MATHEMATICS

A NOTE ON "STANDARD" VERSUS "NON-STANDARD" TOPOLOGY

BY

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A. ROBINSON has recently developed a theory of "non-standard analysis" (see [2] and [3]) which has had many interesting applications to ordinary classical analysis. The use of non-standard models of the real numbers has produced a variety of interesting results, often significantly extending what has been obtained by classical methods. Of equal importance is also the often revealing insight into the structure of the classical theories which the application of non-standard methods has led to, providing, in particular, a precise language for talking about the "infinitesimals" of the founders of the differential and integral calculus.

In a survey paper (ROBINSON [2]) the author sketches an application of non-standard methods to topology. The approach is based upon a formalization of topology within a type-theoretic language. A relation $Q(x, y)$ within this language is called *concurrent* in a (standard) model (i.e. an ordinary topological space) M if for every finite set of entities (which may be individuals, i.e. points of the space, or relations) $a_1, a_2, \ldots, a_n, n \geqslant 1$, which belong to the domain of the first place of Q in M, there exists a $b \in M$ such that $Q(a_i, b)$ holds for $i=1, \ldots, n$ in M. An admissible non-standard extension $^*M$ of M is an elementary extension of M such that for all concurrent relations Q in M, there exists a $b_Q \in {}^*M$ such that $Q(a, b_Q)$ holds in $^*M$ for all $a \in {}^*M$ of the appropriate type. It is an immediate consequence of the Henkin completeness theorem for type theory that admissible non-standard extensions exist. (But note that the sub-set concept may also be non-standard in the extension.)

Let M be any topological space and $^*M$ a non-standard extension. The points $a \in M$ also belong to $^*M$ and will there be called *standard points*. Further the canonical extensions of the open sets in M will be called the *standard open sets* of $^*M$. Note that a standard open set does not in general coincide with the open set in M of which it is the canonical extension.

For any $a \in {}^*M$ the *monad* of a is defined as the intersection of all standard open sets which include a. A point in $^*M$ which belongs to the monad of a standard point is called *near-standard*. The following theorem is basic to the non-standard theory:

Theorem. *A space M is (quasi-) compact if and only if all points of $^*M$ are near-standard.*

$^*M$ can be any admissible non-standard extension. The proof as presented in [2, p. 294] typifies the blend of mathematical and logical arguments which is so characteristic for the non-standard theory as practiced by A. Robinson.

Compactness is usually defined in terms of open coverings. Another approach is through the use of *ultrafilters*. H. Cartan observed in 1937 that compactness is equivalent to the fact that every ultrafilter converges. This means that a space M is compact if and only if for every ultrafilter F on M there is some point $a \in M$ such that F is finer than the neighbourhood filter of a (i.e. every open set containing a belongs to the ultrafilter). The treatise of N. Bourbaki [1] is based upon this approach. In particular an extremely simple proof of Tychonoff's theorem is made possible using ultrafilters.

After having seen [2] I became convinced that the two approaches to compactness were in the main points equivalent, simply because non-standard points and ultrafilters both describe all possible ways how a subfamily of a given family may converge. The purpose of this note is to work out this equivalence explicitly and to point out that the Robinson characterization of compactness in terms of near-standard points is nothing but Cartan's theorem on ultrafilters within a different language.

With every point $^*b$ in $^*M$ we can associate an ultrafilter $F_{^*b}$ in M in the following way:

$$X \in F_{^*b} \text{ iff. } ^*b \in {}^*X,$$

i.e. $F_{^*b}$ is the "trace" on M of a principal ultrafilter on $^*M$, viz. the one consisting of all sets in $^*M$ which contains $^*b$.

Conversely with every ultrafilter F_λ on M one may associate a point $^*b_\lambda \in {}^*M$. The truth of this assertion is an immediate consequence of the fact that an ultrafilter satisfies the finite intersection property, which means that the relation "$A \in F_\lambda$ and $b \in A$" is concurrent.

It follows that $F_\lambda = F_{^*b_\lambda}$. In fact, if $X \in F_\lambda$, then by construction $^*b_\lambda \in {}^*X$, i.e. $X \in F_{^*b_\lambda}$. As F_λ is maximal equality follows.

Let $^*a$ be a standard point in $^*M$. We show that $^*b$ belongs to the monad of $^*a$ if and only if $F_{^*b}$ is finer than the neighbourhood filter $V(a)$ of a in M. But this is rather immediate: If $V(a) \subseteq F_{^*b}$, then in particular $0 \in F_{^*b}$ for every open set 0 containing a. $0 \in F_{^*b}$ means that $^*b \in {}^*0$, and this implies that $^*b$ belongs to every standard open set containing $^*a$. The argument works in reverse, hence *the Robinson characterization theorem reduces to the well-known characterization of compactness in terms of ultrafilters.*

Remark. From [3] it appears that the problem of obtaining a non-standard proof of Tychonoff's theorem presented some difficulties. In [3] a proof due to S. Kripke is given. Granted our equivalence-proof it is easily seen that the proof in [3] is essentially the same as the ultrafilter proof. A slightly different "translation" of the ultrafilter proof is as follows: Let M_λ, $\lambda \in \Lambda$ be an arbitrary family of topological spaces. Let non-standard extensions be given as ultraproducts. Then observe that there is a mapping ψ from the non-standard extension of the product $(\prod_{\lambda \in \Lambda} M_\lambda)^I/D$ to the product of the non-standard extensions $\prod_{\lambda \in \Lambda} (M_\lambda^I/D)$. A non-standard proof of Tychonoff's theorem is easily obtained by appropriately translating the usual proof (i.e. an ultrafilter converges in a product space if and only if all of its projections converges), using the observation that with each $^*b \in (\prod M_\lambda)^I/D$ one may associate for each $\lambda \in \Lambda$ an element $^*b_\lambda \in M_\lambda^I/D$, viz. $^*b_\lambda = pr_\lambda \circ \psi\ (^*b)$.

REFERENCES

1. Bourbaki, N., *Topologie Generale*, Chap. I, Paris 1961 (3rd edition).
2. Robinson, A., *Topics in non-Archimedian mathematics*, in Addison, Henkin, Tarski, *The Theory of Models*, North-Holland, Amsterdam, 1965, pp. 285–298.
3. ———, *Non-standard Analysis*, North-Holland, Amsterdam, 1966.