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Global branches of periodic solutions for forced delay differential equations on compact manifolds

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Abstract

We prove a global bifurcation result for T -periodic solutions of the T -periodic delay differential equation $x'(t) = \lambda f(t, x(t), x(t-1))$ depending on a real parameter $\lambda \geq 0$. The approach is based on the fixed point index theory for maps on ANRs.

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1. Introduction

Let $M \subseteq \mathbb{R}^k$ be a smooth manifold with (possibly empty) boundary, and let

$$f : \mathbb{R} \times M \times M \rightarrow \mathbb{R}^k$$

be a continuous map which is T -periodic in the first variable and tangent to M in the second one; that is

$$f(t+T, p, q) = f(t, p, q) \in T_p M, \quad \forall (t, p, q) \in \mathbb{R} \times M \times M,$$

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where $T_pM \subseteq \mathbb{R}^k$ denotes the tangent space of M at p . Consider the delay differential equation

$$x'(t) = \lambda f(t, x(t), x(t - 1)) \tag{1.1}$$

depending on a nonnegative real parameter λ . By a *T-periodic pair* of the above equation we mean a pair (λ, x) , where $\lambda \geq 0$ and $x : \mathbb{R} \rightarrow M$ is a T -periodic solution of (1.1) corresponding to λ . The set of the T -periodic pairs of (1.1) is regarded as a subset of $[0, +\infty) \times C_T(M)$, where $C_T(M)$ is the set of the continuous T -periodic maps from \mathbb{R} to M with the metric induced by the Banach space $C_T(\mathbb{R}^k)$ of the continuous T -periodic \mathbb{R}^k -valued maps (with the standard supremum norm). A T -periodic pair (λ, x) will be called *trivial* if $\lambda = 0$. In this case x is a constant M -valued map and will be identified with a point of M .

Under the assumptions that M is compact with nonzero Euler–Poincaré characteristic, that $T \geq 1$, and that f satisfies a natural inward condition along the boundary of M (when nonempty), we prove the existence of an unbounded—with respect to λ —connected branch of nontrivial T -periodic pairs whose closure intersects the set of the trivial T -periodic pairs in a nonempty set called *set of bifurcation points*. Our result extends an analogous one of the last two authors for the undelayed case (see [6] and [7]).

This unusual notion of bifurcation goes back to Ambrosetti and Prodi: in [14] they used the expression *atypical bifurcation*, also called *co-bifurcation* in [5].

We point out that the assumption $T \geq 1$ is crucial for the method used here, based on fixed point index theory for locally compact maps on ANR’s and applied to a Poincaré-type T -translation operator. In a forthcoming paper we will tackle the case $0 < T < 1$, in which the T -translation operator is not locally compact (actually, not locally condensing).

2. Preliminary results

Let M be an arbitrary subset of \mathbb{R}^k . We recall the notions of tangent cone and tangent space of M at a given point p in the closure \overline{M} of M . The definition of tangent cone is equivalent to the classical one introduced by Bouligand in [2].

Definition 2.1. A vector $v \in \mathbb{R}^k$ is said to be *inward* to M at $p \in \overline{M}$ if there exist two sequences $\{\alpha_n\}$ in $[0, +\infty)$ and $\{p_n\}$ in M such that

$$p_n \rightarrow p \quad \text{and} \quad \alpha_n(p_n - p) \rightarrow v.$$

The set C_pM of the vectors which are inward to M at p is called the *tangent cone* of M at p . The *tangent space* T_pM of M at p is the vector subspace of \mathbb{R}^k spanned by C_pM . A vector v of \mathbb{R}^k is said to be *tangent* to M at p if $v \in T_pM$.

To simplify some statements and definitions we put $C_pM = T_pM = \emptyset$ whenever $p \in \mathbb{R}^k$ does not belong to \overline{M} (this can be regarded as a consequence of Definition 2.1 if one replaces the assumption $p \in \overline{M}$ with $p \in \mathbb{R}^k$). Observe that T_pM is the trivial subspace $\{0\}$ of \mathbb{R}^k if and only if p is an isolated point of M . In fact, if p is an accumulation point, then, given any $\{p_n\}$ in $M \setminus \{p\}$ such that $p_n \rightarrow p$, the sequence $\{\alpha_n(p_n - p)\}$, with $\alpha_n = 1/\|p_n - p\|$, admits a convergent subsequence whose limit is a unit vector.

One can show that in the special and important case when M is a ∂ -manifold, i.e. a smooth manifold with (possibly empty) boundary ∂M , then T_pM has the same dimension as M for all

$p \in M$. Moreover, C_pM is a closed half-space in T_pM (delimited by $T_p\partial M$) if $p \in \partial M$, and $C_pM = T_pM$ if $p \in M \setminus \partial M$.

Let, as above, M be a subset of \mathbb{R}^k , and let $g : \mathbb{R} \times M \times M \rightarrow \mathbb{R}^k$ be a continuous map. We say that g is *tangent to M in the second variable* or, for short, that g is a *vector field on M* if $g(t, p, q) \in T_pM$ for all $(t, p, q) \in \mathbb{R} \times M \times M$. In particular, g will be said *inward* (to M) if $g(t, p, q) \in C_pM$ for all $(t, p, q) \in \mathbb{R} \times M \times M$. If M is a closed subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$, we will say that g is *away from $N \setminus M$* if $g(t, p, q) \notin C_p(N \setminus M)$ for all $(t, p, q) \in \mathbb{R} \times M \times M$.

Given a vector field $g : \mathbb{R} \times M \times M \rightarrow \mathbb{R}^k$ (on M), consider the following delay differential equation:

$$x'(t) = g(t, x(t), x(t - 1)). \tag{2.1}$$

By a *solution* of (2.1) we mean a continuous function $x : J \rightarrow M$, defined on a (possibly unbounded) real interval with length greater than 1, which is of class C^1 on the subinterval $(\inf J + 1, \sup J)$ of J and verifies $x'(t) = g(t, x(t), x(t - 1))$ for all $t \in J$ with $t > \inf J + 1$.

Given g as above and given a continuous map $\varphi : [-1, 0] \rightarrow M$, consider the following initial value problem:

$$\begin{cases} x'(t) = g(t, x(t), x(t - 1)), \\ x(t) = \varphi(t), \quad t \in [-1, 0]. \end{cases} \tag{2.2}$$

A solution of this problem is a solution $x : J \rightarrow M$ of (2.1) such that $J \supseteq [-1, 0]$ and $x(t) = \varphi(t)$ for all $t \in [-1, 0]$.

The following technical lemma regards the existence of a persistent solution of problem (2.2).

Lemma 2.2. *Let M be a compact subset of a boundaryless smooth manifold $N \subseteq \mathbb{R}^k$ and assume that g is a vector field on M which is away from $N \setminus M$. Then problem (2.2) admits a solution defined on the whole half line $[-1, +\infty)$.*

Proof. First of all, notice that we may extend g to a vector field g_1 on N . Indeed, since M is closed in N , because of the Tietze Extension Theorem, g has an \mathbb{R}^k -valued (continuous) extension to $\mathbb{R} \times N \times N$. It is sufficient to consider the component of this extension which is tangent to N in the second variable.

Now, let us use g_1 to define a suitable new extension $\tilde{g} : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ of g . Let $U \subseteq \mathbb{R}^k$ be a tubular neighborhood of N and let $r : U \rightarrow N$ be the associated retraction (if N is an open set of \mathbb{R}^k , then $U = N$ and r is the identity). Let $\sigma : \mathbb{R}^k \rightarrow [0, 1]$ be a continuous function with compact support, $\text{supp } \sigma$, contained in U and such that $\sigma(p) = 1$ if $p \in M$ (observe that U is an open neighborhood of M in \mathbb{R}^k). Define \tilde{g} by

$$\tilde{g}(t, p, q) = \begin{cases} \sigma(p)\sigma(q)g_1(t, r(p), r(q)) & \text{if } p, q \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Now, consider the following auxiliary problem depending on $n \in \mathbb{N}$:

$$\begin{cases} x'(t) = \tilde{g}(t, x(t - \frac{1}{n}), x(t - 1)), & t > 0, \\ x(t) = \varphi(t), & t \in [-1, 0]. \end{cases} \tag{2.3}$$

Clearly problem (2.3) has a solution defined on $[-1, 1/n]$ and, given a solution on $[-1, \beta]$, one can extend it to the interval $[-1, \beta + 1/n]$. Thus, problem (2.3) has a global solution $x_n : [-1, +\infty) \rightarrow \mathbb{R}^k$.

Define $\mu : [0, +\infty) \rightarrow \mathbb{R}$ by

$$\mu(t) = \max\{\|\tilde{g}(\tau, p, q)\| : \tau \in [0, t], p, q \in \text{supp } \sigma\}.$$

Notice that μ is continuous because of the compactness of $\text{supp } \sigma$. For all $n \in \mathbb{N}$ and all $t > 0$, we have $\|x'_n(t)\| \leq \mu(t)$ and, consequently,

$$\|x_n(t)\| \leq \|\varphi(0)\| + \int_0^t \mu(s) ds, \quad t \geq 0.$$

Thus, by Ascoli’s Theorem, we may assume that, as $n \rightarrow \infty$, $\{x_n(t)\}$ converges to a continuous function $x(t)$, uniformly on compact subsets of $[-1, +\infty)$. Because of this, $\{x'_n(t)\}$ converges to $\tilde{g}(t, x(t), x(t - 1))$, uniformly on compact subsets of $(0, +\infty)$. Therefore, by classical results, one gets $x'(t) = \tilde{g}(t, x(t), x(t - 1))$ for all $t > 0$. Thus, the assertion follows if we show that $x(t)$ lies entirely in M .

Let us show first that $x(t) \in N$ for all $t \geq 0$ (this could be false if \tilde{g} were an arbitrary continuous extension of g). Clearly $x(t)$ belongs (for all $t \geq 0$) to the compact subset $\text{supp } \sigma$ of the tubular neighborhood U . Thus, the C^1 function

$$\delta(t) = \|x(t) - r(x(t))\|^2$$

is well defined for $t \geq 0$ and verifies $\delta(0) = 0$. Assume, by contradiction, that $x(t) \notin N$ for some $t > 0$. This means that $\delta(t) > 0$ for some $t > 0$ and, consequently, its derivative must be positive at some $\tau > 0$. That is,

$$\delta'(\tau) = 2\langle x(\tau) - r(x(\tau)), \tilde{g}(\tau, x(\tau), x(\tau - 1)) - w(\tau) \rangle > 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in \mathbb{R}^k , and $w(\tau)$ is the derivative at $t = \tau$ of the curve $t \mapsto r(x(t))$. This is a contradiction since both the vectors $\tilde{g}(\tau, x(\tau), x(\tau - 1))$ and $w(\tau)$ are tangent to N at $r(x(\tau))$ and, consequently, orthogonal to $x(\tau) - r(x(\tau))$.

It remains to show that $x(t) \in M$ for all $t > 0$. Let $s = \inf\{t > 0 : x(t) \notin M\}$, and assume by contradiction $s < +\infty$ (here we adopt the convention $\inf \emptyset = +\infty$). Let $\{t_n\}$ be a sequence converging to s and such that $x(t_n) \in N \setminus M$. Clearly $x(s) \in M$ and $t_n > s$ for all $n \in \mathbb{N}$. We have

$$\lim_{n \rightarrow \infty} \frac{x(t_n) - x(s)}{t_n - s} = x'(s) = g(s, x(s), x(s - 1)).$$

This implies, because of the definition of tangent cone, that the vector $g(s, x(s), x(s - 1))$ belongs to $C_{x(s)}(N \setminus M)$, contradicting the fact that the vector field g is away from $N \setminus M$. \square

From now on M will be a compact ∂ -manifold in \mathbb{R}^k . In this case one may regard M as a subset of a smooth boundaryless manifold N of the same dimension as M (see e.g. [11]). It is not hard to show that a vector field g on M is away from the complement $N \setminus M$ if and only if it is *strictly inward*; meaning that g is inward and $g(t, p, q) \notin T_p \partial M$ for all $(t, p, q) \in \mathbb{R} \times \partial M \times M$.

Proposition 2.3. *Let $M \subseteq \mathbb{R}^k$ be a compact ∂ -manifold and let g be an inward vector field on M . Then, problem (2.2) admits a solution defined on the whole half line $[-1, +\infty)$.*

Proof. As already pointed out, we may regard M as a subset of a smooth boundaryless manifold N of the same dimension as M . Let $\nu : M \rightarrow \mathbb{R}^k$ be any strictly inward tangent vector field on M . For example, define $\nu(p)$ for any $p \in \partial M$ as the unique unitary vector belonging to $C_p M \cap T_p \partial M^\perp$, and then extend ν to a tangent vector field on the whole manifold M (by removing the normal component of the extension ensured by the Tietze Extension Theorem). For any $n \in \mathbb{N}$, define the strictly inward vector field $g_n : \mathbb{R} \times M \times M \rightarrow \mathbb{R}^k$ by $g_n(t, p, q) = g(t, p, q) + \nu(p)/n$, and let $x_n : [-1, +\infty) \rightarrow M$ be a solution of the initial value problem

$$\begin{cases} x'(t) = g_n(t, x(t), x(t - 1)), & t > 0, \\ x(t) = \varphi(t), & t \in [-1, 0], \end{cases}$$

whose existence is ensured by Lemma 2.2. As in the proof of Lemma 2.2, one can show that $\{x_n(t)\}$ has a subsequence which converges (uniformly on compact subsets of $[-1, +\infty)$) to a solution of problem (2.2), and we are done. \square

The following result regards uniqueness and continuous dependence on data of the solutions of problem (2.2). Its proof is standard and, therefore, will be omitted.

Proposition 2.4. *Let g be as in Proposition 2.3 and assume, moreover, that it is of class C^1 . Then, problem (2.2) admits a unique solution on $[-1, +\infty)$. Moreover, if $\{g_n\}$ is a sequence of C^1 inward vector fields on M which converges uniformly to g and $\{\varphi_n\}$ is a sequence of continuous maps from $[-1, 0]$ to M which converges uniformly to φ , then the sequence of the solutions of the initial value problems*

$$\begin{cases} x'(t) = g_n(t, x(t), x(t - 1)), & t > 0, \\ x(t) = \varphi_n(t), & t \in [-1, 0] \end{cases}$$

converges uniformly on compact subsets of $[-1, +\infty)$ to the solution of (2.2).

3. Fixed point index

This section is devoted to summarizing the main properties of the fixed point index in the context of ANRs. Let X be a metric ANR and consider a locally compact (continuous) X -valued map k defined on a subset $\mathcal{D}(k)$ of X . Given an open subset U of X contained in $\mathcal{D}(k)$, if the set of fixed points of k in U is compact, the pair (k, U) is called *admissible*. It is known that to any admissible pair (k, U) we can associate an integer $\text{ind}_X(k, U)$ —the *fixed point index* of k in U —which satisfies properties analogous to those of the classical Leray–Schauder degree [10]. The reader can see for instance [1,9,12] or [13] for a comprehensive presentation of the index theory for ANR’s. As regards the connection with the homology theory we refer to standard algebraic topology textbooks (e.g. [3,15]).

Let us summarize the main properties of the index.

- (i) (*Existence*) If $\text{ind}_X(k, U) \neq 0$, then k admits at least one fixed point in U .
- (ii) (*Normalization*) If X is compact, then $\text{ind}_X(k, X) = \Lambda(k)$, where $\Lambda(k)$ denotes the Lefschetz number of k .

- (iii) (*Additivity*) Given two open disjoint subsets U_1, U_2 of U such that any fixed point of k in U is contained in $U_1 \cup U_2$, then $\text{ind}_X(k, U) = \text{ind}_X(k, U_1) + \text{ind}_X(k, U_2)$.
- (iv) (*Excision*) Given an open subset U_1 of U such that k has no fixed point in $U \setminus U_1$, then $\text{ind}_X(k, U) = \text{ind}_X(k, U_1)$.
- (v) (*Commutativity*) Let X and Y be metric ANR's. Suppose that U and V are open subsets of X and Y respectively and that $k : U \rightarrow Y$ and $h : V \rightarrow X$ are locally compact maps. Assume that one of the sets of fixed points of hk in $k^{-1}(V)$ or kh in $h^{-1}(U)$ is compact. Then, the other set is compact as well and $\text{ind}_X(hk, k^{-1}(V)) = \text{ind}_Y(kh, h^{-1}(U))$.
- (vi) (*Generalized homotopy invariance*) Let I be a compact real interval and Ω an open subset of $X \times I$. For any $\lambda \in I$, denote $\Omega_\lambda = \{x \in X : (x, \lambda) \in \Omega\}$. Let $H : \Omega \rightarrow X$ be a locally compact map such that the set $\{(x, \lambda) \in \Omega : H(x, \lambda) = x\}$ is compact. Then $\text{ind}_X(H(\cdot, \lambda), \Omega_\lambda)$ is independent of λ .

The last property is actually a slight generalization (and a consequence) of the standard homotopy invariance which deals with maps defined on Cartesian products $U \times I$ (U open in X).

4. Branches of periodic solutions

From now on we will adopt the following notation. By M we mean a compact ∂ -manifold in \mathbb{R}^k and by $C([-1, 0], M)$ the (complete) metric space of the M -valued (continuous) functions defined on $[-1, 0]$ with the metric induced by the Banach space $C([-1, 0], \mathbb{R}^k)$. Given $T > 0$, by $C_T(\mathbb{R}^k)$ we denote the Banach space of the continuous T -periodic maps $x : \mathbb{R} \rightarrow \mathbb{R}^k$ (with the standard supremum norm) and by $C_T(M)$ we mean the metric subspace of $C_T(\mathbb{R}^k)$ of the M -valued maps.

Let $f : \mathbb{R} \times M \times M \rightarrow \mathbb{R}^k$ be an inward vector field on M which is T -periodic in the first variable. Consider the following delay differential equation depending on a parameter $\lambda \geq 0$:

$$x'(t) = \lambda f(t, x(t), x(t - 1)). \tag{4.1}$$

We will say that $(\lambda, x) \in [0, +\infty) \times C_T(M)$ is a T -periodic pair (of (4.1)) if $x : \mathbb{R} \rightarrow M$ is a T -periodic solution of (4.1) corresponding to λ . A T -periodic pair of the type $(0, x)$ is said to be *trivial*. In this case the function x is constant and will be identified with a point of M , and vice versa.

A pair $(\lambda, \varphi) \in [0, +\infty) \times C([-1, 0], M)$ will be called a T -starting pair (of (4.1)) if there exists $x \in C_T(M)$ such that $x(t) = \varphi(t)$ for all $t \in [-1, 0]$ and (λ, x) is a T -periodic pair. A T -starting pair of the type $(0, \varphi)$ will be called *trivial*. Clearly, the map $\rho : (\lambda, x) \mapsto (\lambda, \varphi)$ which associates to a T -periodic pair (λ, x) the corresponding T -starting pair (λ, φ) is continuous (φ being the restriction of x to the interval $[-1, 0]$). Moreover, if f is C^1 , from Proposition 2.4 it follows that ρ is actually a homeomorphism between the set of T -periodic pairs and the set of T -starting pairs.

Given $p \in M$, it is convenient to regard the pair $(0, p)$ both as a trivial T -periodic pair and as a trivial T -starting pair. With this in mind, notice that the restriction of the map ρ to $\{0\} \times M \subseteq [0, +\infty) \times C_T(M)$ as domain and to $\{0\} \times M \subseteq [0, +\infty) \times C([-1, 0], M)$ as codomain is the identity.

An element $p_0 \in M$ will be called a *bifurcation point* of Eq. (4.1) if every neighborhood of $(0, p_0)$ in $[0, +\infty) \times C_T(M)$ contains a nontrivial T -periodic pair (i.e. a T -periodic pair (λ, x)

with $\lambda > 0$). The following result provides a necessary condition for a point $p_0 \in M$ to be a bifurcation point.

Proposition 4.1. *Assume that $p_0 \in M$ is a bifurcation point of Eq. (4.1). Then the tangent vector field $w : M \rightarrow \mathbb{R}^k$ defined by*

$$w(p) = \frac{1}{T} \int_0^T f(t, p, p) dt$$

vanishes at p_0 .

Proof. By assumption there exists a sequence $\{(\lambda_n, x_n)\}$ of T -periodic pairs such that $\lambda_n > 0$, $\lambda_n \rightarrow 0$, and $x_n(t) \rightarrow p_0$ uniformly on \mathbb{R} . Given $n \in \mathbb{N}$, since $x_n(T) = x_n(0)$ and $\lambda_n \neq 0$, we get

$$\int_0^T f(t, x_n(t), x_n(t - 1)) dt = 0,$$

and the assertion follows passing to the limit. \square

Our main result (Theorem 4.6) provides a sufficient condition for the existence of a bifurcation point in M . More precisely, under the assumption that the Euler–Poincaré characteristic of M is nonzero, we will prove the existence of a *global bifurcating branch* for Eq. (4.1); that is, an unbounded and connected set of nontrivial T -periodic pairs whose closure intersects the set $\{0\} \times M$ of the trivial T -periodic pairs. We point out that, $C_T(M)$ being bounded, a global bifurcating branch is necessarily unbounded with respect to λ . In particular, the existence of such a branch ensures the existence of a T -periodic solution of Eq. (4.1) for each $\lambda \geq 0$.

Since M is an ANR, it is not difficult to show (see e.g. [4]) that the metric space $C([-1, 0], M)$ is an ANR as well (clearly of the same homotopy type as M). For the sake of simplicity, from now on, the metric space $C([-1, 0], M)$ will be denoted by X .

Suppose, for the moment, that f is C^1 (this assumption will be removed in Theorem 4.6). Given $\lambda \geq 0$ and $\varphi \in X$, consider in M the following delay differential (initial value) problem:

$$\begin{cases} x'(t) = \lambda f(t, x(t), x(t - 1)), & t > 0, \\ x(t) = \varphi(t), & t \in [-1, 0]. \end{cases} \tag{4.2}$$

When necessary, the unique solution of problem (4.2), ensured by Proposition 2.4, will be denoted by $x_{(\lambda, \varphi)}(\cdot)$ to emphasize the dependence on (λ, φ) . Given $\lambda \in [0, +\infty)$, consider the Poincaré-type operator

$$P_\lambda : X \rightarrow X$$

defined as $P_\lambda(\varphi)(s) = x_{(\lambda, \varphi)}(s + T)$, $s \in [-1, 0]$. The following two propositions regard some crucial properties of P_λ .

Proposition 4.2. *The fixed points of P_λ correspond to the T -periodic solutions of Eq. (4.1) in the following sense: φ is a fixed point of P_λ if and only if it is the restriction to $[-1, 0]$ of a T -periodic solution.*

Proof. (If) Obvious.

(Only if) Let $\varphi \in X$ be such that $P_\lambda(\varphi)(s) = x_{(\lambda,\varphi)}(s + T) = \varphi(s)$ for any $s \in [-1, 0]$. Define $\eta : [-1, +\infty) \rightarrow M$ by $\eta(t) = x_{(\lambda,\varphi)}(t + T)$. Then, if $t \in [-1, 0]$ we have

$$\eta(t) = x_{(\lambda,\varphi)}(t + T) = \varphi(t),$$

and, if $t > 0$,

$$\begin{aligned} \eta'(t) &= x'_{(\lambda,\varphi)}(t + T) \\ &= \lambda f(t + T, x_{(\lambda,\varphi)}(t + T), x_{(\lambda,\varphi)}(t + T - 1)) \\ &= \lambda f(t, \eta(t), \eta(t - 1)). \end{aligned}$$

That is, the function η is a solution of problem (4.2) and, because of the uniqueness of the solution, it follows that

$$x_{(\lambda,\varphi)}(t + T) = \eta(t) = x_{(\lambda,\varphi)}(t), \quad t \in [-1, +\infty).$$

Consequently, the T -periodic extension of $x_{(\lambda,\varphi)}$ to \mathbb{R} is a solution of (4.1). \square

Proposition 4.3. *The map $P : [0, +\infty) \times X \rightarrow X$, defined by $(\lambda, \varphi) \mapsto P_\lambda(\varphi)$, is continuous. Moreover, if $T \geq 1$, then P is locally compact.*

Proof. The continuity of P is a consequence of Proposition 2.4. If $T \geq 1$, the local compactness follows from Ascoli’s Theorem. \square

Let us remark that in the case when $0 < T < 1$ the operator P is still continuous but not locally compact.

If $\lambda = 0$, given $\varphi \in X$, problem (4.2) becomes

$$\begin{cases} x'(t) = 0, & t > 0, \\ x(t) = \varphi(t), & t \in [-1, 0]. \end{cases}$$

In the interval $[0, +\infty)$ the solution of this problem is the constant map $t \mapsto \varphi(0)$. Thus,

$$P_0(\varphi)(s) = \varphi(0), \quad s \in [-1, 0].$$

Hence, P_0 sends X into the subset of the constant functions (which can be identified with M), and its restriction $P_0|_M : M \rightarrow M$ coincides with the identity. By the commutativity property of the fixed point index, using the identification introduced above, we get

$$\text{ind}_X(P_0, X) = \text{ind}_M(P_0|_M, M).$$

Moreover, the normalization property of the fixed point index implies that

$$\text{ind}_M(P_0|_M, M) = \text{ind}_M(I|_M, M) = \Lambda(I|_M) = \chi(M).$$

The latter equality follows from the fact that the Lefschetz number of the identity on a compact ANR coincides with its Euler–Poincaré characteristic. Consequently,

$$\text{ind}_X(P_0, X) = \chi(M). \tag{4.3}$$

The following result (see Lemma 1.4 of [8]) will play a crucial role in the proof of Lemma 4.5 and Theorem 4.6.

Lemma 4.4. *Let K be a compact subset of a locally compact metric space Z . Assume that any compact subset of Z containing K has nonempty boundary. Then $Z \setminus K$ contains a connected set whose closure is not compact and intersects K .*

Lemma 4.5 below regards the existence of an unbounded connected branch of nontrivial T -starting pairs for Eq. (4.1) which emanates from the set of the trivial T -starting pairs. In the undelayed case, the analogue of Lemma 4.5 (see [6, Theorem 1]) is in a finite-dimensional context since, in that case, the Poincaré operator P_λ maps M into itself.

Since we identify M with the subset of X of the constant maps, from now on $\{0\} \times M$ will be regarded as a subset of $[0, +\infty) \times X$. Given a set $G \subseteq [0, +\infty) \times X$ and $\lambda \geq 0$, we will denote by G_λ the slice $\{x \in X: (\lambda, x) \in G\}$.

Lemma 4.5. *Let M be a compact ∂ -manifold with nonzero Euler–Poincaré characteristic, and let f be a C^1 inward vector field on M which is T -periodic in the first variable, with $T \geq 1$. Then, Eq. (4.1) admits a connected branch of nontrivial T -starting pairs whose closure in the set of the T -starting pairs is not compact and intersects $\{0\} \times M$.*

Proof. Let

$$S = \{(\lambda, \varphi) \in [0, +\infty) \times X: (\lambda, \varphi) \text{ is a } T\text{-starting pair of (4.1)}\}.$$

Notice that, as a consequence of Proposition 4.3, the set S is locally compact. Moreover, the slice S_0 coincides with M (regarded as the set of constant functions from $[-1, 0]$ to M).

We apply Lemma 4.4 with $\{0\} \times M$ in place of K and with S in place of Z . Assume, by contradiction, that there exists a compact set $\widehat{S} \subseteq S$ containing $\{0\} \times M$ and with empty boundary in S . Thus, \widehat{S} is also an open subset of the metric space S . Hence, there exists a bounded open subset U of $[0, +\infty) \times X$ such that $\widehat{S} = U \cap S$. Since \widehat{S} is compact, the generalized homotopy invariance property of the fixed point index implies that $\text{ind}_X(P_\lambda, U_\lambda)$ does not depend on $\lambda \in [0, +\infty)$. Moreover, the slice $\widehat{S}_\lambda = U_\lambda \cap S_\lambda$ is empty for some λ . This implies that $\text{ind}_X(P_\lambda, U_\lambda) = 0$ for any $\lambda \in [0, +\infty)$ and, in particular, $\text{ind}_X(P_0, U_0) = 0$.

Now, since U_0 is an open subset of X containing M , by the excision property of the fixed point index, taking into account equality (4.3), we get that

$$\text{ind}_X(P_0, U_0) = \text{ind}_X(P_0, X) = \chi(M) \neq 0,$$

which is a contradiction. Therefore, because of Lemma 4.4, there exists a connected subset of S whose closure in S intersects $\{0\} \times M$ and is not compact. \square

Let S denote the set of the T -starting pairs of (4.1) and let $A \subseteq S$ be a connected branch of nontrivial T -starting pairs as in the assertion of Lemma 4.5. Since the map $P : (\lambda, \varphi) \mapsto P_\lambda(\varphi)$ is continuous, S is a closed subset of $[0, +\infty) \times X$ and, consequently, the closure \bar{A} of A in S is the same as in $[0, +\infty) \times X$. Thus, \bar{A} cannot be bounded since, otherwise, it would be compact because of Ascoli’s Theorem. Moreover, since X is bounded, the set A is necessarily unbounded in λ . This implies, in particular, that, under the assumption that f is C^1 , Eq. (4.1) has a T -periodic solution for any $\lambda \geq 0$.

In Theorem 4.6 below, which deals with T -periodic pairs instead of T -starting pairs, the inward vector field f is assumed to be merely continuous. Under the assumption that the Euler–Poincaré characteristic of M is nonzero, the result asserts the existence of a global bifurcating branch of nontrivial T -periodic pairs, which, $C_T(M)$ being bounded, must be unbounded with respect to λ .

Theorem 4.6. *Let M be a compact ∂ -manifold with nonzero Euler–Poincaré characteristic, and let f be an inward vector field on M , T -periodic in the first variable, with $T \geq 1$. Then, Eq. (4.1) admits an unbounded connected set of nontrivial T -periodic pairs whose closure meets the set of the trivial T -periodic pairs.*

Proof. The proof will be divided into two steps. In the first one f is assumed to be C^1 (so that Lemma 4.5 applies) and in the second one f is merely continuous.

Step 1. Assume that f is of class C^1 . Let $\Sigma \subseteq [0, +\infty) \times C_T(M)$ denote the set of the T -periodic pairs of (4.1) and $S \subseteq [0, +\infty) \times X$ the set of the T -starting pairs (of the same equation). Let $A \subseteq S$ be a connected branch of nontrivial T -starting pairs as in the assertion of Lemma 4.5. As already pointed out, the map $\rho : \Sigma \rightarrow S$, which associates to any T -periodic pair (λ, x) the corresponding T -starting pair (λ, φ) , is a homeomorphism. Moreover, the restriction of ρ to $\{0\} \times M \subseteq \Sigma$ as domain and to $\{0\} \times M \subseteq S$ as codomain is the identity. Thus, the subset $\rho^{-1}(A)$ of Σ is connected, made up of nontrivial T -periodic pairs, its closure in Σ is not compact and meets the set $\{0\} \times M$ of the trivial T -periodic pairs. One can easily check that Σ is closed in $[0, +\infty) \times C_T(M)$ and, because of Ascoli’s Theorem, any bounded subset of Σ is relatively compact. Thus $\rho^{-1}(A)$ must be unbounded and its closure in Σ is the same as in $[0, +\infty) \times C_T(M)$.

Step 2. Suppose now that f is continuous and let, as in the previous step, Σ denote the set of the T -periodic pairs of (4.1). As already pointed out, Σ is a closed, locally compact subset of $[0, +\infty) \times C_T(M)$.

We apply Lemma 4.4 with $\{0\} \times M$ in place of K and with Σ in place of Z . Assume, by contradiction, that there exists a compact set $\widehat{\Sigma} \subseteq \Sigma$ containing $\{0\} \times M$ and with empty boundary in the metric space Σ . Thus, $\widehat{\Sigma}$ is also an open subset of Σ and, consequently, both $\widehat{\Sigma}$ and $\Sigma \setminus \widehat{\Sigma}$ are closed in $[0, +\infty) \times C_T(M)$. Hence, there exists a bounded open subset W of $[0, +\infty) \times C_T(M)$ such that $\widehat{\Sigma} \subseteq W$ and $\partial W \cap \Sigma = \emptyset$.

Let now $\{f_n\}$ be a sequence of C^1 inward vector fields on M , T -periodic in the first variable, and such that $\{f_n(t, p, q)\}$ converges to $f(t, p, q)$ uniformly on $[0, T] \times M \times M$. Given any $n \in \mathbb{N}$, let Σ_n denote the set of the T -periodic pairs of the equation

$$x'(t) = \lambda f_n(t, x(t), x(t - 1)).$$

Since W is bounded and contains $\{0\} \times M$, the previous step implies that for any $n \in \mathbb{N}$ there exists a pair $(\lambda_n, x_n) \in \Sigma_n \cap \partial W$. We may assume $\lambda_n \rightarrow \lambda_0$ and, by Ascoli’s Theorem, $x_n(t) \rightarrow x_0(t)$ uniformly. Since $\{\lambda_n f_n(t, p, q)\}$ converges to $\lambda_0 f(t, p, q)$ uniformly on $[0, T] \times M \times M$, $x_0(t)$ is a T -periodic solution of the equation

$$x'(t) = \lambda_0 f(t, x(t), x(t - 1)).$$

That is, (λ_0, x_0) is a T -periodic pair of (4.1) and, consequently, (λ_0, x_0) belongs to $\partial W \cap \Sigma$, which is a contradiction. Therefore, by Lemma 4.4 one can find a connected branch C of non-trivial T -periodic pairs of (4.1) whose closure in Σ (which is the same as in $[0, +\infty) \times C_T(M)$) intersects $\{0\} \times M$ and is not compact. Finally, C cannot be bounded since, otherwise, because of Ascoli’s Theorem, its closure would be compact. This completes the proof. \square

Observe that from Proposition 4.1 and Theorem 4.6 we can deduce the following well-known consequence of the Poincaré–Hopf Theorem: *If w is an inward tangent vector field on a compact ∂ -manifold with nonzero Euler–Poincaré characteristic, then w must vanish at some point.*

5. Examples

In this section we give three examples illustrating how our main result applies. In the first one $M \subseteq \mathbb{R}^k$ is the closure of an open ball; in the second one M is an annulus in \mathbb{R}^{2n+1} ; and in the third one M is a (two-dimensional) sphere in \mathbb{R}^3 . As before, any point $p \in M$ will be identified with the constant function which assigns p to any $t \in \mathbb{R}$. All the maps are tacitly assumed to be continuous.

Example 5.1. Let $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ be T -periodic in the first variable, with $T \geq 1$. Assume that the inner product $\langle f(t, p, q), p \rangle$ is negative for $\|p\|$ large and all $(t, q) \in \mathbb{R} \times \mathbb{R}^k$.

Let us prove that the equation

$$x'(t) = \lambda f(t, x(t), x(t - 1)) \tag{5.1}$$

admits a connected branch of T -periodic pairs $(\lambda, x) \in (0, +\infty) \times C_T(\mathbb{R}^k)$ which is unbounded with respect to λ and whose closure in $[0, +\infty) \times C_T(\mathbb{R}^k)$ contains a pair of the type $(0, p_0)$ with $p_0 \in \mathbb{R}^k$ such that $w(p_0) = 0$, where $w : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is the average wind velocity defined by

$$w(p) = \frac{1}{T} \int_0^T f(t, p, p) dt.$$

By assumption, there exists $r > 0$ such that $\langle f(t, p, q), p \rangle$ is negative for $\|p\| = r$ and all $(t, q) \in \mathbb{R} \times \mathbb{R}^k$. Let $M = \overline{B(0, r)}$, where $B(0, r)$ denotes the open ball in \mathbb{R}^k centered at 0 with radius r . Clearly, f is an inward vector field on M (it is actually strictly inward). Moreover, $\chi(M) = 1$ since M is contractible. Hence, Proposition 4.1 and Theorem 4.6 apply to Eq. (5.1).

Example 5.2. Let $k \in \mathbb{N}$ be odd and let $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ be T -periodic in the first variable, with $T \geq 1$. Assume that $f(t, p, q)$ is centrifugal for $\|p\| > 0$ small and centripetal for $\|p\|$ large.

Let us show how Theorem 4.6 applies to prove that the equation

$$x'(t) = f(t, x(t), x(t - 1))$$

has a T -periodic solution $x(t)$ satisfying the condition $x(t) \neq 0$ for all $t \in \mathbb{R}$. Incidentally, observe that the above equation admits the trivial solution since, f being continuous, as a consequence of the centrifugal hypothesis on f we must have $f(t, 0, q) = 0$ for all $(t, q) \in \mathbb{R} \times \mathbb{R}^k$.

Because of the centrifugal and centripetal assumptions, there exist $r_1, r_2 > 0$, with $r_1 < r_2$, such that for all $(t, q) \in \mathbb{R} \times \mathbb{R}^k$ the inner product $\langle f(t, p, q), p \rangle$ is positive when $\|p\| = r_1$ and negative when $\|p\| = r_2$. Let M be the annulus $\overline{B(0, r_2)} \setminus B(0, r_1)$. Clearly, f is an inward vector field on M . Moreover, $\chi(M) = 2$ since M is homotopically equivalent to the (even-dimensional) sphere S^{k-1} . Hence, Theorem 4.6 implies that, for any $\lambda \geq 0$, the equation

$$x'(t) = \lambda f(t, x(t), x(t - 1))$$

has a solution lying on the annulus M .

In the above example, the assumption that the dimension k is odd cannot be removed. In fact, if k is any even natural number, we may define a centrifugal-centripetal vector field $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ by

$$f(t, p, q) = Ap + (1 - \|p\|)p,$$

where A is the $k \times k$ matrix associated with the linear operator $(p_1, p_2, \dots, p_k) \mapsto (-p_2, p_1, \dots, -p_k, p_{k-1})$. Observe that f is an autonomous (and undelayed) vector field; therefore, given any $T > 0$, it may be regarded as T -periodic. However, all the periodic solutions of

$$x' = Ax + (1 - \|x\|)x$$

have period 2π since they are as well solutions of the linear differential equation $x' = Ax$. In fact, because of the centrifugal-centripetal property of f , they must lie in the unit sphere S^{k-1} .

Example 5.3. Consider the following system of delay differential equations:

$$\begin{cases} x'_1(t) = -x_2(t)x_3(t - 1), \\ x'_2(t) = x_1(t)x_3(t - 1) - x_3(t) \sin t, \\ x'_3(t) = x_2(t) \sin t. \end{cases}$$

Let us show that this system has a 2π -periodic solution lying on the unit sphere S^2 of \mathbb{R}^3 . Let $f : \mathbb{R} \times S^2 \times S^2 \rightarrow \mathbb{R}^3$ be defined by

$$f(t, p, q) = (-p_2q_3, p_1q_3 - p_3 \sin t, p_2 \sin t),$$

where $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ belong to S^2 . Clearly, f is an inward vector field on S^2 , since $\partial S^2 = \emptyset$ and $\langle f(t, p, q), p \rangle = 0$ for all $(t, q) \in \mathbb{R} \times S^2$. Moreover, it is 2π -periodic with respect to $t \in \mathbb{R}$. We need to prove that the equation

$$x'(t) = \lambda f(t, x(t), x(t - 1))$$

admits a 2π -periodic solution (on S^2) for $\lambda = 1$. This is a consequence of Theorem 4.6, since $\chi(S^2) = 2$.

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