Computers and Mathematics with Applications 61 (2011) 1946–1950

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Solving random diffusion models with nonlinear perturbations by the Wiener–Hermite expansion method

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ARTICLE INFO

Keywords:
Random differential equation
Wiener–Hermite expansion
Perturbation method

ABSTRACT

This paper deals with the construction of approximate series solutions of random nonlinear diffusion equations where nonlinearity is considered by means of a frank small parameter and uncertainty is introduced through white noise in the forcing term. For the simpler but important case in which the diffusion coefficient is time independent, we provide a Gaussian approximation of the solution stochastic process by taking advantage of the Wiener–Hermite expansion together with the perturbation method. In addition, approximations of the main statistical functions associated with a solution, such as the mean and variance, are computed. Numerical values of these functions are compared with respect to those obtained by applying the Runge–Kutta second-order stochastic scheme as an illustrative example.

1. Introduction

Deterministic differential equations of the form $\dot{x}(t) = a(t)x(t)$ constitute the basic form of so-called diffusion or transport problems which appear in relevant models such as: the growth population geometric (or Malthusian) model in biology, where $a(t)$ represents the *per capita* growth rate; the neutron and gamma ray transport model in physics, where coefficient $a(t)$ involves the geometry of the cross-sections of the medium; the continuous composed interest rate models for studying the evolution of an investment under time-variable interest rate $r(t)$ in which case $a(t) = 1+r(t)$; etc. Despite the usefulness of these basic models, they do not often cover all possible situations observed from a practical point of view. In fact, as a simple but illustrative example, if $a(t) = a > 0$, the Malthus model predicts unlimited growth of a species despite the fact that resources are always limited. Then, the logistic (or Verhulst) model introduces a nonlinear term in order to overcome this drawback by considering the differential equation $\dot{x}(t) = ax(t) - b(x(t))^2,\ a, b > 0$, where the nonlinearity intensity is given by parameter b. In many practical situations it is appropriate to assume that the nonlinear term affecting the phenomena under study is small enough; then its intensity is controlled by means of a frank small parameter, say ϵ. Relevant examples in this sense appear for instance in epidemiology, where the so-called SIS models become nonlinear differential equations where the nonlinear term coefficient denoting the contagious rate can be assumed to be a frank small parameter in many situations [1]. In addition to these considerations, diffusion models with nonlinear perturbations can also consider the introduction of a forcing term in order to model external aspects which can become very complex, such as: the environment in biology; unexpected material changes in the surrounding medium in physics; and foreign political events that can affect the markets where an investment has been ordered in finance. Stochastic differential equations based on the white noise process provide a powerful tool for dynamically modeling these complex and uncertain aspects. Over

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doi:10.1016/j.camwa.2010.07.057

the last few years, new and relevant methods for finding the exact solutions of such equations have been developed. They include the homotopy perturbation method [2–4] and the Exp-function method [5,6].

This paper deals with the solution of stochastic differential models of the form

$$\dot{x}(t) = a(t)x(t) - \epsilon(x(t))^2 + \lambda n(t), \quad t > 0, \qquad x(0) = x_0, \tag{1}$$

where the diffusion coefficient $a(t)$ and initial condition x_0 are deterministic, ϵ is a small parameter and $n(t) = n(t)(\omega)$ is the white noise process, whose intensity is given by parameter λ, and ω is a random outcome for a triple probability space $(\Omega, \mathcal{A}, P)$ where Ω is a sample space, $\mathcal{A}$ is a σ-algebra associated with Ω and P is a probability measure.

The paper is organized as follows. Section 2 summarizes the main results of the Wiener–Hermite expansion (WHE) providing a powerful technique for representing any stochastic process in terms of certain deterministic kernels to be determined as well as the so-called Wiener–Hermite (WH) polynomials. In Section 3, the WHE is applied in order to obtain two initial integro-differential equations that are satisfied by these kernels. By taking advantage of the perturbation method the solutions of these equations are obtained in Section 4. Previous development is illustrated for the simpler but important case where the diffusion coefficient is autonomous. In addition, we compute approximations for its main statistical moments such as the mean and variance. A comparison of the results obtained with respect to the Runge–Kutta second-order stochastic scheme for solving stochastic differential equations is also provided. Conclusions are shown in Section 5.

2. The Wiener–Hermite expansion (WHE)

For the sake of clarity in the presentation, we summarize the main ideas of the Wiener–Hermite expansion (WHE) in this section on the basis of the Wiener–Hermite (WH) polynomials. For further details we recommend [7,8,4]. The WH polynomials constitute a complete set of statistically orthogonal stochastic processes, say $H^{(i)} = H^{(i)}(t_1, \dots, t_i)$, defined in terms of white noise $n(t)$ and the Dirac delta function $\delta(\cdot)$ through the following recurrence relations:

$$H^{(i)}(t_1, \dots, t_i) = H^{(i-1)}(t_1, \dots, t_{i-1})H^{(1)}(t_i) - \sum_{j=1}^{i-1} H^{(i-2)}(t_{i_1}, \dots, t_{i_{i-2}})\delta(t_{i-j} - t_i), \quad i \geq 2, \tag{2}$$

starting from $H^{(0)} = 1$ and $H^{(1)}(t_1) = n(t_1)$. Taking into account the following statistical properties of white noise processes:

$$\mathrm{E}[n(t)] = 0, \qquad \mathrm{E}[n(t_1)n(t_2)] = \delta(t_1 - t_2), \tag{3}$$

where $\mathrm{E}[\cdot]$ denotes the expectation operator, one can establish that WH polynomials are centered with respect to the origin (except $\mathrm{E}\left[H^{(0)}\right] = 1$) and they are statistically orthogonal:

$$\mathrm{E}\left[H^{(i)}\right] = 0, \quad \forall i \geq 1; \qquad \mathrm{E}\left[H^{(i)}H^{(j)}\right] = 0, \quad \forall i \neq j. \tag{4}$$

As a consequence of the completeness of the WH set [8], any arbitrary stochastic process, say $x(t) = x(t;\omega)$, $\omega \in \Omega$, can be expanded in terms of a WH polynomial set and this expansion converges to the original stochastic process, i.e.,

$$x(t) = x^{(0)}(t) + \int_{\mathbb{R}} x^{(1)}(t;t_1)H^{(1)}(t_1)\,\mathrm{d}t_1 + \int_{\mathbb{R}^2} x^{(2)}(t;t_1,t_2)H^{(2)}(t_1,t_2)\,\mathrm{d}t_1\mathrm{d}t_2 + \cdots, \tag{5}$$

where $x^{(0)} = x^{(0)}(t)$, $x^{(i)} = x^{(i)}(t;t_1,\dots,t_i)$, $i \geq 1$, are called the (deterministic) kernels of the WHE of $x(t)$. The first two terms of the right-hand side define the Gaussian representation of $x(t)$ (the zeroth-order term being just its mean or average, i.e., $\mathrm{E}[x(t)] = x^{(0)}(t)$), while the second-order and higher terms correspond to the non-Gaussian part. The variance of $x(t)$ can be expressed as follows:

$$\mathrm{Var}[x(t)] = \int_{\mathbb{R}} \left(x^{(1)}(t;t_1)\right)^2 \mathrm{d}t_1 + 2\int_{\mathbb{R}^2} \left(x^{(2)}(t;t_1,t_2)\right)^2 \mathrm{d}t_1\mathrm{d}t_2 + \cdots. \tag{6}$$

3. Application of the WHE to approximate the solution of the nonlinear problem

In this section we will apply the WHE in order to analyze the response of the nonlinear model (1) to the Gaussian stochastic process $n(t)$ with intensity λ. The procedure can be described as follows: first, from the original governing equation (1), we expand the unknown $x(t)$ by means of the WHE given by (5); then, integral–differential deterministic equations are derived for the dynamics of the unknown kernel functions $x^{(i)}$ of the WHE of the response. For that, we take advantage of the stochastic orthogonality properties of WH polynomials.

In practice, the WHE series for the response must be truncated after a few terms. Henceforth, we are only interested in obtaining the Gaussian approximation of the response $x(t)$ of problem (1); then two integral–differential equations for $x^{(0)}(t)$ and $x^{(1)}(t;t_1)$ must be established. For the first one, we just follow the previous procedure: we substitute the WHE (5) of $x(t)$ in model (1); next we take the expectation operator over the resulting expression and, finally, we take advantage of properties (4) as well as the fact that $\mathrm{E}\left[H^{(1)}(t_1)H^{(1)}(t_2)\right] = \delta(t_1 - t_2)$ and $\mathrm{E}\left[H^{(2)}(t_1,t_2)H^{(2)}(t_3,t_4)\right] = \delta(t_1 - t_3)\delta(t_2 - t_4) + \delta(t_1 - t_4)\delta(t_2 - t_3)$. This leads to

$$\dot{x}^{(0)}(t) = a(t)x^{(0)}(t) - \epsilon \left\{ \left(x^{(0)}(t)\right)^2 + \int_{\mathbb{R}} \left(x^{(1)}(t;t_1)\right)^2 \, \mathrm{d}t_1 \right\}, \qquad x^{(0)}(0) = x_0, \tag{7}$$

where the initial condition has been derived by setting $t = 0$ in (5), next applying the expectation operator, and again taking advantage of the first property given by (4). In order to establish another (deterministic) differential equation for $x^{(1)}(t;t_1)$, firstly we multiply WHE (5) of $x(t)$ by $H^{(1)}(t_5)$, next we take the expectation operator, and then we again apply the above properties together with $\mathrm{E}\left[H^{(1)}(t_1)H^{(1)}(t_2)H^{(1)}(t_3)\right] = 0$, $\mathrm{E}\left[H^{(1)}(t_1)H^{(2)}(t_2,t_3)H^{(2)}(t_4,t_5)\right] = 0$ and $\mathrm{E}\left[H^{(1)}(t_1)H^{(1)}(t_2)H^{(2)}(t_3,t_4)\right] = \delta(t_1-t_3)\delta(t_2-t_4) + \delta(t_1-t_4)\delta(t_2-t_3)$. In this way, one gets

$$\dot{x}^{(1)}(t;t_1) = a(t)x^{(1)}(t;t_1) - 2\epsilon x^{(0)}(t)x^{(1)}(t;t_1) + \lambda\delta(t-t_1), \qquad x^{(1)}(0;t_1) = 0, \quad \forall t_1. \tag{8}$$

In this case, the initial condition has been derived by multiplying the WHE (5) by $H^{(1)}(t_2)$, next setting $t = 0$, then taking the expectation operator and, finally, applying the first property of (4) as well as $\mathrm{E}\left[H^{(1)}(t_1)H^{(1)}(t_2)\right] = \delta(t_1-t_2)$.

4. The application of the perturbation method. An illustrative example

In order to compute the Gaussian part of the stochastic process solution of problem (1), we need to solve the nonlinear coupled deterministic problems (7)–(8). Note that both problems depend on the small parameter $\epsilon > 0$. Then a reliable technique for solving them is the so-called perturbation method under which the deterministic kernels can be represented in a first approximation as

$$x^{(0)}(t) = x_0^{(0)}(t) + \epsilon x_1^{(0)}(t), \qquad x^{(1)}(t;t_1) = x_0^{(1)}(t;t_1) + \epsilon x_1^{(1)}(t;t_1). \tag{9}$$

Substituting these representations in Eqs. (7)–(8) and neglecting these powers of ϵ whose exponents are greater than 1, one obtains the following initial value problems:

$$\dot{x}_0^{(0)}(t) = a(t)x_0^{(0)}(t), \qquad x_0^{(0)}(0) = x_0, \tag{10}$$

$$\dot{x}_1^{(0)}(t) = a(t)x_1^{(0)}(t) - \left(x_0^{(0)}(t)\right)^2 - \int_0^\infty \left(x_0^{(1)}(t;t_1)\right)^2 \mathrm{d}t_1, \qquad x_1^{(0)}(0) = 0, \tag{11}$$

$$\dot{x}_0^{(1)}(t;t_1) = a(t)x_0^{(1)}(t;t_1) + \lambda\delta(t-t_1), \qquad x_0^{(1)}(0;t_1) = 0, \quad \forall t_1 \geq 0, \tag{12}$$

$$\dot{x}_1^{(1)}(t;t_1) = a(t)x_1^{(1)}(t;t_1) - 2x_0^{(0)}(t)x_0^{(1)}(t;t_1), \qquad x_1^{(1)}(0;t_1) = 0, \quad \forall t_1 \geq 0. \tag{13}$$

Example 1. Let us consider the frequently encountered situation where the diffusion coefficient does not depend on time, i.e., $a(t) = a$. In this case, we first compute directly the solutions of problems (10) and (12), and after that we solve problems (11) and (13). The results obtained are

$$x_0^{(0)}(t) = x_0 \mathrm{e}^{at}, \qquad x_1^{(0)}(t) = -\frac{\left(\mathrm{e}^{at}-1\right)\left(\mathrm{e}^{at}(2a(x_0)^2+\lambda^2)-\lambda^2\right)}{2a^2}, \tag{14}$$

$$x_0^{(1)}(t;t_1) = \begin{cases} \lambda \mathrm{e}^{a(t-t_1)} & \text{if } t \geq t_1, \\ 0 & \text{if } t < t_1, \end{cases} \tag{15}$$

$$x_1^{(1)}(t;t_1) = \begin{cases} -\dfrac{2\mathrm{e}^{a(t-t_1)}\left(\mathrm{e}^{at}-1\right)\lambda x_0}{a} & \text{if } t \geq t_1, \\ 0 & \text{if } t < t_1. \end{cases} \tag{16}$$

Taking into account that $\mathrm{E}[x(t)] = x^{(0)}(t) = x_0^{(0)}(t) + \epsilon x_1^{(0)}(t)$, one gets the following approximation of the mean of $x(t)$:

$$\mathrm{E}[x(t)] = x_0\mathrm{e}^{at} - \epsilon \frac{\left(\mathrm{e}^{at}-1\right)\left(\mathrm{e}^{at}(2a(x_0)^2+\lambda^2)-\lambda^2\right)}{2a^2}. \tag{17}$$

As regards the variance approximation, note that by the perturbation method and (6) one gets

$$\mathrm{Var}[x(t)] = \int_{-\infty}^{\infty} \left\{ \left(x_0^{(1)}(t;t_1)\right)^2 + 2\epsilon x_0^{(1)}(t;t_1)x_1^{(1)}(t;t_1) + \epsilon^2 \left(x_1^{(1)}(t;t_1)\right)^2 \right\} \mathrm{d}t_1, \tag{18}$$

which leads in our case to

$$\mathrm{Var}[x(t)] = \frac{\lambda^2}{2a}\left(\mathrm{e}^{2at}-1\right) - 2\epsilon\frac{\lambda^2 x_0}{a^2}\left(\mathrm{e}^{at}-1\right)^2\left(\mathrm{e}^{at}+1\right) + 2\epsilon^2\frac{\lambda^2(x_0)^2}{a^3}\left(\mathrm{e}^{at}-1\right)^3\left(\mathrm{e}^{at}+1\right). \tag{19}$$

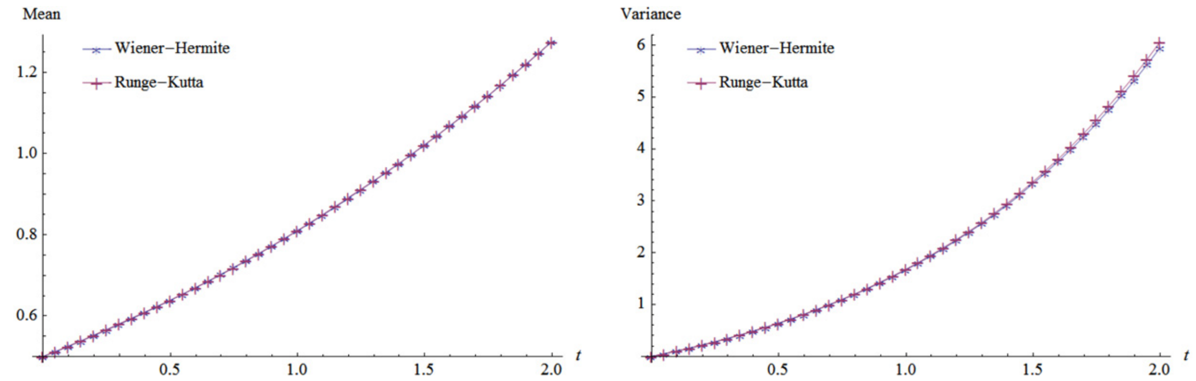

Fig. 1. Comparison of the expectation (left) and the variance (right) obtained by using the Wiener–Hermite expansion technique for problem (1) with $a(t) = 1/2$, $\lambda = 1$, $\epsilon = 10^{-2}$ and $x_0 = 0.5$ on the interval $[0, 2]$ and a Runge–Kutta stochastic scheme considering $m = 100\,000$ simulations and taking as the step $h = 10^{-4}$.

Table 1
Numerical values of the expectation and variance as well as the relative errors obtained by using the Wiener–Hermite expansion technique for problem (1) with $a(t) = 1/2$, $\lambda = 1$, $\epsilon = 10^{-2}$ and $x_0 = 0.5$ on the interval $[0, 2]$ and a Runge–Kutta stochastic scheme considering $m = 100\,000$ simulations and taking as the step $h = 10^{-4}$.

t	$\mathrm{E}\left[x^{\mathrm{WHE}}(t)\right]$	RelErrE	$\mathrm{Var}\left[x^{\mathrm{WHE}}(t)\right]$	RelErrVar
0.00	0.5	0	0	0
0.25	0.565465	0.001495642	0.282515	0.004461187
0.50	0.638576	0.000480526	0.641372	0.004179709
0.75	0.720045	0.0009411	1.09676	0.00440265
1.00	0.810596	0.00063616	1.67398	0.00537721
1.25	0.9110935	0.00100017	2.4046	0.00656068
1.50	1.02172	0.00034244	3.32786	0.01095191
1.75	1.14352	0.0006563	4.49228	0.01562586
2.00	1.27674	0.0006662	5.95747	0.01874725

Columns $\mathrm{E}\left[x^{\mathrm{WHE}}(t)\right]$ and $\mathrm{Var}\left[x^{\mathrm{WHE}}(t)\right]$ of Table 1 show the results for the average and variance obtained from (17) and (19), respectively, for $\lambda = 1$, $\epsilon = 10^{-2}$, $a = 1/2$, $x_0 = 0.5$. In Fig. 1, we compare these results with respect to those obtained by using a Runge–Kutta second-order stochastic scheme [9], where the Brownian motion involved has been simulated taking $m = 100\,000$ simulations and step $h = 10^{-4}$. In addition, the third and fifth columns of Table 1 show the relative errors for the average (RelErrE) and variance (RelErrVar) with respect to the Runge–Kutta scheme results. Note that the approximations obtained from the two approaches agree.

5. Conclusions

This paper shows that the WHE technique constitutes a powerful tool for constructing approximate solutions for the stochastic process for random diffusion models with nonlinear perturbations where uncertainty is considered by means of an additive term defined by white noise. As has been highlighted, such models appear in frequent applications in fields such as physics and epidemiology. Although the success of the WHE method depends heavily on the complexity encountered in dealing with integro-differential equations, a large number of deterministic techniques for solving them are available, including mathematical software [10]. Besides computing the Gaussian approximation of the solution, we have also provided approximations of its average and variance. As we have shown, these results agree with those obtained by applying other stochastic numerical methods. Finally, we remark that in the near future, we will report on the corresponding results for the non-Gaussian approximation.

Acknowledgements

This work was partially supported by the Spanish M.C.Y.T. and FEDER grants MTM2009-08587, TRA2007-68006-C02-02, DPI2010-20891-C02-01 as well as the Universidad Politécnica de Valencia grant PAID-06-09 (ref. 2588).

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