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FURTHER CHARACTERIZATIONS OF CUBIC LATTICE GRAPHS

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Abstract. A cubic lattice graph with characteristic n is a graph whose points can be identified with the ordered triplets on n symbols and two points are adjacent whenever the corresponding triplets have two coordinates in common. An L_2 graph is a graph whose points can be identified with the ordered pairs on n symbols such that two points are adjacent if and only if the corresponding pairs have a common coordinate. The main result of this paper is two new characterizations of cubic lattice graphs. The main result depends on a new L_2 graph characterization and shows the relation between cubic lattice and L_2 graphs. The main result also suggests a conjecture concerning the characterization of interchange graphs of complete m-partite graphs.

1. Introduction

A cubic lattice graph with characteristic n is a graph whose points can be identified with the ordered triplets on n symbols and two points are adjacent whenever the corresponding triplets have two coordinates in common. An L_2 graph is a graph whose points can be identified with the ordered pairs on n symbols such that two points are adjacent if and only if the corresponding pairs have a common coordinate. The main result of this paper is two new characterizations of cubic lattice graphs. The main result depends on a new L_2 graph characterization and shows the relation between cubic lattice and L_2 graphs. The main result suggests a conjecture concerning the characterization of interchange graphs of complete m-partite graphs.

In this paper, we consider only finite undirected graphs without loops or multiple lines or edges. The *degree* of a point is the number of lines incident with that point. A graph is *regular* if all its points have the same degree. A graph is *connected* if every pair of points are joined by a path. The *distance* $d(u, v)$ between points u and v is the length of a shortest path joining them. The number of points w adjacent to both u and v is denoted $\Delta(u, v)$. If u is adjacent to v, $\Delta(u, v)$ is called the *edge degree* of the edge uv.

Shrikhande [8] and Moon [7] showed that the following properties characterize an L_2 graph G except for one exceptional case when $n = 4$:

(A_1) G has n^2 points.
(A_2) G is regular of degree $2(n-1)$.
(A_3) If $d(u, v) = 1$, then $\Delta(u, v) = n-2$.
(A_4) If $d(u, v) = 2$, then $\Delta(u, v) = 2$.

Laskar [5] and Aigner [1] showed that the following properties characterize a cubic lattice graph G except for one exceptional case when $n = 4$:

(B_1) G has n^3 points.
(B_2) G is connected and regular of degree $3(n-1)$.
(B_3) If $d(u, v) = 1$, then $\Delta(u, v) = n-2$.
(B_4) If $d(u, v) = 2$, then $\Delta(u, v) = 2$.
(B_5) If $d(u, v) = 2$, then there exist exactly $n-1$ points w such that $d(u, w) = 1$ and $d(v, w) = 3$.

Note that (A_3) and (B_3), and (A_4) and (B_4) are identical and both characterizations have one exceptional case when $n = 4$.

Laskar [6] has also characterized cubic lattice graphs for $n > 7$ in terms of the eigenvalues of the adjacency matrix.

2. Characterization of L_2 graphs

As mentioned in the introduction, an L_2 *graph* is a graph whose points can be identified with the n^2 ordered pairs on n symbols such that two points are adjacent whenever the corresponding pairs have a common coordinate. A *matrix graph* is a graph whose points can be identified with a set of distinct ordered pairs of positive integers such that two points are adjacent whenever the corresponding pairs have a common coordinate. Thus an L_2 graph is a special case of a matrix graph. Hedetniemi [4] has obtained four characterizations of matrix graphs. The L_2 graph characterization follows easily from one of these characterizations. This L_2 graph characterization will be used in the proof of one of the cubic lattice graph characterizations.

The point set of a graph G will be denoted $V(G)$. A *complete graph* K_p has every pair of its p points adjacent. For any subset S of $V(G)$, the *induced* subgraph $\langle S\rangle$ has point set S and two points of S are adjacent if and only if they are adjacent in G. Let π be a partition of $V(G)$. Then π is a *K-partition* if the subgraph induced by each block of π is a complete graph. The *lines contained in* π are the lines of G contained in the subgraphs induced by the blocks of π. Two K-partitions $\pi_1 = \{V_1, \ldots, V_m\}$ and $\pi_2 = \{W_1, \ldots, W_n\}$ are *orthogonal* if $|V_i \cap W_j| \leq 1$ for every i and j.

Theorem 1 (Hedetniemi). *A graph G is a matrix graph if and only if there exist two orthogonal K-partitions of V(G) containing all the lines of G.*

Theorem 2. *A graph G is an L_2 graph if and only if there exist two orthogonal K-partitions of V(G) containing all the lines of G and each partition consists of n blocks of order n.*

Proof. Let $\pi_1 = \{V_1, \ldots, V_n\}$ and $\pi_2 = \{W_1, \ldots, W_n\}$ be two orthogonal K-partitions of $V(G)$ containing all the lines of G and $|V_i| = |W_i| = n$ for $i = 1, \ldots, n$. Define a mapping f from $V(G)$ into $V(L_2)$ by $f(u) = (k, m)$ if and only if $V_k \cap W_m = \{u\}$. Since π_1 and π_2 are orthogonal, f is $1-1$. From $|V_i| = |W_j| = n$ for any i and j, it follows that $|V_i \cap W_j| = 1$ and f is onto.

To complete the proof that $G \cong L_2$ we must show that f preserves adjacency. Let uv be a line in G and let $\{u\} = V_i \cap W_j$ and $\{v\} = V_k \cap W_m$. Then $f(u) = (i, j)$ and $f(v) = (k, m)$. Since the two K-partitions contain all the lines of G, either $i = k$ or $j = m$, but not both. In either case (i, j) is adjacent to (k, m).

Now assume that $(i, j) = f(u)$ is adjacent to $(s, t) = f(v)$. This implies that either $u, v \in V_i = V_s$ or $u, v \in W_j = W_t$. Since $\langle V_i\rangle$ and $\langle W_j\rangle$ are complete subgraphs, u is adjacent to v.

The necessity of these conditions is obvious if we let the blocks of π_i be the points of L_2 with the same i^{th} coordinate.

3. The main theorem

Recall that a *cubic lattice graph with characteristic n* is a graph whose points can be identified with the n^3 ordered triplets on n symbols such that two points are adjacent whenever the corresponding triplets have two coordinates in common. A *cube graph* is a graph whose points can be identified with a set of distinct ordered triplets of positive integers such that two points are adjacent whenever the corresponding triplets have two common coordinates. Hence a cubic lattice graph is a special case of a cube graph. In [2], the author obtained two cube graph characterizations. The two cubic lattice graph characterizations are modifications of these characterizations.

A *lattice* is a partially ordered set L in which each pair of elements has a greatest lower bound and a least upper bound. If α and β are two elements in L, we denote their greatest lower bound by $\alpha \cdot \beta$ and their least upper bound by $\alpha + \beta$. Let 0 denote the *zero element* of L. The set of partitions of a set S forms a lattice. If L is the set of partitions of S, then $a \equiv b\ (\alpha \cdot \beta)$ if and only if $a \equiv b\ (\alpha)$ and $a \equiv b\ (\beta)$, and $a \equiv b\ (\alpha + \beta)$ if and only if there exists a sequence $a_0, a_1, ..., a_p$ such that $a = a_0$, $b = a_p$, and $a_i \equiv a_{i+1}\ (\alpha)$ or $a_i \equiv a_{i+1}\ (\beta)$ for $0 \leq i \leq p-1$.

Let π be a partition of the points of a graph G. Then π is an *L_2 (M)-partition* if the subgraph induced by each block of π is an L_2 (connected matrix) graph. Three partitions π_1, π_2, and π_3 are *triorthogonal* if $\pi_1 \cdot \pi_2 \cdot \pi_3 = 0$.

The next theorem characterizes cube graphs.

Theorem 3 (Cook). *The following are equivalent*:

(1) *G is a cube graph.*

(2) *There exist three triorthogonal M-partitions, M_1, M_2, and M_3, of $V(G)$ containing each line of G exactly twice and $M_i \cdot M_j$, $i \neq j$, is a K-partition of $V(G)$.*

(3) *There exist three mutually orthogonal K-partitions, π_1, π_2, and π_3, of $V(G)$ containing all the lines of G and $(\pi_i + \pi_j)\cdot(\pi_i + \pi_k) = \pi_i + (\pi_j \cdot \pi_k)$ for $1 \leq i, j, k \leq 3$.*

The proof of the main theorem requires several lemmas.

Lemma 1. *Let L be a lattice with a* 0 *element and let* π_1, π_2, *and* π_3 *be three elements of L with the following properties:*

(1) $\pi_i \cdot \pi_j = 0$ *for* $i \neq j$, $1 \leq i, j \leq 3$.

(2) $(\pi_i + \pi_j) \cdot (\pi_i + \pi_k) = \pi_i + (\pi_j \cdot \pi_k)$ *for* $1 \leq i, j, k \leq 3$.

Then

$$(\pi_i \cdot \pi_j) + (\pi_i \cdot \pi_k) = \pi_i \cdot (\pi_j + \pi_k) \text{ for } 1 \leq i, j, k \leq 3.$$

Proof. The proof will be by cases.

Case 1. $i = j$ or $i = k$,

$(\pi_i \cdot \pi_j) + (\pi_i \cdot \pi_k) = \pi_i = \pi_i \cdot (\pi_j + \pi_k)$.

Case 2. $j = k$,

$(\pi_i \cdot \pi_j) + (\pi_i \cdot \pi_k) = \pi_i \cdot \pi_j = \pi_i \cdot (\pi_j + \pi_k)$.

Case 3. $i \neq j \neq k \neq i$,

$$\begin{aligned}
0 &= (\pi_i \cdot \pi_j) + (\pi_i \cdot \pi_k) \\
&= ((\pi_i \cdot \pi_j) + \pi_i) \cdot ((\pi_i \cdot \pi_j) + \pi_k) && (1) \\
&= ((\pi_i + \pi_i) \cdot (\pi_i + \pi_j)) \cdot ((\pi_k + \pi_i) \cdot (\pi_k + \pi_j)) && (2) \\
&= (\pi_i \cdot (\pi_i + \pi_j)) \cdot ((\pi_i + \pi_k) \cdot (\pi_j + \pi_k)) && \text{commutative laws} \\
&= \pi_i \cdot ((\pi_i + \pi_k) \cdot (\pi_j + \pi_k)) && \text{absorption laws} \\
&= (\pi_i \cdot (\pi_i + \pi_k)) \cdot (\pi_j + \pi_k) && \text{associative laws} \\
&= \pi_i \cdot (\pi_j + \pi_k) && \text{absorption laws}
\end{aligned}$$

In the following assume that G is a cubic lattice graph with characteristic n.

Lemma 2. *The set of n points of G with two common coordinates form a clique.*

Proof. If $n = 1$, the lemma is true. Suppose that $u_1, \ldots, u_n$, $n > 1$, have the same two coordinates in common. Clearly these n points form a complete subgraph of G. Any point v adjacent to these n points must also have these same two coordinates. Therefore the n points must form a clique.

Lemma 3. *Let* π_1, π_2, *and* π_3 *be three K-partitions of V(G) whose blocks consist of the points of G that have the same first and second, first and* *third, and second and third coordinates, respectively. Then the set of points in each block of $\pi_1 + \pi_2$, $\pi_1 + \pi_3$, and $\pi_2 + \pi_3$ have the same first, second, and third coordinates, respectively.*

Proof. Let u and v be two points in a block of $\pi_1 + \pi_2$. From the definition of $\pi_1 + \pi_2$, this implies the existence of a sequence $u = u_0, ..., u_p = v$, where $u_{i-1} \equiv u_i\ (\pi_1)$ or $u_{i-1} \equiv u_i\ (\pi_2)$ for $i = 1, ..., p$. Then for $i = 1, ..., p$, u_{i-1} and u_i have either the same first and second or first and third coordinates. Hence u and v have the same first coordinate.

By an analogous argument, the points in each block of $\pi_1 + \pi_3$ and $\pi_2 + \pi_3$ have the same second and third coordinates.

Theorem 4 (Characterization of cubic lattice graphs). *The following are equivalent:*

(1) *G is a cubic lattice graph with characteristic n.*

(2) *There exist three triorthogonal L_2-partitions of $V(G)$, M_1, M_2, and M_3, such that each M_i contains n blocks of order n^2, the three partitions contain every line of G exactly twice, and $M_i \cdot M_j$, $i \neq j$, is a K-partition of $V(G)$.*

(3) *There exist three mutually orthogonal K-partitions of $V(G)$, π_1, π_2, and π_3, containing all the lines of G, π_i contains n^2 blocks of order n and*

$$\text{(D)} \qquad (\pi_i+\pi_j)\cdot(\pi_i+\pi_k) = \pi_i+(\pi_j\cdot\pi_k) \text{ for } 1 \leq i, j, k \leq 3.$$

Proof. (1) *implies* (3). Let π_1, π_2, and π_3 be three K-partitions of $V(G)$ whose blocks consist of the points of G that agree on the first and second, first and third, and second and third coordinates, respectively. Clearly π_1, π_2, and π_3 are mutually orthogonal and contain all the lines of G. Also each K-partition contains n^2 blocks of order n. In fact the subgraph induced by each block of π_i is a clique by Lemma 2.

All that remains is to show that π_1, π_2, and π_3 satisfy property (D). If $i = j$ or $i = k$, then

$$\pi_i + (\pi_j\cdot\pi_k) = \pi_i = (\pi_i + \pi_j)\cdot(\pi_i + \pi_k).$$

If $j = k$, then

$$\pi_i + (\pi_j \cdot \pi_k) = \pi_i + \pi_j = (\pi_i + \pi_j) \cdot (\pi_i + \pi_k)\,.$$

If $i \neq j \neq k \neq i$, then since π_1, π_2, and π_3 are mutually orthogonal,

$$(\pi_i + \pi_j) \cdot (\pi_i + \pi_k) \geq \pi_i = \pi_i + (\pi_j \cdot \pi_k).$$

Assume without loss of generality that $(\pi_1 + \pi_2) \cdot (\pi_1 + \pi_3) \geq \pi_1$. By Lemma 3 the points in each block of $\pi_1 + \pi_2$ have the same first coordinate and the points in each block of $\pi_1 + \pi_3$ have the same second coordinate. Hence $(\pi_1 + \pi_2) \cdot (\pi_1 + \pi_3) = \pi_1$.

(3) *implies* (2). Let π_1, π_2, and π_3 be three K-partitions of $V(G)$ satisfying the stated conditions. Define three partitions of $V(G)$, M_1, M_2, and M_3, by $\pi_1 + \pi_2$, $\pi_1 + \pi_3$, and $\pi_2 + \pi_3$, respectively.

First we will show that each block of M_i, $i = 1, 2, 3$, is of order n^2. Let $M_i = \pi_j + \pi_k$. Since π_j and π_k are orthogonal and contain n^2 blocks of order n, each block of M_i must be of order n^2 or greater. Suppose a block of one of the M_i's, say M_1 is of order greater than n^2. This implies that M_1 contains $m < n$ blocks and that $M_1 \cdot M_2$ contains $mq < n^2$ blocks where M_2 contains $q \leq n$ blocks. But this contradicts property (D) as

$$M_1 \cdot M_2 = (\pi_1 + \pi_2) \cdot (\pi_1 + \pi_3) = \pi_1 + (\pi_2 \cdot \pi_3) = \pi_1$$

and π_1 contains n^2 blocks of order n.

To show that the lines contained in $M_i = \pi_j + \pi_k$ are the lines contained in either π_j or π_k, suppose that M_i contains a line not contained in either π_j or π_k. Then this line must be contained in π_p, $j \neq p \neq k$, i.e., $\pi_p \cdot (\pi_j + \pi_k) > 0$. But this contradicts Lemma 1 as

$$\pi_p \cdot (\pi_j + \pi_k) = (\pi_p \cdot \pi_j) + (\pi_p \cdot \pi_k) = 0.$$

Clearly M_1, M_2, and M_3 contain each line of G exactly twice.

We will use Theorem 2 to show that $M_i = \pi_j + \pi_k$ is an L_2-partition. From the preceding, each block X of M_i is of order n^2 and M_i contains the lines contained in either π_j or π_k. The n blocks of order n of π_j and π_k which have a nonempty intersection with X are orthogonal K-partitions of $V(\langle X \rangle)$. Hence by Theorem 2 the subgraph induced by each block of M_i is an L_2 graph.

It follows almost immediately from (D) that M_1, M_2, and M_3 are triorthogonal:

$$\begin{aligned} M_1 \cdot M_2 \cdot M_3 &= (\pi_1 + \pi_2)\cdot(\pi_1 + \pi_3)\cdot(\pi_2 + \pi_3) \\ &= (\pi_1 + \pi_2)\cdot(\pi_1 + \pi_3)\cdot(\pi_1 + \pi_3)\cdot(\pi_2 + \pi_3) \\ &= ((\pi_1 + \pi_2)\cdot(\pi_1 + \pi_3))\cdot((\pi_3 + \pi_1)\cdot(\pi_3 + \pi_2)) \\ &= (\pi_1 + (\pi_2 \cdot \pi_3)) \cdot (\pi_3 + (\pi_1 \cdot \pi_2)) \\ &= \pi_1 \cdot \pi_3 \\ &= 0. \end{aligned}$$

From (D) and the fact that π_1, π_2, and π_3 are orthogonal K-partitions, it follows that $M_i \cdot M_j$, $i \neq j$, is a K-partition of $V(G)$.

(2) *implies* (1). Let M_1, M_2, and M_3 be three L_2-partitions satisfying the stated conditions, where $M_1 = \{U_1, \ldots, U_n\}$, $M_2 = \{V_1, \ldots, V_n\}$, and $M_3 = \{W_1, \ldots, W_n\}$ and $|U_i| = |V_j| = |W_k| = n^2$. Define a mapping f from $V(G)$ into a cubic lattice graph H with characteristic n by $f(v) = (i, j, k)$ if and only if $v \in U_i \cap V_j \cap W_k$. Clearly f is 1–1 and onto.

We must show that f preserves adjacency. Let uv be a line. Since M_1, M_2, and M_3 contain each line of G exactly twice, $u \equiv v\ (M_i)$ and $u \equiv v\ (M_j)$, $f(u)$ and $f(v)$ must agree on two coordinates and hence are adjacent. Conversely, if $f(u)$ is adjacent to $f(v)$, then $f(u)$ and $f(v)$ must agree on two coordinates. This implies that u and v are in the same block of $M_i \cdot M_j$, $i \neq j$. But every block of $M_i \cdot M_j$, $i \neq j$, induces a complete subgraph in G and hence u is adjacent to v.

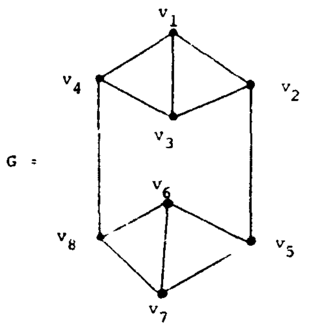
Therefore $G \cong H$.

One might conjecture that property (D) is superfluous. But Fig. 1 shows that this is not the case.

The three partitions, π_1, π_2, and π_3, are mutually orthogonal, contain each line of G, and each partition contains 2^2 blocks of order 2. But these partitions do not satisfy (D) as

$$(\pi_1 + \pi_2)\cdot(\pi_1 + \pi_3) = \pi_1 + \pi_3 > \pi_1 = \pi_1 + (\pi_2 \cdot \pi_3).$$

The graph G is not a cubic lattice graph as points v_2 and v_3, and v_6 and v_7 do not satisfy property (B_3) (see Introduction).



$$\pi_1 = \{\overline{v_1v_2}, \overline{v_3v_4}, \overline{v_5v_6}, \overline{v_7v_8}\}$$
$$\pi_2 = \{\overline{v_1v_3}, \overline{v_2v_5}, \overline{v_4v_8}, \overline{v_6v_7}\}$$
$$\pi_3 = \{\overline{v_1v_4}, \overline{v_2v_3}, \overline{v_5v_7}, \overline{v_6v_8}\}$$

Fig. 1.

4. Conjecture

The n^{th} *interchange graph* $I_n(G)$ of G is a graph whose points are the complete subgraphs of order $n + 1$ of G and two points of $I_n(G)$ are adjacent if and only if the corresponding K_{n+1}'s have a K_n in common. The *line graph* $L(G)$ of G is $I_1(G)$. An *m-partite graph* G is a graph whose points can be partitioned into m subsets $V_1, ..., V_m$ such that every line joins V_i with V_j, $i \neq j$. A *bigraph* is a 2-partite graph. A *complete m-partite graph* contains every line joining V_i with V_j. We write $G = K_{p_1,...,p_m}$ if V_i has p_i points for $i = 1, ..., m$.

It follows immediately that an L_2 graph is the line graph of $K_{n,n}$. We have a similar result for cubic lattice graphs.

Theorem 5. *The cubic lattice graph with characteristic n is isomorphic to* $I_2(K_{n,n,n})$.

Proof. Let the points of $K_{n,n,n}$ be $U \cup V \cup W$, where $U = \{u_1, ... u_n\}$, $V = \{v_1, ..., v_n\}$, and $W = \{w_1, ..., w_n\}$. Then u_i, v_j, w_k form a K_3 for $1 \leq i, j, k \leq n$. Let the point $z_{i,j,k}$ denote this K_3 in $I_2(K_{n,n,n})$. Let G be the cubic lattice graph with characteristic n. Then the mapping f from $V(I_2(K_{n,n,n}))$ into $V(G)$ defined by $f(z_{i,j,k}) = (i, j, k)$ is clearly 1–1, onto, and preserves adjacency. Hence $I_2(K_{n,n,n}) \cong G$.

In [3], Grünbaum mentions that for $m > 3$ interchange graphs of complete m-partite graphs do not seem to have been investigated. Theorem 4 seems capable of being extended to these graphs. Let $L_m(n)$ denote the $(m-1)^{\text{st}}$ interchange graph of the complete m-partite graph

$K_{n,\ldots,n}$. Then an L_2 graph is denoted $L_2(n)$ and a cubic lattice graph with characteristic n is denoted $L_3(n)$. That is, the points of $L_m(n)$ can be identified with the n^m ordered m-tuples on n symbols such that two points are adjacent if and only if the corresponding m-tuples have $m-1$ coordinates in common. A partition π of $V(G)$ is an $L_m(n)$*-partition* if the subgraph induced by each block of π is an $L_m(n)$ graph.

Conjecture. The following are equivalent.

(1) G is the interchange graph of the complete m-partite graph $K_{n,\ldots,n}$ for $m > 3$.

(2) There exist m m-orthogonal $L_{m-1}(n)$-partitions $M_1, \ldots, M_m$ of $V(G)$ such that each M_i contains n blocks of order n^{m-1}, $M_1, \ldots, M_{m-1}$, and M_m contain every line of G exactly $m-1$ times, and the subgraph induced by each block of the partition formed by the product of $m-1$ distinct M_i's is K_n.

(3) There exist m mutually orthogonal K-partitions $\pi_1, \ldots, \pi_m$ of $V(G)$ containing all the lines of G, each partition contains n^{m-1} blocks of order n and the lattice generated by these partitions is distributive.

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