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Prime power divisors of binomial coefficients

Paul Erdős, Grigori Kolesnik *

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Abstract

It is known that for sufficiently large n and m and any r the binomial coefficient $\binom{n}{m}$ which is close to the middle coefficient is divisible by p^r where p is a 'large' prime. We prove the exact divisibility of $\binom{n}{m}$ by p^r for $p>c(n)$. The lower bound is essentially the best possible. We also prove some other results on divisibility of binomial coefficients.

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1. Introduction

Paul Erdős asked many questions about the divisibility of binomial coefficients by prime powers [1,2]. One of his well-known conjectures was that $\binom{n}{m}$ is not square-free if $n>4$. This conjecture was proved by Sárközy [11] for sufficiently large n and proved independently by Velammal [12] and by Granville and Ramare [4] for all $n>4$. Granville and Ramare have, in fact, proved more-they showed that $p^2 \| \binom{n}{m}$ for some $p \geqslant \sqrt[2]{n/5}$, among some other results. Another well-known conjecture of Erdős is that for any n and any $m \in [3, n/2]$ there exists a prime $p \geqslant m$ such that $p \| \binom{n}{m}$. Granville [3] proved that this is true for a large n and $(\log m)^3 \gg (\log n)^2 \log\log n$. Sander considered divisibility of $\binom{n}{m}$ by prime powers [6–10]. He proved that for any $a \in N$ and any $\varepsilon \in (0,1)$ there exists $m_0 = m_o(a,\varepsilon)$ such that for all $m>m_0$ and all n satisfying $|n-2m| < m^{(1-\varepsilon)}$, $p^a \| \binom{n}{m}$ for some prime $p > 1/2m^{1/(1+a)}$. He also proved [6] that for every j satisfying $c_0 \leqslant j^{10}(\log j)^3 \leqslant \log n$ there exist $\gg (\log n/(\log\log n)^3)^{1/10}$ primes $p \geqslant n^{1/(1+j)}$ such that $p^j \| \binom{n}{m}$. Sander also proved that if $m_i + m_j \geqslant c_0$ and $|m_i - m_j| < (m_i + m_j)^{(1-\varepsilon)}$ for some $1 \leqslant i < j \leqslant k$ then there exists $p \geqslant p_0$ such that $p^\alpha \mid [m_1, \ldots, m_k] = (m_1 + \cdots + m_k)!/(m_1! \cdots m_k!)$. The second author became interested

* Corresponding author. E-mail: gkolesn@calstatela.edu.

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in the problem after Andrew Granville gave him a preprint of his paper [3]. Erdős and Kolesnik had several conversations regarding the problems and had an extensive correspondence in which we shared our ideas how to solve the problems. Unfortunately, Erdős died before our paper was completed so that the second author had to finish it alone. We improved and generalized all mentioned above results as well as answered a question of Erdős on how large is an integer d such that $d^r \| \binom{n}{m}$. We proved

Theorem 1. *Let n be a sufficiently large integer. Assume that $r \leqslant \sqrt[4]{\log n/\log\log n}$ and $n^{1-a/r} \leqslant m \leqslant n/2$ for some $a < \frac{1}{4}$. Then every subinterval $[P, P_1]$ of $[(n-m)^{1/r}, n^{1/r}]$ with $P_1 - P \geqslant n^{a_1/r}$ and $a_1 > 1 - a$ contains $\approx 2^{1-r}[\pi(P_1) - \pi(P)]$ primes such that $p^r \| \binom{n}{m}$. Also, if d is the largest square-free number such that $d^r \| \binom{n}{m}$, then $d = \exp(2^{1-r} m^{1/r}(c_0 + o(1)))$, where $c_0 = c_0(b) = \sum_{v=1}^{\infty}[1-(1-\delta(v))^{1/r}] > \frac{1}{8}$, $b = m/n$, $\delta(v) = \delta(b, v) = (v - \lfloor bv \rfloor)$ if $\{bv\} \leqslant 1 - b$ and $\delta(v) = \|bv\|/(bv + \|bv\|)$ otherwise. If b is small then $c_0 = (1/r)\int_0^\infty \{x\} x^{1-1/r}\,dx + O(1/rb^{1-1/r})$ if $r > 1$ and $c_0 = \log 1/b + 1 + O(b)$ if $r = 1$. If $b = \frac{1}{2} + \varepsilon$ for a small ε then $c_0 = \sum_{n=0}^{\infty}(n+\frac{1}{2})^{-1/r}[1-(1-1/(2n+2))^{1/r}] + O(\varepsilon)$.*

Theorem 2. *Let r, m, n be positive integers satisfying*

$$1 \leqslant r^4 \ll (\log m)^3/((\log n)^2 \log\log n) \text{ and } m \leqslant n/2.$$

Then there are $\gg r m^{1/r}/(4^r \log m)$ primes $p \in [m^{1/r}, n^{1/r}]$ such that $p^r \| \binom{n}{m}$.

A simple corollary of Theorem 2 is:

Theorem 3. *Let $r \in N$. Assume that for some i, j m_i is a sufficiently large integer such that $(\log m_i)^3 \gg (\log m_j)^2 \log\log m_j$. Then there are $\gg (m_i)^{1/r}/\log m_i$ primes $p \in [(m_i)^{1/r}, (m_i + m_j)^{1/r}]$ such that $p^r | [m_1, \ldots, m_k]$.*

Proof. To prove Theorem 3 we notice that

$$[m_1, \ldots, m_k] = [m_i, m_j][m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_{j-1}, m_{j+1}, \ldots, m_k]$$

so that Theorem 3 follows from Theorem 2.

For the sake of completeness we prove:

Theorem 4. *Let $2 \leqslant m \leqslant n/2$ and let n be large. Then there exist $\gg m/\log m$ primes $p > m$ such that $p | \binom{n}{m}$.*

Proof. To prove Theorem 4 we write

$$\log(n!/(n-m)!) = \sum_{p \leqslant m} \alpha_p \log p, \tag{1}$$

where

$$\alpha_p = \sum_{j=1}^{J} (\lfloor n/p^j \rfloor - \lfloor (n-m)/p^j \rfloor - \lfloor m/p^j \rfloor)$$
$$= \sum (\{(n-m)/p^j\} - \{n/p^j\} + \{m/p^j\})$$

$J = \lfloor \log n / \log P \rfloor$. Since

$$\sum_{p \leqslant m} \alpha_p \log p < m \sum_{k \leqslant m} \Lambda(k)/k + \pi(m) \log n$$

and

$$\sum_{k=n-m+1}^{n} \log k > m \log(n-m) - n \log(1 - m/n) - m$$

$-\frac{1}{2}\log(1-m/n) + 1/(6n)$ and, for $p > m$, $\alpha_p \log p \leqslant \log n$, (1) implies that the number of primes $> m$ which divide $n!/(n-m)!$ (and, therefore, $\binom{n}{m}$) is $\geqslant 1/\log n (m \log(n - m) - m - n \log(1 - m/n) - \frac{1}{2}\log(1 - m/n) - 1/(6n) - m \sum_{k \leqslant m} \Lambda(k)/k - \pi(m) \log n)$. The above expression is $\gg m$ if $m < cn$ for appropriate c. If $cn \leqslant m \leqslant n/2$ then $\alpha_p = 1$ for all $p \in (n-m, n]$; the number of such primes, as well-known, is $\gg m/\log m$. □

2. Notation

Throughout the paper we will use the following notation: $\{x\}$ — fractional part of x; $\lfloor x \rfloor = x - \{x\}$; $\lceil x \rceil = -\lfloor -x \rfloor$; $\|x\| = \inf\{|n-x|: n \in N\}$; $e(x) = \exp(2\pi i x)$; $f(x) \ll g(x)$ means $f(x) = O(g(x))$; $f(x) \sim g(x)$ means $1 \ll |f(x)/g(x)| \ll 1$; $f(x) \approx g(x)$ means $f(x) = g(x)(1 + o(1))$; $|A|$ — the cardinality of the set A.

3. Lemmas

Lemma 1. *Let $\chi(x) = \chi_{[\alpha,\beta)}(x)$ be the characteristic function of $[\alpha, \beta) \subseteq [0,1)$ modulo 1. Then for any $\delta > 0$ and any positive integer k we have $\chi(x) \leqslant \beta - \alpha + 3\delta + \sum_{1 \leqslant |l| \leqslant L} a_l e(lx)$ and $\chi(x) \leqslant \beta - \alpha - 3\delta + \sum_{1 \leqslant |l| \leqslant L} b_l e(lx)$, where $L = \lceil k/(2\pi\delta)(2/(\pi k \delta))^{1/k} \rceil$, $a_l = (e(\delta l - \alpha l) - e(-\beta l - \delta l))/(2\pi i l)(\sin(2\pi\delta l/k)/(2\pi\delta l/k))^k$ and $b_j = -a_j$ with α and β interchanged.*

Proof. Taking $\Delta = \delta/k$, we obtain

$$\chi(x) \leqslant (2\Delta)^{-k} \int_{-\Delta}^{\Delta} \cdots \int_{-\Delta}^{\Delta} \chi_{[\alpha-\delta,\beta+\delta)}(x + t_1 + \cdots + t_k)\, dt_1 \cdots dt_k$$

and

$$\chi(x) \geqslant (2\Delta)^{-k} \int_{-\Delta}^{\Delta} \cdots \int_{-\Delta}^{\Delta} \chi_{[\alpha+\delta,\beta-\delta)}(x+t_1+\cdots+t_k)\,\mathrm{d}t_1 \cdots \mathrm{d}t_k. \tag{2}$$

Expanding the characteristic function inside the integral sign into Fourier series we obtain

$$\chi(x) \leqslant (2\Delta)^{-k} \int_{-\Delta}^{\Delta} \cdots \int_{-\Delta}^{\Delta} \sum_{j=-\infty}^{\infty} \frac{e(\delta j - \alpha j) - e(\beta j + \delta j)}{2\pi i j}$$
$$\times e(j(x+t_1+\cdots+t_k))\,\mathrm{d}t_1 \cdots \mathrm{d}t_k = \beta - \alpha + 2\delta + \sum a_j e(jx).$$

Evaluating subsum of the last sum with $|j| > L$ trivially, we obtain the upper bound for $\chi(x)$. Using (2), we similarly obtain the lower bound. □

Lemma 2. *Let N be a sufficiently large number and $f(x)$ be $(2r+1)$ times differentiable function such that for all $x \in [N, N_1]$ we have $|\mathrm{d}^j f(x)/\mathrm{d}x^j| \leqslant F x^{-j}(A+j)^j$, $0 < j < 2r+2$ with some $0 < A < \exp((\log N)^2/6\log F)$ and some large F, and that for all but at most one $j \in [1, 2r]$ we have $|\mathrm{d}^j f(x)/\mathrm{d}x^j| \geqslant F x^{-j}(A+j)^{-j}$, where $r = \lceil \log F/\log N \rceil \geqslant 2$. Assume that $N \geqslant N_1 - N \geqslant N^{1+u}F^{-1/3}$ and $N_1 - N \geqslant N^{1/2+u}$ for some $0 < u < \frac{1}{3}$. Then $S = \sum_{N<x<N_1} e(f(x) \ll A(N_1-N)N^{-c/r^2}(\log N)^{1/(64r^3)}$, where $c = \min\{u/49; \frac{1}{64}\}$ and the implied in $\ll$constant is absolute.*

Proof. The proof is similar to the proof of the corresponding result of [5]. However, since the conditions of the lemma are slightly different, we have to adjust the proof. We can obviously assume that $r \leqslant \sqrt{\log N}$. If $r \leqslant 12$, then we use van der Corput's estimates in the form

$$S \ll (N_1-N)(\lambda_k^{1/(K-2)} + (N_1-N)^{-2/K}$$
$$+ (\min\{[\lambda_k(N_1-N)^{4-8/K}]^{-2/K}; [\lambda_{k+1}(N_1-N)^{5-8/K}]^{-4/(3K)}\}), \tag{3}$$

where $k \geqslant 1, K = 2^k$ and $|\mathrm{d}^j f(x)/\mathrm{d}x^j| \sim \lambda_j$. Since this form of van der Corput's estimates is not well known, we will give the proof. If $k = 2$, we use Poisson's summation formula and obtain $S = \sum_n \int_N^{N_1} e(f(x) - nx)\,\mathrm{d}x + O(1)$, where the sum is over $n \in [f'(N) - \frac{1}{3}, f'(N_1) + \frac{1}{3}]$. By the mean value Theorem,

$$\int_N^{N_1} e(f(x) - nx)\,\mathrm{d}x \ll \min\{\lambda_2^{-1/2}; \lambda_3^{-1/3}\}$$

and

$$\sum_n 1 \leqslant f'(N_1) - f'(N) + 2 \ll (N_1 - N)\lambda_2 + 1$$

so we obtain

$$S \ll (N_1 - N)\lambda_2^{1/2} + \min\{\lambda_2^{-1/2}; \lambda_3^{-1/3}\}. \tag{4}$$

If $k \geqslant 2$ then we use Weyl–van der Corput inequality $l = k - 2$ times to get

$$|S|^L \ll [(N_1 - N)/\sqrt[2]{Q}]^L + (N_1 - N)^L Q^{1-L} \sum_{q_1=1}^{Q} \cdots \sum_{q_l=1}^{Q^{L/2}} \left| \sum_x e(f_1(x)) \right|,$$

where $L = 2^l$, $f_1(x) = \int_0^1 \cdots \int_0^1 \partial^l f(x + t_1 q_1 + \cdots + t_l q_l)/(\partial t_1 \ldots \partial t_l)\, dt_1 \cdots dt_l$ and the last sum is over $N \leqslant x \leqslant N_1 - q_1 - \cdots - q_l$. Here $d^2 f_1(x)/dx^2 \sim q_1 \ldots q_l \lambda_k$. Using (4) to evaluate the sum over x and choosing an optimal $Q \leqslant (N_1 - N)^2$ we prove (3). We estimate S using (3) with $k = r$ if $\{\log F/\log N\} \leqslant 1 - u$ or $k = r + 1$ otherwise. If $k = r = 2$ then, since either

$$\begin{aligned}\lambda_2(N_1 - N)^2 &\geqslant F(N_1 - N)^2 N^{-2}(A+2)^{-2}\\ &\geqslant (N_1 - N)^2(A+2)^{-2} \geqslant N^{2u-1}(A+2)^{-2}\end{aligned}$$

or

$$\lambda_3(N_1 - N)^3 \geqslant F(N_1 - N)^3 N^{-3}(A+3)^{-3} \geqslant N^{3u}(A+3)^{-3}$$

and since $\lambda_2 \leqslant F(A+2)^2 N^{-2} \leqslant x(A+2)^2 N^{-u}$, we obtain

$$S \ll A(N_1 - N)N^{-u/2} \ll A(N_1 - N)N^{-2u/r^2}.$$

If $k = 3 = r + 1$, then $\lambda_3 \leqslant F(A+3)^3 N^{-3} \leqslant (A+3)^3/N$ and either

$$\lambda_3(N_1 - N)^3 \geqslant F(N_1 - N)^3 N^{-3}(A+3)^{-3} \geqslant N^{3u}(A+3)^{-3}$$

or

$$\begin{aligned}\lambda_4(N_1 - N)^4 &\geqslant F(N_1 - N)^4 N^{-4}(A+4)^{-4}\\ &\geqslant (N_1 - N)^4 N^{-2-u}(A+4)^{-4} \geqslant N^{3u}(A+4)^{-4},\end{aligned}$$

so that

$$S \ll (N_1 - N)AN^{-u/2} \ll A(N_1 - N)N^{-2u/r^2}.$$

If $r = k = 3$ then, as above, $\lambda_3 \leqslant (A+3)^3 N^{-u}$ and either

$$\lambda_3(N_1 - N)^{-3} \geqslant (A+3)^3 N^{3u}$$

or

$$\lambda_4(N_1 - N)^4 \geqslant (A+4)^{-4}(N_1 - N)^4 F N^{-4} \geqslant (A+4)^{-4} N^{4u}$$

and therefore

$$S \ll A(N_1 - N)N^{-u/6} \ll A(N_1 - N)N^{-3u/(2r^2)}.$$

Now we can assume that $3<k<13$ or $k=13=r+1$. If $r=k$ then $\lambda_k \leqslant (A+k)^k N^{-u}$ and either

$$\begin{aligned}\lambda_k(N_1-N)^{4-8/K} &\geqslant F(A+k)^{-k}N^{-k}(N_1-N)^{4-8/K}\\ &\geqslant (N_1-N)^3(A+k)^{-k}/N \geqslant N^u(A+k)^{-k}\end{aligned}$$

or

$$\begin{aligned}\lambda_{k+1}(N_1-N)^{5-8/K} &\geqslant F(N_1-N)^4N^{-k-1}(A+k+1)^{-k-1}\\ &\geqslant (N_1-N)^4N^{-2}(A+k+1)^{-k-1} \geqslant N^{4u}(A+k+1)^{-k-1}.\end{aligned}$$

If $r=k-1$ then $\lambda_k \leqslant (A+k)^k/N$ and either

$$\begin{aligned}\lambda_k(N_1-N)^{4-8/K} &\geqslant F(N_1-N)^{4-8/K}N^{-k}(A+k)^{-k}\\ &\geqslant (N_1-N)^3N^{-1-u}(A+k)^{-k} \geqslant N^{2u}(A+k)^{-k}\end{aligned}$$

or

$$\begin{aligned}\lambda_{k+1}(N_1-N)^{5-8/K} &\geqslant F(N_1-N)^{5-8/K}N^{-k-1}(A+k+1)^{-k-1}\\ &\geqslant N^{3u}(A+k+1)^{-k-1}.\end{aligned}$$

In both cases we have

$$S \ll A(N_1-N)N^{-uk^2r^{-2}/(K-2)} \ll A(N_1-N)N^{-u/(29r^2)}.$$

Now we assume that $r\geqslant 12$. We take $X=N^{1/2}F^{-1/(4r)}$ and write

$$S=X^{-2}\sum_{N\leqslant n\leqslant N_1}\sum_{1\leqslant x,y\leqslant X} e(f(xy+n))+O(N^{1/4}) \ll X^{-2}\sum_n |S(n)|+N^{1/4},$$

where

$$\begin{aligned}S(n)&=\sum_{x,y} e(f(n+xy))\\ &=\sum_{x,y} e\left(\sum_{j=1}^{2r}\alpha_j x^j y^j\right)+\theta(A+2r+1)^{2r+1}FX^{4r+4}N^{-2r-1}(2r+1)!,\end{aligned}$$

and $|\theta|\leqslant 1$, $\alpha_j=(\mathrm{d}^j f(n)/\mathrm{d}n^j)/(j!)$. Here $FX^{4r+4}N^{-2r-1}\leqslant X^2N^{-1/4}$. We take $k=16r^2$ and obtain

$$\begin{aligned}|S_1|^{2k} &\equiv \left|\sum_{x,y} e\left(\sum \alpha_j x^j y^j\right)\right|^{2k} \leqslant \left(\sum_{x=1}^{X} 1\right)^{2k-1}\sum_x\left|\sum_y e\left(\sum_j \alpha_j x^j y^j\right)\right|^{2k}\\ &= X^{2k-1}\sum_x\sum_{\underline{\lambda}} J_{k,2r}(\underline{\lambda})e\left(\sum_j \alpha_j\lambda_j x^j\right),\end{aligned}$$

where $J_{k,m}(\underline{\lambda})$ denotes the number of solutions of the system

$$\sum_{i=1}^{k}(y_i^j - y_{i+k}^j) = \lambda_j (j=1,\dots,m),\ 1 \leqslant y_i \leqslant X (i=1,\dots,2k),\ |\lambda_j| \leqslant kX^j.$$

It is well known (see, for example, [5]) that if $k \geqslant m^2(1+d)$ then

$$J_{k,m}(\underline{\lambda}) \leqslant J_{k,m}(0) \leqslant (4m)^{4kdm} X^{2k-m(m+1)/2+(1-1/m)^{md}m(m+1)/2}. \tag{5}$$

We use Cauchy's inequality again:

$$\begin{aligned} |S_1|^{2k^2} &\leqslant X^{2k^2-k}\left(\sum_{\underline{\lambda}} J_{k,2r}(\underline{\lambda})\right)^{k-1} \sum_{\underline{\lambda}} J_{k,2r}(\underline{\lambda}) \left|\sum_x e\left(\sum_j \alpha_j \lambda_j x^j\right)\right|^k \\ &= X^{4k^2-3k} \sum_{\underline{\lambda}} J_{k,2r}(\underline{\lambda}) \left|\sum_x e\left(\sum \alpha_j \lambda_j x^j\right)\right|^k \end{aligned}$$

and

$$\begin{aligned} |S_1|^{4k^2} &\leqslant X^{8k^2-6k} \sum_{\underline{\lambda}} J_{k,2r}(\underline{\lambda})^2 \sum_{\underline{\lambda}} \left|\sum_x e(\alpha_j \lambda_j x^j)\right|^{2k} \\ &= X^{8k^2-6k} J_{2k,2r}(X) \sum_{\underline{\lambda}} \sum_{\underline{\mu}} J_{k,2r}(\underline{\mu}) e\left(\sum \alpha_j \lambda_j \mu_j\right) \\ &\leqslant X^{8k^2-6k} J_{2k,2r}(X) J_{k,2r}(X) \prod_{j=1}^{2r} \sum_{\mu_j} \min\{2kX^j; 1/\|\alpha_j \mu_j\|\}. \end{aligned} \tag{6}$$

If α_j is small then we divide the interval $[-kX^j, kX^j]$ into $\leqslant 2kX^j\alpha_j+1$ subintervals of the length $\leqslant 1/\alpha_j$. The sum over μ_j in a subinterval is $\leqslant 4kX^j + 2\sum_{1 \leqslant x \leqslant 1/(2\alpha_j)} 1/(\alpha_j x) \leqslant 4kX^j + 2\log(1/\alpha_j)/\alpha_j$, and

$$\sum_{\mu_j} \min\{2kX^j; 1/\|\alpha_j \mu_j\|\} \leqslant 4k^2X^{2j}\alpha_j + 4kX^j \log(1/\alpha_j) + 2kX^j + 2/\alpha_j \log(1/\alpha_j),$$

where for all but at most one j we have $FN^{-j}(A+j)^{-j} \leqslant \alpha_j \leqslant FN^{-j}(A+j)^j$. Substituting this and (5) into (6), we get

$$\begin{aligned} &|S_1|^{4k^2} \\ &\leqslant X^{8k^2-6k} J_{2k,2r}(X) J_{k,2r}(X) \prod_{j=1}^{r} 4k^2 X^{2j} \prod_{j=r+1}^{2r-1} (\alpha_j + X^{-j}\log N + X^{-2j}\log N/\alpha_j) \\ &\ll X^{8k^2-6k} J_{2k,2r}(X) J_{k,2r}(X) 4^{2r} k^{4r} X^{4r^2+2r} (2\log N)^{r-1} (A+2r)^{2r} \\ &\quad \times \prod_{j=r+1, j\neq j_0}^{2r-1} (FN^{-j} + N^j X^{-2j}/F) \end{aligned}$$

$$\ll (A+2r)^{2r}k^{4r}X^{8k^2-6k}(8r)^{80kr}5^{4r}X^{4k-4r^2-2r+r(2r+1)(1-1/(2r))^{14r}}$$
$$\times(8r)^{40kr}X^{2k-2r^2-r+r(r+1)(1-1/(2r))^{6r}}X^{4r^2+2r}N^{4/3r-7/6r^2}(\log N)^{r-1}$$
$$\ll (48k^2r)^{2r}(8r)^{120kr}X^{8k^2}N^{r(r+1)(e^{-3}+e^{-7})/4+4/3r-7/6r^2}(\log N)^r,$$

$$S_1 \ll AX^2N^{-1/(64r^2)}(\log N)^{1/(64r^3)},$$

and

$$S \ll A(N_1-N)N^{-cr^{-2}}(\log N)^{1/(64r^3)}$$

with $c=\min\{\frac{1}{64};u/29\}$. $\square$

Lemma 3. *Let $f(x)$ be a function satisfying the conditions of Lemma 2. Assume also that for all $j<2r+2$ we have $|j\,\mathrm{d}^jf(x)/\mathrm{d}x^j+x\,\mathrm{d}^{j+1}f(x)/\mathrm{d}x^{j+1}|\leqslant F(A+j)^jN^{-j}$ and for all but at most one $j<2r+1$, $|j\,\mathrm{d}^jf(x)/\mathrm{d}x^j+x\,\mathrm{d}^{j+1}f(x)/\mathrm{d}x^{j+1}|\geqslant F(A+j)^{-j}N^{-j}$, where F,N,A are as in Lemma 2 and $r=\lceil 2\log F/\log N\rceil$. Let $N\geqslant N_1-N\geqslant N^{3/4+u}$ for some $u\in(0,\frac{1}{4}]$ and $N_1-N\geqslant N^{1+u}F^{-1/4}$. Then*

$$S=\sum_{N\leqslant p\leqslant N_1} e(f(p)) \ll A(N_1-N)N^{-c_1/r^2}(\log N)^{1/(64r^3)}r^4/u^2,$$

where $c_1=c/4=\min\{\frac{1}{256};u/120\}$ and the implied in $\ll$constant is absolute.

Proof. We can obviously assume that $\log N\geqslant(\log F)^{2/3}$. Using Abel's summation formula and Vaughan's identity we obtain

$$S\ll\sqrt{N}+1/\log N\left|\sum_{N\leqslant n\leqslant N_2\leqslant N_1}\Lambda(n)e(f(n))\right|\leqslant\sqrt{N}+1/\log N(|S_1|+|S_2|+|S_3|),$$

where

$$S_1=\sum_{N\leqslant mk\leqslant N_2,m\leqslant M}\mu(m)(\log k)e(f(mk)),$$

$$S_2=\sum_{N\leqslant mk\leqslant N_2,m\leqslant M^2}b(m)e(f(mk)),$$

$$S_3=\sum_{N\leqslant mk\leqslant N_2,k\geqslant M,m\geqslant M}a(m)\Lambda(k)e(f(mk)),$$

$$b(m)=\sum_{rk=m,r\leqslant M,k\leqslant M}\mu(r)\Lambda(k),\qquad M=N^{1/4},\qquad a(m)=\sum_{rk=m,r\leqslant M}\mu(r).$$

Here

$$\sum_{X \ll m \ll X} a(m)^2 = \sum_{r_1,r_2 \leqslant M} \mu(r_1)\mu(r_2) \sum_{r_1k_1=r_2k_2} 1 \ll \sum_{r_1r_2r_3 \leqslant N} \frac{X}{r_1r_2r_3} \ll X \log^3 N,$$

$$\sum_{Y \ll y \ll Y} b^2(y) = \sum_{r \leqslant M} \sum_{r_1,r_2 \leqslant M/r} \mu(r_1r_2r) \sum_{r_1k_1=r_2k_2} \Lambda(k_1)\Lambda(k_2) \ll Y \log^2 N,$$

$$\sum_{X \ll m \ll X} \Lambda^2(m) \ll X \log N.$$

We can obviously assume that $N_2 - N \geqslant (N_1 - N)N^{-c_1/r^2}$.

To estimate S_1 we use Abel's summation formula and Lemma 2 to the sum over k. Here

$$(N_2 - N)/m > (N/m)^{1/2+u} \quad \text{and} \quad (N_2 - N)/m > (N/m)^{1+u/2}F^{-1/3}$$

so that by Lemma 2 we get

$$S_1 \ll \sum_{m \leqslant M} |\mu(m)| A(N_2 - N)/m(N/m)^{-c/(2r^2)}(\log N)^{1/(64r^3)}$$
$$\ll A(N_1 - N)N^{-c/(64r^3)}r^2/u.$$

We can similarly estimate S_2 and obtain

$$S_2 \ll A(N_1 - N)N^{-c/(4r^2)}(\log N)^{1/(64r^3)}r^4/u^2.$$

To estimate S_3 we divide it into $\ll \log N$ subsums with $M \leqslant X \leqslant m \leqslant 2X \leqslant N_1/M$ (and $M \leqslant Y \leqslant N/(2X) \leqslant k \leqslant N_1/X = Y_1$). We assume first that $X \geqslant \sqrt{N}$ and, assuming that X corresponds to the largest subsum, we obtain

$$|S_3|^2 \ll \log^2 N \sum_{X \leqslant m \leqslant 2X} a^2(m) \sum_{X \leqslant m \leqslant 2X} \left| \sum_k e(f(mk)) \right|^2$$
$$\leqslant X \log^3 N \sum_{Y \leqslant k_1 \leqslant k_2 \leqslant Y_1} \Lambda(k_1)\Lambda(k_2) \left| \sum_m e(f(mk_1) - f(mk_2)) \right|, \tag{7}$$

where the last sum is over $m \in [N/k_1, N_2/k_2]$. Here $(k_2 - k_1)/k_1 \leqslant (N_2 - N)/N$ and $N_2/k_2 - N/k_1 \leqslant (N_1 - N)/k_2$; also,

$$\mathrm{d}^j/\mathrm{d}m^j(f(mk_2) - f(mk_1)) = k_2^j(f^{(j)}(mk_2) - k_1^j f^{(j)}(mk_1))$$
$$= (k_2 - k_1)m^{1-j}[jx^{j-1}f^{(j)}(x) + x^j f^{(j+1)}(x)]$$

for some $x \in (N, N_1)$. If $|k_2 - k_1| \leqslant (N_1 - N)X^{-1}N^{-2c_1/r^2}$ or if $N_2/k_2 - N/k_1 \leqslant (N_1 - N)Y^{-1}N^{-2c_1/r^2}$ then we denote the corresponding part of the sum in (7) by S_4

and obtain

$$|S_4| \leqslant X\log^4 N \sum_{Y\leqslant k\leqslant Y_1} \Lambda(k)(N_1-N)X^{-1}N^{-2c_1/r^2}(N_1-N)/k$$
$$\leqslant (N_1-N)^2N^{-2c_1/r^2}\log^4 N.$$

If $k_2-k_1>(N_1-N)X^{-1}N^{-2c_1/r^2}$ then the function $g(m)=f(mk_1)-f(mk_2)$ satisfies the conditions of Lemma 2 with $(k_2-k_1)F/Y$ instead of F; using it,we get

$$|S_3|^2 \ll X\log^3 N\sum_{k_1,k_2}[\Lambda^2(k_1)+\Lambda^2(k_2)](N_1-N)/NX^{1-c/r^2}$$
$$\ll (N_1-N)^2X^{2-c/r^2}YN^{-2}\sum_{Y\leqslant k\leqslant Y_1}\Lambda^2(k)\log^3 N \ll (N_1-N)^2N^{-2c_1/r^2}\log^4 N.$$

If $Y\geqslant\sqrt{N}$ then we change the order of summation and, as above, we obtain the same estimate. □

4. The main results

To prove Theorem 1 we use the well known formula for α_p such that $p^{\alpha_p}\|\binom{n}{m}$:

$$\alpha_p=\sum_{j=1}^{J}(\lfloor n/p^j\rfloor-\lfloor m/p^j\rfloor-\lfloor (n-m)/p^j\rfloor)$$
$$=\sum_{j=1}^{J}(\{(n-m)/p^j\}+\{m/p^j\}-\{n/p^j\})$$

with $J=\lfloor \log n/\log p\rfloor$. If $n^{1/(r+1)}<p\leqslant n^{1/r}$ then $\alpha_p=r$ if and only if

$$\{m/p^j\}+\{(n-m)/p^j\}-\{n/p^j\}=1 \quad \text{for } j=1,2,\ldots,r. \tag{8}$$

Also, since $\{(n-m)/p^j\}=\{\{n/p^j\}-\{m/p^j\}\}$, (8) holds if and only if $\{n/p^j\}<\{m/p^j\}$. We denote $A=\{p\in[P,P_1]\colon \alpha_p=r\}$. If $p\in[P,P_1]$, then $\{n/p^r\}=n/p^r-1$, $\{m/p^r\}=m/p^r$ and $\{(n-m)/p^r\}=(n-m)/p^r$, so that (8) holds for $j=r$, and

$$|A|=\sum_{P\leqslant p\leqslant P_1}\prod_{j=1}^{r-1}(\{m/p^r\}+\{(n-m)/p^j\}-\{n/p^j\}).$$

We take $\delta = n^{-1/(10r)}$ and denote $B = \{p \in [P, P_1]\colon \ \|n_i/p^j\| \leqslant \delta\}$ for some $j < r$ and $n_i = m, n$ or $n - m$. Since for $\|t\| > \delta$ we have

$$\{t\} = 1/(2\delta)\int_{-\delta}^{\delta}\{t+u\}\,\mathrm{d}u = 1/(2\delta)\int_{-\delta}^{\delta}\left(1/2 - \sum_{|k|=1}^{\infty} 1/(2\pi i k)e(k(t+u))\right)\mathrm{d}u$$

$$= 1/2 + \sum_{|k|=1}^{\infty} a_k e(kt),$$

where $a_k = -\sin(2\pi k\delta)/(4\pi^2 k^2 i\delta)$. Using this we obtain

$$|A| = \mathrm{O}(|B|) + \sum_{P \leqslant p \leqslant P_1}\prod_{j=1}^{r-1}\left(1/2 + \sum_{|k|=1}^{\infty} a_k[e(km/p^j) + e(k(n-m)/p^j) - e(kn/p^j)]\right)$$

$$= \mathrm{O}(|B|) + 2^{1-r}[\pi(P_1) - \pi(P)]$$

$$+ \sum_{u=1}^{r-1}\sum_{\underline{j}} 2^{u+1-r}\sum_{\underline{k}} a_{k_1}\ldots a_{k_u}\sum_{P \leqslant p \leqslant P_1} e(k_1 n_1 p^{-j_1} + \cdots + k_u n_u p^{-j_u}), \qquad (9)$$

where $n_i = m$, n or $n - m$ and $1 \leqslant j_1 < j_2 < \ldots < j_u \leqslant r - 1$. Denoting with $\sum_{\underline{k}}^{(1)}$ the sum over $\underline{k}$ such that $|k_i| \geqslant (1/\delta)\log^r n$ for at least one i we trivially obtain

$$\left|\sum_{\underline{k}}^{(1)}\sum_{p} e(f(p))\right| \leqslant \delta[\pi(P_1) - \pi(P)] \approx \delta(P_1 - P)/\log P,$$

where

$$f(x) = k_1 n_1 x^{-j_1} + \cdots + k_u n_u x^{-j_u};$$

if $|k_i| \leqslant (1/\delta)\log^r n$ for $i = 1, \ldots, u$, then $f(x)$ satisfies the conditions of Lemma 3 with $F = k_1 n_1 P^{-j_1}$ and $A = 2$. Using it we obtain

$$\left|\sum_{\underline{k}}\sum_{p} e(f(p))\right| \ll \delta(P_1 - P)/\log P + (P_1 - P)n^{-c/r^3}\log^{r+4} n,$$

where $c = \min\{1/1024; 1/480(a_1 - (a+2)/3)\}$. To evaluate $|B|$ we use Lemma 1 with $n = 1$ and get

$$|B| = \delta[\pi(P_1) - \pi(P)] + \sum_{|k|=1}^{\infty} a_k \sum_{P \leqslant p \leqslant P_1} e(n_1 k/p^j),$$

where $n_1 = m, n$ or $n - m$ and $0 < j < r$. Using Lemma 3 or the trivial estimate to evaluate the last sum and the well known result $\pi(P_1) - \pi(P) \approx (P_1 - P)/\log P$ we obtain $|B| \ll (P_1 - P)n^{-c/r^3}$, and

$$|A| = 2^{1-r}(P_1 - P)/\log\, p + \mathrm{O}((P_1 - P)n^{-c/r^3}).$$

If $r \leqslant (\log n/\log\log n)^{1/4}$ then the O-term is smaller than the main term. To prove the second part of Theorem 1 we write $d_1 = \prod_p p$, where the product is taken over all p with $\alpha_p = r$, and since

$$\prod_{p \leqslant m^{(1-\varepsilon)/r}} p \leqslant \exp\left(\sum_{k \leqslant m^{(1-\varepsilon)/r}} \Lambda(k)\right) \leqslant \exp(2m^{(1-\varepsilon)/r}),$$

we need to show that

$$\prod_p{}^{(1)} p = \exp\left(\sum_p{}^{(1)} \log p\right) = \exp(2^{1-r} m^{1/r}(c_0 + o(1))),$$

where $\prod_p^{(1)}$ and $\sum_p^{(1)}$ are over $p \in [m^{(1-\varepsilon_1)/r}, n^{1/r}] = [P, P_1]$ with $\alpha_p = r$ and a sufficiently small constant ε_1. Here

$$p^{r+1} \geqslant m^{(r+1)(1-\varepsilon_1)/r} \geqslant n^{(1+1/r)(1-1/(4r))} > N$$

so that $\alpha_p = r$ if and only if $\{m/p^j\} + \{(n-m)/p^j\} - \{n/p^j\} = 1$ for $j = 1, \ldots, r$. Denoting with $\sum_p^{(2)}$ the sum over all $p \in [P, P_1]$ satisfying

$$\{n/p^j\} < \{m/p^j\} \tag{10}$$

we obtain

$$\sum_p{}^{(1)} \log p = \sum_p{}^{(2)} \log p \prod_{j=1}^{r-1} (\{m/p^j\} + \{(n-m)/p^j\} - \{n/p^j\}). \tag{11}$$

We divide the set of all p satisfying (10) into subsets with $u \leqslant m/p^r < u+1$ and $v \leqslant n/p^r < v+1$ for some non-negative integers u, v satisfying

$$u \leqslant v \leqslant nm^{\varepsilon_1 - 1} \equiv V.$$

Then (10) becomes $(n-m)/p^r < u - v$, and

$$\max\{(n-m)/(v-u); m/(u+1)\} < p^r \leqslant \min\{m/u; n/v\}, \tag{12}$$

where we assume that $m/u = \infty$ if $u = 0$. The set of p satisfying (12) is non-empty if and only if $u < bv < u+1$, and $u = \lfloor bv \rfloor$, so that (11) becomes

$$\max\{(1-b)/(v - \lfloor bv \rfloor); b/(\lfloor bv \rfloor + 1)\} < p^r/n \leqslant 1/v,$$

and

$$1 - \delta(v) < p^r v/n \leqslant 1, \; P(v) < p \leqslant P_1(v),$$

where $P_1(v)=(n/v)^{1/r}$ and $P(v)=P_1(v)(1-\delta(v))^{1/r}$. Substituting this into (11) we get

$$\sum_p^{(1)} \log p = \sum_{v\leqslant V}\ \sum_{P(v)\leqslant p\leqslant P_1(v)} \log p \prod_{j=1}^{r-1}\left(\left\{\frac{m}{p^j}\right\}+\left\{\frac{n-m}{p^j}\right\}-\left\{\frac{n}{p^j}\right\}\right). \qquad (13)$$

If v is such that $P_1(v)-P(v)\leqslant (n/v)^{(7-4a)/(8r)}$ then the subsum over all such v in (13) is

$$\leqslant \sum_{v\leqslant V}(n/v)^{(7-4a)/(8r)}\log n \ll n^{1+\varepsilon_1}m^{(7-4a)/(8r)-1}=\mathrm{o}(m^{(1-a_1)/r})$$

for some positive a_1 if ε_1 is sufficiently small. The remaining part of the sum in (13) can be evaluated in the same way as in the proof of the first part of the theorem. We obtain

$$\begin{aligned}\sum_p^{(1)}\log p &= \sum_{v\leqslant V}[P_1(v)-P(v))(2^{1-r}+\mathrm{O}(P^{-c_1/r^2}(v))]+\mathrm{O}(m^{(1-a_1)/r})\\ &= 2^{1-r}\left[\sum_{v\leqslant V}(P_1(v)-P(v))+\mathrm{o}(m^{1/r}/r)\right].\end{aligned}$$

Here

$$\sum_{v\leqslant V}[P_1(v)-P(v)]=m^{1/r}\sum_{v\leqslant V}(bv)^{-1/r}[1-\delta(v))^{1/r}]=m^{1/r}(c_0+\mathrm{o}(1)).$$

If $b=1/q$ for some integer q then

$$\delta(v)=\{v/q\}/(v-\lfloor v/q\rfloor)=j/(q(qk+j-k))$$

for $v=qk+j$, $k\geqslant 0$, $0<j<q$, and

$$\begin{aligned}c_0 &= \sum_{k=0}^{\infty}(k+j/q)^{-1/r}(1-(1-j/(q(qk-k+j)))^{1/r}]\\ &\geqslant \left(\frac{1}{rq}\right)\sum_{j=1}^{q-1} j/q\sum_{k=0}^{\infty}(k+j/q)^{-1-1/r} > 1/(rq)\sum_{j=1}^{q-1}(j/q)^{1-1/r}r\\ &\geqslant \int_0^{1-1/q} x^{1-1/r}\,\mathrm{d}x=(1-1/q)^{2-1/r}/(2-1/r)>1/8.\end{aligned}$$

If $b=1/Q$ with some non-integer $Q>2$ then we divide v's into intervals $[Qk, Q(k+1))$ and within such interval we write $v=\lfloor Qk\rfloor + j$. Since $k\leqslant bv<k+1$, we have $\{bv\}=bv-k$, which is $\leqslant(1-b)$ if $v\leqslant(k+1)/b-1$, so that $\delta(v)=(bv-k)/(v-k)$ for

$$0<j\leqslant \lfloor (k+1)Q\rfloor-\lfloor kQ\rfloor-1\equiv J=\lfloor\lfloor kQ\rfloor+\lfloor Q\rfloor\rfloor-\lfloor kQ\rfloor-1$$

if $kQ<1-Q$ and $J=\lfloor Q\rfloor$ otherwise, we obtain for $Q>3$:

$$\begin{aligned} c_0 &\geqslant (Q^{1/r}/r)\sum_{k=0}^{\infty}\sum_{j=1}^{J}(b\lfloor Qk\rfloor+bj-k)/(\lfloor Qk\rfloor+j-k)(\lfloor Qk\rfloor+j)^{1/r}\\ &=(Q^{1/r-1}/r)\sum_{k=0}^{\infty}\sum_{j=1}^{J}(j-\{Qk\})/(\lfloor Qk\rfloor+j-k)(\lfloor Qk\rfloor+j)^{-1/r}\\ &\geqslant (Q^{1/r-1}/r)\sum_{j=2}^{\lfloor Q\rfloor-1}\sum_{k=0}^{\infty}(j-1)/(Qk+j-k-1)(Qk+j-1)^{-1/r}\end{aligned}$$

$$\begin{aligned}&\sum_{k=1}^{\infty}\min\left\{\frac{\{Q\}(Qk+1)^{-1/r}}{(Qk+1-k)};\frac{(Qk+Q-1)^{-1/r}(\lfloor Q\rfloor-1)}{(Qk+Q-k-1)}\right\}\\ &\quad>\left[\sum_{j=1}^{\lfloor Q\rfloor-2}j\sum_{k=0}^{\infty}(k+j/Q)^{-1/r}(k+j/(Q-1))^{-1}+\min\{\{Q\};2/3\}\right.\\ &\qquad\left.\times\sum_{k=1}^{\infty}[(k+1/Q)^{-1/r}]/[k+1/(Q-1)]\right]\Big/(Q(Q-1)r).\end{aligned}$$

Since

$$\sum_{k=0}^{\infty}(k+j/Q)^{-1/r}/(k+j/(Q-1))>\int_1^{\infty}x^{-1-1/r}\,dx=r$$

and

$$\sum_{k=1}^{\infty}(k+1/Q)^{-1/r}/(k+1/(Q-1))>\int_1^{\infty}(x+1/(Q-1))^{-1-1/r}\,dx=r(1-1/Q)^{1/r},$$

we get

$$c_0>\left(\sum_{j=1}^{\lfloor Q\rfloor-2}j+\min\{\{Q\};2/3\}(1-1/Q)^{1/r}\right)/(Q(Q-1))>1/8$$

if $Q>3$. Now we assume that $2<Q<3$ and $Q=1/Q_1$. As above,

$$\begin{aligned}c_0\geqslant Q^{1/r-1}&\left[\sum_{k=0}^{\infty}\frac{(1-\{k/Q_1\})(Qk+1-\{k/Q_1\})^{-1/r}}{Qk+1-k-\{k/Q_1\}}\right.\\ &\left.+\sum_{k=1}^{\infty}\frac{(2-\{k/Q_1\})(Qk+2-\{k/Q_1\})^{-1/r}}{Qk+2-k-\{k/Q_1\}}\right]\Big/r,\end{aligned}$$

where the last sum is the sum over all positive integers k such that $\{k/Q_1\}\geqslant 1-1/Q_1$. Trivially, $c_0>1/8$ if $r<4$, so we assume that $r>3$. We divide the k's into subsets with $iQ_1 \leqslant k < (i+1)Q_1$, $i = 0, 1, \dots$ and write $k = \lfloor iQ_1 \rfloor + j$, $1 \leqslant j \leqslant J$, where

$$J = \lfloor (i+1)Q_1 \rfloor - \lfloor iQ_1 \rfloor = \lfloor Q_1 \rfloor$$

if $\{iQ_1\} + \{Q_1\} < 1$ and $J = \lfloor Q_1 \rfloor + 1$ otherwise, so that

$$\{k/Q_1\} = (j - \{iQ_1\})/Q_1 \geqslant 1 - 1/Q_1$$

if and only if $j = \lfloor Q_1 \rfloor$ and $\{iQ_1\} + \{Q_1\} \leqslant 1$ or $j = \lfloor Q_1 \rfloor + 1$ (and $\{iQ_1\} + \{Q_1\} \geqslant 1$). We obtain:

$$c_0 \geqslant (Q^{1/r-1}/r)\left[1 + \sum_{i=0}^{\infty}\sum_{j=1}^{\lfloor Q_1 \rfloor}\frac{(1 - j/Q_1)(Qk_1 + 1)^{-1/r}}{((Q-1)k_1 + 1)} + \sum_{i=0}^{\infty}\frac{(Qk_2 + 1)^{-1/r}}{(Qk_2 - k_2 + 1)}\right],$$

where $k_1 = \lfloor iQ_1 \rfloor + j$ and $k_2 = \lfloor (i+1)Q_1 \rfloor$ so that, as above, we get

$$\begin{aligned}
c_0 &\geqslant Q^{1/r-1}\left[1 + \sum_{j=1}^{\lfloor Q_1 \rfloor}\frac{(1 - j/Q_1)\sum_{i=0}^{\infty}(Q(iQ_1 + j) + 1)^{-1/r}}{(Q-1)(iQ_1 + j) + 1}\right.\\
&\qquad\left. + \sum_{i=0}^{\infty}\frac{(QQ_1(i+1) + 1)^{-1/r}}{(Q-1)Q_1(i+1) + 1}\right]\Big/ r\\
&\geqslant Q^{1/r-1}\left[1/r + \sum_{j=1}^{\lfloor Q_1 \rfloor}\frac{(1 - j/Q_1)(j/Q_1 + 1/(Q_1(Q-1)))^{-1/r}}{(Q-1)Q_1(QQ_1)^{1/r}}\right.\\
&\qquad\left. + \frac{(1 + 1/((Q-1)Q_1))^{-1/r}}{(Q-1)Q_1(Q_1Q)^{1/r}}\right]\\
&= \frac{Q^{1/r}}{Qr} + \frac{\sum_{j=1}^{\lfloor Q_1 \rfloor}(1 - j/Q_1)(j + 1/(Q-1))^{-1/r}}{Q(Q-1)Q_1}\\
&\qquad + \frac{1}{Q(Q-1)Q_1(Q_1 + 1/(Q-1))^{1/r}} > 1/8
\end{aligned}$$

for all $Q_1 > 1$. If b is small then, as above, we get: if $r > 1$ then

$$\begin{aligned}
c_0 &= (1/r)\sum_{v=1}^{\infty}(bv)^{-1/r}\delta(v) + O(b/r)\\
&= (1/r)\sum_{v \leqslant Q} b^{1-1/r}v^{-1/r} + (1/(Q^2 r))\sum_{k=0}^{\infty}\sum_{1 \leqslant j \leqslant Q} j(k + j/Q)^{-1-1/r} + O(b/r)\\
&= (1/r)\int_0^1 \sum_{k=0}^{\infty} x(k + x)^{-1-1/r}\,dx + O(b^{1-1/r}/r)\\
&= (1/r)\int_0^{\infty}\{x\}x^{-1-1/r}\,dx + O(b^{1-1/r}/r),
\end{aligned}$$

which is $<r/(2r-1)$ and $>r(r-2^{1-1/r})/((2r-1)(r-1))$; if $r=1$ then $c_0=\log Q+1+O(b)$.

To prove Theorem 2 we can assume that $m<n^{1-a/r}$ for some $a<\frac{1}{4}$, otherwise the theorem follows from Theorem 1. We denote $A=\{p\in(P,P_1]:\alpha_p=r\}$, where $P=m^{1/r}$, $P_1=(2m)^{1/r}$. Here $p\in A$ if and only if

$$\sum_{j=1}^{J}(\{m/p^j\}+\{(n-m)/p^j\}-\{n/p^j\})=r,$$

where $J=\lfloor\log n/\log P\rfloor$. Since $\{m/p^j\}+\{(n-m)/p^j\}-\{n/p^j\}=1$ if and only if $\{n/p^j\}<\{m/p^j\}$, we have

$$\begin{aligned}|A|&\geqslant\left|\left\{p\in(P,P_1]:\left\{\frac{n}{p^j}\right\}<\left\{\frac{m}{p^j}\right\},\right.\right.\\&\qquad\left.\left.1\leqslant j\leqslant r\text{ and }\left\{\frac{n}{p^j}\right\}\geqslant\left\{\frac{m}{p^j}\right\},\,r+1\leqslant j\leqslant J\right\}\right|\\&\geqslant\left|\left\{p\in(P,P_1]:\{\frac{n}{p^j}\}<\{m/p^j\},\ 1\leqslant j\leqslant r\right\}\right|-|B|,\end{aligned}$$

where $B=\{p\in(P,P_1]:\{np^{-j-r}\}\geqslant 1/m^j$ for some $j\in[1,J-r]\}$. Denoting $\chi_1(x)=\chi_{[0,1/2]}(x)$, $\chi_2(x)=1-\chi_1(x)$ and $\chi_3(x)=\chi_{[0,\eta)}(x)$, $\eta=P^{-a/3}$ (see Lemma 1), we obtain

$$|A|\geqslant\sum_{P<p\leqslant P_1}\prod_{j=1}^{r}\chi_1(n/p^j)\prod_{j=1}^{r-1}\chi_2(m/p^j)-\sum_{j=1}^{J-r-1}\sum_{P<p\leqslant P_1}\chi_3(nx^{-j-r})-|B_1|,$$

where $B_1=\{x\in(P,P_1]:\{n/x^J\}\leqslant 1/m\}$. If $\{\log n/\log P\}\leqslant\frac{3}{4}$ then we divide $(P,P_1]$ into $\leqslant(P_1-P)\,JnP^{-J-1}+1$ subintervals such that $\lfloor n/x^J\rfloor$ does not change within each interval, so that the number of $x\in B_1$ in each subinterval is $\leqslant P^{J+1}/(Jnm)+1$, and

$$|B_1|\leqslant(P_1-P)/m+P^{J+1}/(Jnm)+(P_1-P)JnP^{-J-1}+1\ll(P_1-P)P^{-1/6}.$$

If $\{\log n/\log P\}>\frac{3}{4}$ then we write

$$|B_1|\leqslant|\sum_{P\leqslant p\leqslant P_1}\chi_3(n/x^J)|$$

and estimate the sum as below. Using Lemma 1 with $\delta=\eta$, $L=P^{a/2}$ and $k=3$, we obtain

$$\begin{aligned}|A|\ \geqslant&\sum_{P<p\leqslant P_1}\prod_{j=1}^{r}\left(1/2-3/\log^3 n+\sum_{|k|=1}^{L}a_ke(nk/p^j)\right)\prod_{j=1}^{r-1}\left(1/2-3/\log^3 n\right.\\&\left.+\sum_{|k|=1}^{L}a_ke(mk/p^j)\right)-\sum_{r<j<J}\sum_{P<p\leqslant P_1}\left(4\eta+\sum_{|k|=1}^{L}a_ke(nk/x^j)\right)-|B_1|\end{aligned}$$

$$\gg (\pi(P_1) - \pi(P) + \mathrm{O}(P/\log^3 P))4^{-r}$$
$$-\sum_{\underline{j}} A_{\underline{j}} \left| \sum_{P<p\leqslant P_1} e(j_1 n/p + \cdots + j_r n/p^r + j_{r+1} m/p + \cdots + j_{2r-1} m p^{1-r}) \right|$$
$$-\sum_{j=1}^{J_1} \sum_{k=1}^{L} |a_k| \left| \sum_{P<x\leqslant P_1} e(nk/x^j) \right|,$$

where $A_{\underline{j}} = |a_{j_1} \dots a_{j_{2r-1}}|$, $\underline{j} = (j_1, \dots, j_{2r-1}) \neq (0, \dots, 0)$ and $J_1 = J$ if $\{\log n/\log P\} > 3/4$, $J_1 = J - 1$ otherwise. The function

$$f(x) = j_1 n/x + \cdots + j_r n/x^r + j_{r+1} m/x + \cdots + j_{2r-1} x^{1-r}$$

satisfies the conditions of Lemma 3. Indeed, let i and k be the smallest integers in $[1, r]$ and $[r+1, 2r-1]$, respectively, such that $j_i j_k \neq 0$. Then $f^{(u)}(x) = \mathrm{d}^u/\mathrm{d}x^u (j_i n/x^i + j_k m x^{r-k} + \cdots)$ and, because $m = \mathrm{o}(n/L)$ and either $i \neq j - r$ or $f(x) = j_i n/x^i(1 + \mathrm{o}(1))$, the last derivative is $\sim (|j_i| n x^{-i-u} + |j_k| m x^{r-k-u})$for all but at most one u. Using Lemmas 3 and 2 to evaluate the sums over p and x respectively, we obtain

$$|A| \gg 4^{-r} P/\log P - \mathrm{O}(P\eta + P^{1 - c\log^2 m/(r \log n)^2} \log^{5+2r} n) \gg r m^{1/r} 4^{-r}/\log m.$$

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