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Transmutations, L-bases and complete families of solutions of the stationary Schrödinger equation in the plane ☆

Hugo M. Campos, Vladislav V. Kravchenko *, Sergii M. Torba

Department of Mathematics, CINVESTAV del IPN, Unidad Queretaro, Libramiento Norponiente No. 2000, Fracc. Real de Juriquilla, Queretaro, Qro. C.P. 76230, Mexico

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ABSTRACT

An L-basis associated to a linear second-order ordinary differential operator L is an infinite sequence of functions $\{\varphi_k\}_{k=0}^{\infty}$ such that $L\varphi_k = 0$ for $k = 0, 1$, $L\varphi_k = k(k-1)\varphi_{k-2}$, for $k = 2, 3, \ldots$ and all φ_k satisfy certain prescribed initial conditions. We study the transmutation operators related to L in terms of the transformation of powers of the independent variable $\{(x-x_0)^k\}_{k=0}^{\infty}$ to the elements of the L-basis and establish a precise form of the transmutation operator realizing this transformation. We use this transmutation operator to establish a completeness of an infinite system of solutions of the stationary Schrödinger equation from a certain class. The system of solutions is obtained as an application of the theory of bicomplex pseudoanalytic functions and its completeness was a long sought result. Its use for constructing reproducing kernels and solving boundary and eigenvalue problems has been considered even without the required completeness justification. The obtained result on the completeness opens the way for further development and application of the tools of pseudoanalytic function theory.

1. Introduction

Transmutation operators are a widely used tool in the theory of linear differential equations (see, e.g., [1,6,22,24,29] and the recent review [27]). It is well known that under certain quite general conditions the transmutation operator transmuting the operator $A = -\frac{d^2}{dx^2} + q(x)$ into $B = -\frac{d^2}{dx^2}$ is a Volterra integral operator with good properties. Its kernel can be obtained as a solution of the Goursat problem for the Klein–Gordon equation with the variable coefficient. In the book [11] another approach to the transmutation was developed. It was shown that to every (regular) linear second-order ordinary differential operator L one can associate a linear space spanned on a so-called L-basis — an infinite family of functions $\{\varphi_k\}_{k=0}^{\infty}$ such that $L\varphi_k = 0$ for $k = 0, 1$, $L\varphi_k = k(k-1)\varphi_{k-2}$, for $k = 2, 3, \ldots$ and all φ_k satisfy certain prescribed initial conditions. Then the operator of transmutation was introduced as an operation transforming functions from one such linear space corresponding to a certain operator L to functions from another linear space corresponding to another operator M, and the transformation consists in substituting the L-basis with the M-basis preserving the same coefficients in the expansion.

In the present work we find out how the canonical Volterra integral transmutation operator acts on powers of the independent variable x^k (which represent a basis associated with the operator $\frac{d^2}{dx^2}$), introduce a parametrized family of transmutation operators and construct a transmutation operator which transforms the powers x^k into the functions φ_k from the L-basis. We prove that it is indeed a transmutation and can be written in the form of a Volterra integral operator.

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* Corresponding author.
E-mail address: vkravchenko@qro.cinvestav.mx (V.V. Kravchenko).

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We apply this result to prove the completeness of certain families of solutions of linear two-dimensional elliptic equations with variable complex-valued coefficients. These families of solutions were obtained earlier [7,17] as scalar parts of bicomplex pseudoanalytic formal powers and used for solving boundary value and eigenvalue problems [5]. Nevertheless no result on their completeness even in simplest cases was known due to profound differences between complex and bicomplex pseudoanalytic function theories and inapplicability of many classical results and techniques in the bicomplex situation. The use of the constructed transmutation operators and their extremely fortunate transformation properties regarding the L-bases allow us to observe that the infinite families of solutions mentioned above are nothing but the transmuted harmonic polynomials. Using their well-known completeness properties together with the properties of the Volterra integral transmutation operators we obtained several results on the completeness of families of solutions for equations with variable complex-valued coefficients.

In Section 2 we introduce the definition and some basic facts about transmutations together with an example which we constructed for illustrating some results of the present work. In Section 3 we introduce the L-basis as a system of recursive integrals. In Section 4 we study the action of the transmutation operators on the recursive integrals and construct the transmutation operator which transforms powers of x into functions of the L-basis. In Section 5 we introduce several definitions and results from the recently developed bicomplex pseudoanalytic function theory and explain its relation to linear second-order elliptic equations with variable complex-valued coefficients. In Section 6 we construct infinite families of solutions for a class of such equations and show that they are images of harmonic polynomials under the transmutation operator. We use this fact to prove their completeness under certain additional conditions. Section 7 contains some concluding remarks.

2. Transmutation operators for Sturm–Liouville equations

According to the definition given by Levitan [22], let E be a linear topological space, A and B be linear operators: $E \to E$. Let E_1 and E_2 be closed subspaces of E.

Definition 1. A linear invertible operator T defined on the whole E and acting from E_1 to E_2 is called a transmutation operator for the pair of operators A and B if it fulfills the following two conditions.

1. Both the operator T and its inverse T^{-1} are continuous in E;
2. The following operator equality is valid

$$AT = TB \tag{1}$$

or which is the same

$$A = TBT^{-1}.$$

Our main interest concerns the situation when $A = -\frac{d^2}{dx^2} + q(x)$, $B = -\frac{d^2}{dx^2}$, and q is a continuous complex-valued function. Hence for our purposes it will be sufficient to consider the functional space $E = C^2[a,b]$ with the topology of uniform convergence. One of the possibilities to introduce a transmutation operator on the whole C^2-space was considered by Lions [23] and later on in other references (see, e.g., [24]), and consists in constructing a Volterra integral operator corresponding to a midpoint of the segment of interest. As we begin with this transmutation operator it is convenient to consider a symmetric segment $[-a,a]$ and hence the functional space $E = C^2[-a,a]$. It is worth mentioning that other well-known ways to construct the transmutation operators (see, e.g., [22,29]) imply imposing initial conditions on the functions and consequently lead to transmutation operators satisfying (1) only on subclasses of $C^2[a,b]$.

Thus, we consider the space $E = C^2[-a,a]$ and an operator of transmutation for the defined above A and B can be realized in the form (see, e.g., [22] and [24])

$$Tu(x) = u(x) + \int_{-x}^{x} K(x,t)u(t)\,dt \tag{2}$$

where $K(x,t)$ is a unique solution of the Goursat problem

$$\left(\frac{\partial^2}{\partial x^2} - q(x)\right)K(x,t) = \frac{\partial^2}{\partial t^2}K(x,t), \tag{3}$$

$$K(x,x) = \frac{1}{2}\int_0^x q(s)\,ds, \qquad K(x,-x) = 0. \tag{4}$$

An important property of this transmutation operator consists in the way how it maps solutions of the equation

$$v'' + \omega^2 v = 0 \tag{5}$$

into solutions of the equation

$$u'' - q(x)u + \omega^2 u = 0 \tag{6}$$

where ω is a complex number. Denote by $e_0(i\omega, x)$ the solution of (6) satisfying the initial conditions

$$e_0(i\omega, 0) = 1 \quad \text{and} \quad e_0'(i\omega, 0) = i\omega. \tag{7}$$

The subindex "0" indicates that the initial conditions correspond to the point $x = 0$ and the letter "e" reminds us that the initial values coincide with the initial values of the function $e^{i\omega x}$.

The transmutation operator (2) maps $e^{i\omega x}$ into $e_0(i\omega, x)$,

$$e_0(i\omega, x) = T\left[e^{i\omega x}\right] \tag{8}$$

(see [24, Theorem 1.2.1]).

Following [24] we introduce the notations

$$K(x, t; h) = h + K(x, t) + K(x, -t) + h \int_t^x \left\{K(x, \xi) - K(x, -\xi)\right\} d\xi$$

where h is a complex number, and

$$K(x, t; \infty) = K(x, t) - K(x, -t).$$

Theorem 2. *(See [24].) Solutions $c(\omega, x; h)$ and $s(\omega, x; \infty)$ of Eq.* (6) *satisfying the initial conditions*

$$c(\omega, 0; h) = 1, \qquad c_x'(\omega, 0; h) = h, \qquad s(\omega, 0; \infty) = 0, \qquad s_x'(\omega, 0; \infty) = 1$$

can be represented in the form

$$c(\omega, x; h) = \cos\omega x + \int_0^x K(x, t; h)\cos\omega t \, dt \tag{9}$$

and

$$s(\omega, x; \infty) = \frac{\sin\omega x}{\omega} + \int_0^x K(x, t; \infty)\frac{\sin\omega t}{\omega}\, dt. \tag{10}$$

The operators

$$T_c u(x) = u(x) + \int_0^x K(x, t; h)u(t)\, dt$$

and

$$T_s u(x) = u(x) + \int_0^x K(x, t; \infty)u(t)\, dt$$

are not transmutations on the whole space $C^2[-a, a]$, they even do not map all solutions of (5) into solutions of (6). For example, as we show below

$$\left(-\frac{d^2}{dx^2} + q(x)\right)T_s[1] \neq T_s\left[-\frac{d^2}{dx^2}(1)\right] = 0$$

when q is constant.

Example 3. Transmutation operator for operators $A := \frac{d^2}{dx^2} + c$, c is a constant, and $B := \frac{d^2}{dx^2}$. According to [24, (1.2.25), (1.2.26)], finding the kernel of transmutation operator is equivalent to finding the function $H(s, t) = K(s + t, s - t)$, satisfying the Goursat problem

$$\frac{\partial^2 H(s,t)}{\partial s \partial t} = -cH(s,t), \qquad H(s,0) = -\frac{cs}{2}, \qquad H(0,t) = 0.$$

The solution of this problem is given by [12, (4.85)]

$$H(s,t) = -\frac{c}{2}\int_0^s J_0\bigl(2\sqrt{ct(s-\xi)}\bigr)\,d\xi = -\frac{\sqrt{cst}\,J_1(2\sqrt{cst}\,)}{2t},$$

where J_0 and J_1 are Bessel functions of the first kind, and the formula is valid even if the radicand is negative. Hence,

$$K(x,y) = H\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = -\frac{1}{2}\frac{\sqrt{c(x^2-y^2)}\,J_1(\sqrt{c(x^2-y^2)}\,)}{x-y}. \tag{11}$$

From (11) we get the ‘sine’ kernel

$$K(x,t;\infty) = -\frac{t\sqrt{c(x^2-t^2)}\,J_1(\sqrt{c(x^2-t^2)}\,)}{x^2-t^2}, \tag{12}$$

and can check the above statement about the operator T_s,

$$T_s[1](x) = 1 - \int_0^x \frac{t\sqrt{c(x^2-t^2)}\,J_1(\sqrt{c(x^2-t^2)}\,)}{x^2-t^2}\,dt = J_0(x\sqrt{c}\,), \qquad \left(\frac{d^2}{dx^2}+c\right)T_s[1] = \frac{\sqrt{c}\,J_1(x\sqrt{c}\,)}{x} \neq 0.$$

3. A complete system of recursive integrals

Let $f \in C^2(a,b) \cap C^1[a,b]$ be a complex valued function and $f(x) \neq 0$ for any $x \in [a,b]$. The interval (a,b) is supposed to be finite. Let us consider the following auxiliary functions

$$\tilde{X}^{(0)}(x) \equiv X^{(0)}(x) \equiv 1, \tag{13}$$

$$\tilde{X}^{(n)}(x) = n\int_{x_0}^x \tilde{X}^{(n-1)}(s)\bigl(f^2(s)\bigr)^{(-1)^{n-1}}\,ds, \tag{14}$$

$$X^{(n)}(x) = n\int_{x_0}^x X^{(n-1)}(s)\bigl(f^2(s)\bigr)^{(-1)^{n}}\,ds, \tag{15}$$

where x_0 is an arbitrary fixed point in $[a,b]$. We introduce the infinite system of functions $\{\varphi_k\}_{k=0}^{\infty}$ defined as follows

$$\varphi_k(x) = \begin{cases} f(x)X^{(k)}(x), & k \text{ odd}, \\ f(x)\tilde{X}^{(k)}(x), & k \text{ even}, \end{cases} \tag{16}$$

where the definition of $X^{(k)}$ and $\tilde{X}^{(k)}$ is given by (13)–(15) with x_0 being an arbitrary point of the interval $[a,b]$.

Example 4. Let $f \equiv 1$, $a = 0$, $b = 1$. Then it is easy to see that choosing $x_0 = 0$ we have $\varphi_k(x) = x^k$, $k \in \mathbb{N}_0$ where by $\mathbb{N}_0$ we denote the set of non-negative integers.

In [18] it was shown that the system $\{\varphi_k\}_{k=0}^{\infty}$ is complete in $L_2(a,b)$ and in [19] its completeness in the space of piecewise differentiable functions with respect to the maximum norm was obtained and the corresponding series expansions in terms of the functions φ_k were studied.

The system (16) is closely related to the notion of the L-basis introduced and studied in [11]. Here the letter L corresponds to a linear ordinary differential operator. This becomes more transparent from the following result obtained in [16] (for additional details and simpler proof see [17] and [20]) establishing the relation of the system of functions $\{\varphi_k\}_{k=0}^{\infty}$ to Sturm–Liouville equations.

Theorem 5. *(See [16].) Let q be a continuous complex valued function of an independent real variable $x \in [a,b]$, λ be an arbitrary complex number. Suppose there exists a solution f of the equation*

$$f'' - qf = 0 \tag{17}$$

on (a,b) such that $f \in C^2[a,b]$ and $f \neq 0$ on $[a,b]$. Then the general solution of the equation

$$u'' - qu = \lambda u \tag{18}$$

on (a,b) has the form

$$u = c_1 u_1 + c_2 u_2$$

where c_1 and c_2 are arbitrary complex constants,

$$u_1 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k)!} \varphi_{2k} \quad \text{and} \quad u_2 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} \varphi_{2k+1} \tag{19}$$

and both series converge uniformly on $[a,b]$.

Remark 6. It is easy to see that by definition the solutions u_1 and u_2 satisfy the following initial conditions

$$u_1(x_0) = f(x_0), \qquad u_1'(x_0) = f'(x_0), \tag{20}$$

$$u_2(x_0) = 0, \qquad u_2'(x_0) = 1/f(x_0). \tag{21}$$

4. Transmutations and systems of recursive integrals

Let us obtain the expansion of the solution $e_0(i\omega, x)$ from Section 2 in terms of the functions φ_k. We suppose that f is a solution of (17) fulfilling the condition of Theorem 5 on a finite interval $(-a,a)$. We normalize f in such a way that $f(0) = 1$ and let $f'(0) = h$ where h is some complex constant. Then according to Remark 6 the solutions (19) of Eq. (18) have the following initial values

$$u_1(0) = 1, \qquad u_1'(0) = h, \qquad u_2(0) = 0, \qquad u_2'(0) = 1.$$

Hence due to (7) we obtain $e_0(i\omega, x) = u_1(x) + (i\omega - h)u_2(x)$. From (8) and (19) we have the equality

$$\sum_{k=0}^{\infty} \frac{(i\omega)^{2k}}{(2k)!} \varphi_{2k}(x) + (i\omega - h) \sum_{k=0}^{\infty} \frac{(i\omega)^{2k}}{(2k+1)!} \varphi_{2k+1}(x) = \sum_{j=0}^{\infty} \frac{(i\omega)^j x^j}{j!} + \int_{-x}^{x} \left(K(x,t) \sum_{j=0}^{\infty} \frac{(i\omega)^j t^j}{j!} \right) dt.$$

As the series under the sign of integral converges uniformly and the kernel $K(x,t)$ is at least continuously differentiable (for a continuous q [24]) we obtain the following relation

$$\sum_{k=0}^{\infty} \frac{(i\omega)^{2k}}{(2k)!} \varphi_{2k}(x) + \sum_{k=0}^{\infty} \frac{(i\omega)^{2k+1}}{(2k+1)!} \varphi_{2k+1}(x) - h \sum_{k=0}^{\infty} \frac{(i\omega)^{2k}}{(2k+1)!} \varphi_{2k+1}(x) = \sum_{j=0}^{\infty} \frac{(i\omega)^j}{j!} \left(x^j + \int_{-x}^{x} K(x,t) t^j \, dt \right).$$

The equality holds for any ω hence we obtain the term wise relations

$$\varphi_k = T\left[x^k\right] \quad \text{when } k \text{ is odd} \tag{22}$$

and

$$\varphi_k - \frac{h}{k+1} \varphi_{k+1} = T\left[x^k\right] \quad \text{when } k \in \mathbb{N}_0 \text{ is even}. \tag{23}$$

Taking into account the first of these relations the second can be written also as follows

$$\varphi_k = T\left[x^k + \frac{h}{k+1} x^{k+1}\right] \quad \text{when } k \in \mathbb{N}_0 \text{ is even}. \tag{24}$$

Thus, we proved the following statement.

Theorem 7. *Let q be a continuous complex valued function of an independent real variable $x \in [-a,a]$, and f be a particular solution of* (17) *such that $f \in C^2[-a,a]$, $f \neq 0$ on $[-a,a]$ and normalized as $f(0) = 1$. Denote $h := f'(0) \in \mathbb{C}$. Suppose T is the operator defined by* (2) *where the kernel K is a solution of the problem* (3), (4) *and φ_k, $k \in \mathbb{N}_0$ are functions defined by* (16). *Then equalities* (22)–(24) *hold.*

Thus, we clarified what is the result of application of the transmutation T to the powers of the independent variable. This is very useful due to the fact that as a rule the construction of the kernel $K(x,t)$ in a more or less explicit form up to now is impossible. Our result gives an algorithm for transmuting functions which can be represented or at least approximated by finite or infinite polynomials in the situation when $K(x,t)$ is unknown.

Remark 8. Let f be the solution of (17) satisfying the initial conditions

$$f(0) = 1, \quad \text{and} \quad f'(0) = 0. \tag{25}$$

If it does not vanish on $[-a, a]$ then from Theorem 7 we obtain that $\varphi_k = T[x^k]$ for any $k \in \mathbb{N}_0$. In general, of course there is no guaranty that the solution with such initial values have no zeros on $[-a, a]$ and hence the operator T transmutes the powers of x into φ_k whose construction is based on the solution f satisfying (25) only in some neighborhood of the origin.

Similarly to Theorem 7 we obtain the following statement.

Theorem 9. *Under the conditions of Theorem 7 the following equalities are valid*

$$\varphi_k = T_c\big[x^k\big] \quad \text{when } k \in \mathbb{N}_0 \text{ is even} \tag{26}$$

and

$$\varphi_k = T_s\big[x^k\big] \quad \text{when } k \in \mathbb{N} \text{ is odd}. \tag{27}$$

Proof. It is easy to see that $c(\omega, x; h) = u_1(x)$ and $s(\omega, x; \infty) = u_2(x)$ where u_1 and u_2 are defined by (19). From here and from (9), (10) by expanding $\cos\omega x$ and $\sin\omega x$ into their Taylor series we obtain (26) and (27). □

Now, for a given nonvanishing solution of (17) on $(-a, a)$ satisfying the initial conditions $f(0) = 1$ and $f'(0) = h$ where $h \in \mathbb{C}$ and for the corresponding system of functions (16) we construct a transmutation operator for the pair $\frac{d^2}{dx^2}$ and $\frac{d^2}{dx^2} - q(x)$ such that x^k are transformed into $\varphi_k(x)$ on the whole segment $[-a, a]$ for any $k \in \mathbb{N}_0$. For this we introduce the following projectors acting on any continuous function (defined on $[-a, a]$) according to the rules $P_e f(x) = (f(x) + f(-x))/2$ and $P_o f(x) = (f(x) - f(-x))/2$. Consider the following operator

$$\mathbf{T} = T_c P_e + T_s P_o.$$

It is easy to see that by construction for an even k we obtain $\mathbf{T}[x^k] = T_c P_e[x^k] = T_c[x^k] = \varphi_k$ and analogously for an odd k, $\mathbf{T}[x^k] = \varphi_k$ due to (27). Thus, $\varphi_k = \mathbf{T}[x^k]$ for any $k \in \mathbb{N}_0$. Moreover, the operator $\mathbf{T}$ can be written as a Volterra operator in a form similar to (2). We have

$$\mathbf{T}u(x) = u(x) + \int_{-x}^{x} \mathbf{K}(x, t; h) u(t)\, dt \tag{28}$$

where

$$\mathbf{K}(x, t; h) = \frac{h}{2} + K(x, t) + \frac{h}{2} \int_t^x \big(K(x, s) - K(x, -s)\big)\, ds. \tag{29}$$

Let us notice that $\mathbf{K}(x, t; 0) = K(x, t)$ and that the expression

$$\mathbf{K}(x, t; h) - \mathbf{K}(x, -t; h) = K(x, t) - K(x, -t) + \frac{h}{2} \int_{-t}^{t} \big(K(x, s) - K(x, -s)\big)\, ds = K(x, t) - K(x, -t)$$

does not depend on h. Thus, we obtain a way to compute $\mathbf{K}(x, t; h)$ for any h by a given $\mathbf{K}(x, t; h_1)$ for some particular value h_1.

Theorem 10. *The integral kernels $\mathbf{K}(x, t; h)$ and $\mathbf{K}(x, t; h_1)$ are related by the expression*

$$\mathbf{K}(x, t; h) = \frac{h - h_1}{2} + \mathbf{K}(x, t; h_1) + \frac{h - h_1}{2} \int_t^x \big(\mathbf{K}(x, s; h_1) - \mathbf{K}(x, -s; h_1)\big)\, ds. \tag{30}$$

Let us prove that $\mathbf{T}$ is indeed a transmutation.

Theorem 11. *Under the conditions of Theorem* 7 *let us assume additionally that $q \in C^1[-a,a]$. Then the operator* (28) *with the kernel defined by* (29) *transforms x^k into $\varphi_k(x)$ for any $k \in \mathbb{N}_0$ and*

$$\left(-\frac{d^2}{dx^2}+q(x)\right)\mathbf{T}[u]=\mathbf{T}\left[-\frac{d^2}{dx^2}(u)\right] \tag{31}$$

for any $u \in C^2[-a,a]$.

Proof. Under the condition $q \in C^1[-a,a]$, the kernel $K(x,t)$ in (2) is twice continuously differentiable with respect to both x and t [24, Theorem 1.2.2]. Hence, the kernel $\mathbf{K}(x,t;h)$ is also twice continuously differentiable with respect to both x and t, and $C^2[-a,a]$ is invariant under the operator $\mathbf{T}$, that is, the left-hand side of (31) is well defined for all $u \in C^2[-a,a]$.

Since $(-\frac{d^2}{dx^2}+q(x))\varphi_k = k(k-1)\varphi_{k-2}$, $k \geqslant 2$, and $(-\frac{d^2}{dx^2}+q(x))\varphi_k = 0$, $k=0,1$ (see [16]), the equality (31) is valid for all powers x^k and, by linearity, for all polynomials. Let $u \in C^2[-a,a]$. Then, $u'' \in C[-a,a]$ and by the Weierstrass theorem there exists a sequence of polynomials Q_n such that

$$Q_n \to u'', \quad n \to \infty \text{ uniformly on } [-a,a]. \tag{32}$$

Integrating (32) twice we conclude that the sequence of polynomials defined by

$$P_n(x) = u(0) + u'(0)x + \int_0^x \int_0^t Q_n(s)\,ds\,dt$$

is such that

$$P_n \to u, \qquad P_n' \to u' \quad \text{and} \quad P_n'' \to u'', \quad n \to \infty$$

uniformly in $[-a,a]$. Since the kernel $\mathbf{K}(x,t;h)$ is twice continuously differentiable, it is easy to see that also

$$\mathbf{T}[P_n] \to \mathbf{T}[u], \qquad \big(\mathbf{T}[P_n]\big)' \to \big(\mathbf{T}[u]\big)' \quad \text{and} \quad \big(\mathbf{T}[P_n]\big)'' \to \big(\mathbf{T}[u]\big)'', \quad n \to \infty$$

uniformly in $[-a,a]$. Therefore,

$$\left(-\frac{d^2}{dx^2}+q(x)\right)\mathbf{T}[u] = \lim_{n\to\infty}\left(-\frac{d^2}{dx^2}+q(x)\right)\mathbf{T}[P_n] = \lim_{n\to\infty}\mathbf{T}\left[-\frac{d^2}{dx^2}(P_n)\right] = \mathbf{T}\left[-\frac{d^2}{dx^2}(u)\right]. \qquad \square$$

It is possible to give another final part of the proof, revealing some properties of the adjoint operator $\mathbf{T}^*$.

Proof. Consider $u \in C^2[-a,a]$ and let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of polynomials such that $p_n \to u$, $n \to \infty$. Consider $f_n = (-\frac{d^2}{dx^2}+q(x))\mathbf{T}[p_n]$. Since p_n is a polynomial, we also have $f_n = \mathbf{T}[-\frac{d^2}{dx^2}(p_n)]$. Let $f = (-\frac{d^2}{dx^2}+q(x))\mathbf{T}[u]$, $\tilde{f} = \mathbf{T}[-\frac{d^2}{dx^2}(u)]$. From now on consider u, p_n, f_n, f, $\tilde{f}$ as elements of the Hilbert space $L_2[-a,a]$. Since $\mathbf{T}$ is the Volterra operator, it is continuous in $L_2[-a,a]$ space. Hence, $\mathbf{T}p_n \to \mathbf{T}u$, $n \to \infty$. Consider any function $\psi \in C_0^2[-a,a]$, that is twice continuously differentiable and supported on some $[\alpha,\beta] \subset (-a,a)$. Then we have

$$(f_n - f, \psi) = \left(\left(-\frac{d^2}{dx^2}+q(x)\right)(\mathbf{T}p_n - \mathbf{T}u), \psi\right) = \left(\mathbf{T}p_n - \mathbf{T}u, \left(-\frac{d^2}{dx^2}+\overline{q(x)}\right)\psi\right) \to 0, \quad n \to \infty. \tag{33}$$

For the right-hand side of (31), we need the adjoint operator $\mathbf{T}^*$. Since $\mathbf{T}$ is the Volterra operator, its adjoint is given by the expression [13, Chapter 3, Example 3.17] $\mathbf{T}^*u(x) = u(x) + \int_{-a}^{-|x|}\overline{\mathbf{K}(t,x;h)}u(t)\,dt + \int_{|x|}^{a}\overline{\mathbf{K}(t,x;h)}u(t)\,dt$, from which it is easy to see that $\mathbf{T}^*\psi \in C_0^2[-a,a]$ for any $\psi \in C_0^2[-a,a]$. Hence,

$$(f_n - \tilde{f}, \psi) = \left(-\frac{d^2}{dx^2}(p_n - u), \mathbf{T}^*\psi\right) = \left(p_n - u, -\frac{d^2}{dx^2}(\mathbf{T}^*\psi)\right) \to 0, \quad n \to \infty. \tag{34}$$

It follows from (33) and (34) that $(f - \tilde{f}, \psi) = 0$ for any $\psi \in C_0^2[-a,a]$. Since the set $C_0^2[-a,a]$ is dense in $L_2[-a,a]$, we have $f = \tilde{f}$ as elements of $L_2[-a,a]$ as well as continuous functions. $\square$

Example 12. Consider the same operators A and B as in Example 3. Then the transmutation operator $\mathbf{T}$ is defined by the integration kernel (29)

$$\mathbf{K}(x,y;h)=-\frac{1}{2}\frac{\sqrt{c(x^2-y^2)}\,J_1(\sqrt{c(x^2-y^2)})}{x-y}+\frac{h}{2}J_0\big(\sqrt{c\big(x^2-y^2\big)}\big)=K(x,y)+\frac{h}{2}J_0\big(\sqrt{c\big(x^2-y^2\big)}\big).$$

If we consider the function $f(x)=e^{i\kappa x}$, $\kappa^2=c$ as a solution of $Af=0$ satisfying $f(0)=1$, $f'(0)=i\kappa$, then the first four functions φ_k are

$$\varphi_0(x)=e^{i\kappa x},\qquad \varphi_1(x)=\frac{\sin(\kappa x)}{\kappa},\qquad \varphi_2(x)=\frac{\kappa xe^{i\kappa x}-\sin(\kappa x)}{i\kappa^2},\qquad \varphi_3(x)=\frac{3(\sin(\kappa x)-\kappa x\cos(\kappa x))}{\kappa^3}$$

and from Theorem 9 we obtain the integrals

$$\varphi_k(x)=x^k-\frac{\kappa}{2}\int_{-x}^{x}\left(\frac{\sqrt{(x^2-y^2)}\,J_1(\kappa\sqrt{(x^2-y^2)})y^k}{x-y}+iy^kJ_0\big(\kappa\sqrt{\big(x^2-y^2\big)}\big)\right)dy$$

which validity can be checked numerically.

5. Bicomplex numbers and pseudoanalytic functions

Together with the imaginary unit i we consider another imaginary unit j, such that

$$j^2=i^2=-1\quad\text{and}\quad ij=ji. \tag{35}$$

We have then two copies of the algebra of complex numbers: $\mathbb{C}_i:=\{a+ib,\ \{a,b\}\subset\mathbb{R}\}$ and $\mathbb{C}_j:=\{a+jb,\ \{a,b\}\subset\mathbb{R}\}$. The expressions of the form $w=u+jv$ where $\{u,v\}\subset\mathbb{C}_i$ are called bicomplex numbers. The conjugation with respect to j we denote as follows $\bar{w}=u-jv$. The components u and v will be called the scalar and the vector part of w respectively. We will use the notation $u=\operatorname{Sc}w$ and $v=\operatorname{Vec}w$.

The set of all bicomplex numbers with a natural operation of addition and with the multiplication defined by the laws (35) represents a commutative ring with unit. We denote it by $\mathbb{B}$. It contains zero divisors: the nonzero elements w such that $w\bar{w}=0$. Introducing the pair of idempotents $P^+=\frac{1}{2}(1+ij)$ and $P^-=\frac{1}{2}(1-ij)$ ($(P^\pm)^2=P^\pm$) it is easy to see (e.g., [17, p. 154]) that $w=u+jv$ is a zero divisor if and only if $w=2P^+u$ or $w=2P^-u$. For other algebraic properties of bicomplex numbers we refer to [25,26].

We consider $\mathbb{B}$-valued functions of two real variables x and y. Denote $\bar{\partial}=\frac{1}{2}(\frac{\partial}{\partial x}+j\frac{\partial}{\partial y})$ and $\partial=\frac{1}{2}(\frac{\partial}{\partial x}-j\frac{\partial}{\partial y})$. An equation of the form

$$\bar{\partial}w=aw+b\bar{w}, \tag{36}$$

where w, a and b are $\mathbb{B}$-valued functions is called a bicomplex Vekua equation. When all the involved functions have their values in $\mathbb{C}_j$ only, Eq. (36) becomes the well-known complex Vekua equation (see [17,30]). We will assume that $w\in C^1(\Omega)$ where $\Omega\subset\mathbb{R}^2$ is an open domain and a, b are Hölder continuous in Ω.

When $a\equiv 0$ and $b=\frac{\bar{\partial}\phi}{\phi}$ where $\phi:\bar{\Omega}\to\mathbb{C}_i$ possesses Hölder continuous partial derivatives in Ω and $\phi(x,y)\neq 0$, $\forall(x,y)\in\bar{\Omega}$ we will say that the bicomplex Vekua equation

$$\bar{\partial}w=\frac{\bar{\partial}\phi}{\phi}\bar{w} \tag{37}$$

is a Vekua equation of the main type or the main Vekua equation.

For classical complex Vekua equations Bers introduced [3] the notions of a generating pair, generating sequence, formal powers and Taylor series in formal powers. As was shown in [7,17] the definition of these notions can be extended onto the bicomplex situation. Here we briefly recall the main definitions.

Definition 13. A pair of $\mathbb{B}$-valued functions F and G possessing Hölder continuous partial derivatives in Ω with respect to the real variables x and y is said to be a generating pair if it satisfies the inequality

$$\operatorname{Vec}(\bar{F}G)\neq 0\quad\text{in }\Omega. \tag{38}$$

Condition (38) implies that every bicomplex function w defined in a subdomain of Ω admits the unique representation $w=\phi F+\psi G$ where the functions ϕ and ψ are scalar ($\mathbb{C}_i$-valued).

Remark 14. When $F\equiv 1$ and $G\equiv j$ the corresponding bicomplex Vekua equation is

$$\bar{\partial}w=0, \tag{39}$$

and its study in fact reduces to the complex analytic function theory. This is due to the fact that the functions P^+w and P^-w are necessarily antiholomorphic and holomorphic respectively. Indeed, application of P^+ and P^- to (39) gives us

$$\partial_z P^+ w = 0 \quad \text{and} \quad \partial_{\bar z} P^- w = 0 \tag{40}$$

where $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\partial_{\bar z} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Moreover, $P^+w = P^+(u + jv) = P^+(u - iv)$ and $P^-w = P^-(u + iv)$. Due to (40) the scalar functions $w^+ := u - iv$ and $w^- := u + iv$ are antiholomorphic and holomorphic respectively. We stress that w^+ is not necessarily a complex conjugate of w^- (u and v are $\mathbb{C}_i$-valued).

In general a reduction of the bicomplex Vekua equation (36) to a pair of decoupled complex Vekua equations is impossible. Application of P^+ and P^- to (36) reduces it to the following system of equations

$$\partial_z w^+ = a^+ w^+ + b^+ w^-$$

and

$$\partial_{\bar z} w^- = a^- w^- + b^- w^+$$

for two complex functions w^+ and w^- with complex coefficients $a^\pm$, $b^\pm$.

Assume that (F, G) is a generating pair in a domain Ω.

Definition 15. Let the $\mathbb{B}$-valued function w be defined in a neighborhood of $z_0 \in \Omega \subset \mathbb{C}_j$. In a complete analogy with the complex case we say that at z_0 the function w possesses the (F, G)-derivative $\dot w(z_0)$ if the (finite) limit

$$\dot w(z_0) = \lim_{z \to z_0} \frac{w(z) - \lambda_0 F(z) - \mu_0 G(z)}{z - z_0} \tag{41}$$

exists where λ_0 and μ_0 are the unique scalar constants such that $w(z_0) = \lambda_0 F(z_0) + \mu_0 G(z_0)$.

Similarly to the complex case (see, e.g., [17, Chapter 2]) it is easy to show that if $\dot w(z_0)$ exists then at z_0, $\bar\partial w$ and ∂w exist and equations

$$\bar\partial w = a_{(F,G)} w + b_{(F,G)} \bar w \tag{42}$$

and

$$\dot w = \partial w - A_{(F,G)} w - B_{(F,G)} \bar w \tag{43}$$

hold, where $a_{(F,G)}$, $b_{(F,G)}$, $A_{(F,G)}$ and $B_{(F,G)}$ are the *characteristic coefficients* characteristic coefficients of the pair (F, G) defined by the formulas

$$a_{(F,G)} = -\frac{\bar F \bar\partial G - \bar G \bar\partial F}{F\bar G - \bar F G}, \qquad b_{(F,G)} = \frac{F \bar\partial G - G \bar\partial F}{F\bar G - \bar F G}, \qquad A_{(F,G)} = -\frac{\bar F \partial G - \bar G \partial F}{F\bar G - \bar F G}, \qquad B_{(F,G)} = \frac{F \partial G - G \partial F}{F\bar G - \bar F G}.$$

Note that $F\bar G - \bar F G = -2j\,\mathrm{Vec}(\bar F G) \neq 0$.

If $\bar\partial w$ and ∂w exist and are continuous in some neighborhood of z_0, and if (42) holds at z_0, then $\dot w(z_0)$ exists, and (43) holds. Let us notice that F and G possess (F, G)-derivatives, $\dot F \equiv \dot G \equiv 0$ and the following equalities are valid which determine the characteristic coefficients uniquely

$$\bar\partial F = a_{(F,G)} F + b_{(F,G)} \bar F, \qquad \bar\partial G = a_{(F,G)} G + b_{(F,G)} \bar G, \qquad \partial F = A_{(F,G)} F + B_{(F,G)} \bar F, \qquad \partial G = A_{(F,G)} G + B_{(F,G)} \bar G.$$

If the (F, G)-derivative of a $\mathbb{B}$-valued function $w = \phi F + \psi G$ (where the functions ϕ and ψ are scalar) exists, besides the form (43) it can also be written as follows $\dot w = \partial\phi F + \partial\psi G$.

Definition 16. Let (F, G) and (F_1, G_1) — be two generating pairs in Ω. (F_1, G_1) is called successor of (F, G) and (F, G) is called predecessor of (F_1, G_1) if

$$a_{(F_1,G_1)} = a_{(F,G)} \quad \text{and} \quad b_{(F_1,G_1)} = -B_{(F,G)}.$$

By analogy with the complex case we have the following statement.

Theorem 17. *Let w be a bicomplex (F, G)-pseudoanalytic function and let (F_1, G_1) be a successor of (F, G). Then $\dot w$ is a bicomplex (F_1, G_1)-pseudoanalytic function.*

Definition 18. Let (F, G) be a generating pair. Its adjoint generating pair $(F, G)^* = (F^*, G^*)$ is defined by the formulas

$$F^* = -\frac{2\bar{F}}{F\bar{G} - \bar{F}G}, \qquad G^* = \frac{2\bar{G}}{F\bar{G} - \bar{F}G}.$$

The (F, G)-integral is defined as follows

$$\int_\Gamma W\, d_{(F,G)}z = F(z_1)\,\mathrm{Sc}\int_\Gamma G^* W\, dz + G(z_1)\,\mathrm{Sc}\int_\Gamma F^* W\, dz$$

where Γ is a rectifiable curve leading from z_0 to z_1.

If $W = \phi F + \psi G$ is a bicomplex (F, G)-pseudoanalytic function where ϕ and ψ are complex valued functions then

$$\int_{z_0}^{z} \dot{W}\, d_{(F,G)}z = W(z) - \phi(z_0)F(z) - \psi(z_0)G(z), \tag{44}$$

and this integral is path-independent and represents the (F, G)-antiderivative of $\dot{W}$.

Definition 19. A sequence of generating pairs $\{(F_m, G_m)\}$, $m = 0, \pm 1, \pm 2, \ldots$, is called a generating sequence if (F_{m+1}, G_{m+1}) is a successor of (F_m, G_m). If $(F_0, G_0) = (F, G)$, we say that (F, G) is embedded in $\{(F_m, G_m)\}$.

Let W be a bicomplex (F, G)-pseudoanalytic function. Using a generating sequence in which (F, G) is embedded we can define the higher derivatives of W by the recursion formula

$$W^{[0]} = W; \qquad W^{[m+1]} = \frac{d_{(F_m,G_m)} W^{[m]}}{dz}, \quad m = 1, 2, \ldots.$$

Definition 20. The formal power $Z_m^{(0)}(a, z_0; z)$ with center at $z_0 \in \Omega$, coefficient a and exponent 0 is defined as the linear combination of the generators F_m, G_m with scalar constant coefficients λ, μ chosen so that $\lambda F_m(z_0) + \mu G_m(z_0) = a$. The formal powers with exponents $n = 0, 1, 2, \ldots$ are defined by the recursion formula

$$Z_m^{(n+1)}(a, z_0; z) = (n+1)\int_{z_0}^{z} Z_{m+1}^{(n)}(a, z_0; \zeta)\, d_{(F_m,G_m)}\zeta. \tag{45}$$

This definition implies the following properties.

1. $Z_m^{(n)}(a, z_0; z)$ is an (F_m, G_m)-pseudoanalytic function of z.
2. If a' and a'' are scalar constants, then
$$Z_m^{(n)}\big(a' + ja'', z_0; z\big) = a' Z_m^{(n)}(1, z_0; z) + a'' Z_m^{(n)}(j, z_0; z).$$
3. The formal powers satisfy the differential relations
$$\frac{d_{(F_m,G_m)} Z_m^{(n)}(a, z_0; z)}{dz} = n Z_{m+1}^{(n-1)}(a, z_0; z).$$
4. The asymptotic formulas
$$Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \quad z \to z_0$$
hold.

The case of the main bicomplex Vekua equation is of a special interest due to the following relation with the stationary Schrödinger equation.

Theorem 21. *(See [14].) Let $W = W_1 + jW_2$ be a solution of the main bicomplex Vekua equation*

$$\bar{\partial} W = \frac{\bar{\partial}\phi}{\phi}\bar{W} \quad \text{in } \Omega \tag{46}$$

where $W_1 = \mathrm{Sc}\, W$, $W_2 = \mathrm{Vec}\, W$ and the $\mathbb{C}_i$-valued function ϕ is a nonvanishing solution of the equation

$$-\Delta u+q_1(x,y)u=0 \quad \text{in } \Omega \tag{47}$$

where q_1 is a continuous $\mathbb{C}_i$-valued function. Then W_1 is a solution of (47) *in Ω and W_2 is a solution of the associated Schrödinger equation*

$$-\Delta v+q_2(x,y)v=0 \quad \text{in } \Omega \tag{48}$$

where $q_2=8\frac{\bar\partial\phi\partial\phi}{\phi^2}-q_1$.

We need the following notation. Let w be a $\mathbb{B}$-valued function defined on a simply connected domain Ω with $w_1=\operatorname{Sc} w$ and $w_2=\operatorname{Vec} w$ such that

$$\frac{\partial w_1}{\partial y}-\frac{\partial w_2}{\partial x}=0, \quad \forall(x,y)\in\Omega, \tag{49}$$

and let $\Gamma\subset\Omega$ be a rectifiable curve leading from (x_0,y_0) to (x,y). Then the integral

$$\bar{A}w(x,y):=2\left(\int_\Gamma w_1\,dx+w_2\,dy\right)$$

is path-independent, and all $\mathbb{C}_i$-valued solutions φ of the equation $\bar\partial\varphi=w$ in Ω have the form $\varphi(x,y)=\bar{A}w(x,y)+c$ where c is an arbitrary $\mathbb{C}_i$-constant. In other words the operator $\bar{A}$ denotes the well-known operation for reconstructing the potential function from its gradient.

Theorem 22. *(See [14].) Let W_1 be a $\mathbb{C}_i$-valued solution of the Schrödinger equation* (47) *in a simply connected domain Ω. Then a $\mathbb{C}_i$-valued solution W_2 of the associated Schrödinger equation* (48) *such that W_1+jW_2 is a solution of* (46) *in Ω can be constructed according to the formula*

$$W_2=\frac{1}{\phi}\bar{A}\left(j\phi^2\bar\partial\left(\frac{W_1}{\phi}\right)\right)+\frac{c_1}{\phi}$$

where c_1 is an arbitrary $\mathbb{C}_i$-constant.

Vice versa, given a solution W_2 of (48)*, the corresponding solution W_1 of* (47) *such that W_1+jW_2 is a solution of* (46) *has the form*

$$W_1=-\phi\bar{A}\left(\frac{j}{\phi^2}\bar\partial(\phi W_2)\right)+c_2\phi$$

where c_2 is an arbitrary $\mathbb{C}_i$-constant.

As was shown in [15] (see also [17]) a generating sequence can be obtained in a closed form, for example, in the case when ϕ has a separable form $\phi=S(s)T(t)$ where s and t are conjugate harmonic functions and S, T are arbitrary twice continuously differentiable functions. In practical terms this means that whenever the Schrödinger equation (47) admits a particular nonvanishing solution having the form $\phi=f(\xi)\,g(\eta)$ where (ξ,η) is one of the encountered in physics orthogonal coordinate systems in the plane a generating sequence corresponding to (46) can be obtained explicitly [17, Section 4.8]. The knowledge of a generating sequence allows one to construct the formal powers following Definition 20. This construction is a simple algorithm which can be quite easily and efficiently realized numerically [5,8]. Moreover, in the case of a complex main Vekua equation which in the notations admitted in the present paper corresponds to the case of ϕ being a real-valued function (then the main bicomplex Vekua equation decouples into two main complex Vekua equations) the completeness of the system of formal powers was proved [5] in the sense that any pseudoanalytic in Ω and Hölder continuous on $\partial\Omega$ function can be approximated uniformly and arbitrarily closely by a finite linear combination of the formal powers. The real parts of the complex pseudoanalytic formal powers represent then a complete system of solutions of one Schrödinger equation meanwhile the imaginary parts give us a complete system of solutions of the associated Schrödinger equation.

In the bicomplex case the system of formal powers is constructed in the same way as in the complex situation and the system of functions

$$\left\{\operatorname{Sc} Z_0^{(n)}(1,z_0;z),\ \operatorname{Sc} Z_0^{(n)}(j,z_0;z)\right\}_{n=0}^{\infty} \tag{50}$$

is an infinite system of solutions of (47). Nevertheless up to now no result on the completeness of the system of bicomplex formal powers or of the family of solutions (50) has been proved. The reason is that such important basic facts which are in the core of pseudoanalytic function theory as the similarity principle are not valid for bicomplex Vekua equations.

In the present work in order to establish such completeness for a certain class of Schrödinger equations we implement the transmutation operators.

6. Complete families of solutions

First we consider an important special case. Assume that ϕ has the form $\phi(x, y) = f(x)g(y)$ where f and g are arbitrary $\mathbb{C}_i$-valued twice continuously differentiable and nonvanishing functions defined on the segments $[a_1, b_1]$ and $[a_2, b_2]$ respectively. In this case there exists a periodic generating sequence with a period two corresponding to the main Vekua equation (46):

$$(F, G) = \left(fg, \frac{j}{fg}\right), \qquad (F_1, G_1) = \left(\frac{g}{f}, \frac{jf}{g}\right), \qquad (F_2, G_2) = (F, G), \qquad (F_3, G_3) = (F_1, G_1), \qquad \ldots,$$

and the corresponding formal powers admit the following elegant representation [3]. We consider the formal powers with the centre at the point $(x_0, y_0) \in [a_1, b_1] \times [a_2, b_2]$. We assume that $f(x_0) = g(y_0) = 1$ and define the recursive integrals according to (13)–(15) as well as the system of functions $\{\varphi_k\}_{k=0}^{\infty}$ according to (16). In a similar way we define a system of functions $\{\psi_k\}_{k=0}^{\infty}$ corresponding to g,

$$\psi_k(y) = \begin{cases} g(y)Y^{(k)}(y), & k \text{ odd}, \\ g(y)\tilde{Y}^{(k)}(y), & k \text{ even}, \end{cases} \tag{51}$$

where

$$\tilde{Y}^{(0)}(y) \equiv Y^{(0)}(y) \equiv 1, \tag{52}$$

$$\tilde{Y}^{(n)}(y) = n\int_{y_0}^{y} \tilde{Y}^{(n-1)}(s)\big(g^2(s)\big)^{(-1)^{n-1}}\, ds, \tag{53}$$

$$Y^{(n)}(y) = n\int_{y_0}^{y} Y^{(n-1)}(s)\big(g^2(s)\big)^{(-1)^{n}}\, ds. \tag{54}$$

Then the formal powers corresponding to (46) can be defined as follows. For $\alpha = \alpha' + i\alpha''$ and $z_0 = x_0 + jy_0$ we have

$$Z^{(n)}(\alpha, z_0, z) = f(x)g(y)\,\mathrm{Sc}\,{}_*Z^{(n)}(\alpha, z_0, z) + \frac{j}{f(x)g(y)}\,\mathrm{Vec}\,{}_*Z^{(n)}(\alpha, z_0, z) \tag{55}$$

where

$${}_*Z^{(n)}(\alpha, z_0, z) = \alpha'\sum_{k=0}^{n}\binom{n}{k}X^{(n-k)}j^k\tilde{Y}^{(k)} + j\alpha''\sum_{k=0}^{n}\binom{n}{k}\tilde{X}^{(n-k)}j^kY^{(k)} \quad \text{for an odd } n \tag{56}$$

and

$${}_*Z^{(n)}(\alpha, z_0, z) = \alpha'\sum_{k=0}^{n}\binom{n}{k}\tilde{X}^{(n-k)}j^k\tilde{Y}^{(k)} + j\alpha''\sum_{k=0}^{n}\binom{n}{k}X^{(n-k)}j^kY^{(k)} \quad \text{for an even } n. \tag{57}$$

Remark 23. Formulae (55)–(57) clearly generalize the binomial representation for the analytic powers $\alpha(z - z_0)^n$. If one chooses $f \equiv 1$ and $g \equiv 1$ then $Z^{(n)}(\alpha, z_0, z) = \alpha(z - z_0)^n$.

Consider the family of solutions of (47) obtained from the scalar parts of the formal powers (50). We have

$$\mathrm{Sc}\,Z^{(n)}(1, z_0; z) = \phi(x, y)\,\mathrm{Sc}\sum_{k=0}^{n}\binom{n}{k}X^{(n-k)}j^k\tilde{Y}^{(k)} = f(x)g(y)\sum_{\text{even}\,k=0}^{n}\binom{n}{k}X^{(n-k)}j^k\tilde{Y}^{(k)} \quad \text{for an odd } n,$$

$$\mathrm{Sc}\,Z^{(n)}(1, z_0; z) = f(x)g(y)\sum_{\text{even}\,k=0}^{n}\binom{n}{k}\tilde{X}^{(n-k)}j^k\tilde{Y}^{(k)} \quad \text{for an even } n,$$

$$\mathrm{Sc}\,Z^{(n)}(j, z_0; z) = f(x)g(y)\sum_{\text{odd}\,k=1}^{n}\binom{n}{k}\tilde{X}^{(n-k)}j^{k+1}Y^{(k)} \quad \text{for an odd } n,$$

$$\mathrm{Sc}\,Z^{(n)}(j, z_0; z) = f(x)g(y)\sum_{\text{odd}\,k=1}^{n}\binom{n}{k}X^{(n-k)}j^{k+1}Y^{(k)} \quad \text{for an even } n.$$

Taking into account the definition of the functions φ_k and ψ_k (Eqs. (16) and (51)) it is easy to rewrite the last four equalities as follows

$$\operatorname{Sc} Z^{(n)}(1, z_0; z) = \sum_{\text{even}\, k=0}^{n} (-1)^{\frac{k}{2}} \binom{n}{k} \varphi_{n-k}(x)\psi_k(y)$$

and

$$\operatorname{Sc} Z^{(n)}(j, z_0; z) = \sum_{\text{odd}\, k=1}^{n} (-1)^{\frac{k+1}{2}} \binom{n}{k} \varphi_{n-k}(x)\psi_k(y).$$

Thus, for every n we have two nontrivial exact solutions of (47) (except for $n=0$ for which we observe that by construction $\operatorname{Sc} Z^{(0)}(j, z_0; z) \equiv 0$). This infinite family of solutions can be written as follows

$$u_0(x, y) = f(x)g(y), \tag{58}$$

$$u_m(x, y) = \operatorname{Sc} Z^{(\frac{m+1}{2})}(1, z_0; z) = \sum_{\text{even}\, k=0}^{\frac{m+1}{2}} (-1)^{\frac{k}{2}} \binom{\frac{m+1}{2}}{k} \varphi_{\frac{m+1}{2}-k}(x)\psi_k(y) \quad \text{for an odd } m, \tag{59}$$

$$u_m(x, y) = \operatorname{Sc} Z^{(\frac{m}{2})}(j, z_0; z) = \sum_{\text{odd}\, k=1}^{\frac{m}{2}} (-1)^{\frac{k+1}{2}} \binom{\frac{m}{2}}{k} \varphi_{\frac{m}{2}-k}(x)\psi_k(y) \quad \text{for an even } m. \tag{60}$$

Remark 24. In the case when $f \equiv 1$ and $g \equiv 1$ we obtain that the system $\{u_m\}_{m=0}^{\infty}$ is the system of harmonic polynomials $\{\operatorname{Sc}(z-z_0)^n, \operatorname{Sc}(j(z-z_0)^n)\}_{n=0}^{\infty}$. Theorems about its completeness like the Runge theorem and further related results are well known (see, e.g., [9,28,31,32]). It is convenient to introduce the notation

$$p_0(x, y) = 1, \qquad p_m(x, y) = \operatorname{Sc}(z-z_0)^{\frac{m+1}{2}} = \sum_{\text{even}\, k=0}^{\frac{m+1}{2}} (-1)^{\frac{k}{2}} \binom{\frac{m+1}{2}}{k} (x-x_0)^{\frac{m+1}{2}-k}(y-y_0)^k \quad \text{for an odd } m,$$

$$p_m(x, y) = \operatorname{Sc}\bigl(j(z-z_0)^{\frac{m}{2}}\bigr) = \sum_{\text{odd}\, k=1}^{\frac{m}{2}} (-1)^{\frac{k+1}{2}} \binom{\frac{m}{2}}{k} (x-x_0)^{\frac{m}{2}-k}(y-y_0)^k \quad \text{for an even } m.$$

Remark 25. Every u_m is a result of application of an operator of transmutation to the corresponding harmonic polynomial p_m. Indeed, consider for simplicity $z_0 = 0$ and suppose that f is defined on the segment $[-a, a]$ and g is defined on $[-b, b]$. Both functions are assumed to be $\mathbb{C}_i$-valued twice continuously differentiable and nonvanishing. Let $\mathbf{T}_f$ be the operator $\mathbf{T}$ defined by (28) and $\mathbf{T}_g$ be its equivalent associated with the function g. That is,

$$\mathbf{T}_g v(y) = v(y) + \int_{-y}^{y} \mathbf{K}_g\bigl(y, t; g'(0)\bigr) v(t)\, dt \tag{61}$$

where

$$\mathbf{K}_g\bigl(y, t; g'(0)\bigr) = \frac{g'(0)}{2} + K_g(y, t) + \frac{g'(0)}{2} \int_t^y \bigl(K_g(y, s) - K_g(y, -s)\bigr)\, ds$$

and K_g is a solution of the Goursat problem

$$\left(\frac{\partial^2}{\partial y^2} - q_g(y)\right) K_g(y, t) = \frac{\partial^2}{\partial t^2} K_g(y, t), \qquad K_g(y, y) = \frac{1}{2}\int_0^y q_g(s)\, ds, \qquad K_g(y, -y) = 0$$

with $q_g := g''/g$. Then

$$u_m(x, y) = \mathbf{T}_f \mathbf{T}_g p_m(x, y). \tag{62}$$

This relation follows from Theorem 11 according to which the operator $\mathbf{T}_f$ maps a k-th power of x into $\varphi_k(x)$ for any $k \in \mathbb{N}_0$ and similarly the operator $\mathbf{T}_g$ maps a k-th power of y into $\psi_k(y)$. Moreover, $\mathbf{T}_f \mathbf{T}_g\, p_m = \mathbf{T}_g \mathbf{T}_f\, p_m$.

This observation together with the Runge approximation theorem for harmonic functions allows us to prove the following Runge-type theorem for the family of solutions $\{u_m\}_{m=0}^{\infty}$.

Theorem 26. *Let $\Omega \subset \bar{R} = [-a,a] \times [-b,b]$ be a simply connected domain such that together with any point (x,y) belonging to Ω the rectangle with the vertices (x,y), $(-x,y)$, $(x,-y)$ and $(-x,-y)$ also belongs to Ω. Let the equation*

$$\big(-\Delta + q(x,y)\big)u(x,y) = 0 \tag{63}$$

in Ω admit a particular solution of the form $\phi(x,y) = f(x)g(y)$ where f and g are $\mathbb{C}_i$-valued functions, $f \in C^2[-a,a]$, $g \in C^2[-b,b]$, $f(x) \neq 0$, $g(y) \neq 0$ for any $x \in [-a,a]$ and $y \in [-b,b]$ (obviously, q has the form $q(x,y) = q_1(x) + q_2(y)$ with $q_1 = f''/f$ and $q_2 = g''/g$). Then any solution u of (63) in Ω can be approximated arbitrarily closely on any compact subset K of Ω by a finite linear combination of the functions u_m. That is for any $\varepsilon > 0$ there exists such a number $M \in \mathbb{N}$ and such coefficients $\{\alpha_m\}_{m=0}^{M} \subset \mathbb{C}_i$ that $|u(x,y) - \sum_{m=0}^{M} \alpha_m u_m(x,y)| < \varepsilon$ for any point $(x,y) \in K$.

Proof. Let u be a solution of (63) in Ω. Consider a compact subset K of Ω. Let $K_1 \supset K$ be a compact subset of Ω possessing the same symmetry as Ω, that is with any point (x,y) belonging to K_1 the rectangle with the vertices (x,y), $(-x,y)$, $(x,-y)$ and $(-x,-y)$ also belongs to K_1. We have [31] that the harmonic function $v := \mathbf{T}_f^{-1}\mathbf{T}_g^{-1}u$ can be approximated in K_1 with respect to the maximum norm by a harmonic polynomial, $\|v - P_M\| < \varepsilon_1$, where $P_M = \sum_{m=0}^{M} \alpha_m p_m$. Now we use the fact that $\mathbf{T}_f$ and $\mathbf{T}_g$ are bounded Volterra operators possessing bounded inverse operators. We have

$$\|u - \mathbf{T}_f\mathbf{T}_g P_M\| = \|\mathbf{T}_f\mathbf{T}_g v - \mathbf{T}_f\mathbf{T}_g P_M\| \leqslant \varepsilon_1 \|\mathbf{T}_f\|\|\mathbf{T}_g\| = \varepsilon \tag{64}$$

where the norms $\|\mathbf{T}_f\|$ and $\|\mathbf{T}_g\|$ can be estimated in terms of the maximum values of the corresponding (continuous) kernels $\mathbf{K}_f$ and $\mathbf{K}_g$. Now the inequality (64) in K_1 implies the inequality $|u(x,y) - \sum_{m=0}^{M} \alpha_m u_m(x,y)| < \varepsilon$ in K. □

The restricting condition on the shape of the domain Ω is due to the necessity to have well defined the function v as the image of u under the application of the transmutation operators. Let one of the functions f or g be real-valued, for example, f. Then to prove the completeness of the system $\{u_m\}_{m=0}^{\infty}$ using a transmutation operator one can assume the symmetry of the domain only with respect to the variable y.

Theorem 27. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain such that together with any point (x,y) belonging to Ω the point $(x,-y)$ and the segment joining (x,y) with $(x,-y)$ belong to Ω as well. Let Eq. (63) in Ω admit a particular solution of the form $\phi(x,y) = f(x)g(y)$ where f is a real-valued and g is a $\mathbb{C}_i$-valued, both twice continuously differentiable and nonvanishing up to the boundary functions. Then any solution u of (63) in Ω can be approximated arbitrarily closely on any compact subset K of Ω by a finite linear combination of the functions u_m.*

Proof. Consider the equation

$$\big(-\Delta + q_1(x)\big)v(x,y) = 0 \quad \text{in } \Omega \tag{65}$$

where $q_1 = f''/f$ is real valued. Denote by $\hat{u}_m$ the functions defined by (58)–(60) with $g \equiv 1$. Then as was proved in [5] any solution v of (65) can be approximated arbitrarily closely on any compact subset of Ω by linear combinations of the functions $\hat{u}_m$. As $u_m = \mathbf{T}_g\hat{u}_m$, $m \in \mathbb{N}_0$ (where g is the factor in ϕ depending on y) once again using the boundedness of $\mathbf{T}_g$ and $\mathbf{T}_g^{-1}$ we obtain the completeness of $\{u_m\}_{m=0}^{\infty}$. □

Extension of the results of the preceding two theorems onto arbitrary simply connected domains is possible if Eq. (63) has the Runge property (see, e.g., [21,4,10]).

Definition 28. Equation $Lu = 0$ is said to have the Runge approximation property if, whenever, Ω_1 and Ω_2 are two simply connected domains, Ω_1 a subset of Ω_2, any solution in Ω_1 can be approximated uniformly in compact subsets of Ω_1 by a sequence of solutions which can be extended as solutions to Ω_2.

It is known [4,10] that the Runge property in the case of elliptic equations with real-valued coefficients is equivalent to the (weak) unique continuation property (if every solution of $Lu = 0$ which vanishes in an open set vanishes identically) and is true, e.g., for second-order elliptic equations with real-analytic coefficients. Without going into further details concerning the Runge approximation property which is beyond the scope of the present work, we prove that if Eq. (63) has this property then the family of solutions $\{u_m\}_{m=0}^{\infty}$ is complete in any simply connected domain.

Theorem 29. *Let Eq. (63) in a rectangle $R = (-a,a) \times (-b,b)$ admit a particular solution of the form $\phi(x,y) = f(x)g(y)$ where f and g are arbitrary $\mathbb{C}_i$-valued twice continuously differentiable and nonvanishing functions in $[-a,a]$ and $[-b,b]$ respectively. Let $\Omega \subset R$ be a simply connected domain. Assume Eq. (63) has the Runge property. Then any solution u of (63) in Ω can be approximated arbitrarily closely on any compact subset K of Ω by a finite linear combination of the functions u_m defined by (58)–(60).*

Proof. Consider a solution u in Ω which due to the Runge property can be approximated on K by a solution v of (63) in R. Due to Theorem 26, v in its turn can be approximated on K by the functions u_m from where we obtain the required approximation of the solution u in terms of the solutions u_m. □

In the rest of the present section we show that solutions of (63) sufficiently smooth up to the boundary of the domain of interest Ω can be approximated by functions u_m in $\bar{\Omega}$. By Σ_α^q we denote the linear space of solutions of (63) in Ω satisfying the following regularity requirement $u \in C^2(\Omega) \cap C^{1+\alpha}(\bar{\Omega})$, $0 \leqslant \alpha \leqslant 1$. This linear space can be equipped with one of the following scalar products (we assume that zero is neither a Dirichlet nor a Neumann eigenvalue)

$$\langle u, v\rangle_1 = \int_{\partial\Omega} uv^*\,ds \quad \text{and} \quad \langle u, v\rangle_2 = \int_{\partial\Omega} \frac{\partial u}{\partial n}\frac{\partial v^*}{\partial n}\,ds \tag{66}$$

where by "$*$" we denote the complex conjugation in $\mathbb{C}_i$, and $\frac{\partial}{\partial n}$ is the outer normal derivative. With the aid of the scalar products (66) two Bergman-type reproducing kernels [2] can be introduced for solving the Dirichlet and Neumann problems respectively as well as the corresponding eigenvalue problems [5].

A complete orthonormal system of functions in Σ_α^q with respect to $\langle\cdot,\cdot\rangle_1$ or $\langle\cdot,\cdot\rangle_2$ allows one to construct a corresponding "Dirichlet" or "Neumann" reproducing kernel respectively. In [5] the completeness of the family of solutions $\{u_m\}_{m=0}^\infty$ of (63) obtained as real parts of complex pseudoanalytic formal powers was proved in the case when (63) admits a particular solution in a separable form $\varphi(s,t) = S(s)T(t)$ where S and T are arbitrary twice continuously differentiable nonvanishing real-valued functions, $\Phi = s + it$ is a conformal mapping defined in $\bar{\Omega}$ and Ω is a domain bounded by a Jordan curve. In the same paper it was shown that the completeness of $\{u_m\}_{m=0}^\infty$ in Σ_α^q in the case when the particular solution φ is complex-valued is an important open problem and its solution is required not only for solving boundary value problems for (63) with a complex-valued coefficient but also for solving spectral problems for (63) even in the situation when the coefficient is real-valued. Here by means of the developed results concerning the transmutation operators we obtain the completeness of the family of solutions $\{u_m\}_{m=0}^\infty$ in Σ_α^q under the conditions of Theorems 26 and 27.

Theorem 30. *Let $\Omega \subset \bar{R} = [-a,a]\times[-b,b]$ be a simply connected domain such that together with any point (x,y) belonging to Ω the rectangle with the vertices (x,y), $(-x,y)$, $(x,-y)$ and $(-x,-y)$ also belongs to Ω. Let Eq.* (63) *in Ω admit a particular solution of the form $\phi(x,y) = f(x)g(y)$ where f and g are $\mathbb{C}_i$-valued functions, $f \in C^2[-a,a]$, $g \in C^2[-b,b]$, $f(x) \neq 0$, $g(y) \neq 0$ for any $x \in [-a,a]$ and $y \in [-b,b]$. Then the family of solutions $\{u_m\}_{m=0}^\infty$ is complete in Σ_α^q, $\alpha > 0$ with respect to both norms generated by the scalar products* (66)*.*

Proof. Let $u \in \Sigma_\alpha^q$. Consider the harmonic function $v := \mathbf{T}_f^{-1}\mathbf{T}_g^{-1}u$ which belongs to Σ_α^0 due to the fact that the kernels in both transmutation operators are at least C^1-functions and hence the operator $\mathbf{T}_f^{-1}\mathbf{T}_g^{-1}$ transforms Σ_α^q into Σ_α^0. There exists (see, e.g., [5]) a sequence of harmonic polynomials P_M such that when $M \to \infty$, $P_M \to v$ uniformly in $\bar{\Omega}$ together with their first partial derivatives. It is easy to see that this implies the uniform convergence in $\bar{\Omega}$ of the sequences $\mathbf{T}_f\mathbf{T}_g P_M = U_M \to u = \mathbf{T}_f\mathbf{T}_g v$, $\frac{\partial U_M}{\partial x} \to \frac{\partial u}{\partial x}$ and $\frac{\partial U_M}{\partial y} \to \frac{\partial u}{\partial y}$. Indeed, the uniform convergence of U_M to u follows directly from the boundedness of the transmutation operators, see the proof of Theorem 26, and the verification of the uniform convergence of the partial derivatives is straightforward. Consider

$$\begin{aligned}\frac{\partial}{\partial x}\mathbf{T}_f P_M(x,y) &= \frac{\partial}{\partial x}P_M(x,y) + \frac{\partial}{\partial x}\int_{-x}^{x}\mathbf{K}_f(x,t;h)P_M(t,y)\,dt \\ &= \frac{\partial}{\partial x}P_M(x,y) + \int_{-x}^{x}\frac{\partial}{\partial x}\mathbf{K}_f(x,t;h)P_M(t,y)\,dt + \mathbf{K}_f(x,x;h)P_M(x,y) + \mathbf{K}_f(x,-x;h)P_M(-x,y)\end{aligned}$$

from where due to the uniform convergence of P_M to v together with their partial derivatives and due to the fact that $\frac{\partial}{\partial x}\mathbf{K}_f(x,t;h)$ is continuous, it follows that $\frac{\partial U_M}{\partial x} \to \frac{\partial u}{\partial x}$ uniformly in $\bar{\Omega}$. The proof of the uniform convergence of $\frac{\partial U_M}{\partial y}$ to $\frac{\partial u}{\partial y}$ is analogous.

Now, the completeness of $\{u_m\}_{m=0}^\infty$ in Σ_α^q with respect to the norm generated by $\langle\cdot,\cdot\rangle_1$ follows from the uniform convergence of U_M to u in $\bar{\Omega}$ and hence from the completeness of $\{u_m\}_{m=0}^\infty$ with respect to the maximum norm in $\bar{\Omega}$. To verify the completeness of $\{u_m\}_{m=0}^\infty$ in Σ_α^q with respect to the norm generated by $\langle\cdot,\cdot\rangle_2$ consider the following chain of relations

$$\|u - U_M\|^2 = \int_{\partial\Omega}\frac{\partial(u-U_M)}{\partial n}\frac{\partial(u^*-U_M^*)}{\partial n}\,ds \leqslant \int_{\partial\Omega}\left|\nabla(u-U_M)\right|^2 ds \leqslant L\sup_{\partial\Omega}\left|\nabla(u-U_M)\right|^2 \to 0. \qquad \square$$

Theorem 31. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain such that together with any point (x, y) belonging to Ω the point $(x, -y)$ and the segment joining (x, y) with $(x, -y)$ belong to Ω as well. Let Eq.* (63) *in Ω admit a particular solution of the form $\phi(x, y) = f(x)g(y)$ where f is a real-valued and g is a $\mathbb{C}_i$-valued, both twice continuously differentiable and nonvanishing up to the boundary functions. Then $\{u_m\}_{m=0}^{\infty}$ is complete in Σ_α^q, $\alpha > 0$ with respect to both norms generated by the scalar products* (66)*.*

Proof. The first part of the proof is similar to that of Theorem 27. We have that $u_m = \mathbf{T}_g \hat{u}_m$, $m \in \mathbb{N}_0$ where $\{\hat{u}_m\}_{m=0}^{\infty}$ is complete in $\Sigma_\alpha^{q_1}$ (see [5]) with respect to the required norms. Then the completeness of $\{u_m\}_{m=0}^{\infty}$ is proved analogously to the proof of Theorem 30. $\square$

Remark 32. In order to prove the completeness of the family of solutions $\{u_m\}_{m=0}^{\infty}$ in Σ_α^q with respect to both norms generated by the scalar products (66) under less restrictive conditions on the shape of the domain Ω in fact we need a result on the existence of an infinite system of solutions of (63) in $R \supset \Omega$ and complete in $\Sigma_\alpha^q(\partial\Omega)$ or in a maximum norm in $\bar{\Omega}$. If such a system exists then according to Theorem 26 every element of it can be approximated arbitrarily closely by linear combinations of functions u_m which would allow one to prove the completeness of $\{u_m\}_{m=0}^{\infty}$. Thus, if such a complete system exists then $\{u_m\}_{m=0}^{\infty}$ is precisely such system. The question on the existence requires further study.

7. Conclusions

Transmutation operators for Sturm–Liouville equations are considered and their new properties concerning the transformation of certain infinite systems of functions generated by the Sturm–Liouville operators are presented. These infinite systems of functions slightly generalize the notion of L-bases [11] and play an important role in the theory of linear differential equations. We show how a transmutation operator can be constructed mapping one such basis into another and give an application of this result obtaining several theorems on the completeness of certain families of solutions of two-dimensional stationary Schrödinger equations which are obtained as scalar parts of bicomplex pseudoanalytic formal powers. To our best knowledge this is the first result of this kind in bicomplex pseudoanalytic function theory. Its importance is in the fact that it opens the way for construction of Bergman-type reproducing kernels for corresponding second-order elliptic equations with variable complex-valued coefficients and hence for solving boundary and eigenvalue problems.

References

[1] H. Begehr, R. Gilbert, Transformations, Transmutations and Kernel Functions, vols. 1–2, Longman, Pitman, 1992.
[2] S. Bergman, M. Schiffer, Kernel Functions and Elliptic Differential Equations in Mathematical Physics, Academic Press, 1953.
[3] L. Bers, Theory of Pseudo-Analytic Functions, New York University, 1952.
[4] L. Bers, J. Fritz, M. Schechter, Partial Differential Equations, American Mathematical Society, Providence, RI, 1979.
[5] H. Campos, R. Castillo, V.V. Kravchenko, Construction and application of Bergman-type reproducing kernels for boundary and eigenvalue problems in the plane, Complex Var. Elliptic Equ., iFirst, doi:10.1080/17476933.2011.611941.
[6] R. Carroll, Transmutation Theory and Applications, North-Holland, Amsterdam, 1986.
[7] A. Castañeda, V.V. Kravchenko, New applications of pseudoanalytic function theory to the Dirac equation, J. Phys. A: Math. Gen. 38 (2005) 9207–9219.
[8] R. Castillo, V.V. Kravchenko, R. Reséndiz, Solution of boundary value and eigenvalue problems for second order elliptic operators in the plane using pseudoanalytic formal powers, Math. Methods Appl. Sci. 34 (2011) 455–468.
[9] D.L. Colton, Solution of Boundary Value Problems by the Method of Integral Operators, Pitman Publ., London, 1976.
[10] D.L. Colton, Analytic Theory of Partial Differential Equations, Monogr. Stud. Math., vol. 8, Pitman (Advanced Publishing Program), Boston, MA–London, 1980.
[11] M.K. Fage, N.I. Nagnibida, The Problem of Equivalence of Ordinary Linear Differential Operators, Nauka, Novosibirsk, 1987 (in Russian).
[12] P.R. Garabedian, Partial Differential Equations, John Willey and Sons, New York–London, 1964.
[13] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1966.
[14] V.V. Kravchenko, On a factorization of second order elliptic operators and applications, J. Phys. A: Math. Gen. 39 (2006) 12407–12425.
[15] V.V. Kravchenko, Recent developments in applied pseudoanalytic function theory, in: A. Escassut, W. Tutschke, C.C. Yang (Eds.), Some Topics on Value Distribution and Differentiability in Complex and p-Adic Analysis, Science Press, 2008, pp. 293–328.
[16] V.V. Kravchenko, A representation for solutions of the Sturm–Liouville equation, Complex Var. Elliptic Equ. 53 (2008) 775–789.
[17] V.V. Kravchenko, Applied Pseudoanalytic Function Theory, Front. Math., Birkhäuser, Basel, 2009.
[18] V.V. Kravchenko, On the completeness of systems of recursive integrals, Commun. Math. Anal. Conf. 03 (2011) 172–176.
[19] V.V. Kravchenko, S. Morelos, S. Tremblay, Complete systems of recursive integrals and Taylor series for solutions of Sturm–Liouville equations, Math. Methods Appl. Sci., in press.
[20] V.V. Kravchenko, R.M. Porter, Spectral parameter power series for Sturm–Liouville problems, Math. Methods Appl. Sci. 33 (2010) 459–468.
[21] P.D. Lax, A stability theorem for solutions of abstract differential equations, and its application to the study of the local behavior of solutions of elliptic equations, Comm. Pure Appl. Math. 9 (1956) 747–766.
[22] B.M. Levitan, Inverse Sturm–Liouville Problems, VSP, Zeist, 1987.
[23] J.L. Lions, Solutions élémentaires de certains opérateurs différentiels à coefficients variables, J. Math. 36 (1) (1957) 57–64.
[24] V.A. Marchenko, Sturm–Liouville Operators and Applications, Birkhäuser, Basel, 1986.
[25] D. Rochon, M.V. Shapiro, On algebraic properties of bicomplex and hyperbolic numbers, An. Univ. Oradea Fasc. Mat. 11 (2004) 71–110.
[26] D. Rochon, S. Tremblay, Bicomplex quantum mechanics: I. The generalized Schrödinger equation, Adv. Appl. Clifford Algebr. 14 (2004) 231–248.
[27] S.M. Sitnik, Transmutations and applications: A survey, in: Yu.F. Korobeinik, A.G. Kusraev (Eds.), Advances in Modern Analysis and Mathematical Modeling, Vladikavkaz Scientific Center of the Russian Academy of Sciences and Republic of North Ossetia–Alania, Vladikavkaz, 2008, pp. 226–293, arXiv:1012.3741v1 [math.CA].
[28] P.K. Suetin, Orthogonal Polynomials in Two Variables, Nauka, Moscow, 1988 (in Russian).
[29] K. Trimeche, Transmutation Operators and Mean-Periodic Functions Associated with Differential Operators, Harwood Academic Publishers, London, 1988.

[30] I.N. Vekua, Generalized Analytic Functions, Nauka, Moscow, 1959 (in Russian), English translation: Pergamon Press, Oxford, 1962.
[31] J.L. Walsh, The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, Bull. Amer. Math. Soc. 35 (4) (1929) 499–544.
[32] J.L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, Colloq. Publ., American Mathematical Society, 1956.