Topology Vol. 14, pp. 91–107. Pergamon Press, 1975. Printed in Great Britain

CONFIGURATION SPACES OF POSITIVE AND NEGATIVE PARTICLES

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(*Received* 4 *May* 1974)

§1. INTRODUCTION

THE AIM of this paper is to investigate the topology of two "configuration spaces" associated to a smooth manifold M. The first is the space $C(M)$ of all finite subsets of M. Its points can be thought of as sets of indistinguishable particles moving about on M, and it is topologised so that particles cannot collide. (cf. [4], where configurations of distinguishable particles on a manifold are studied.) The second space, denoted $C^{\pm}(M)$, has as its points pairs of finite subsets of M, to be thought of as "positive" and "negative" particles. It is topologised so that particles of the same sign cannot collide, but a pair of particles of opposite sign can collide and annihilate each other. (More precise definitions are given in §2.)

When M is euclidean space $\mathbb{R}^n$ the space $C(M)$ has been studied in [1], [5] and [7]. If $C_k(M)$ is the part of $C(M)$ consisting of sets of k particles, then there is a map $C_k(M) \to F_k$, where F_k is the space of base-point preserving maps of degree k from the sphere S^n to itself, which induces an isomorphism of homology groups up to a dimension tending to ∞ with k. Thinking of S^n as obtained from $\mathbb{R}^n$ by adding a point at ∞, one can as well describe F_k as the space of maps from $\mathbb{R}^n$ to $\mathbb{R}^n \cup \infty$ which have compact support and degree k. If one likes, one can think of the map $C_k(M) \to F_k$ as assigning to a set of particles a "field" which it produces (cf. [7]). For a general n-dimensional manifold M (without boundary) the maps $R^n \to R^n \cup \infty$ should be replaced by tangent vector fields on M which are allowed to become infinite. Let E_M be the space obtained from the tangent bundle of M by adding a point at infinity to each fibre (thus E_M is a fibre bundle on M with fibre S^n), and let $\Gamma(M)$ be the space of cross-sections of E_M with compact support. Such a cross-section has a degree: let $\Gamma_k(M)$ be the cross-sections of degree k. Our first main results are:

THEOREM 1.1. *Let M be a closed compact manifold. Then there are maps $C_k(M) \to \Gamma_k(M)$ which, for each n, induce isomorphisms $H_n(C_k(M)) \to H_n(\Gamma_k(M))$ when k is sufficiently large.*

THEOREM 1.2. *Let M be an open, paracompact manifold* (*i.e. it has no closed components*). *Then there are maps $C_k(M) \to \Gamma_k(M)$ which induce an isomorphism*

$$\varinjlim_{k\to\infty} H_*(C_k(M)) \cong \varinjlim_{k\to\infty} H_*(\Gamma_k(M)).$$

Moreover, if M is the interior of a compact manifold with boundary, $H_n(C_k(M)) \to H_n(\Gamma_k(M))$ is an isomorphism for large k.

(The sense in which $H_*(C_k(M))$ and $H_*(\Gamma_k(M))$ form a direct system will be explained in §4. Roughly speaking, one maps $C_k(M)$ to $C_{k+1}(M)$ by "adding a particle from infinity" in a standard way.)

The theorem about $C^{\pm}(M)$ is both stronger and simpler to state. The reason is that the spaces $C_k(M)$ are different for different k, and become more complicated as k increases, while if M is open the homotopy-type of $\Gamma_k(M)$ is independent of k, so that one can at most expect to prove something as k tends to infinity. But when one allows oppositely charged particles and annihilation, the situation is different: for open M the homotopy-type of $C_k^{\pm}(M)$, the part of $C^{\pm}(M)$ where the total charge is k, is independent of k. For this reason it was first conjectured that $C^{\pm}(M) \simeq \Gamma(M)$. But this is false. There is a fibre bundle $E_M^{\pm}$ on M whose fibre is a $2n$-dimensional space X_n obtained by attaching a $2n$-cell to S^n, and one has

THEOREM 1.3. *If M is a manifold without boundary, there is a homotopy equivalence $C^{\pm}(M) \to \Gamma^{\pm}(M)$, where $\Gamma^{\pm}(M)$ is the space of cross-sections of $E_M^{\pm}$ with compact support.*

This result is new even if $M = \mathbb{R}^n$, when it can be restated as $C^{\pm}(\mathbb{R}^n) \simeq \Omega^n X_n$, where $\Omega^n X_n$ is the n-fold loop space of X_n (cf. §2).

The proofs of these theorems make no appeal to the known results in the case of euclidean space. Thus they include in particular a new proof of the theorem about $\Omega^n S^n$ which is perhaps simpler and more conceptual than the known ones. The idea of the proof can be described very simply.

First let us remark that in the foregoing M was a manifold without boundary, not necessarily compact. If M is compact but has a boundary ∂M then it is clear that $C(M) \simeq C(\text{Int } M)$ and $C^{\pm}(M) \simeq C^{\pm}(\text{Int } M)$, where $\text{Int } M = M - \partial M$ is the interior of M. Also $\Gamma(\text{Int } M)$ is contained in, and homotopy equivalent to, $\Gamma(M, \partial M)$, the sections of E_M which vanish on the boundary ∂M. And similarly $\Gamma^{\pm}(\text{Int } M) \simeq \Gamma^{\pm}(M, \partial M)$. In future it will be more convenient to work entirely with compact manifolds, with or without boundary. This suffices to prove Theorems 1.2 and 1.3, for even though a given open manifold M may not be the interior of a compact manifold with boundary it is still the countable union of such manifolds, and the theorems about $C(M)$ and $C^{\pm}(M)$ can be obtained by a simple limit argument.

The proof depends on considering more general configuration spaces in which one allows particles to be created or annihilated in some closed subset L of M, which in practice will be contained in the boundary. Thus $C(M, L)$ will denote the quotient space of $C(M)$ by the equivalence relation which identifies two finite subsets s and s' of M if $s \cap (M - L) = s' \cap (M - L)$. $C(M, \partial M)$ will be abbreviated to $\tilde{C}(M)$. One defines $C^{\pm}(M, L)$ and $\tilde{C}^{\pm}(M)$ similarly.

The spaces $\tilde{C}(N)$, where N is a submanifold of M of the same dimension as M, are in some sense local versions of $C(M)$, for the restriction map which takes $s \subset M$ to $s \cap N$ maps $C(M)$ to $\tilde{C}(N)$, and not $C(N)$. They have very simple properties. For instance, if M is a closed disc in $\mathbb{R}^n$, then by radial expansion, pushing particles out to the boundary where they vanish, one has a deformation of $\tilde{C}(M)$ onto the part consisting of configurations with at most one particle. This is homeomorphic to $M/\partial M$, an n-sphere. Furthermore, if M is the union of two parts M_1 and M_2 with intersection M_{12}, then under suitable hypotheses (see (3.6)) $\tilde{C}(M)$ is the homotopy fibre product of $\tilde{C}(M_1)$ and $\tilde{C}(M_2)$ over $\tilde{C}(M_{12})$. Using these two remarks one can find the homotopy type of $\tilde{C}(M)$ by building M up step by step out of simple pieces, more precisely by induction on the number of handles in a handle-decomposition of M. Thus one obtains

THEOREM 1.4. *If M is a compact manifold with no closed components, then* $C(M, \partial M) \simeq \Gamma(M)$.

There is an analogous result for $C^{\pm}(M, \partial M)$, but one need not require $\partial M \neq \phi$. From this one deduces Theorem 1.3. The proofs of Theorems 1.1 and 1.2 are also based on this, but require considerably more work (see §4 and §5).

It is perhaps worth mentioning that the methods of this paper prove also the following generalisations of these theorems. Suppose that the particles have some kind of internal structure expressed by a parameter with values in a parameter space P. Without loss of generality one can suppose that P has a zero-point p_o such that when its parameter becomes p_o a particle annihilates itself. In the case of positive and negative particles one supposes that two opposite particles can annihilate each other only if their parameters are the same. Then the theorems still hold for such structured particles but the bundles E_M and $E_M{}^{\pm}$ must be replaced by bundles $E_M(P)$ and $E_M{}^{\pm}(P)$ whose fibres are S^nP and $S^nP \times S^nP/(\text{diag})$ respectively. (S^nP is the reduced n-fold suspension of P.) One could also let the space P vary from point to point of M forming a fibre bundle on M, with the same result.

Finally, the author wishes to thank Graeme Segal for suggesting the problem, and discussing it with her.

§2. THE CONFIGURATION SPACES $\tilde{C}(M)$ AND $\tilde{C}^{\pm}(M)$

We will always suppose that M is a compact smooth manifold with possibly empty boundary and perhaps also with corners, unless it is explicitly stated to the contrary. Let $C_k(M) = \{s \subset M: \text{card } s = k\}$. Then $C_k(M)$ may be identified with

$$\{(m_1, \ldots, m_k) \in M^k: m_i \neq m_j \quad \text{if} \quad i \neq j\}/\sim,$$

where $(m_1, \ldots, m_k) \sim (m_1', \ldots, m_k')$ iff the sets $\{m_1, \ldots, m_k\}$ and $\{m_1', \ldots, m_k'\}$ are equal. With $C_k(M)$ topologised as this quotient of $M^k - \Delta$, where Δ denotes $\{(m_1, \ldots, m_k) \in M^k: m_i = m_j \text{ for some } i \neq j\}$, $C(M)$ is given the topology of the disjoint union $\bigcup_{k \geq 0} C_k(M)$.

If L is any closed subset of ∂M, the points of the space $C(M, L)$ are equivalence classes $[s]$ of finite subsets s of M, where $s \sim s'$ iff $s \cap (M - L) = s' \cap (M - L)$. $C(M, L)$ is topologised as a quotient of $C(M)$. It can be considered either as the configuration space where particles are annihilated and created in L or as the space where particles may vanish, or appear, by crossing the boundary at L. We write $\tilde{C}(M)$ for $C(M, \partial M)$.

LEMMA 2.1. *$C(M)$ and $\tilde{C}(M)$ have the homotopy-type of CW complexes.*

Proof. This is clear for $C(M)$ since $C_k(M)$ is obviously a manifold. Now $C_k(M)$ has a subcomplex $D_k = \{s : s \cap \partial M \neq \phi\}$. And if $\tilde{C}^k = \{[s] \in \tilde{C}(M): \text{card } s \leq k\}$, then $\tilde{C}^k$ is obtained from $\tilde{C}^{k-1}$ by attaching $C_k(M)$ to $\tilde{C}^{k-1}$ by a map $D_k \to \tilde{C}^{k-1}$. Therefore each $\tilde{C}^k$ is homotopic to a CW complex, and since $\tilde{C}(M) = \varinjlim_k \tilde{C}^k$, $\tilde{C}(M)$ is too.

The configuration space $C^{\pm}(M)$ of positive and negative particles is defined to be $C(M) \times C(M)/\sim$ where $(s, t) \sim (s', t')$ iff $s - t = s' - t'$ and $t - s = t' - s'$. Similarly $C^{\pm}(M, L)$ is $C(M) \times C(M)/\sim$ where $(s, t) \sim (s', t')$ iff $(s - t) \cap (M - L) = (s' - t') \cap (M - L)$ and $(t - s) \cap (M - L) = (t' - s') \cap (M - L)$. These spaces are given the quotient topology. As before we write $\tilde{C}^{\pm}(M)$ for $C^{\pm}(M, \partial M)$.

LEMMA 2.2. *$C^{\pm}(M)$ and $\tilde{C}^{\pm}(M)$ have the homotopy-type of CW complexes.*

Proof. We prove this for $C^{\pm}(M)$. Let $C^{\pm}_{k,l} = \{[s, t] \in C^{\pm}(M) : \text{card } s = k, \text{card } t = l\}$. Each $C^{\pm}_{k,l}$ is closed in $C^{\pm}(M)$, and clearly $C^{\pm}(M) = \varinjlim_{k,l} C^{\pm}_{k,l}$. By Lemma 2.1 each $C^{\pm}_{k,0}$, $k \geq 0$, and $C^{\pm}_{0,l}$, $l \geq 0$, is homotopic to a CW complex. Now the group $\Sigma_k \times \Sigma_l$ acts freely on $(M^k - \Delta) \times (M^l - \Delta)$ (Σ_k is the symmetric group on k letters), preserving the subspace $D_{k,l} = \{((m_1, \ldots, m_k), (m_1', \ldots, m_l')) : m_i = m_j' \text{ for some } i, j\}$. Therefore $D'_{k,l} = D_{k,l}/(\Sigma_k \times \Sigma_l)$ is a subcomplex of the CW complex $C'_{k,l} = (M^k - \Delta) \times (M^l - \Delta)/(\Sigma_k \times \Sigma_l)$. But $C^{\pm}_{k,l}$ is obtained by attaching $C'_{k,l}$ to $C^{\pm}_{k-1,l-1}$ by a map $D'_{k,l} \to C^{\pm}_{k-1,l-1}$. It follows by induction on k, l that each $C^{\pm}_{k,l}$ is homotopic to a CW complex. Therefore $C^{\pm}(M)$ is also.

One knows similarly that all configuration spaces discussed here are homotopic to CW complexes.

Now suppose that M has a Riemannian metric d, and, if $\varepsilon > 0$ write M_ε for $\{x \in M: d(x, \partial M) \geq \varepsilon\}$. (If $\partial M = \phi$, take $M_\varepsilon = M$.) We will always assume that $\varepsilon > 0$ is chosen so that $M - M_{2\varepsilon}$ is a collar neighbourhood of ∂M, homeomorphic to $\partial M \times [0, 2\varepsilon)$.

Let $\tilde{C}(M, \varepsilon')$ be the closed subset of $\tilde{C}(M)$ where the particles have pairwise separation $\geq 2\varepsilon'$. Then $\tilde{C}(M, \varepsilon')$ can be considered to be the configuration space of disjoint open discs $\delta_{\varepsilon}'(x) = \{y : d(x, y) < \varepsilon'\}$ in M_{ε}' which can disappear across the boundary, by associating to each particle the centre of a disc (cf. [7]). Form the disc space

$$\tilde{C}_\varepsilon(M) = \bigcup_{0 < \varepsilon' \leq \varepsilon} \tilde{C}(M, \varepsilon') \times \{\varepsilon'\} \subseteq \tilde{C}(M) \times \mathbb{R}.$$

There is an obvious projection $p: \tilde{C}_\varepsilon(M) \to \tilde{C}(M)$.

LEMMA 2.3. *$p: \tilde{C}_\varepsilon(M) \to \tilde{C}(M)$ is a homotopy equivalence.*

Proof. It is easy to see that $\tilde{C}(M)$ is topologically the direct limit $\varinjlim_{\delta > 0} \tilde{C}(M, \delta)$. On the other hand $\tilde{C}_\varepsilon(M)$ is $\varinjlim_{\varepsilon \geq \delta > 0} \tilde{C}(M, \delta) \times [\delta, \varepsilon]$. Hence the projection $p : \tilde{C}_\varepsilon(M) \to \tilde{C}(M)$ is a weak homotopy equivalence. But both spaces are homotopic to CW complexes, so p is a homotopy equivalence.

A similar result is true for positive and negative particles. Let $\tilde{C}^{\pm}(M, \varepsilon')$ be the closed subset of $\tilde{C}^{\pm}(M)$ where particles of the same sign have pairwise separation $\geq 2\varepsilon'$, and put $\tilde{C}_\varepsilon^{\pm}(M) = \bigcup_{0 < \varepsilon' \leq \varepsilon} \tilde{C}^{\pm}(M, \varepsilon') \times \{\varepsilon'\} \subseteq \tilde{C}^{\pm}(M) \times \mathbb{R}$ as before.

LEMMA 2.4. *The projection $p : \tilde{C}_\varepsilon^{\pm}(M) \to \tilde{C}^{\pm}(M)$ is a homotopy equivalence.*

Proof as for (2.3).

Our object is to show that $\tilde{C}(M)$ and $\tilde{C}^{\pm}(M)$ have the same homotopy type as spaces of continuous sections of certain bundles E_M and $E_M{}^{\pm}$ over M. It is convenient to define these bundles in the following way. Let D_x be the unit disc in the tangent space $T(M)_x$ to M at x, and let S_x be $D_x/\partial D_x$. Then E_M (resp. $E_M{}^{\pm}$) is the bundle associated to $T(M)$ with fibre S_x (resp. $(S_x \times S_x)/\Delta_x$) at x. Here Δ_x is the diagonal $\{(y, y) : y \in S_x\}$. The fibres of E_M (resp. $E_M{}^{\pm}$) are based, with base point $*_x$ equal to the image of ∂D_x (resp. Δ_x) in the fibre.

If N is some subset of M we write $\Gamma(N)$ (resp. $\Gamma^{\pm}(N)$) for the space of continuous sections of E_M (resp. $E_M{}^{\pm}$) over N.

Now suppose that $\varepsilon > 0$ is small enough so that for each $x \in M_\varepsilon$ $\exp_x : \varepsilon D_x \to \delta_x(\varepsilon) = \{y : d(y, x) \leq \varepsilon\}$ is a diffeomorphism. Then define $\phi_\varepsilon : \tilde{C}_\varepsilon(M) \to \Gamma(M_\varepsilon)$ as follows. For $([s], \varepsilon') \in \tilde{C}(M, \varepsilon') \times \{\varepsilon'\}$, and $x \in M_\varepsilon$ put $\phi_\varepsilon([s], \varepsilon')(x) = *_x$ unless $d(x, x_i) < \varepsilon'$ for some $x_i \in s$. In this case put $\phi_\varepsilon([s], \varepsilon')(x) = d(x, x_i)/\varepsilon' \, . \, t(x, x_i)$, where $t(x, x_i)$ is the unit tangent at x to the minimal geodesic from x_i to x. (This geodesic exists by our assumption on ε.) Note that ϕ_ε is well defined. For a disc of radius ε and centre on ∂M does not intersect M_ε, and, also, $\tilde{C}_\varepsilon(M)$ consists of sets of disjoint discs. Sometimes we will write $\phi_{M,\varepsilon}$ for ϕ_ε if we wish to emphasize that it has been defined relative to M. It is clear that different choices of metric d on M give rise to homotopic maps ϕ_ε.

THEOREM 2.5. *If M is compact and has no closed components, then $\phi_\varepsilon : \tilde{C}_\varepsilon(M) \to \Gamma(M_\varepsilon)$ is a homotopy equivalence.*

Note that $\tilde{C}_\varepsilon(M)$ is homotopic to $\tilde{C}(M)$ by Lemma 2.3. Also the restriction map $\Gamma(M) \to \Gamma(M_\varepsilon)$ is an equivalence for small ε. Therefore this theorem is equivalent to Theorem 1.4.

Proof of 2.5. When $M = D$, the unit disc in $\mathbb{R}^n$.

There is a commutative diagram

$$\begin{array}{ccc} \tilde{C}(D, \varepsilon') & \longrightarrow & \Gamma(D_\varepsilon) \\ \big\downarrow & & \big\downarrow \simeq \\ \tilde{C}(D, 2) & \xrightarrow[\cong]{} & S^n \end{array}$$

where the horizontal maps are induced by ϕ_ε, the right hand vertical map is evaluation at the origin, and the left hand vertical map is given by the radial expansion $[s] \to [(2/\varepsilon')s]$. Since the other three maps are equivalences, $\phi_\varepsilon : \tilde{C}(D, \varepsilon') \to \Gamma(D_\varepsilon)$ is too. Thus $\phi_\varepsilon : \tilde{C}_\varepsilon(D) \to \Gamma(D_\varepsilon)$ is an equivalence.

The proof for general M will be given in §3.

For the spaces $C(M, L)$ one can prove the following.
Let M be a compact manifold (without corners), and suppose that its boundary ∂M is the union of two submanifolds L and L' of the same dimension as ∂M and with boundaries $\partial L = \partial L' = L \cap L'$. Let $\Gamma(M, L')$ be the space of sections of E_M over M which vanish (i.e. equal $*_x$) on L'.

THEOREM 2.6. *If M is connected and $L \neq \phi$, there is an equivalence $C(M, L) \to \Gamma(M, L')$.*

What the map is here will be clear from the proof in §3. Note that this is false if L consists of a single point, or is empty (see (4.2), (4.5)). Also, since $C(M, L) \simeq \tilde{C}(M - L')$, the result may be reformulated as $\tilde{C}(M - L') \xrightarrow{\simeq} \Gamma(M, L')$, for certain $L' \subseteq \partial M$.

The results for positive and negative particles are simpler than those for positive particles alone since there are no exceptional cases. We define $\phi_\varepsilon^{\pm} : \tilde{C}_\varepsilon^{\pm}(M) \to \Gamma^{\pm}(M_\varepsilon)$ by $\phi_\varepsilon^{\pm}(([s, t], \varepsilon'))(x)$ is the image in $(S_x \times S_x)/\Delta_x$ of $(\phi_\varepsilon([s], \varepsilon')(x), \phi_\varepsilon([t], \varepsilon')(x))$. This is well defined since particles in $s \cap t$ give rise to elements of Δ_x, and so make no contribution to $\phi_\varepsilon^{\pm}$.

THEOREM 2.7. *$\phi_\varepsilon^{\pm} : \tilde{C}_\varepsilon^{\pm}(M) \to \Gamma^{\pm}(M_\varepsilon)$ is a homotopy equivalence.*

Similarly, if $\partial M = L \cup L'$ as above, one has

THEOREM 2.8. *For any L, there is an equivalence $C^{\pm}(M, L) \to \Gamma^{\pm}(M, L')$.*

Theorem 1.3 is an immediate consequence of this. For putting $L = \phi$ gives $C^{\pm}(M) \xrightarrow{\simeq} \Gamma^{\pm}(M, \partial M)$, which implies that $C^{\pm}(\text{Int } M) \xrightarrow{\simeq} \Gamma^{\pm}(\text{Int } M)$, where Int M is $M - \partial M$. By taking limits, one sees that $C^{\pm}(M) \xrightarrow{\simeq} \Gamma(M)$ for all manifolds without boundary.

When M is D, the unit disc in $\mathbb{R}^n$, this theorem says that $C^{\pm}(D)$, or $C^{\pm}(\mathbb{R}^n)$, is homotopic to $\text{Map}_0(S^n, (S^n \times S^n)/\Delta)$, the space of base-point preserving maps $S^n \to (S^n \times S^n)/\Delta$ (where Δ is the diagonal $\{(y, y) : y \in S^n\}$). Now Segal's theorem in [7] may be interpreted as saying that the monoid $C(\mathbb{R}^n)$, where composition is by "juxtaposition", has classifying space $\Omega^{n-1}S^n$, and hence that the H-space obtained by "adjoining homotopy inverses" to $C(\mathbb{R}^n)$ is homotopy equivalent to $\Omega^n S^n \cong \text{Map}_0(S^n, S^n)$. One might hope that adding negative particles to $C(\mathbb{R}^n)$ would be equivalent to forming this H-space, but Theorem 2.8 says this is not so. However $C^{\pm}(\mathbb{R}^n)$ is related to $\Omega^n S^n$. In fact there are inclusions

$$\Omega^n S^n \subsetneq C^{\pm}(\mathbb{R}^n) \subsetneq \Omega^{n+1} S^{n+1}$$

whose composition is the suspension. These maps are induced by inclusions

$$S^n \subsetneq (S^n \times S^n)/\Delta \subsetneq \Omega S^{n+1},$$

where the first map is $x \mapsto (x, *) \in (S^n \times S^n)/\Delta$, and the second is obtained as follows. Think of S^n as included in ΩS^{n+1} and define $f : S^n \times S^n \to \Omega S^{n+1}$ by $(\lambda, \mu) \mapsto \lambda \vee (-\mu)$ where $\vee$ denotes addition of loops. Since $f|_\Delta$ is null-homotopic, f is homotopy equivalent to a map f', constant on Δ.

§3. PROOFS OF THE THEOREMS OF §2

These theorems are proved in the following way. First we show that for suitable submanifolds N of M the restriction maps $\tilde{C}(M) \to \tilde{C}(N)$ and $\tilde{C}^{\pm}(M) \to \tilde{C}^{\pm}(N)$ are quasifibrations. Then, using this and the fact that Theorems 2.5 and 2.7 are true when M is a disc, we deduce them for general M by induction on the number of handles in a handle decomposition of M. Theorems 2.6 and 2.8 now follow fairly easily, since $C^{\pm}(M, L)$, for instance, is the fibre of a quasifibration $\tilde{C}^{\pm}(M \cup_{L'} (L' \times I)) \to \tilde{C}^{\pm}(L' \times I)$.

Let N and N' be two manifolds of the same dimension as M, and with boundaries ∂N and $\partial N'$. We suppose that they are embedded as closed submanifolds of M in such a way that $M = N \cup N'$, and $N \cap N' = \partial N \cap \partial N' = B$ say. If, also, B is a submanifold of ∂N and of $\partial N'$, we will say that the inclusion $N \subsetneq M$ is nice. Sometimes we will require that the following condition is satisfied:

(*) each connected component of B has non-empty intersection with ∂M.

Define the restriction map $r : \tilde{C}(M) \to \tilde{C}(N)$ by $[s] \mapsto [s \cap N]$, and $r^{\pm} : \tilde{C}^{\pm}(M) \to \tilde{C}^{\pm}(N)$ by $[s, t] \mapsto [s \cap N, t \cap N]$. Clearly each fibre $r^{-1}([s])$ of r is homeomorphic to $C(N', \overline{\partial N' - B})$ by the map $[t] \mapsto [t \cap N']$. Similarly the fibres of $r^{\pm}$ are homeomorphic to $C^{\pm}(N', \overline{\partial N' - B})$.

Recall that the homotopy theoretic fibre $F(r, x)$ of a map $r : Y \to X$ at the point $x \in X$ consists of all pairs (y, γ) where $y \in Y$ and γ is a path in X from $f(y)$ to x. There is an inclusion $r^{-1}(x) \subsetneq F(r, x)$ given by $y \mapsto (y, \gamma)$, where γ is the constant path. The map r is called a quasifibration if this inclusion induces an isomorphism of homotopy groups for all $x \in X$.

PROPOSITION 3.1. *If $N \subseteq M$ is a nice inclusion for which condition (*) is satisfied, then $r : \tilde{C}(M) \to \tilde{C}(N)$ is a quasifibration with fibre $F = C(N', \overline{\partial N' - B})$.*

PROPOSITION 3.2. *For all nice inclusions $N \subseteq M$, $r^{\pm} : \tilde{C}^{\pm}(M) \to \tilde{C}^{\pm}(N)$ is a quasifibration with fibre $F = C^{\pm}(N', \overline{\partial N' - B})$.*

These propositions are proved in a standard way, using the following lemma (see [3], (2.2), (2.10), (2.15)):

LEMMA 3.3. *Let $X = \bigcup X_k$, where each X_k is closed and $X_1 \subset X_2 \subset \ldots$. Let $r : Y \to X$ be a map. Suppose that for each k*

(i) *$r : r^{-1}(X_k - X_{k-1}) \to X_k - X_{k-1}$ is a fibration with fibre F, and*

(ii) *there is an open subset U_k of X_k which contains X_{k-1} and there are homotopies $h_t : U_k \to U_k$ and $H_t : r^{-1}(U_k) \to r^{-1}(U_k)$ such that*

(a) $h_o = \text{id}$, $h_t(X_{k-1}) \subseteq X_{k-1}$, $h_1(U_k) \subseteq X_{k-1}$;

(b) $H_o = \text{id}$, $r \circ H_t = h_t \circ r$;

(c) $H_1 : r^{-1}(x) \to r^{-1}(h_1(x))$ is a homotopy equivalence for all $x \in U_k$.

Then $r : Y \to X$ is a quasifibration with fibre F.

As in May [5, p. 62], we will filter $\tilde{C}(N)$ by the sets $\tilde{C}^k(N)$ where $\tilde{C}^k(N)$ consists of all configurations of $\leq k$ particles. Write V_k for $\tilde{C}^k(N) - \tilde{C}^{k-1}(N)$, and F for $C(N', \overline{\partial N' - B})$. Embed F in $\tilde{C}(M)$ as $\{[s] : s \subset N'\}$. Since $V_k = \{[s] \in \tilde{C}(M) : s \subset N - \partial N, \text{ card } s = k\}$, the map $([s], [s']) \mapsto [s \cup s']$ is a homeomorphism $V_k \times F \to r^{-1}(V_k)$. Thus condition (i) of Lemma 3.3 holds.

By our assumptions on B, we may choose $\varepsilon > 0$ so that the set $\{y \in M: d(y, B) < 2\varepsilon\}$ is homeomorphic to $B \times (-2\varepsilon, 2\varepsilon)$. Also, for small $\varepsilon > 0$, $N - N_{2\varepsilon}$ is homeomorphic to $\partial N \times [0, 2\varepsilon)$. It follows that there is a homotopy $f_t : (M, \partial M) \to (M, \partial M)$ such that $f_0 = \text{id}$, $f_1(N_\varepsilon) = N$ and $f_t | f_t^{-1}(M - \partial M)$ is always injective. Because of this injectivity of f_t, f_t induces a homotopy $(f_t)_* : \tilde{C}(M) \to \tilde{C}(M)$.

Now let U_k be $\{[s] \in \tilde{C}^k(N): \text{card } (s \cap N_\varepsilon) \leq k - 1\}$, and let h_t, H_t be the appropriate restrictions of $(f_t)_*$. Clearly conditions (a) and (b) of Lemma 3.3 are satisfied. Thus Proposition 3.1 will follow once we have proved:

LEMMA 3.4. *If (*) is satisfied, then* $H_1 : r^{-1}([s]) \to r^{-1}(h_1[s])$ *is a homotopy equivalence for all* $[s]$ *in* U_k.

Proof. As was pointed out before, each fibre $r^{-1}([s])$ is canonically isomorphic to $F = C(N', \overline{\partial N' - B})$. In terms of this identification $H_1 : r^{-1}([s]) \to r^{-1}(h_1[s])$ is a map $H_1 : F \to F$. In fact it is $[t] \mapsto [f_1(t)] \cup [w]$ where $[w] = [f_1(s) \cap N']$ is a configuration in the neighborhood $W = f_1(N) \cap N'$ of B in N'. Now $[f_1(t)]$ lies outside W for all $[t] \in F$. Also condition (*) implies that each component of B has non-empty intersection with ∂M, so that $[w]$ can be connected to the empty configuration through configurations in W. Therefore the map $[t] \mapsto [f_1(t)] \cup [w]$ is homotopic to $[t] \mapsto [f_1(t)]$, and this is homotopic to id_F, since $f_1 \simeq \text{id}$.

The proof of Proposition 3.2 is essentially the same as that of 3.1. $\tilde{C}^{\pm}(N)$ is filtered by the sets $\tilde{C}_k^{\pm}(N) = \tilde{C}^k(N) \times \tilde{C}^k(N)/\sim$, U_k is taken to be $\{([s], [t]) \subseteq \tilde{C}_k^{\pm}(N) : \text{card } ((s \cup t) \cap N_\varepsilon) \leq 2k - 1\}$, and h_t and H_t are induced by f_t as before. Then parts (a) and (b) of condition (ii) in Lemma 3.3 are clearly satisfied. Also condition (i) holds with $F = C^{\pm}(N', \overline{\partial N' - B})$ by the same argument as for positive particles. Thus we need only prove

LEMMA 3.5. *For all* γ *in* U_k, $H_1 : r^{-1}(\gamma) \to r^{-1}(h_1(\gamma))$ *is a homotopy equivalence.*

Proof. As in the proof of (2.4) we may identify $r^{-1}(\gamma)$ and $r^{-1}(h_1(\gamma))$ canonically with F, thus getting a map $H_1 : F \to F$ given by $\delta \mapsto h_1(\delta) \cup w$. As before, w is a configuration in W (w is the restriction of $h_1(\gamma)$ to N'), while $h_1(\delta)$ lies outside W for all $\delta \in F$. Suppose that $w = [s, t]$, and let $w' = [t, s]$. Then w and $h_1(w')$ are disjoint. Also, $w \cup h_1(w')$ is a configuration in $W \cup h_1(W)$ which may be contracted to the empty configuration inside $W \cup h_1(W)$. For w' has negative particles where w has positive ones, and vice versa, so that the particles of $w \cup h_1(w')$ cancel each other in pairs. Similar remarks apply to $w' \cup h_1(w)$. It follows easily that the map $k : F \to F$ given by $\delta \mapsto h_1(\delta) \cup w'$ is homotopy inverse to H_1.

Suppose now that M is the union of two submanifolds M_1, M_2 and that all four inclusions $M_1 \cap M_2 \subsetneq M_i$, $M_i \subsetneq M$ (where $i = 1, 2$) satisfy the conditions of Proposition

3.1. Write M_{12} for $M_1 \cap M_2$ and consider the square

$$\begin{array}{ccc} \tilde{C}(M) & \longrightarrow & \tilde{C}(M_1) \\ \downarrow & & \downarrow \\ \tilde{C}(M_2) & \longrightarrow & \tilde{C}(M_{12}) \end{array}$$

where all maps are restrictions, and so are quasifibrations. This square is commutative, and cartesian. It follows from [3], Satz (2.7) that $\tilde{C}(M)$ is weakly equivalent to the homotopy theoretic fibre product of $\tilde{C}(M_1)$ and $\tilde{C}(M_2)$ over $\tilde{C}(M_{12})$. Therefore, since all these spaces are homotopic to CW complexes (see (2.1)) this is a proper equivalence and the square is homotopy cartesian.

LEMMA 3.6. *Let M, M_1 and M_2 be as above. Then, if Theorem* 2.5 *is true for $M_1 \cap M_2$, M_1 and M_2 it is also true for M.*

Proof. Consider the following diagram:

$$\begin{pmatrix} \tilde{C}(M) \to \tilde{C}(M_1) \\ \downarrow \qquad\quad \downarrow \\ \tilde{C}(M_2) \to \tilde{C}(M_{12}) \end{pmatrix} \xleftarrow{p} \begin{pmatrix} \tilde{C}_\varepsilon(M) \to \tilde{C}_\varepsilon(M_1) \\ \downarrow \qquad\quad \downarrow \\ C_\varepsilon(M_2) \to C_\varepsilon(M_{12}) \end{pmatrix} \xrightarrow{\phi_\varepsilon} \begin{pmatrix} \Gamma(M_\varepsilon) \to \Gamma(M_{1,\varepsilon}) \\ \downarrow \qquad\quad \downarrow \\ \Gamma(M_{2,\varepsilon}) \to \Gamma(M_{12,\varepsilon}) \end{pmatrix}$$

where all maps inside the squares are restrictions, and those between squares are as indicated. The diagram commutes strictly. Furthermore, all maps between squares are homotopy equivalences except, possibly, for ϕ_ε on $\tilde{C}_\varepsilon(M)$ (see Lemma 2.3 for the maps p). The right-hand square is obviously homotopy cartesian. By the remarks above, so is the left-hand square. Hence so is the one in the middle. It follows that $\phi_\varepsilon : \tilde{C}_\varepsilon(M) \to \Gamma(M_\varepsilon)$ is an equivalence.

LEMMA 3.7. *Theorem* 2.5 *is true for $S^k \times D^{n-k}$, where $0 \leq k < n$.*

Proof (by induction on k). We have already shown that the theorem holds for $S^0 \times D^n$. Suppose it holds when $k = k_0 - 1$, and let $k = k_0 < n$. Then S^k is the union of two copies of D^k with intersection $\cong S^{k-1} \times I$. Also, if $M = S^k \times D^{n-k}$, $M_1 = M_2 = D^k \times D^{n-k}$ and $M_1 \cap M_2 = S^{k-1} \times I \times D^{n-k}$, it is clear that the conditions of Lemma 3.6 are satisfied. (Note that k must be $< n$ if condition (*) is to hold.) Thus the result follows.

LEMMA 3.8. *Suppose that Theorem* 2.5 *is true for M, and let M' be M with a handle of index k attached, where $k < n$. Then the theorem holds for M'.*

Proof. $M' = M \cup (D^k \times D^{n-k})$ where $D^k \times D^{n-k}$ is attached to M by an embedding $\psi : S^{k-1} \times D^{n-k} \subsetneq \partial M$. Since ∂M has a collar $\partial M \times I$ in M, ψ may be extended to an embedding $(S^{k-1} \times I) \times D^{n-k} \subsetneq \partial M \times I$. Thus M' may be considered as $M \cup (D^k \times D^{n-k})$ where $M \cap (D^k \times D^{n-k})$ is $(S^{k-1} \times I) \times D^{n-k}$. If $k < n$, this decomposition of M' satisfies the conditions of Lemma 3.6. Therefore, since the theorem holds for $S^{k-1} \times D^{n-k+1}$ (Lemma 3.7), for D^n and for M, it holds also for M'.

Now recall that any compact connected n-dimensional manifold M with non-empty boundary may be built up from the disc D^n by attaching a finite number of handles of index $< n$ (see, for instance, [6]). Therefore, Lemma 3.8 implies that Theorem 2.5 holds for all finite unions of such manifolds.

Theorem 2.7 is proved in exactly the same way. The only change is that since $\tilde{C}^{\pm}(M) \to \tilde{C}^{\pm}(N)$ is a quasifibration whether or not condition (*) holds, there is no need to restrict the attached handles to those of index $< n$. Thus the theorem holds for all compact manifolds.

Proof of Theorem 2.6. Suppose first that L' has no closed components. We may attach $L' \times I$ to M by identifying $L' \times \{0\}$ with $L' \subseteq \partial M$, so getting a manifold with corners $M \cup_{L'} (L' \times I)$, called X say. Then the inclusion $L' \times I \subsetneq X$ is nice, and it satisfies condition (*) since L' has no closed components. Consider the diagram

$$\begin{array}{ccccccc} \tilde{C}(X) & \xleftarrow{p} & \tilde{C}_\varepsilon(X) & \xrightarrow{\phi_{X,\varepsilon}} & \Gamma(X_\varepsilon) & \xleftarrow{\rho} & \Gamma(X) \\ \downarrow r & & \downarrow & & \downarrow & & \downarrow \\ \tilde{C}(L' \times I) & \xleftarrow{p} & \tilde{C}_\varepsilon(L' \times I) & \xrightarrow{\phi_{L' \times I,\varepsilon}} & \Gamma((L' \times I)_\varepsilon) & \xleftarrow{\rho} & \Gamma(L' \times I), \end{array}$$

where the vertical maps and the maps ρ are appropriate restriction maps. It is homotopy commutative, and all horizontal maps are equivalences. Therefore these maps induce equivalences of the homotopy fibres of the vertical maps. But, since by Proposition 3.1 r is a quasifibration where the spaces and all fibres are homotopic to CW complexes, its homotopy fibres are equivalent to its actual fibres and these, in turn, are equivalent to $C(M, L)$. Also the fibre of $\Gamma(X) \to \Gamma(L' \times I)$ is $\Gamma(M, L')$. Therefore there is an equivalence $C(M, L) \to \Gamma(M, L')$.

In the general case let $L' = A' \cup B'$ where A' is the union of the closed components of L'. Then $\partial M - A'$ is closed. Let us call it A. It will suffice to prove that $C(M, A) \xrightarrow{\simeq} \Gamma(M, A')$. This is proved in the same way as Theorem 2.5. For the result is obviously true if $M \cong A' \times I$, with A' identified with $A' \times \{0\}$ and A identified with $A' \times \{1\}$. And if M is any connected manifold whose boundary ∂M is the disjoint union of A' with $\partial M - A'$, there is a finite sequence of triples $(M_i; A', \partial M_i - A')$ which starts with $(A' \times I; A' \times \{0\}, A' \times \{1\})$ and ends with $(M; A', \partial M - A')$ and where M_{i+1} is obtained from M_i attaching a handle of index $<$ dim M to $\partial M_i - A'$ (see [6]). By using the argument of Lemma 3.8 one shows that each $C(M_i, \partial M_i - A') \to \Gamma(M_i, A')$ is an equivalence. The result follows.

The proof of Theorem 2.8 is the same, except that no conditions need to be placed on L since $\tilde{C}(X) \to \tilde{C}(L' \times I)$ is always a quasifibration.

§4. THE CONFIGURATION SPACE $C(M)$

In this section we prove Theorems 1.1 and 1.2.

Let M be a compact connected, manifold with non-empty connected boundary ∂M. As in the case of positive and negative particles (see the proof of (2.6) above), the restriction map $r : C(M \cup_{\partial M} (\partial M \times I)) \to C(\partial M \times I)$ has fibres each homeomorphic to $C(M)$. However the inclusion $\partial M \times I \subsetneq M \cup_{\partial M} (\partial M \times I)$ does not satisfy the condition (*) of §3, so that r is not a quasifibration. This is remedied by "making a hole" in $\partial M \times I$, which gives us something more nearly a quasifibration.

Choose a point m in ∂M and let A be the "annulus with a hole" $(\partial M - m) \times [0, 1]$. Then $\partial A = A_0 \cup A_1$, where A_i is $(\partial M - m) \times \{i\}$, $i = 0, 1$. Further, let X be $M \cup_{A_0} A$, where A_0 is identified with $\partial M - m \subseteq M$, with the amalgamation topology (so that U is open in X iff $U \cap M$ and $U \cap A$ are open). Its boundary ∂X is $m \cup A_1$. One checks that the restriction map $r : \tilde{C}(X) \to \tilde{C}(A)$ given by $[s] \mapsto [s \cap A]$ is well-defined and continuous. Its fibres are each homeomorphic to $C(M, m)$, the space of subsets s of M, modulo $s \sim s'$ iff $s \cap (M - m) = s' \cap (M - m)$.

LEMMA 4.1. *$r : \tilde{C}(X) \to \tilde{C}(A)$ is a homology fibration. That is, the canonical inclusions of the actual fibres of r into its homotopy fibres induce isomorphisms on homology.*

The proof is given in §5.

Recall that $C_k(M)$ is $\{s \subset M : \text{card } s = k\}$, and define $g : C_k(M) \to C_{k+1}(M)$ by $g(s) = \{f_1(s)\} \cup \{m\}$, where $f_t : M \to M$ is an injective homotopy such that $f_0 = \text{id}, f_t = \text{id}$ except near m, and $m \notin f_1(M)$. Let Tel $C_k(M)$ be the telescope formed from $C_0(M) \xrightarrow{g} C_1(M) \xrightarrow{g} C_2(M) \to \dots$. There is an obvious projection π : Tel $C_k(M) \to C(M, m)$, up to homotopy.

LEMMA 4.2. *π : Tel $C_k(M) \to C(M, m)$ is a homotopy equivalence.*

Proof. Let $C^k(M, m) = \{[s] \in C(M, m) : \text{card } s \leq k\}$. Consider the diagram

$$\begin{array}{ccc} C_k(M) & \hookrightarrow & C^k(M, m) \\ \downarrow g & & \downarrow \\ C_{k+1}(M) & \hookrightarrow & C^{k+1}(M, m) \end{array}$$

where all maps except for g are inclusions. Since the diagram commutes up to homotopy, and since $C(M, m) = \varinjlim C^k(M, m)$, it suffices to show that the inclusion $j : C_k(M) \hookrightarrow C^k(M, m)$ is an equivalence.

Define $g' : C(M - m) \to C(M - m)$ by $g'(s) = \{f_1(s)\} \cup \{f_1(m)\}$ (where f_t is as in the definition of g), and let $h : C^k(M, m) \to C_k(M)$ be given by:

$$h(s) = g' \circ \dots \circ g'(f_t(s \cap (M - m))),$$

where g' is applied $k - \text{card}(s \cap (M - m))$ times, and $t \in [0, 1]$ is related to the distance $d = d(m, s \cap (M - m))$ by $t = 0$ if $d \geq$ some ε, and $t \to 1$ as $d \to 0$. One checks that h is continuous, and is a homotopy inverse to j.

It follows from this that $H_*(C(M, m)) = \varinjlim_k H_*(C_k(M))$. Therefore, to prove Theorem 1.2 we need only relate the spaces $\tilde{C}(X)$ and $\tilde{C}(A)$ to certain spaces of cross-sections. Now, using Theorem 2.6 one sees that $\tilde{C}(A)$ is homotopic to $\Gamma(\partial M, m)$, the sections of the bundle E_M over ∂M which vanish at m. However, $\tilde{C}(X)$ does not have a simple description in terms of section spaces. Therefore we consider certain covering spaces $\bar{C}(A)$ and $\bar{C}(X)$ of $\tilde{C}(A)$ and $\tilde{C}(X)$.

A point of $\bar{C}(A)$ is a pair $(s, n) : s \subset A, n \in \mathbb{Z}$ modulo the relation $(s, n) \sim (s', n')$ iff $s \cap (A - \partial A) = s' \cap (A - \partial A)$ and card $(s \cap A_0) - n =$ card $(s' \cap A_0) - n'$. Similarly, a point of $\bar{C}(X)$ is $(s, n) : S \subset X$, $n \in \mathbb{Z}$ modulo $(s, n) \sim (s', n')$ iff $s \cap (X - \partial X) = s' \cap (X - \partial X)$

and card $(s \cap \{m\}) - n =$ card $(s' \cap \{m\}) - n'$. It is trivial that each $\bar{C} \to \tilde{C}$ is a covering. Define $\bar{r} : \bar{C}(X) \to \bar{C}(A)$ by $(s, n) \mapsto (s \cap (A - \partial A)$, card $(s \cap M) - n)$. This is well-defined and continuous. Its fibres are homeomorphic to $C(M, m)$, and, by Lemma 4.1, it is a homology fibration.

LEMMA 4.3. $\bar{C}(X) \simeq \Gamma(M, m)$.

Proof. Let $\bar{C}_n(X)$ be that part of $\bar{C}(X)$ which is representable as (s, n). Then $\bar{C}_n(X) \simeq$, $\tilde{C}(M - m)$, as may be proved by an explicit deformation of particles. By Theorem 2.6 $\tilde{C}(M - m) \simeq \Gamma(M, m)$. Therefore we need only prove that $\bar{C}(X) \simeq \bar{C}_n(X)$. This will follow, if we show that the inclusion $j : \bar{C}_n(X) \subsetneq \bar{C}_n(X) \cup \bar{C}_{n+1}(X)$ (the union being taken in $\bar{C}(X)$) is an equivalence. However, let $k : \bar{C}_n(X) \cup \bar{C}_{n+1}(X) \to \bar{C}_{n+1}(X)$ be the map which adds a particle at $f_1(m)$ to all elements of $\bar{C}_n(X)$ (cf. the definition of h in the proof of (4.2)). And let $l : \bar{C}_{n+1}(X) \to \bar{C}_n(X)$ be $(s, n + 1) \mapsto (s, n)$. Clearly $l \circ k$ is a homotopy inverse to j.

LEMMA 4.4. $\bar{C}(A) \simeq \tilde{\Gamma}(\partial M, m)$, *the universal cover of* $\Gamma(\partial M, m)$.

Proof. We have already remarked that $\tilde{C}(A) \simeq \Gamma(\partial M, m)$. Now it follows from obstruction theory that $\pi_1(\Gamma(\partial M, m)) \cong \mathbb{Z}$, and that the generator of this group is a loop corresponding to the loop in $\tilde{C}(A)$ which is described by a particle moving along the ray $m' \times I$ in $A = (\partial M - m) \times I$ from $m' \times \{0\}$ to $m' \times \{1\}$, (where $m' \in \partial M - m$). Therefore $\bar{C}(A)$ is the universal cover of $\tilde{C}(A)$, and the result follows.

We are now ready to prove the version of Theorem 1.2 which is valid for compact manifolds. $\Gamma_k(M, \partial M)$ is the space of sections of E_M over M which vanish on ∂M and have degree k. (This degree can be defined even for non-orientable M using the Thom isomorphism for the sphere bundle E_M.)

THEOREM 4.5. *Let M be a connected compact manifold with non-empty boundary ∂M. Then there is a map $C(M) \to \Gamma(M, \partial M)$ which takes $C_k(M)$ into $\Gamma_k(M, \partial M)$ for each k, and induces an isomorphism*

$$\varinjlim_k H_*(C_k(M)) \cong \varinjlim_k H_*(\Gamma_k(M, \partial M)).$$

Moreover, for each n, $H_n(C_k(M)) \to H_n(\Gamma_k(M, \partial M))$ is an isomorphism for sufficiently large k.

Proof. Suppose first that ∂M is connected. Then it follows from Lemmas 4.3 and 4.4 that there is a homotopy commutative diagram

$$\begin{array}{ccc} \bar{C}(X) & \xrightarrow{\simeq} & \Gamma(M, m) \\ {\scriptstyle \bar{r}}\downarrow & & \downarrow \\ \bar{C}(A) & \xrightarrow{\simeq} & \Gamma(\partial M, m) \end{array}$$

where the right hand vertical map comes from the restriction $\Gamma(M, m) \to \Gamma(\partial M, m)$, and the horizontal maps are equivalences. Now, the fibre of $\Gamma(M, m) \to \Gamma(\partial M, m)$ is $\Gamma(M, \partial M)$, and, by obstruction theory, $\pi_0(\Gamma(M, \partial M) = \mathbb{Z}$. Also, since $\Gamma(M, m) \simeq \bar{C}(X)$ is connected and $\tilde{\Gamma}(\partial M, m)$ is simply connected (see (4.4)), the fibre of $\Gamma(M, m) \to \tilde{\Gamma}(\partial M, m)$ is connected. Therefore it is just one component, $\Gamma_0(M, \partial M)$ say, of $\Gamma(M, \partial M)$ and, by Lemmas 4.1 and

4.2 there are induced isomorphisms

$$\varinjlim_k H_*(C_k(M)) \xrightarrow[\cong]{\pi_*} H_*(C(M, m)) \xrightarrow{\cong} H_*(\Gamma_o(M, \partial M)).$$

To finish the proof of the first statement, we show how the $\Gamma_k(M, \partial M)$ form a direct system compatible with $C_o(M) \xrightarrow{g} C_1(M) \xrightarrow{g} \dots$. For each k there is a map γ_k: $\Gamma_k(M, \partial M) \to \Gamma_{k+1}(M, \partial M)$ which makes the diagram

$$\begin{array}{ccccc} C_{k,\varepsilon}(M) & \xrightarrow{\phi_\varepsilon} & \Gamma_k(M_\varepsilon, \partial M_\varepsilon) & \xleftarrow[\simeq]{} & \Gamma_k(M, \partial M) \\ \downarrow g & & & & \downarrow \gamma_k \\ C_{k+1,\varepsilon}(M) & \xrightarrow{\phi_\varepsilon} & \Gamma_{k+1}(M_\varepsilon, \partial M_\varepsilon) & \xleftarrow[\simeq]{} & \Gamma_{k+1}(M, \partial M) \end{array}$$

commute up to homotopy. (γ_k alters sections near m by "adding a section of degree 1 near m". More precisely, let $s_1 \in \Gamma_1(M, \partial M)$ vanish except in the disc $\{y \in M : d(y, m) < \varepsilon\}$. Then γ_k can be defined as the composite

$$\Gamma_k(M, \partial M) \longrightarrow \Gamma_k(M_\varepsilon, \partial M_\varepsilon) \xrightarrow{+s_1} \Gamma_{k+1}(M, \partial M),$$

where, for $s \in \Gamma_k(M_\varepsilon, \partial M_\varepsilon)$, $s + s_1$ is the section over M which equals s on M_ε and s_1 on $M - M_\varepsilon$.) Notice that γ_k is an equivalence: its inverse corresponds to "adding a section of degree -1 near m". Therefore, if $\varinjlim \Gamma_k(M, \partial M)$ is formed with respect to the γ_k, the inclusion $\Gamma_o(M, \partial M) \hookrightarrow \varinjlim \Gamma_k(M, \partial M)$ is an equivalence. One now verifies that the map $\varinjlim H_*(C_k(M)) \to \varinjlim H_*(\Gamma_k(M, \partial M)) \cong H_*(\Gamma_o(M, \partial M))$ obtained in this way is the same as that obtained previously.

To prove that $H_n(C_k(M)) \to H_n(\Gamma_k(M, \partial M))$ is an isomorphism for large k one argues as follows.† First recall that a "many-valued function $f : X \to Y$", i.e. a continuous map $\tilde{f} : X \to P^m Y$ for some m, where P^m denotes the m-fold symmetric product, induces a homomorphism $f_* : H_*(X) \to H_*(Y)$. For f extends to a homomorphism $AX \to AY$, where AX denotes the free abelian group on X, and $\pi_i(AX) \cong H_i(X)$ by the results of [3]. In particular there is a map $\tau_{k,l} : H_*(C_l(M)) \to H_*(C_k(M))$ for $k \le l$ induced by the $\binom{l}{k}$-valued function $s \mapsto \mathscr{P}_k(s)$, where $\mathscr{P}_k(s)$ is the set of subsets of $s \subset M$ of cardinality k. Notice that if $g_l : C_{l-1}(M) \to C_l(M)$ is the map which adds a particle at $m \in \partial M$ then $\tau_{k,l} \circ (g_l)_* = \tau_{k,l-1} + (g_k)_* \circ \tau_{k-1,l-1}$, because $\mathscr{P}_k(s \cup \{m\}) = \mathscr{P}_k(s) + \mathscr{P}_{k-1}(s) \cup \{m\}$. It follows from Lemma 2 of [2] that $(g_k)_*: H_*(C_{k-1}(M)) \to H_*(C_k(M))$ is the embedding of a direct summand. However we know that $\varinjlim_k H_*(C_k(M)) \xrightarrow{\cong} \varinjlim_k H_*(\Gamma_k(M, \partial M))$, and that the right-hand $\varinjlim$ is actually constant. Therefore $H_n(C_k(M)) \xrightarrow{\cong} H_n(\Gamma_k(M, \partial M))$ for all large enough k, provided that $H_n(\Gamma_k(M, \partial M))$ is finitely generated. One sees that this is true by induction over the number of n-simplices in a smooth triangulation of $(M, \partial M)$. For it is clearly true for the n-simplex $(\Delta^n, \partial\Delta^n)$, and the result when $(M, \partial M) = (M' \cup \Delta^n, \partial M)$ follows from that for $(M', \partial M')$ by considering the fibration

$$\Gamma_k(M', \partial M') \to \Gamma_k(M, \partial M) \to \Gamma(\Delta^n, \partial\Delta^n \cap \partial M).$$

† This argument and the proof of (1.1) are due to G. Segal.

This completes the proof of the theorem in the case when ∂M is connected. If ∂M is not connected, choose a component L of ∂M and let $m \in L$. The result follows by replacing A and $\tilde{C}(X)$ in the above argument by $A' = (L - m) \times [0, 1]$ and $C(M \cup A', A_1' \cup m)$ respectively.

Proof of Theorem 1.2. Here M is a paracompact open manifold without boundary. We may suppose that $M = \bigcup_{n \geq 0} M_n$, where $M_n \subseteq M_{n+1}$ and each M_n is a compact connected manifold. Choose an end of M, that is an element, E say, of $\varprojlim_n \pi_0(M - M_n)$. Thus $E = \{E_n\}$ where, for each n, E_n is a connected component of $M - M_n$ and $E_n \subset E_{n-1}$. We may choose for each n a connected component L_n of ∂M_n in such a way that $L_n \cap \bar{E}_n \neq \phi$ and L_n and L_{n-1} belong to the same connected component of $M_n - \operatorname{Int} M_{n-1}$. Pick a point m_n in L_n, and let $g_n : C_k(M_n) \to C_{k+1}(M_n)$ be the map which adds a particle at m_n. Then the diagram

$$\begin{array}{ccc} C_k(M_n) & \xrightarrow{g_n} & C_{k+1}(M_n) \\ \downarrow & & \downarrow \\ C_k(M_{n+1}) & \xrightarrow{g_{n+1}} & C_{k+1}(M_{n+1}) \end{array}$$

(where the vertical maps are induced by the inclusions $M_n \subsetneq M_{n+1}$) commutes up to homotopy, because m_n and m_{n+1} belong to the same component of $M_{n+1} - \operatorname{Int} M_n$. Therefore the g_n induce a map $g_*: \varinjlim_n H_*(C_k(M_n)) \to \varinjlim_n H_*(C_{k+1}(M_n))$. But, for each k, $H_*(C_k(M)) = \varinjlim_n H_*(C_k(M_n))$. So we have $g_* : H_*(C_k(M)) \to H_*(C_{k+1}(M))$. If $\varinjlim_k H_*(C_k(M))$ is defined using these maps g_* and $\varinjlim_k H_*(\Gamma_k(M))$ is defined similarly, we see that the theorem holds for M, since (4.5) holds for each M_n.

Notice that since $\varinjlim_k H_*(\Gamma_k(M)) = H_*(\Gamma_0(M))$ does not depend on the choice of end E, the same is true for $\varinjlim_k H_*(C_k(M))$. Also, if $H_*(\Gamma_0(M))$ is finitely generated, $H_n(\Gamma_0(M)) \cong H_n(\Gamma_0(M_l))$ for some l, and it follows that $H_n(C_k(M)) \to H_n(\Gamma_k(M))$ is an isomorphism for large k. In particular this is true if M is the interior of a compact manifold with boundary.

Proof of Theorem 1.1. Choose $\varepsilon > 0$ so that $C_k(M, \varepsilon)$, the closed subset of $C(M)$ where particles have pairwise separation $\geq 2\varepsilon$, is homotopic to $C_k(M)$. Let U be an open disc in M of radius $< \varepsilon$ and let $N = M - U$. Then the restriction map $C_k(M, \varepsilon) \to \tilde{C}(U, \varepsilon) \cong U^+ \cong S^n$ gives rise to a map $p : C_k(M) \to S^n$. Over $S^n - \infty = U$, p is a fibration with fibre $C_{k-1}(N)$, while $p^{-1}(\infty) = C_k(N)$. Therefore the homology exact sequence for $(C_k(M), C_k(N))$ has as relative group $\tilde{H}_*(S^n(C_{k-1}(N)^+)) = H_{*-n}(C_{k-1}(N))$. Now consider the commutative diagram

$$\begin{array}{ccccccccc} \dots \to & H_{q-n+1}(C_{k-1}(N)) & \to & H_q(C_k(N)) & \to & H_q(C_k(M)) & \to & H_{q-n}(C_{k-1}(N)) & \to \dots \\ & \downarrow{\scriptstyle (*)} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle (*)} & \\ \dots \to & H_{q-n+1}(\Gamma_k(N, \partial N)) & \to & H_q(\Gamma_k(N, \partial N)) & \to & H_q(\Gamma_k(M)) & \to & H_{q-n}(\Gamma_k(N, \partial N)) & \to \dots \end{array}$$

where the bottom sequence is that of the pair $(\Gamma_k(M), \Gamma_k(M, \bar{U}) \cong \Gamma_k(N, \partial N))$ identified with the Wang sequence of the fibration $\Gamma_k(M, \bar{U}) \to \Gamma_k(M) \to S^n$, and the vertical maps are the obvious ones, except that (*) is the obvious one composed with adding a section of degree $+1$ at a point of ∂N. By Theorem 4.5 we may apply the 5-lemma to conclude that $H_n(C_k(M)) \xrightarrow{\cong} H_n(\Gamma_k(M))$ for large n.

§5. ON HOMOLOGY FIBRATIONS

A map $r\colon Y \to X$ is called a homology fibration if the canonical inclusions of its real fibres $r^{-1}(x)$ into its homotopy fibres $F(r, x)$ induce isomorphisms on homology for all $x \in X$. We prove a result for such fibrations similar to that for quasifibrations stated as Lemma 3.3 above. To do this, it seems to be necessary to assume that X has nice local properties. Recall that X is said to be uniformly locally connected (ULC) if there is a neighbourhood V of the diagonal in $X \times X$ and a map $\lambda : V \times I \to X$ such that $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$ and $\lambda(x, x, t) = x$ for all $(x, y) \in V$, $t \in I$ (see [8, p. 490]).

PROPOSITION 5.1. *Let $X = \bigcup X_k$ where each X_k is closed and $X_1 \subseteq X_2 \subseteq X_3 \ldots$. Let $r : Y \to X$ be a map. Suppose that*

(i) *all spaces X_k, $X_k - X_{k-1}$, $r^{-1}(X_k)$, $r^{-1}(X_k - X_{k-1})$, have the homotopy type of CW complexes*;

(ii) *each X_k is ULC*;

(iii) *each $x \in X$ has a basis of contractible neighbourhoods U such that the contraction of U lifts to a deformation retraction of $r^{-1}(U)$ into $r^{-1}(x)$*;

(iv) *each $r\colon r^{-1}(X_k - X_{k-1}) \to X_k - X_{k-1}$ is a fibration with fibre F*;

(v) *for each k, there is an open subset U_k of X_k such that $X_{k-1} \subseteq U_k$, and there are homotopies $h_t\colon U_k \to U_k$ and $H_t\colon r^{-1}(U_k) \to r^{-1}(U_k)$ satisfying*

(a) $h_o = \mathrm{id}$, $h_t(X_{k-1}) \subseteq X_{k-1}$, $h_1(U_k) \subseteq X_{k-1}$;

(b) $H_o = \mathrm{id}$, $r \circ H_t = h_t \circ r$;

(c) $H_1 : r^{-1}(x_k) \to r^{-1}(H_1(x))$ *induces an isomorphism on homology for all* $x \in U_k$.

Then $r : Y \to X$ is a homology fibration with fibre F.

The proof follows that of the lemma for quasifibrations (see [3], (2.2), (2.10), (2.15)). Using (v) one shows that if $r|r^{-1}(X_k)$ is a homology fibration, so is $r|r^{-1}(U_k)$. Also, if $r|r^{-1}(X_k)$ is a homology fibration for each k, $r : Y \to X$ is too, because its homotopy fibre is just $\varinjlim_k F_k$, where F_k is the homotopy fibre of $r|r^{-1}(X_k)$. The proof is completed by showing that if $r|r^{-1}(U_k)$ and $r|r^{-1}(X_k - X_{k-1})$ are homology fibrations, then their union $r|r^{-1}(X_k)$ is too. It is for this that the extra conditions (i), (ii), (iii) of (5.1) are needed.

LEMMA 5.2. *Let $X = V_1 \cup V_2$ (where each V_i is open), and let $r : Y \to X$ be a map such that $r|r^{-1}(V_i)$ is a homology fibration for $i = 1, 2$. Suppose that*

(i) *all the spaces X, Y, V_i, $r^{-1}(V_i)$, have the homotopy type of CW complexes*;

(ii) *X is ULC*;

(iii) *condition* (iii) *of* (5.1) *is satisfied.*

Then $r: Y \to X$ is a homology fibration.

Proof.† Choose $x_0 \in X$, and let P be the space of paths in X beginning at x_0. Let $p: P \to X$ be the endpoint map. Then p is open, and, using (ii) and (iii) one sees that condition (iii) of (5.1) is satisfied for the map $p^* Y \to P$ and all $\xi \in P$. The fibres of $p^* Y \to P$ are just those of p. Also, $p^* Y$ is the homotopy fibre of r at x_0. Therefore, it suffices to prove that the map $p^* Y \to P$, which we will call g, is a homology fibration. Note if $P_i = p^{-1}(V_i)$, g is a homology fibration over P_1 and P_2, since the maps $P_i \to V_i$ are fibrations.

By (i), all the spaces we are concerned with have the homotopy type of CW complexes. Therefore g will induce an isomorphism on homology if and only if it induces an isomorphism on Čech cohomology with coefficients in any abelian group. Now, one has the Leray spectral sequence $H^p(P, H^q g) \Rightarrow H^*(p^* Y)$, where $H^q g$ is the sheaf associated to the presheaf $U \mapsto H^q(g^{-1}(U))$ on P. It will suffice to show that $H^q g$ is locally constant with stalk $H^q(g^{-1}(\xi))$ at $\xi \in P$, since then the spectral sequence collapses. That the stalk is as asserted follows at once from condition (iii) (for g), for the stalk is $\varinjlim_{\varepsilon} H^q(g^{-1}(U))$. As for local constancy, let U be a neighbourhood of ξ satisfying (iii), and suppose that $U \subset P_1$, say. Then we need only show that $H^q(g^{-1}(U)) \xrightarrow{\cong} H^q(g^{-1}(\eta))$ for any $\eta \in U$. Consider the diagram

$$\begin{array}{ccccc} g^{-1}(\eta) & \longrightarrow & g^{-1}(U) & \xleftarrow{H\simeq} & g^{-1}(\xi) \\ {\scriptstyle H\simeq}\downarrow & & \downarrow & & \downarrow{\scriptstyle H\simeq} \\ F(\eta) & \xrightarrow{\simeq} & \Pi & \xleftarrow{\simeq} & F(\xi) \end{array}$$

where $F(\xi)$ and $F(\eta)$ are the homotopy fibres of $g|g^{-1}(P_1)$ at ξ and η, Π is the homotopy theoretic fibre product of U and $g^{-1}(P_1)$ over P_1, and all maps are the obvious inclusions. We wish to show that $g^{-1}(\eta) \to g^{-1}(U)$ is a homology isomorphism. But the diagram commutes, and the maps marked $H\simeq$ are known to be homology isomorphisms. Thus the result follows.

Proof of Lemma 4.1. The proof is much the same as that of Proposition 3.1. $\tilde{C}(A)$ is filtered by the sets $\tilde{C}^k(A) = \{s \subset A: \text{card } s \leq k\}$, and U_k, h_t and H_t are defined as before. It is clear that conditions (iv) and (v) (a), (b) of (5.1) are satisfied. We must check that the others hold. Now (i) follows from Lemma 2.1, and it is not difficult to see that (ii) holds. Also (iii) follows from the remark that if $U \subset \tilde{C}(A)$, any homotopy $c_t: U \to U$ with $c_0 = \text{id}$ may be lifted to $r^{-1}(U)$ as long as it satisfies the condition: for all $x \in U$ and $t_0 \in I$, $c_{t_0}(x) \in \tilde{C}^k(A)$ implies that $c_t(x) \in \tilde{C}^k(A)$ for all $t \geq t_0$. It remains to prove that $H_1: r^{-1}(x) \to r^{-1}(h_1(x))$ is a homology isomorphism for all $x \in U_k$. By arguing as in the proof of Lemma 3.4 one sees that this follows from

LEMMA 5.3. *Let $n \neq m$ be a point of ∂M, and define $h: C(M, m) \to C(M, m)$ by $[s] \to [g_1(s)] \cup \{n\}$, where $g_t: M \to M$ is an injective homotopy such that $g_0 = \text{id}$, $g_t = \text{id}$ except near n, and $n \notin g_1(M)$. Then h induces an isomorphism on homology.*

† This proof is an adaptation by G. Segal of an unpublished proof by D. Quillen of the group completion theorem for a topological monoid.

Proof. It is sufficient to prove this for the corresponding map h : Tel $C_k(M) \to$ Tel $C_k(M)$ (see (4.2)). Let Tel′ $\bigcup_{\text{even } k} C_k \times [k, k+1]$. Then the inclusion i : Tel′$\hookrightarrow$ Tel is homotopic to $h \circ i$ and induces a surjection on homology. It follows that h induces an isomorphism on homology.

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