{ab} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad (\gamma^3)_{ab} = (\sigma^3 \otimes \sigma^1)_{ab} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{5.7}$$

Use of these matrices then leads to the results

$$\begin{aligned}
\{D_A, \, D_A\} &= i\left[(\gamma^\mu)_{1\,1}\partial_\mu - (\gamma^\mu)_{4\,4}\partial_\mu\right] - 2\,(\gamma^\mu)_{1\,4}\partial_\mu \\
&= i\left[(\gamma^0)_{1\,1}\partial_0 + (\gamma^1)_{1\,1}\partial_1 - (\gamma^0)_{4\,4}\partial_0 - (\gamma^1)_{4\,4}\partial_1\right] \\
&= 0 \,,
\end{aligned}$$

$$\begin{aligned}
\{D_A, \, D_B\} &= i\left[(\gamma^\mu)_{1\,2}\partial_\mu + (\gamma^\mu)_{3\,4}\partial_\mu\right] - (\gamma^\mu)_{2\,4}\partial_\mu + (\gamma^\mu)_{1\,3}\partial_\mu \\
&= i\left[(\gamma^3)_{1\,2}\partial_3 + (\gamma^3)_{3\,4}\partial_3\right] - (\gamma^2)_{2\,4}\partial_2 + (\gamma^2)_{1\,3}\partial_2 \\
&= 0 \,,
\end{aligned}$$

$$\begin{aligned}
\{D_B, \, D_B\} &= i\left[(\gamma^\mu)_{2\,2}\partial_\mu - (\gamma^\mu)_{3\,3}\partial_\mu\right] + 2\,(\gamma^\mu)_{2\,3}\partial_\mu \\
&= i\left[(\gamma^0)_{2\,2}\partial_0 + (\gamma^1)_{2\,2}\partial_1 - (\gamma^0)_{3\,3}\partial_0 - (\gamma^1)_{3\,3}\partial_1\right] \\
&= 0 \,,
\end{aligned}$$

$$\begin{aligned}
\{D_A, \, \overline{D}_{\dot{B}}\} &= -i\left[(\gamma^3)_{1\,2}\partial_3 - (\gamma^3)_{3\,4}\partial_3\right] - (\gamma^2)_{2\,4}\partial_2 - (\gamma^2)_{1\,3}\partial_2 \\
&= -i\,2\left[\partial_3 - i\,\partial_2\right],
\end{aligned}$$

$$\begin{aligned}
\{D_A, \, \overline{D}_{\dot{A}}\} &= -i\left[(\gamma^\mu)_{1\,1}\partial_\mu + (\gamma^\mu)_{4\,4}\partial_\mu\right] \\
&= -i\left[(\gamma^0)_{1\,1}\partial_0 + (\gamma^1)_{1\,1}\partial_1 + (\gamma^0)_{4\,4}\partial_0 + (\gamma^1)_{4\,4}\partial_1\right] \\
&= -i\,2\left[\partial_0 + \partial_1\right],
\end{aligned}$$

$$\begin{aligned}
\{D_B, \, \overline{D}_{\dot{B}}\} &= -i\left[(\gamma^\mu)_{2\,2}\partial_\mu + (\gamma^\mu)_{3\,3}\partial_\mu\right] \\
&= -i\left[(\gamma^0)_{2\,2}\partial_0 + (\gamma^1)_{2\,2}\partial_1 + (\gamma^0)_{3\,3}\partial_0 + (\gamma^1)_{3\,3}\partial_1\right] \\
&= -i\,2\left[\partial_0 - \partial_1\right]. \tag{5.8}
\end{aligned}$$

If we define

$$\partial_{A\dot{A}} = -2\left[\partial_0 + \partial_1\right], \qquad \partial_{A\dot{B}} = -2\left[\partial_3 - i\,\partial_2\right], \qquad \partial_{B\dot{B}} = -2\left[\partial_0 - \partial_1\right], \tag{5.9}$$

then the operators D_A, D_B, $\overline{D}_{\dot{A}}$, $\overline{D}_{\dot{B}}$, $\partial_{A\dot{A}}$, $\partial_{A\dot{B}}$, and $\partial_{B\dot{B}}$ satisfy the exact algebraic and hermiticity properties of the corresponding objects defined in "*Superspace*", and we thus have an explicit dictionary.

We define the 2×2 matrix $\partial_{\alpha\dot\alpha}$ as

$$\partial_{\alpha\dot\alpha} = \begin{pmatrix} \partial_{A\dot A} & \partial_{B\dot B} \\ \partial_{B\dot A} & \partial_{B\dot B} \end{pmatrix}, \tag{5.10}$$

where $\partial_{B\dot A}$ is the complex conjugate of $\partial_{A\dot B}$, i.e.

$$\partial_{B\dot A} \equiv \overline{\partial_{B\dot A}} = -2 \left[\partial_3 + i\,\partial_2 \right]. \tag{5.11}$$

We have then explicitly

$$\partial_{\alpha\dot\alpha} = -2 \begin{pmatrix} \partial_0 + \partial_1 & \partial_3 - i\partial_2 \\ \partial_3 + i\partial_2 & \partial_0 - \partial_1 \end{pmatrix}. \tag{5.12}$$

Defining the soldering forms as

$$\tilde\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\tilde\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tilde\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.13}$$

we may neatly package $\partial_{\alpha\dot\alpha}$ as

$$\partial_{\alpha\dot\alpha} = -2\tilde\sigma^\mu \partial_\mu. \tag{5.14}$$

Finally, the two-component Weyl spinor operators denoted by D_α and $\overline{D}_{\dot\alpha}$ in *Superspace* [39] are given by

$$D_\alpha = \begin{bmatrix} D_A \\ D_B \end{bmatrix}, \qquad \overline{D}_{\dot\alpha} = \begin{bmatrix} \overline{D}_{\dot A} \\ \overline{D}_{\dot B} \end{bmatrix}. \tag{5.15}$$

With this completed dictionary, we note that in the four dimensional $\mathcal{N} = 1$ conventions of *Superspace*, one can define a quantum number $\widehat{\chi}_{\rm o}$ that appears in the following definition

$$\widehat{\chi}_{\rm o}\,\Box = -\left[2\,D^\alpha \overline{D}^2 D_\alpha + \Box \right], \tag{5.16}$$

and we wish to calculate the value of this quantum number on the spinor component fields that appear in the chiral, vector and tensor superfields. It is appropriate here to note that in principle and on a general superfield there may be no value $\widehat{\chi}_{\rm o}$ for which this equation possesses a solution. However, whenever a superfield is subject to a sufficient number of spinorial differential constraints, this is not a concern. In particular, for superfields that represent *irreducible* supermultiplets, such constraints are enforced. This is most certainly the case for the chiral (Φ), vector (W_α) and tensor supermultiplets (G).

We recall that these superfields can be described in the following manner by use of the respective pre-potentials U, V, and Υ_α:

(a.) $\Phi = \overline{D}^2\,U$, where $U \neq \overline{U}$,

(b.) $W_\alpha = i\,\overline{D}^2\,D_\alpha V$ where $V = \overline{V}$, and

(c.) $G = D^\alpha\,\overline{D}^2\,\Upsilon_\alpha + \text{h.c.}$

which will permit a rapid determination of the value of $\widehat{\chi}_{\text{o}}$ on the spinor component in each supermultiplet. These spinor components are given respectively by

$$\psi_\alpha \equiv D_\alpha\,\Phi\,\big|\,, \qquad \lambda_\alpha \equiv W_\alpha\,\big|\,, \qquad \chi_\alpha \equiv D_\alpha\,G\,\big|\,, \tag{5.17}$$

which leads us to three calculations:

$$
\begin{aligned}
\widehat{\chi}_{\text{o}}\,\square\big(D_\beta\,\Phi\,\big|\big) &= -\big[2\,D^\alpha\overline{D}^2 D_\alpha + \square\big]\big(D_\beta\,\Phi\,\big|\big) \\
&= \big[-2\,D^\alpha\overline{D}^2 D_\alpha\big(D_\beta\,\Phi\,\big|\big) - \square\,\big(D_\beta\,\Phi\,\big|\big)\big] \\
&= \big[2\,D_\beta\overline{D}^2\big(D^2\,\Phi\,\big|\big) - \square\,\big(D_\beta\,\Phi\,\big|\big)\big] \\
&= \square\,\big(D_\beta\,\Phi\,\big|\big) &&\rightarrow\ \widehat{\chi}_{\text{o}} = +1\,,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\chi}_{\text{o}}\,\square\big(W_\beta\,\big|\big) &= -\big[2\,D^\alpha\overline{D}^2 D_\alpha + \square\big]\big(W_\beta\,\big|\big) \\
&= \big[-2\,D^\alpha\overline{D}^2 D_\alpha\big(W_\beta\,\big|\big) - \square\,\big(W_\beta\,\big|\big)\big] \\
&= \big[-i2\,\big(D^\alpha\overline{D}^2 D_\alpha\,\overline{D}^2\,D_\beta\,V\big) - \square\,\big(W_\beta\,\big|\big)\big] \\
&= -\square\,\big(W_\beta\,\big|\big) &&\rightarrow\ \widehat{\chi}_{\text{o}} = -1\,,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\chi}_{\text{o}}\,\square\big(D_\beta\,G\,\big|\big) &= -\big[2\,D^\alpha\overline{D}^2 D_\alpha + \square\big]\big(D_\beta\,G\,\big|\big) \\
&= \big[-2\,D^\alpha\overline{D}^2 D_\alpha\big(D_\beta\,G\,\big|\big) - \square\,\big(D_\beta\,G\,\big|\big)\big] \\
&= \big[-2\,D^\alpha\overline{D}^2 D_\alpha\,D_\beta\big(D^\gamma\,\overline{D}^2\,\Upsilon_\gamma + \text{h.c.}\,\big|\big) - \square\,\big(D_\beta\,G\,\big|\big)\big] \\
&= \big[2\,D_\beta\overline{D}^2 D^2\big(D^\gamma\,\overline{D}^2\,\Upsilon_\gamma + \text{h.c.}\,\big|\big) - \square\,\big(D_\beta\,G\,\big|\big)\big] \\
&= -\square\,\big(D_\beta\,G\,\big|\big) &&\rightarrow\ \widehat{\chi}_{\text{o}} = -1\,, \tag{5.18}
\end{aligned}
$$

where respectively we have used the identities,

$$\overline{D}^2 D^2\,\Phi = \square\Phi\,, \qquad D^\alpha\overline{D}^2 D_\alpha\,\overline{D}^2 = 0\,, \qquad D^2 D^\gamma = 0\,. \tag{5.19}$$

These calculations beautifully demonstrate the result that $\chi_{\text{o}} = \widehat{\chi}_{\text{o}}$ on the three respective valise adinkras on one side of the calculation and the three respective supermultiplets on the other. In other words, this is another example of SUSY holography at work.

The result of this section shows that the valise adinkra-based calculation (4.2) leads to the same result as the 4D, $\mathcal{N} = 1$ superfield calculation of the operator defined in (5.16). In other words, the information in the operator in (5.16) is the same as the information in (4.2). Thus, for some operators acting on 4D, $\mathcal{N} = 1$ superfields, equivalent operators can be found to act on valise adinkras. This opens up the possibility that there may be other such operators for which this statement holds.

However, the real power of the valise adinkra viewpoint in these examples has been to easily identify the quantum number $\widehat{\chi}_{\text{o}}$ that exists among 4D, $\mathcal{N} = 1$ superfields that determines when these form an off-shell representation and to explain 'why' $\widehat{\chi}_{\text{o}}$ must a priori vanish when summed over 4D, $\mathcal{N} = 1$ superfields to construct 4D, $\mathcal{N} = 2$ superfields.

6 A Garden Algebra/unconstrained superspace prepotential formulation no-go conjecture

The results of section five also provide the basis for making a conjecture about the relation of representations of $\mathcal{GR}(\text{d},\,N)$, representations of $\mathcal{GR}(\text{d}_L,\,\text{d}_R,\,N)$, and unconstrained

prepotential formulations of higher dimensional supermultiplets. As can be seen from the models studied earlier in this paper, whenever a supermultiplet is a representation of the $\mathcal{GR}(\mathrm{d}, N)$ algebras, it *does not* possess an off-shell central charge. Alternately, whenever a supermultiplet is a representation of the $\mathcal{GR}(\mathrm{d}_L, \mathrm{d}_R, N)$ algebras, it *does* possess an off-shell central charge.

There is another observation about supermultiplets that is interesting to note in the context of unconstrained Salam-Strathdee superfields. An unconstrained Salam-Strathdee superfield is one that is not subject to any type of differential (either spacetime nor D-operator) constraint. All superfields that are quantizable, can be expressed in terms of unconstrained Salam-Strathdee superfields.

There is a direct relation between the component field formulation of a supermultiplet that does not possess off-shell central charges and their expression in terms of unconstrained Salam-Strathdee superfields. The component fields of a supermultiplet come in different engineering dimensions. In the adinkra representation, this assignment of engineering dimension corresponds to the height at which a node associated with a component field appears in the adinkra.

When one identifies the highest fields in the adinkra representing a supermultiplet with no off-shell central charge, one has identified the unconstrained Salam-Strathdee superfields that describes the supermultiplet.

This brings us to a conjecture:

> *Only supermultiplets that do* not *contain off-shell central charges are representations of $\mathcal{GR}(\mathrm{d}, N)$ algebras that can be described by unconstrained Salam-Strathdee superfields uniquely determined by the highest engineering dimension component fields with no spacetime derivatives.*

7 Conclusion

We hope to have convinced the reader that our efforts have uncovered a *new* quantum number (χ_{o}) in supersymmetrical field theory. The value of this quantum number for some familiar 4D, $\mathcal{N} = 1$ supermultiplets is shown in the table below.

This table implies that there are two distinct ways to construct Dirac particles in supersymmetrical theories. A standard approach to embedding Dirac particles into 4D, $\mathcal{N} = 1$ models is to use a pair of chiral superfields that may be denoted by Φ_+ and Φ_- corresponding to a $\chi_{\mathrm{o}} = 2$ system. In a number of our past works [28, 31–33], and [34–38], it has been advocated that an alternate approach to embedding Dirac particles into 4D, $\mathcal{N} = 1$ models is to use a 'CMN pair' consisting of one chiral superfield Φ and one complex linear superfield Σ corresponding to a $\chi_{\mathrm{o}} = 0$ system.

One of the amusing analogies to note is that with respect to off-shell 4D, $\mathcal{N} = 2$ supersymmetry, the adinkra quantum number χ_{o} defined on 4D, $\mathcal{N} = 1$ supermultiplets acts just like color in hadronic physics! It seems likely that off-shell 4D, $\mathcal{N} = 2$ supersymmetry representations must have vanishing adinkra quantum number χ_{o} just as baryons must have vanishing color.

	CM	VM	TM	RSS	CLS	cSG	mSG	nSG
χ_0	1	-1	-1	0	-1	-2	-1	-3

Table 1. The new quantum number χ_0 for the 4D, $\mathcal{N} = 1$ chiral (CM), vector (VM), tensor (TM), real scalar (RSS), complex linear (CLM), conformal supergravity (cSG), old-minimal supergravity (mSG), and non-minimal supergravity (nSG) multiplets [18, 29, 30].

Our present work shows that with regard to 4D, $\mathcal{N} = 2$ SUSY this new quantum number matters. As far as we can tell, all Dirac fermions in off-shell 4D, $\mathcal{N} = 2$ systems have $\chi_\text{o} = 0$. This raises numbers of questions. Does this have implications for 4D, $\mathcal{N} = 1$ SUSY systems, including phenomenology? It is known that there exist 4D, $\mathcal{N} = 1$ duality transformations between $\chi_\text{o} = 0$ and $\chi_\text{o} = 2$ systems. Do our results imply that no such 4D, $\mathcal{N} = 2$ duality transformations exist? Needless to say all of this is very strange and 'funny'.

> *"The most exciting phrase to hear in science,*
> *the one that heralds new discoveries, is not 'Eureka!'*
> *but 'That's funny...'"* — Isaac Asimov

Acknowledgments

A Chiral + chiral L, R matrices

In the following series of appendices, we give the explicit forms of the L-matrices and R-matrices discussed in the third section. We use the compact 'Boolean Factor/Cycle' notation introduced in our work of [19]. The explicit form of the 8×8 L-matrices and R-matrices that appear in (3.22) are found to be:

$$(L_1) = \begin{bmatrix} (10)_b(243) & 0 \\ 0 & (10)_b(243) \end{bmatrix}, \qquad (L_2) = \begin{bmatrix} (12)_b(123) & 0 \\ 0 & (12)_b(123) \end{bmatrix},$$

$$(L_3) = \begin{bmatrix} (6)_b(134) & 0 \\ 0 & (6)_b(134) \end{bmatrix}, \qquad (L_4) = \begin{bmatrix} (0)_b(142) & 0 \\ 0 & (0)_b(142) \end{bmatrix},$$

$$(L_5) = \begin{bmatrix} 0 & (15)_b(243) \\ (0)_b(243) & 0 \end{bmatrix}, \qquad (L_6) = \begin{bmatrix} 0 & (9)_b(123) \\ (6)_b(123) & 0 \end{bmatrix},$$

$$(L_7) = \begin{bmatrix} 0 & (3)_b(134) \\ (12)_b(134) & 0 \end{bmatrix}, \qquad (L_8) = \begin{bmatrix} 0 & (5)_b(142) \\ (10)_b(142) & 0 \end{bmatrix}.$$

B Chiral + vector L, R matrices

The explicit form of the 8×8 L-matrices and R-matrices derived from the case of case of the chiral + vector supermultiplets and that are analogous to those that appear in (3.22) are found to be:

$$(L_1) = \begin{bmatrix} (10)_b(243) & 0 \\ 0 & (10)_b(1243) \end{bmatrix}, \qquad (L_2) = \begin{bmatrix} (12)_b(123) & 0 \\ 0 & (12)_b(23) \end{bmatrix},$$

$$(L_3) = \begin{bmatrix} (6)_b(134) & 0 \\ 0 & (0)_b(14) \end{bmatrix}, \qquad (L_4) = \begin{bmatrix} (0)_b(142) & 0 \\ 0 & (6)_b(1342) \end{bmatrix},$$

$$(L_5) = \begin{bmatrix} 0 & (2)_b(243) \\ (13)_b(1243) & 0 \end{bmatrix}, \qquad (L_6) = \begin{bmatrix} 0 & (4)_b(123) \\ (11)_b(23) & 0 \end{bmatrix},$$

$$(L_7) = \begin{bmatrix} 0 & (14)_b(134) \\ (7)_b(14) & 0 \end{bmatrix}, \qquad (L_8) = \begin{bmatrix} 0 & (8)_b(142) \\ (1)_b(1342) & 0 \end{bmatrix}.$$

C Chiral + tensor L, R matrices

The explicit form of the 8×8 L-matrices and R-matrices derived from the case of case of the chiral + tensor supermultiplets and that are analogous to those that appear in (3.22)

are found to be:

$$
(\mathrm{L}_1) = \begin{bmatrix} (10)_b(243) & 0 \\ 0 & (14)_b(234) \end{bmatrix}, \qquad (\mathrm{L}_2) = \begin{bmatrix} (12)_b(123) & 0 \\ 0 & (4)_b(124) \end{bmatrix},
$$

$$
(\mathrm{L}_3) = \begin{bmatrix} (6)_b(134) & 0 \\ 0 & (8)_b(132) \end{bmatrix}, \qquad (\mathrm{L}_4) = \begin{bmatrix} (0)_b(142) & 0 \\ 0 & (2)_b(143) \end{bmatrix},
$$

$$
(\mathrm{L}_5) = \begin{bmatrix} 0 & (11)_b(243) \\ (0)_b(234) & 0 \end{bmatrix}, \qquad (\mathrm{L}_6) = \begin{bmatrix} 0 & (13)_b(123) \\ (10)_b(124) & 0 \end{bmatrix},
$$

$$
(\mathrm{L}_7) = \begin{bmatrix} 0 & (7)_b(134) \\ (6)_b(132) & 0 \end{bmatrix}, \qquad (\mathrm{L}_8) = \begin{bmatrix} 0 & (1)_b(142) \\ (12)_b(143) & 0 \end{bmatrix}.
$$

D   Vector + vector L, R matrices

The explicit form of the 8×8 L-matrices and R-matrices derived from the case of the vector + vector supermultiplets and that are analogous to those that appear in (3.22) are found to be

$$
(\mathrm{L}_1) = \begin{bmatrix} b_+(10)_b(1243) & 0 \\ 0 & a_+(10)_b(1243) \end{bmatrix}, \qquad (\mathrm{L}_2) = \begin{bmatrix} b_+(12)_b(23) & 0 \\ 0 & a_+(12)_b(23) \end{bmatrix},
$$

$$
(\mathrm{L}_3) = \begin{bmatrix} b_+(0)_b(14) & 0 \\ 0 & a_+(0)_b(14) \end{bmatrix}, \qquad (\mathrm{L}_4) = \begin{bmatrix} b_+(6)_b(1342) & 0 \\ 0 & a_+(6)_b(1342) \end{bmatrix},
$$

$$
(\mathrm{L}_5) = \begin{bmatrix} 0 & b_-(10)_b(1243) \\ a_-(10)_b(1243) & 0 \end{bmatrix}, \qquad (\mathrm{L}_6) = \begin{bmatrix} 0 & b_-(12)_b(23) \\ a_-(12)_b(23) & 0 \end{bmatrix},
$$

$$
(\mathrm{L}_7) = \begin{bmatrix} 0 & b_-(0)_b(14) \\ a_-(0)_b(14) & 0 \end{bmatrix}, \qquad (\mathrm{L}_8) = \begin{bmatrix} 0 & b_-(6)_b(1342) \\ a_-(6)_b(1342) & 0 \end{bmatrix},
$$

where

$$
a_\pm = \cos\left(\frac{m\pi}{2}\right) \pm \sin\left(\frac{m\pi}{2}\right), \qquad b_\pm = \cos\left(\frac{n\pi}{2}\right) \pm \sin\left(\frac{n\pi}{2}\right).
$$

E   Tensor + tensor L, R matrices

The explicit form of the 8×8 L-matrices and R-matrices derived from the case of the tensor + tensor supermultiplets and that are analogous to those that appear in (3.22) are

found to be

$$(L_1) = \begin{bmatrix} b_+(14)_b(234) & 0 \\ 0 & a_+(14)_b(234) \end{bmatrix}, \qquad (L_2) = \begin{bmatrix} b_+(4)_b(124) & 0 \\ 0 & a_+(4)_b(124) \end{bmatrix},$$

$$(L_3) = \begin{bmatrix} b_+(8)_b(132) & 0 \\ 0 & a_+(8)_b(132) \end{bmatrix}, \qquad (L_4) = \begin{bmatrix} b_+(2)_b(143) & 0 \\ 0 & a_+(2)_b(143) \end{bmatrix},$$

$$(L_5) = \begin{bmatrix} 0 & b_-(14)_b(234) \\ a_-(14)_b(234) & 0 \end{bmatrix}, \qquad (L_6) = \begin{bmatrix} 0 & b_-(4)_b(124) \\ a_-(4)_b(124) & 0 \end{bmatrix},$$

$$(L_7) = \begin{bmatrix} 0 & b_-(8)_b(132) \\ a_-(8)_b(132) & 0 \end{bmatrix}, \qquad (L_8) = \begin{bmatrix} 0 & b_-(2)_b(143) \\ a_-(2)_b(143) & 0 \end{bmatrix}.$$

F Vector + tensor L, R matrices

The explicit form of the 8×8 L-matrices and R-matrices derived from the case of the vector + tensor supermultiplets and that are analogous to those that appear in (3.22) are found to be

$$(L_1) = \begin{bmatrix} b_+(14)_b(234) & 0 \\ 0 & a_+(10)_b(1243) \end{bmatrix}, \qquad (L_2) = \begin{bmatrix} b_+(4)_b(124) & 0 \\ 0 & a_+(12)_b(23) \end{bmatrix},$$

$$(L_3) = \begin{bmatrix} b_+(8)_b(132) & 0 \\ 0 & a_+(0)_b(14) \end{bmatrix}, \qquad (L_4) = \begin{bmatrix} b_+(2)_b(143) & 0 \\ 0 & a_+(6)_b(1342) \end{bmatrix},$$

$$(L_5) = \begin{bmatrix} 0 & b_-(14)_b(234) \\ a_-(10)_b(1243) & 0 \end{bmatrix}, \qquad (L_6) = \begin{bmatrix} 0 & b_-(4)_b(124) \\ a_-(12)_b(23) & 0 \end{bmatrix},$$

$$(L_7) = \begin{bmatrix} 0 & b_-(8)_b(132) \\ a_-(0)_b(14) & 0 \end{bmatrix}, \qquad (L_8) = \begin{bmatrix} 0 & b_-(2)_b(143) \\ a_-(6)_b(1342) & 0 \end{bmatrix}.$$

G $q\mathcal{GR}$ bracket example calculations

In this appendix, we will simply demonstrate two example of how the $q\mathcal{GR}$ bracket defined in section two can be used. In the first case we show it leads to a very different perspective using the usual Pauli matrices.

We, of course, use their conventional definitions

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{G.1}$$

implying

$$\sigma^i \sigma^j = \delta^{ij} \mathbf{I} + i\, \epsilon^{ijk} \sigma^k \quad \to \quad \left[\sigma^i,\, \sigma^j\right] = i2\, \epsilon^{ijk} \sigma^k, \tag{G.2}$$

the usual commutator algebra. We note that

$$(\sigma^1)^t = +(\sigma^1), \qquad (\sigma^2)^t = -(\sigma^2), \qquad (\sigma^3)^t = +(\sigma^3). \tag{G.3}$$

Under the action of the $q\mathcal{GR}$ brackets for $q = 1$, we have

$$\left[\sigma^1,\ \sigma^2\right]_{(1)\mathcal{GR}>} = 0\,, \qquad \left[\sigma^2,\ \sigma^3\right]_{(1)\mathcal{GR}>} = 0\,, \qquad \left[\sigma^3,\ \sigma^1\right]_{(1)\mathcal{GR}>} = i2\,\sigma^2\,, \qquad \text{(G.4)}$$

$$\left[\sigma^1,\ \sigma^2\right]_{(1)\mathcal{GR}<} = 0\,, \qquad \left[\sigma^2,\ \sigma^3\right]_{(1)\mathcal{GR}<} = 0\,, \qquad \left[\sigma^3,\ \sigma^1\right]_{(1)\mathcal{GR}<} = i2\,\sigma^2\,. \qquad \text{(G.5)}$$

The results in (G.4) and the ones in (G.5) each separately result imply that a Jacobi-like condition is satisfied by the $q\mathcal{GR}$ bracket for $q = 1$ and the Pauli matrices. So a structure not dissimilar to a Lie algebra emerges. Since the Pauli matrices can be identified as the generators of the su(2) algebra, replacing them by the generators for su(3) leads to more interesting results. It might be of interest to investigate whether such a replacement also lead to a structure not dissimilar to a Lie algebra.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] S.J. Gates Jr. and L. Rana, *A theory of spinning particles for large-N extended supersymmetry*, *Phys. Lett.* **B 352** (1995) 50 [hep-th/9504025] [INSPIRE].

[2] S.J. Gates Jr. and L. Rana, *A theory of spinning particles for large-N extended supersymmetry (II)*, *Phys. Lett.* **B 369** (1996) 262 [hep-th/9510151] [INSPIRE].

[3] S.J. Gates Jr., W.D. Linch III, J. Phillips and L. Rana, *The fundamental supersymmetry challenge remains*, *Grav. Cosmol.* **8** (2002) 96 [hep-th/0109109] [INSPIRE].

[4] S.J. Gates Jr., W.D. Linch III and J. Phillips, *When superspace is not enough*, UMDEPP-02-054, CALT-68-2387 [hep-th/0211034] [INSPIRE].

[5] M. Faux and S.J. Gates Jr., *Adinkras: a graphical technology for supersymmetric representation theory*, *Phys. Rev.* **D 71** (2005) 065002 [hep-th/0408004] [INSPIRE].

[6] C.F. Doran et al., *Topology types of adinkras and the corresponding representations of N-extended supersymmetry*, UMDEPP-08-010, SUNY-O-667 [arXiv:0806.0050] [INSPIRE].

[7] C.F. Doran et al., *Relating doubly-even error-correcting codes, graphs and irreducible representations of N-extended supersymmetry*, in *Discrete and computational mathematics*, F. Liu et al. eds., Nova Science Pub. Inc., Hauppauge U.S.A. (2008), pp. 53–71 [arXiv:0806.0051] [INSPIRE].

[8] C.F. Doran et al., *Codes and supersymmetry in one dimension*, *Adv. Theor. Math. Phys.* **15** (2011) 1909 [arXiv:1108.4124] [INSPIRE].

[9] S.J. Gates Jr., J. Hallett, T. Hübsch and K. Stiffler, *The real anatomy of complex linear superfields*, *Int. J. Mod. Phys.* **A 27** (2012) 1250143 [arXiv:1202.4418] [INSPIRE].

[10] M.G. Faux, K.M. Iga and G.D. Landweber, *Dimensional enhancement via supersymmetry*, *Adv. Math. Phys.* **2011** (2011) 259089 [arXiv:0907.3605] [INSPIRE].

[11] M.G. Faux and G.D. Landweber, *Spin holography via dimensional enhancement*, *Phys. Lett.* **B 681** (2009) 161 [arXiv:0907.4543] [INSPIRE].

[12] T. Hübsch, *Weaving worldsheet supermultiplets from the worldlines within*, `arXiv:1104.3135` [ɪɴSPIRE].

[13] S.J. Gates Jr. and T. Hübsch, *On dimensional extension of supersymmetry: from worldlines to worldsheets*, *Adv. Theor. Math. Phys.* **16** (2012) 1619 [`arXiv:1104.0722`] [ɪɴSPIRE].

[14] M. Gell-Mann, *A schematic model of baryons and mesons*, *Phys. Lett.* **8** (1964) 214 [ɪɴSPIRE].

[15] G. Zweig, *An* SU(3) *model for strong interaction symmetry and its breaking*, CERN-8182-TH-401, CERN-8419-TH-412 (1964).

[16] F. Gursey and L.A. Radicati, *Spin and unitary spin independence of strong interactions*, *Phys. Rev. Lett.* **13** (1964) 173 [ɪɴSPIRE].

[17] O.W. Greenberg, *Spin and unitary spin independence in a paraquark model of baryons and mesons*, *Phys. Rev. Lett.* **13** (1964) 598 [ɪɴSPIRE].

[18] S.J. Gates Jr. et al., *4D, $\mathcal{N} = 1$ supersymmetry genomics (I)*, *JHEP* **12** (2009) 008 [`arXiv:0902.3830`] [ɪɴSPIRE].

[19] I. Chappell II, S.J. Gates Jr. and T. Hübsch, *Adinkra (in)equivalence from Coxeter group representations: a case study*, *Int. J. Mod. Phys.* **A 29** (2014) 1450029 [`arXiv:1210.0478`] [ɪɴSPIRE].

[20] O.W. Greenberg, *Quon statistics*, *Phys. Rev.* **D 43** (1991) 4111 [ɪɴSPIRE].

[21] O.W. Greenberg, *Quons: an interpolation between Bose and Fermi oscillators*, UMD-PP-93-097 [`cond-mat/9301002`] [ɪɴSPIRE].

[22] S.J. Gates Jr., S. Randall and K. Stiffler, *Reduction redux of adinkras*, UMDEPP-013-019, *Int. J. Mod. Phys.* **A 29** (2014) 1450070 [`arXiv:1312.2000`] [ɪɴSPIRE].

[23] P. Fayet, *Fermi-Bose hypersymmetry*, *Nucl. Phys.* **B 113** (1976) 135 [ɪɴSPIRE].

[24] J. Wess, *Fermi-Bose supersymmetry*, lecture given at *International Summer Institute for Theoretical Physics: Trends in Elementary Particle Theory*, Bonn Germany, 29 Jul–9 Aug 1974 [ɪɴSPIRE].

[25] S.J. Gates Jr., J. Parker, V.G.J. Rodgers, L. Rodriguez and K. Stiffler, *A detailed investigation of first and second order supersymmetries for off-shell $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supermultiplets*, UMDEPP-11-009 [`arXiv:1106.5475`] [ɪɴSPIRE].

[26] M.F. Sohnius, K.S. Stelle and P.C. West, *Off mass shell formulation of extended supersymmetric gauge theories*, *Phys. Lett.* **B 92** (1980) 123 [ɪɴSPIRE].

[27] M.F. Sohnius, K.S. Stelle and P.C. West, *Dimensional reduction by Legendre transformation generates off-shell supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 173** (1980) 127 [ɪɴSPIRE].

[28] B.B. Deo and S.J. Gates Jr., *Comments on nonminimal $\mathcal{N} = 1$ scalar multiplets*, *Nucl. Phys.* **B 254** (1985) 187 [ɪɴSPIRE].

[29] S.J. Gates Jr., J. Hallett, J. Parker, V.G.J. Rodgers and K. Stiffler, *4D, $\mathcal{N} = 1$ supersymmetry genomics (II)*, *JHEP* **06** (2012) 071 [`arXiv:1112.2147`] [ɪɴSPIRE].

[30] I. Chappell et al., *4D, $\mathcal{N} = 1$ supergravity genomics*, *JHEP* **10** (2013) 004 [`arXiv:1212.3318`] [ɪɴSPIRE].

[31] S.J. Gates Jr., *Why auxiliary fields matter: the strange case of the 4D, $\mathcal{N} = 1$ supersymmetric QCD effective action (II)*, *Nucl. Phys.* **B 485** (1997) 145 [`hep-th/9606109`] [INSPIRE].

[32] S.J. Gates Jr., *Why auxiliary fields matter: the strange case of the 4D, $\mathcal{N} = 1$ supersymmetric QCD effective action*, *Phys. Lett.* **B 365** (1996) 132 [`hep-th/9508153`] [INSPIRE].

[33] S.J. Gates Jr., M.T. Grisaru, M.E. Knutt-Wehlau, M. Roček and O.A. Soloviev, *$\mathcal{N} = 1$ supersymmetric extension of the QCD effective action*, *Phys. Lett.* **B 396** (1997) 167 [`hep-th/9612196`] [INSPIRE].

[34] S.J. Gates Jr. and S.M. Kuzenko, *The CNM hypermultiplet nexus*, *Nucl. Phys.* **B 543** (1999) 122 [`hep-th/9810137`] [INSPIRE].

[35] S.J. Gates Jr., T. Hübsch and S.M. Kuzenko, *CNM models, holomorphic functions and projective superspace c-maps*, *Nucl. Phys.* **B 557** (1999) 443 [`hep-th/9902211`] [INSPIRE].

[36] S.J. Gates Jr. and S.M. Kuzenko, *4D, $\mathcal{N} = 2$ supersymmetric off-shell σ-models on the cotangent bundles of Kähler manifolds*, *Fortschr. Phys.* **48** (2000) 115 [`hep-th/9903013`] [INSPIRE].

[37] S.J. Gates Jr., S. Penati and G. Tartaglino-Mazzucchelli, *6D supersymmetric nonlinear σ-models in 4D, $\mathcal{N} = 1$ superspace*, *JHEP* **09** (2006) 006 [`hep-th/0604042`] [INSPIRE].

[38] S.J. Gates Jr., S. Penati and G. Tartaglino-Mazzucchelli, *6D supersymmetry, projective superspace and 4D, $\mathcal{N} = 1$ superfields*, *JHEP* **05** (2006) 051 [`hep-th/0508187`] [INSPIRE].

[39] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [`hep-th/0108200`] [INSPIRE].