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A class of rotationally symmetric quantum layers of dimension 4

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ABSTRACT

Under several geometric conditions imposed below, the existence of the discrete spectrum below the essential spectrum is shown for the Dirichlet Laplacian on the quantum layer built over a spherically symmetric hypersurface with a pole embedded in R^4. At the end of this paper, we also show the advantage and independence of our main result comparing with those existent results for higher dimensional quantum layers or quantum tubes.

1. Introduction

The study of the spectral properties of the Dirichlet Laplacian in infinitely stretched regions has attracted so much attention, since it has applications in elasticity, acoustics, electromagnetism, etc. It also has application in the quantum physics. Since Duclos et al. considered the existence of the discrete spectrum of the Dirichlet Laplacian of the quantum layers built over surfaces in [5], many similar results have been obtained for the quantum layers whose reference manifolds are surfaces. However, very little was known about the existence of the discrete spectrum of the Dirichlet Laplacian on the quantum layers of dimension greater than 3.

In [6,7], under some geometric assumptions therein, Lin and Lu have successfully proved the the existence of the discrete spectrum of the Dirichlet Laplacian on the quantum layers built over submanifolds of the Euclidean space R^m $(3 \leq m < \infty)$. However, the parabolicity of the reference submanifold and nonpositivity of the integration of $\mathscr{K}_{m-2}$ defined by (4.1) are necessary. Is the parabolicity of the reference submanifold necessary for the existence of the discrete spectrum? We try to give a negative answer here. In general, it is not easy to judge whether a prescribed manifold is parabolic or not. However, Grigor'yan has shown a sufficient and necessary condition, which is related to the area of the boundary of the geodesic ball and could be easily computed, of parabolicity for spherically symmetric manifolds in [1]. Hence, we guess maybe we can expect to get the existence of the the discrete spectrum of the quantum layer built over some spherically symmetric submanifold, which is non-parabolic, of the Euclidean space R^m $(3 \leq m < \infty)$.

In order to state our main result, we define two quantities σ_0 and σ_{ess} as follows.

Definition 1.1. Let M be a manifold whose Laplacian Δ can be extended to a self-adjoint operator. Let

$$\sigma_0 = \inf_{f\in C_0^\infty(M)} \frac{-\int_M f\Delta f dV_M}{\int_M f^2 dV_M}, \tag{1.1}$$

$$\sigma_{ess} = \sup_K \inf_{f\in C_0^\infty(M\setminus K)} \frac{-\int_M f\Delta f dV_M}{\int_M f^2 dV_M}, \tag{1.2}$$

where K is running over all compact subsets of M, and dV_M denotes the volume element of M.

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In fact, σ_0 and σ_{ess} are the lower bound of the spectrum and the lower bound of the essential spectrum of the Laplacian $\triangle$ on M, respectively. In general case, $\sigma_0 \leq \sigma_{ess}$. If $\sigma_0 < \sigma_{ess}$, then the existence of the discrete spectrum is obvious. In mathematical physics, points in the discrete spectrum are called bound states, and moreover, the lowest bound state is also called the ground state.

We want to show that $\sigma_0 < \sigma_{ess}$ holds for the Laplacian $\triangle$ of the class of quantum layers considered in the sequel. In fact, by using this strategy we can prove the following.

Theorem 1.2 (*Main Theorem*)**.** *Assume Σ is a spherically symmetric hypersurface with a pole embedded in R^4, and Σ is not a hyperplane, if in addition $\mathscr{K}_2$ is integrable on Σ and*

$$\int_{\Sigma} \mathscr{K}_2 d\Sigma \leq 0$$

with $\mathscr{K}_2$ defined by (4.1), then under assumptions A1, A2 *and* A3 *given in Section 2, the ground state of the quantum layer Ω built over Σ exists.*

The paper is organized as follows. The fundamental geometric properties of the quantum layers built over spherically symmetric hypersurfaces will be discussed in the next section. The fact that the Laplacian $\triangle$ on the quantum layers can be extended to a self-adjoint operator will be explained in Section 3. The main theorem above will be proved in Section 4.

2. Geometry of rotationally symmetric quantum layers

Let m $(3 \leq m < \infty)$ be an integer and let Σ be a C^2-smooth hypersurface with a pole embedded in R^m. The existence of a pole on Σ is a nontrivial assumption under which Σ is necessarily diffeomorphic to R^{m-1} leading to the simple connectedness and non-compactness of Σ. Under this assumption, we can also set up the global geodesic polar coordinates to parametrize the hypersurface Σ by a unique patch $p : \Sigma_0 \to R^m$, where $\Sigma_0 := (0, \infty) \times \mathbb{S}^{m-2}$ with $\mathbb{S}^{m-2}$ the unit sphere in R^{m-1}. Naturally, Σ can be identified with the image of Σ_0. The tangent vectors $p_{,\mu} := \frac{\partial p}{\partial q^{\mu}}$ are linearly independent and span the tangent space at every point of Σ, correspondingly, the unit normal vector field $\vec{n}$ can be determined. Let $\Omega_0 := \Sigma_0 \times (-a, a)$, then the quantum layer $\Omega := \Phi(\Omega_0)$ of width $2a$ built over Σ can be defined by a natural mapping $\Phi : \Omega_0 \to R^m$ as follows

$$\Phi(q, u) := p(q) + u\vec{n}(q), \quad (q, u) \in \Sigma_0 \times (-a, a). \tag{2.1}$$

We make an agreement on the indices range, $1 \leq \mu, \nu, \ldots, \leq m-1$ and $1 \leq i, j, \ldots, \leq m$. Denote the pole on Σ by o, we know that the exponential map $\exp_0 : \mathscr{D}_o \to \Sigma$ is a diffeomorphism, where $\mathscr{D}_o = \{s\xi \,|\, 0 \leq s < \infty,\ \xi \in S_o^{m-2}\}$ with S_o^{m-2} the unit sphere in the tangent space $T_o(\Sigma)$. For a fixed vector $\xi \in T_oM$, $|\xi| = 1$, let $\xi^{\perp}$ be the orthogonal complement of $\{\mathbb{R}\xi\}$ in T_oM and let $\tau_s : T_oM \to T_{\exp_o(s\xi)}M$ be the parallel translation along the geodesic $\gamma_{\xi}(s) := \exp_o(s\xi)$ with $\gamma'(0) = \xi$. Define the path of linear transformations $\mathbb{A}(s, \xi) : \xi^{\perp} \to \xi^{\perp}$ by

$$\mathbb{A}(s, \xi)\eta = (\tau_s)^{-1} Y(s),$$

where $Y(s)$ is the Jacobi field along γ_{ξ} satisfying $Y(0) = 0$, $(\nabla_s Y)(0) = \eta$. Moreover, for $\eta \in \xi^{\perp}$, set

$$\mathscr{R}(s)\eta = (\tau_s)^{-1}\mathbb{R}(s)(\tau_s\eta) = (\tau_s)^{-1} R(\gamma_{\xi}'(s), \tau_s\eta)\gamma_{\xi}'(s),$$

then $\mathscr{R}(s)$ is a self-adjoint map of $\xi^{\perp}$, where we use the following signature for the curvature tensor $R(X, Y)Z = -[\nabla_X, \nabla_Y]Z + \nabla_{[X,Y]}Z$. Obviously, the map $\mathbb{A}(s, \xi)$ satisfies the Jacobi equation $\mathbb{A}'' + \mathscr{R}\mathbb{A} = 0$ with initial conditions $\mathbb{A}(0, \xi) = 0$, $\mathbb{A}'(0, \xi) = I$, and by applying the Gauss lemma the Riemannian metric of M can be expressed by

$$dt^2(\exp_o(s\xi)) = ds^2 + |\mathbb{A}(s, \xi)d\xi|^2$$

on the set $\exp_o(\mathscr{D}_0)$. Hence, the induced metric $g_{\mu\nu}$ in the geodesic polar coordinates satisfies

$$\sqrt{\det[g_{\mu\nu}]} = \det \mathbb{A}(s, \xi).$$

Define a function $J > 0$ on $\mathscr{D}_0$ by

$$J^{m-2} = \sqrt{\det[g_{\mu\nu}]},$$

that is $dV_{\Sigma} = J^{m-2} ds d\xi$. We know that the function $J(s, \xi)$ satisfies (cf. [8, p. 244])

$$\begin{aligned} &J'' + \frac{1}{(m-2)} \mathrm{Ricci}\left(\frac{d}{ds}, \frac{d}{ds}\right) J \leq 0 \\ &J(s, \xi) = s + O(s^2) \\ &J'(s, \xi) = 1 + O(s), \end{aligned}$$

where Ricci denotes the Ricci curvature tensor on Σ and $\frac{d}{ds}$ is the radial unit tangent vector along the geodesic $\gamma_\xi(s)$. So, we have

$$J'' + \frac{1}{(m-2)}\mathrm{Ricci}\left(\frac{d}{ds}, \frac{d}{ds}\right)J \le 0 \quad \text{with} \quad J(0,\xi) = 0, \quad J'(0,\xi) = 1. \tag{2.2}$$

Consider now layers which are invariant with respect to rotations around a fixed axis in R^m. We could thus suppose that Σ is a rotational hypersurface parametrized by $p : \Sigma_0 \to R^m$,

$$\begin{aligned} p(s,\theta_1,\ldots,\theta_{m-2}) &:= (r(s)\cos(\theta_1), r(s)\sin(\theta_1)\cos(\theta_2), r(s)\sin(\theta_1)\sin(\theta_2)\cos(\theta_3),\ldots,\\ &\quad r(s)\sin(\theta_1)\ldots\sin(\theta_{m-3})\cos(\theta_{m-2}), r(s)\sin(\theta_1)\ldots\sin(\theta_{m-3})\sin(\theta_{m-2}), z(s)), \end{aligned} \tag{2.3}$$

where $r, z \in C^2((0,\infty))$, $r > 0$ and $(\theta_1,\ldots,\theta_{m-2}) \in \mathbb{S}^{m-2}$. This parametrization will be the geodesic polar coordinate chart if we additionally require

$$\left(r'(s)\right)^2 + \left(z'(s)\right)^2 = 1, \tag{2.4}$$

since by direct calculation the induced metric tensor on Σ can be written as $ds^2 + r^2|d\xi|^2$ with

$$|d\xi|^2 := d\theta_1^2 + (\sin\theta_1)^2 d\theta_2^2 + (\sin\theta_1)^2(\sin\theta_2)^2 d\theta_3^2 + \cdots + (\sin\theta_1)^2(\sin\theta_2)^2\cdots(\sin\theta_{m-3})^2 d\theta_{m-2}^2,$$

the round metric on $\mathbb{S}^{m-2}$, provided the requirement (2.4) is satisfied. So, we have $dV_\Sigma = r^{m-2}dsd\xi$, which implies the function J defined above satisfies $J = r$ in this case. Moreover, under the parametrization (2.3) with the requirement (2.4), Σ is a spherically symmetric hypersurface with a pole, and its Weingarten tensor is given by $(h_{\mu\nu}) = \mathrm{diag}(k_s, k_{\theta_1}\ldots, k_{\theta_{m-2}})$ with the principle curvatures

$$k_s = r'z'' - r''z' \quad \text{and} \quad k_\theta := k_{\theta_1} = \cdots = k_{\theta_{m-2}} = \frac{z'}{r}. \tag{2.5}$$

As pointed out in [5], it is sufficient to know the function $s \to k_s(s)$ only, since r, z can be constructed from the relations

$$r(s) = \int_0^s \cos b(\vartheta)d\vartheta, \qquad z(s) = \int_0^s \sin b(\vartheta)d\vartheta,$$

with $b(\vartheta) := \int_0^s k_s(\vartheta)d\vartheta$.

By (2.2), (2.4), (2.5) and the facts $J = r$ and *Ricci* $\left(\frac{d}{ds}, \frac{d}{ds}\right) = (m-2)k_s k_\theta$, we know that the function $r(s)$ satisfies

$$r'' + k_s k_\theta r = 0 \quad \text{with } r(0) = 0,\ r'(0) = 1, \tag{2.6}$$

This equation will make an important role in the proof of Theorem 1.2.

In the sequel, we impose the following assumptions on Σ.

A1. Σ is not self-intersecting, i.e., Φ is injective.
A2. The half width a of the layer satisfies $a < \rho_m := (\max\{\|k_s\|_\infty, \|k_\theta\|_\infty\})^{-1}$, where $\|\cdot\|_\infty$ denotes the L^∞-norm.
A3. For $x \in \Sigma$, $\|A\|(x) \to 0$ as $d(x, x_0) \to \infty$, where x_0 is a fixed point on the spherically symmetric hypersurface Σ. This means that Σ is asymptotically flat.

3. Self-adjoint extension of the Laplacian on the quantum layers

As in [5,6], from the definition (2.1), the metric tensor of the layer as a submanifold of R^m satisfies

$$G_{ij} = \begin{cases} (\delta_i^\sigma - uh_i^\sigma)(\delta_\sigma^\rho - uh_\sigma^\rho)g_{\rho j}, & 1 \le i,j \le m-1,\\ 0, & i \text{ or } j = m,\\ 1, & i = j = m, \end{cases} \tag{3.1}$$

which implies the metric matrix has the block form

$$(G_{ij}) = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with } G_{\mu\nu} = (\delta_\mu^\sigma - uh_\mu^\sigma)(\delta_\sigma^\rho - uh_\sigma^\rho)g_{\rho\nu},\ 1 \le \mu,\nu \le m-1.$$

Then by (3.1), we obtain

$$\det(G_{AB}) = [\det(1 - uA)]^2 \det(g_{\mu\nu}). \tag{3.2}$$

Since the eigenvalues of the matrix of the Weingarten map are the principle curvatures k_s, k_θ, we have

$$\det(1 - uA) = (1 - uk_s)(1 - uk_\theta)^{m-2}. \tag{3.3}$$

where k_s and k_θ are given by (2.5). By assumption A2, the entries $G_{\mu\nu}$ of the matrix G can be estimated by

$$C_- g_{\mu\nu} \leq G_{\mu\nu} \leq C_+ g_{\mu\nu}, \tag{3.4}$$

where $C_\pm := (1 \pm a\rho_m^{-1})^2$ with $0 < C_- \leq 1 \leq C_+ < 4$. So, assumption A2 makes sure that the mapping Φ is nonsingular, which implies the mapping Φ induces a Riemannian metric G on Ω. Hence, we know that the mapping Φ is a diffeomorphism under assumptions A1 and A2.

There is an interesting truth we would like to point out here. From the last section, we know that the Riemannian metric of the spherically symmetric hypersurface Σ can be expressed as $ds^2 + r^2|d\xi|^2$ with $|d\xi|^2$ the round metric on $\mathbb{S}^{m-2}$ under the parametrization (2.3), then by (3.1) the Riemannian metric of the quantum layer Ω built over Σ can be written as $du^2 + ds^2 + r^2|d\xi|^2$, which implies Ω is also cylindrically symmetric.

For convenience, let $x_1 := s, x_2 := \theta_1, \ldots, x_{m-1} := \theta_{m-2}, x_m := u$, then in the coordinate system $\{x_1, \ldots, x_m\}$ on Ω, the Laplacian $\Delta = \Delta_\Omega$ can be written as

$$\Delta = \frac{1}{\sqrt{\det(G_{ij})}} \sum_{\mu,\nu=1}^{m-1} \frac{\partial}{\partial x_\mu}\left(G^{\mu\nu}\sqrt{\det(G_{ij})}\frac{\partial}{\partial x_\nu}\right) + \frac{1}{\sqrt{\det(G_{ij})}}\frac{\partial}{\partial u}\left(G^{mm}\sqrt{\det(G_{ij})}\frac{\partial}{\partial u}\right).$$

Using (3.3) we could split $\triangle$ into a sum of two parts, $\triangle = \triangle_1 + \triangle_2$, given by

$$\Delta_1 := \frac{1}{\sqrt{\det(G_{ij})}}\frac{\partial}{\partial u}\left(G^{mm}\sqrt{\det(G_{ij})}\frac{\partial}{\partial u}\right) = \frac{\partial^2}{\partial u^2} - \left(\frac{k_s}{1-uk_s} + \frac{(m-2)k_\theta}{1-uk_\theta}\right)\frac{\partial}{\partial u}$$

and

$$\Delta_2 := \Delta - \Delta_1 = \frac{1}{\sqrt{\det(G_{ij})}} \sum_{\mu,\nu=1}^{m-1} \frac{\partial}{\partial x_\mu}\left(G^{\mu\nu}\sqrt{\det(G_{ij})}\frac{\partial}{\partial x_\nu}\right).$$

In the rest part of this section, we will show that this Laplacian $\Delta = \Delta_\Omega$ can be extended to a self-adjoint operator. For any $E, F \in C_0^\infty(\Omega)$, the set of all smooth functions with compact support on Ω, we define the L^2 inner product $(\cdot,\cdot)$ as follows

$$(F, G) = \int_\Omega FG d\Omega,$$

where $d\Omega$ is the volume element of the quantum layer Ω. Correspondingly, the norm $\|E\|$ could be defined by $\|E\| := \sqrt{(E, E)}$. Moreover, if E, F are differentiable, we define

$$(\nabla E, \nabla F) = \int_\Omega \left(\sum_{\mu,\nu=1}^{m-1} G^{\mu\nu}\frac{\partial E}{\partial x_\mu}\frac{\partial F}{\partial x_\nu} + \frac{\partial E}{\partial u}\frac{\partial F}{\partial u}\right) d\Omega.$$

Also, we define $\|\nabla E\| = \sqrt{(\nabla E, \nabla E)}$. Then as the proof of Proposition 2.1 in [6], for any $E, F \in W_0^{1,2}(\Omega)$, the space which is the closure of the space $C_0^\infty(\Omega)$ under the norm

$$\|E\|_{W_0^{1,2}(\Omega)} = \sqrt{\|E\|^2 + \|\nabla E\|^2},$$

the sesquilinear form $Q_1(E, F) := (\nabla E, \nabla F)$ is a quadratic form of a unique self-adjoint operator. Such an operator is an extension of Δ, which we still denote as Δ. Hence we can use (1.1) and (1.2) to compute σ_0 and σ_{ess} for the quantum layer Ω, respectively.

However, generally it is complicated to construct trial functions on the quantum layer Ω directly, our strategy to solve this difficulty is the following: by introducing the unitary transformation $\psi \to \psi\Phi$ with Φ defined by (2.1), we may identify the Hilbert space $L^2(\Omega)$ with $\mathscr{H} := L^2(\Omega_0, d\Omega)$ and the Laplacian $\Delta = \Delta_\Omega$ with the self-adjoint operator H associated with the quadratic form Q_2 on $\mathscr{H}$ defined by

$$Q_2(\psi,\psi) := \int_{\Omega_0} \overline{\psi_{,i}} G^{ij}\psi_{,j} d\Omega,$$

$$\psi \in \mathrm{Dom}Q_2 := \left\{\psi \in W^{1,2}(\Omega_0, d\Omega) | \psi(q,u) = 0 \text{ for a.e. } (q,u) \in \Sigma_0 \times \{\pm a\}\right\},$$

here $\psi(x)$ for $x \in \partial\Omega_0$ means the corresponding trace of the function ψ on the boundary.

4. Proof of main theorem

Under assumptions A2 and A3, as the proof of Theorem 3.1 in [6], we can prove the following.

Theorem 4.1. *Assume Ω is a quantum layer built over an oriented hypersurface immersed in R^m ($3 \leq m < \infty$), then under assumptions* A2 *and* A3, *we have $\sigma_{ess} \geq (\frac{\pi}{2a})^2$.*

In order to prove our main theorem later, we need the following lemma.

Lemma 4.2 ([6]). *Let $a > 0$ be a positive number and let $k_1 = \frac{\pi}{2a}$. Let $\chi_1(u) = \cos(k_1 u)$, let*

$$\eta_k = \int_{-a}^{a} u^k(\chi_{1,u}^2 - k_1^2\chi_1^2)du, \quad \forall k \geq 0,$$

where $\chi_{1,u}$ denotes the derivative of χ_1 with respect to u. Then

$$\eta_k = \begin{cases} 0, & \text{if } k \text{ is odd, or } k = 0; \\ \dfrac{1}{2}\dfrac{(k)!}{(2k_1)^{k-1}}\displaystyle\sum_{l=1}^{k/2}\dfrac{(-1)^{k/2-l}\pi^{2l-1}}{(2l-1)!}, & \text{if } k \neq 0 \text{ is even}. \end{cases}$$

Furthermore, $\eta_k > 0$ if $k \neq 0$ is even.

For the spherically symmetric hypersurface $\Sigma \subseteq R^m$ $(3 \leq m > \infty)$ with a pole, we define a quantity $\mathscr{K}_{m-2}$ by

$$\mathscr{K}_{m-2} := \sum_{k=1}^{[(m-1)/2]} \eta_{2k}c_{2k}(A), \quad 3 \leq m < \infty, \tag{4.1}$$

where η_k for $k \geq 1$ is given in Lemma 4.3, $[(m-1)/2]$ is the integer part of $(m-1)/2$, and $c_k(A)$ is the kth elementary symmetric polynomial of the second fundamental form A of Σ. When $m = 4$, we can obtain the following lemma.

Lemma 4.3. *If $\mathscr{K}_2$ defined by (4.1) is integrable on a 3-dimensional spherically symmetric hypersurface Σ with a pole embedded in R^4, and Σ is not a hyperplane, then we have*

(1) *Σ is non-parabolic,*
(2) *$\lim_{s\to\infty} \frac{r(s)}{s} = 1$,*
(3) *$\int_0^\infty k_s(s)k_\theta(s)r(s)ds = 0$, which implies there exists at least one domain on Σ such that k_s and k_θ have the same sign on this domain, here $r(s)$ is given by (2.3) satisfying (2.4), and k_s, k_θ are given by (2.5).*

Proof. Since $\mathscr{K}_2$ is integrable on Σ which can be parametrized by (2.3) with the requirement (2.4), then we know that $\int_{\Sigma_0} k_s(s)k_\theta(s)d\Sigma$ and $\int_{\Sigma_0} k_\theta^2(s)d\Sigma$ are finite, which implies $\int_0^\infty k_s(s)k_\theta(s)r^2(s)ds$ and $\int_0^\infty k_\theta^2(s)r^2(s)ds$ are finite. By (2.6), we could obtain

$$r'(s)r(s) = \int_0^s \left(r'(v)\right)^2 dv - \int_0^s k_s(v)k_\theta(v)r^2(v)dv,$$

together with (2.4) and (2.5), it follows that

$$r'(s)r(s) = s - \int_0^s k_\theta^2(v)r^2(v)dv - \int_0^s k_s(v)k_\theta(v)r^2(v)dv. \tag{4.2}$$

Let D be

$$D := \int_0^\infty k_s(s)k_\theta(s)r^2(s)ds + \int_0^\infty k_\theta^2(s)r^2(s)ds,$$

then there exists a constant $s_0 > 1$ such that for any $s \geq s_0$, we have

$$\left|\int_0^s k_s(v)k_\theta(v)r^2(v)dv + \int_0^s k_\theta^2(v)r^2(v)dv - D\right| \leq \frac{1}{100}.$$

Integrating (4.2) from s_0 to s results in

$$s^2 - s_0^2 - \left(2D + \frac{1}{50}\right)(s - s_0) + r^2(s_0) \leq r^2(s) \leq s^2 - s_0^2 + \left(2|D| + \frac{1}{50}\right)(s - s_0) + r^2(s_0), \tag{4.3}$$

for any $s \geq s_0$.

On the other hand, from (4.2), we also have

$$\lim_{s\to\infty} \frac{r'(s)r(s)}{s} = 1 - \lim_{s\to\infty} s^{-1}\left[\int_0^s k_\theta^2(v)r^2(v)dv + \int_0^s k_s(v)k_\theta(v)r^2(v)dv\right] = 1,$$

together with (4.3), it follows that

$$r'(\infty) := \lim_{r\to\infty} r'(s) = 1. \tag{4.4}$$

By (2.4), (2.6) and (4.4), we have

$$\int_0^\infty k_s(s)k_\theta(s)r(s)ds = 0 \quad \text{and} \quad \lim_{s\to\infty} z'(s) = 0. \tag{4.5}$$

Now, we would like to prove the first assertion by using the estimate (4.3), however, before that some useful facts about parabolicity should be given first.

Definition 4.4. A complete manifold is said to be non-parabolic if it admits a non-constant positive superharmonic function. Otherwise it is said to be parabolic.

Lemma 4.5 ([2–4]). *Let Riemannian manifold M be geodesically complete, and for some $x \in M$,*

$$\int_1^\infty \frac{1}{S(x,\rho)}d\rho = \infty \tag{4.6}$$

with $S(x,\rho)$ the boundary area of the geodesic sphere $\partial B(x,\rho)$. Then M is parabolic.

In general, (4.6) is not necessary for parabolicity, however, in [1], Grigor'yan has shown that for a spherically symmetric manifold $\widetilde{M}$ with a pole, (4.6) is also a necessary condition for $\widetilde{M}$ being parabolic. Hence, if we want to show Σ is non-parabolic here, it suffices to prove there exists some $x \in \Sigma$ such that

$$\int_1^\infty \frac{1}{S(x,t)}dt < \infty$$

with $S(x,t)$ the area of the boundary of the geodesic ball $B(x,t)$ centered at x with radius t. Now, for the 3-dimensional spherically symmetric hypersurface Σ with a pole o, choose x to be the pole o, then the area $S(o,t)$ can be expressed by $S(o,t) = w_2 r^2(t)$ with w_2 the 2-volume of the unit sphere in R^3. So, by applying (4.3), we have

$$\int_1^\infty \frac{1}{S(o,t)}dt \le \int_1^{1+s_1} \frac{1}{w_2 r^2(t)}dt + \frac{1}{w_2}\int_{1+s_1}^\infty \frac{1}{s^2 - s_0^2 - \left(2D + \frac{1}{50}\right)(s - s_0) + r^2(s_0)}ds < \infty,$$

where s_1 is chosen to be

$$s_1 := \begin{cases} s_0, & \text{if } \aleph \le 0,\\ \max\left\{s_0, \dfrac{1}{100} + D + \sqrt{\left(D + \dfrac{1}{100}\right)^2 - \left(2D + \dfrac{1}{50}\right)s_0 + s_0^2 - r^2(s_0)}\right\}, & \text{if } \aleph > 0, \end{cases}$$

with $\aleph := -r^2(s_0) - (2D + \frac{1}{50})s_0 + s_0^2 + (D + \frac{1}{100})^2$. Our proof is finished. $\square$

By using Lemma 4.3, we could obtain a result on the growth speed of the volume of a geodesic ball of a 3-dimensional spherically symmetric hypersurface related to the integrability of $\mathscr{K}_2$ as follows.

Corollary 4.6. *Let Σ be a 3-dimensional spherically symmetric manifold with a pole o embedded in R^4, if in addition $\mathscr{K}_2$ defined by (4.1) is integrable on Σ, then the volume $V(o,s)$ of the geodesic ball $B(o,s)$ with center o and radius s has cubic growth as s large enough.*

Proof. We can set up the global geodesic polar coordinate chart centered at o for Σ as before, consequently, the volume of the geodesic ball $B(o,s)$ is given by

$$V(o,s) = \int_0^s \int_{\mathbb{S}^2} r^2(v)d\mathbb{S}^2 dv,$$

where r satisfies (2.6). By applying (4.3), we have

$$\frac{w_2(s^3 - s_0^3)}{3} - w_2\cdot\left(D + \frac{1}{50}\right)(s^2 - s_0^2) + c_1(s - s_0) + V(o,s_0) \le V(o,s) \le \frac{w_2(s^3 - s_0^3)}{3}$$
$$+ w_2\cdot\left(|D| + \frac{1}{50}\right)(s - s_0^2) + c_2(s - s_0) + V(o,s_0),$$

for any $s \ge s_0$, where $c_1 := \left[r_0^2 - s_0^2 + (2D + 1/50)\,s_0\right]w_2$ and $c_2 := \left[r_0^2 - s_0^2 - (2\mid D\mid + 1/50)\,s_0\right]w_2$. This implies $V(o,s)$ has the cubic growth as s large enough. $\square$

Let (M, g) be an n-dimensional complete Riemannian manifold, denote by $B(q, r)$ the open geodesic ball centered at a point $q \in M$ with radius r and by $\mathrm{vol}(B(q, r))$ its volume. Define

$$\alpha_M := \lim_{r\to\infty} \frac{\mathrm{vol}(B(q, r))}{v_n(1)r^n},$$

with $v_n(1)$ the volume of the unit ball in R^n. It is not difficult to prove α_M is independent of the choice of q, which implies α_M is a global geometric invariant. We say that (M, g) has large volume growth provided $\alpha_M > 0$. For the spherically symmetric hypersurface Σ with a pole o embedded in R^4 with $\mathscr{K}_2$ integrable, by Corollary 4.6 we have

$$\alpha_\Sigma = \lim_{r\to\infty} \frac{V(o, r)}{v_3(1)r^3} = 1 > 0,$$

which implies Σ has large volume growth provided $\mathscr{K}_2$ is integrable.

Large volume growth assumption is common in deriving a prescribed manifold with nonnegative Ricci curvature to be of finite topological type. However, recently the author proved that a complete open manifold with nonnegative Ricci curvature is of finite topological type without the large volume growth assumption in [10].

By using Lemmas 4.2 and 4.3, we can prove the following conclusion.

Theorem 4.7. *Assume Ω is the quantum layer built over a spherically symmetric hypersurface Σ with a pole embedded in R^4, and Σ is not a hyperplane, if in addition $\mathscr{K}_2$ is integrable on Σ and*

$$\int_\Sigma \mathscr{K}_2 d\Sigma \le 0$$

with $\mathscr{K}_2$ defined by (4.1), then under assumptions A1 and A2, we have $\sigma_0 < (\frac{\pi}{2a})^2$.

Proof. Here we use a similar method as that of Theorem 5.1 in [5]. Set $\chi(u) := \sqrt{\frac{1}{a}}\cos(\frac{\pi u}{2a}) = \sqrt{\frac{1}{a}}\chi_1(u)$. We divide the proof into two steps:

(1) If $\int_\Sigma \mathscr{K}_2 d\Sigma < 0$, construct a trial function $\Psi(s, u) := \varphi_\sigma(s)\chi(u)$, where $\sigma \in (0, 1]$ and

$$\varphi_\sigma(s) := \begin{cases} 1, & \text{if } 0 < s \le s_0, \\ \min\left\{1, \dfrac{K_0(\sigma s)}{K_0(\sigma s_0)}\right\}, & \text{if } s > s_0, \end{cases} \tag{4.7}$$

with $K_0(s)$ the Macdonald function (see [9], Section 9.6). Obviously, $\Psi(s, u)$ is continuous on Ω_0, which implies $\Psi \in DomQ_2$. By (1.1) and the strategy explained at the end of the last section, if we want prove $\sigma_0 < (\frac{\pi}{2a})^2$, it suffices to show that

$$-\int_{\Omega_0} \Psi(s, u)\Delta\Psi(s, u)d\Omega - \left(\frac{\pi}{2a}\right)^2 \int_{\Omega_0} \Psi^2(s, u)d\Omega$$

is strictly negative.

By applying (3.2), (3.3) and Lemma 4.2, we know that

$$-\int_{\Omega_0} \Psi(s, u)\Delta_2\Psi(s, u)d\Omega - \left(\frac{\pi}{2a}\right)^2 \int_{\Omega_0} \Psi^2(s, u)d\Omega = \int_{\Sigma_0} (2k_s k_\theta + k_\theta^2)(\varphi_\sigma(s))^2 d\Sigma.$$

Since $\mathscr{K}_2$ is integrable on Σ, $|\varphi_\sigma(s)| \le 1$, and $\varphi_\sigma \to 1$ pointwise as $\sigma \to 0+$, then by the dominated convergence theorem, we know that

$$-\int_{\Omega_0} \Psi(s, u)\Delta_2\Psi(s, u)d\Omega - \left(\frac{\pi}{2a}\right)^2 \int_{\Omega_0} \Psi^2(s, u)d\Omega \to \int_{\Sigma_0} (2k_s k_\theta + k_\theta^2)d\Sigma = \int_\Sigma \mathscr{K}_2 d\Sigma \tag{4.8}$$

as $\sigma \to 0+$.

On the other hand, an integration of (4.2) together with the fact that $\mathscr{K}_2$ is integrable on Σ yields that for any $s > 0$, there exists a constant c_3 depending on the value of $\int_\Sigma \mathscr{K}_2 d\Sigma$ such that

$$r^2(s) \le s^2 + c_3 s. \tag{4.9}$$

So, by (3.2), (3.3), (3.4) and (4.9), we have

$$\begin{aligned} -\int_{\Omega_0} \Psi(s, u)\Delta_1\Psi(s, u)d\Omega &= -\int_{-a}^{a}\int_{\Sigma_0} \left(\varphi_\sigma'(s)\chi(u)\right)^2 \frac{(1-uk_\theta(s))^2}{1-uk_s(s)} d\Sigma du \\ &\le \frac{w_2 C_+}{\sqrt{C_-}} \int_0^\infty \left(\varphi_\sigma'(s)\right)^2 (s^2 + c_3 s)ds. \end{aligned} \tag{4.10}$$

However, by using Mathematica and properties of the Macdonald function given by

$$-2K_v'(z) = K_{v-1}(z) + K_{v+1}(z),$$
$$-\frac{2v}{z}K_v(z) = K_{v-1}(z) - K_{v+1}(z),$$
$$K_0(z) = -\log z + O(1), \quad \text{as } z \to 0,$$
$$K_1(z) = \frac{1}{z} + O(\log z), \quad \text{as } z \to 0,$$

it follows that as $\sigma \to 0+$, there exists a constant c_4 such that

$$\int_0^\infty \left(\varphi_\sigma'(s)\right)^2 s^2 ds = \frac{1}{(K_0(\sigma s_0))^2}\int_{\sigma s_0}^\infty \left(K_0'(t)\right)^2 t^2 dt \to \frac{3\pi^2}{32(K_0(\sigma s_0))^2} \to 0$$

and

$$\int_0^\infty \left(\varphi_\sigma'(s)\right)^2 s ds \le \frac{c_4}{|\log \sigma s_0|} \to 0.$$

Substituting the above estimates in (4.10) results in

$$-\int_{\Omega_0} \Psi(s,u) \triangle_1 \Psi(s,u) d\Omega \to 0 \tag{4.11}$$

as $\sigma \to 0+$. So, from (4.8) and (4.11), we have

$$-\int_{\Omega_0} \Psi(s,u)\Delta\Psi(s,u)d\Omega - \left(\frac{\pi}{2a}\right)^2 \int_{\Omega_0} \Psi^2(s,u)d\Omega \to \int_\Sigma \mathscr{K}_2 d\Sigma < 0$$

as $\sigma \to 0+$, which implies $\sigma_0 < (\frac{\pi}{2a})^2$.

(2) If $\int_\Sigma \mathscr{K}_2 d\Sigma = 0$, construct a trial function $\Psi_{\sigma,\varepsilon} := (\varphi_\sigma(s) + \varepsilon j(q)u)\chi(u)$ with $\varphi_\sigma(s)$ defined by (4.7) and $j \in C_0^\infty\left((0,s_0)\times \mathbb{S}^2\right)$. Obviously, $\Psi_{\sigma,\varepsilon} \in DomQ_2$. For convenience, for any function $f \in DomQ_2$, let

$$Q_3[f] := -\int_{\Omega_0} f\Delta f d\Omega - \left(\frac{\pi}{2a}\right)^2 \int_{\Omega_0} f^2 d\Omega.$$

By applying Lemma 4.2, we have

$$Q_3[\Psi_{\sigma,\varepsilon}] = Q_3[\varphi_\sigma(s)\chi(u)] - 2\varepsilon\int_{\Omega_0} j(k_s + 2k_\theta)d\Omega + \varepsilon^2 Q_3[j(q)u\chi(u)]. \tag{4.12}$$

The second term on the right hand side of (4.12) can be made nonzero by choosing j supported on a compact subset of Σ_0 where $(k_s + 2k_\theta)$ does not change sign. The existence of this compact subset could be assured by Lemma 4.3(3) and the fact that we could choose s_0 arbitrarily large. So, if we choose the sign of ε in such a way that the second term on the right hand side of (4.12) is negative, then, for sufficiently small ε, the sum of the last two terms of the right hand side of (4.12) will be negative. On the other hand, by the argument in (1), we know that

$$Q_3[\varphi_\sigma(s)\chi(u)] \to \int_\Sigma \mathscr{K}_2 d\Sigma$$

as $\sigma \to 0+$. Hence, we have $Q_3[\Psi_{\sigma,\varepsilon}] < 0$ as $\sigma \to 0+$ and ε sufficiently small, which implies $\sigma_0 < (\frac{\pi}{2a})^2$.

Our proof is finished. □

So, by Theorems 4.1 and 4.7, we have

Corollary 4.8. *Theorem 1.2 is true.*

Remark 4.9. The existence of the ground state of quantum layers built over submanifolds of high dimensional Euclidean space has been obtained in [6,7] under some assumptions therein, but the parabolicity of the reference submanifold is necessary in those assumptions, however, here our 3-dimensional reference hypersurface Σ of R^4 is non-parabolic by Lemma 4.3. So, the existence of the ground state of the cylindrically symmetric quantum layers considered here can not be obtained by the results in [6,7], which indicates that Theorem 1.2 can be seen as a complement to those existent results for higher dimensional quantum layers or quantum tubes.

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